TAIWANESE JOURNAL OF MATHEMATICS
Vol. 21, No. 6, pp. 1381–1411, December 2017
DOI: 10.11650/tjm/8048

Nonlinear Stability of Traveling Waves in a Monostable Epidemic Model with Delay

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Abstract. This paper is concerned with the nonlinear stability of traveling waves of a delayed monostable epidemic model with quasi-monotone condition. We prove that the traveling wave front is exponentially stable by means of the weighted-energy method and the comparison principle to perturbation in some exponentially weighted L^∞ spaces, when the difference between initial data and traveling wave front decays exponentially as $x \to -\infty$, but the initial data can be suitable large in other locations. Finally, we present two examples to support our theoretical results.

1. Introduction

In this paper, we investigate the nonlinear stability of traveling waves for a general delayed monostable epidemic model arising from the spread of an epidemic by oral-faecal transmission. This epidemic model can be described as

$$(1.1)\qquad \begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - \alpha_1 u_1(x,t) + \widetilde{g}_2(u_2(x,t-\tau_2)), \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2 \dfrac{\partial^2 u_2(x,t)}{\partial x^2} - \alpha_2 u_2(x,t) + \widetilde{g}_1(u_1(x,t-\tau_1)). \end{cases}$$

Here, d_1, d_2, α_1 and α_2 are the positive constants, $u_1(x,t)$ denotes the spatial concentration of the bacteria at the point x in the habitat $\Omega = \mathbb{R}$ and time $t \geq 0$, and $u_2(x,t)$ denotes the spatial density of the infective human population at the point x and time $t \geq 0$. In this model, $\alpha_1 u_1$ is the natural death rate of the bacterial population, $\widetilde{g}_2(u_2)$ is the contribution of the infective humans to the growth rate of the bacteria, $\alpha_2 u_2$ is the natural diminishing rate of the infective population, and $\widetilde{g}_1(u_1)$ is the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of the epidemic. Mathematically, for simplification, if

$$\widetilde{t} = \alpha_1 t,\ \widetilde{u}_1 = u_1,\ \widetilde{u}_2 = u_2,\ \widetilde{x} = \sqrt{\frac{\alpha_1}{d_1}}x,\ \widetilde{\tau}_1 = \alpha_1\tau_1,\ \widetilde{\tau}_2 = \alpha_2\tau_2,$$

Received October 13, 2016; Accepted March 27, 2017.
Communicated by Yingfei Yi.
2010 *Mathematics Subject Classification.* 35R10, 35B40, 34K30, 58D25.
Key words and phrases. stability, delayed reaction diffusion system, traveling waves, weighted energy method.
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(1.1) can be rewritten as the following system (dropping the tildes on x, t, u_1, u_2, τ_1, τ_2 for notational convenience)

$$(1.2)\qquad \begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + g_2(u_2(x,t-\tau_2)), \\ \dfrac{\partial u_2(x,t)}{\partial t} = d\dfrac{\partial^2 u_2(x,t)}{\partial x^2} - \beta u_2(x,t) + g_1(u_1(x,t-\tau_1)), \end{cases}$$

where $d = \frac{d_2}{d_1}$, $\beta = \frac{\alpha_2}{\alpha_1}$, $g_1(u) = \frac{1}{\alpha_1}\widetilde{g}_1(u)$, $g_2(u) = \frac{1}{\alpha_1}\widetilde{g}_2(u)$.

Throughout this article, we assume that (1.2) satisfies the initial conditions

$$(1.3)\qquad \begin{cases} u_1(x,s) = u_{10}(x,s), & (x,s) \in \mathbb{R} \times [-\tau_1, 0], \\ u_2(x,s) = u_{20}(x,s), & (x,s) \in \mathbb{R} \times [-\tau_2, 0]. \end{cases}$$

We make the following conditions.

(A1) $g_1 \in C^2([0,K_1],\mathbb{R})$, $g_1(0) = g_2(0) = 0$, $K_2 = \frac{g_1(K_1)}{\beta} > 0$, $g_2 \in C^2([0,K_2],\mathbb{R})$, $g_2(\frac{g_1(K_1)}{\beta}) = K_1$ and $g_2(\frac{g_1(u)}{\beta}) > u$ for $u \in (0,K_1)$, where K_1 is a positive constant;

(A2) $0 \le g_1'(u_1) \le g_1'(0)$ for $u_1 \in [0,K_1]$, $0 \le g_2'(u_2) \le g_2'(0)$ for $u_2 \in [0,K_2]$.

Wu and Hsu [25] showed the existence and qualitative features of entire solutions for the system (1.2) under the assumptions (A1) and (A2). It is easy to see that the system (1.2) has two equilibria $\boldsymbol{u}_- = (u_{1-}, u_{2-}) = (0,0)$ and $\boldsymbol{u}_+ = (u_{1+}, u_{2+}) = (K_1, K_2)$ and (A1) is a basic assumption to ensure that the system (1.2) is monostable on $[\boldsymbol{u}_-, \boldsymbol{u}_+]$.

The theory of traveling wave solutions of reaction-diffusion systems has attracted much attention due to its significant nature in biology, chemistry, epidemiology and physics. For the system (1.1), the spatial dynamics of some special cases have been extensively studied. If $\tau_1 = \tau_2 = 0$, system (1.1) is rewritten as

$$(1.4)\qquad \begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1\dfrac{\partial^2 u_1(x,t)}{\partial x^2} - \alpha_1 u_1(x,t) + \widetilde{g}_2(u_2(x,t)), \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2\dfrac{\partial^2 u_2(x,t)}{\partial x^2} - \alpha_2 u_2(x,t) + \widetilde{g}_1(u_1(x,t)). \end{cases}$$

Hsu and Yang [4] proved the existence, uniqueness, monotonicity, and asymptotic behavior of traveling wave fronts of (1.4) in the monostable case. If $\tau_2 = d_2 = 0$, $\tau_1, d_1 > 0$ and $\widetilde{g}_2(u) = \gamma u$, system (1.1) is rewritten as

$$(1.5)\qquad \begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1\dfrac{\partial^2 u_1(x,t)}{\partial x^2} - \alpha_1 u_1(x,t) + \gamma u_2(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = -\alpha_2 u_2(x,t) + \widetilde{g}_1(u_1(x,t-\tau_1)). \end{cases}$$

Zhao and Thieme [19] proved the existence of spreading speed and minimal wave speed of (1.5) in the quasi-monotone case. By constructing two auxiliary monotone integral equations, these results were then extended by Wu and Liu [26] to the non-quasi-monotone case. If $\tau_1 = 0$, the existence of monotone traveling waves and the minimal wave speed was established in [20] for system (1.5) in the monostable case. Moreover, it was proven in [21] that this minimal wave speed coincides with the asymptotic speed of spread for solutions with initial functions having compact supports. Later, Zhao and Wang [37] proved the existence of Fisher type monotone traveling waves and the minimal wave speed for system (1.5) in the monostable case via the method of upper and lower solutions, and Xu and Zhao [28, 29] proved the existence, uniqueness, and globally exponential stability of traveling wave fronts of (1.5) in the bistable case by the monotone semi-flows approach and spectrum analysis.

Among the basic problems in the theory of traveling wave solutions, the stability of traveling wave solutions is an extremely important subject. Let us draw the background on the progress of the study in this subject. By the spectral analysis, Sattinger [17] investigated a reaction diffusion system without delay and proved that the traveling wave fronts were stable under the perturbations in some exponentially weighted L^∞ spaces. Using the semigroup estimates, Kapitula [6] also studied a reaction diffusion system without delay and obtained that the traveling wave front is stable in polynomially weighted L^∞ spaces. The stability of traveling waves of a quasi-monotone reaction-diffusion bistable equation was obtained by Smith and Zhao [18] through the method of the upper-lower solutions and squeezing technique developed by Chen [1] (see also [22, 23] for this technique). For the monostable case, the study of the stability of traveling waves is not the same as the bistable case and the main difficulty is caused by the unstable equilibrium. The first study of this case was obtained by Mei et al. [16] by using weighted energy method. They studied the diffusive Nicholson's blowflies equation with delay and obtained that if the solution is sufficiently close to a traveling wave front initially, it converges exponentially to the wavefront as $t \to \infty$. By means of the weighted-energy method and the comparison principle, Lin and Mei [8] investigated Nicholson's blowflies equation with diffusion and found that the wavefront is time-asymptotically stable when the delay-time is sufficiently small and the initial perturbation around the wavefront decays to zero exponentially in space as $x \to \infty$, but it can be large in other locations. In [5], Huang et al. used the anti-weighted-energy method developed by Chern et al. [2], considered a nonlocal dispersion equation with time-delay, and proved that all non-critical traveling waves (the wave speed is greater than the minimum speed), including those oscillatory waves, are time-exponentially stable, when the initial perturbations around the waves are small. The other related results on the stability of traveling wave solutions can refer Guo and Johannes [3], Mei et al. [7,10–15], Lv and Wang [9,24], Wu et al. [27], Yang and Liu [32], Yu and Mei [33] and Zhang et al. [34–36].

Recently, Yang et al. [31] obtained that the traveling waves of (1.5) without quasi-monotonicity are exponentially stable by using weighted energy method when the initial perturbation around the traveling waves is suitably small in a weighted norm. If $d_1 = \gamma = \alpha_1 = 1$ and $\widetilde{g}_1(u) = pue^{-au}$, (1.5) is

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + u_2(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = -\alpha_2 u_2(x,t) + pu_1(x,t-\tau_1)e^{-au_1(x,t-\tau_1)}. \end{cases} \tag{1.6}$$

In [30], Yang et al. proved that the traveling wave fronts of (1.6) are exponentially stable by means of the weighted-energy method and the comparison principle when the initial perturbation around the traveling wave decays exponentially as $x \to -\infty$.

In this paper, we will further consider (1.2) with a general reaction term and prove that all noncritical traveling wave fronts with sufficiently large speeds $c \gg 1$ are globally exponentially stability by means of the weighted-energy method and the comparison principle. The shortcoming of this paper is that we do not prove any nonlinear stability result for the slower waves with $c > c_{\min}$ (c can be arbitrarily close to $c_{\min}$), where $c_{\min}$ denotes critical wave speed, and particularly, for the critical traveling waves with $c_{\min}$. We leave this problem for further research.

The rest of this paper is organized as follows. In Section 2, we establish the comparison principle of solutions to Cauchy problem (1.2) and (1.3), present the proof of global existence and uniqueness for the corresponding initial value problem in a Sobolev space, and state our stability result. In Section 3, we devote to the a priori estimates, which are the core of the paper. In Section 4, based on the a priori estimates, we shall prove our main result on the exponential stability of traveling wave fronts. In Section 5, we present two examples to support our theoretical results.

2. Preliminaries and main result

First, we introduce some notations throughout this paper. Let $C > 0$ denote a generic constant and $C_i > 0$ ($i = 1, 2, \ldots$) be a specific constant. I is an interval, typically $I = \mathbb{R}$. Denote by $L^2(I)$ the space of square integrable functions defined on I and $H^k(I)$ ($k \geq 0$) the Sobolev space of the L^2-function $f(x)$ defined on the interval I whose derivatives $\frac{\mathrm{d}^i}{\mathrm{d}x^i} f$ ($i = 1, 2, \ldots, k$) also belong to $L^2(I)$. $L^2_w(I)$ denotes the weighted L^2-space with a weight function $w(x) > 0$ and its norm is defined by

$$\|f\|_{L^2_w} = \left(\int_I w(x) \, |f(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

Let $H_w^k(I)$ be the weighted Sobolev space with the norm

$$\|f\|_{H_w^k} = \left(\sum_{i=0}^{k} \int_I w(x) \left|\frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)\right|^2 \mathrm{d}x\right)^{1/2}.$$

If $T > 0$ is a number and $\mathcal{B}$ is a Banach space, we denote by $C^0([0,T],\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous function on $[0,T]$ and by $L^2([0,T],\mathcal{B})$ the space of the $\mathcal{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of the $\mathcal{B}$-valued L^2-functions on $[0,\infty)$ are defined similarly. Denote

$$L_{ij} := \max_{u\in[0,u_{i+}]} g_i^{(j)}(u), \quad i = 1,2,\ j = 1,2,3, \quad M_1 := g_2(u_{2+}), \quad M_2 := \sqrt{\frac{d}{\beta}} g_1(u_{1+}).$$

A traveling wave solution of (1.2) is a solution with the form

$$u_1(x,t) = \phi_1(\xi), \quad u_2(x,t) = \phi_2(\xi), \quad \xi = x + ct,$$

where $c > 0$ is called the wave speed and $(\phi_1,\phi_2) \in C(\mathbb{R},\mathbb{R})$ is called the profile function. Furthermore, (ϕ_1,ϕ_2) with $c > 0$ satisfies

$$\begin{aligned} c\phi_1'(\xi) &= \phi_1''(\xi) - \phi_1(\xi) + g_2(\phi_2(\xi - c\tau_2)), \\ c\phi_2'(\xi) &= d\phi_2''(\xi) - \beta\phi_2(\xi) + g_1(\phi_1(\xi - c\tau_1)), \end{aligned} \tag{2.1}$$

and the following asymptotic boundary conditions

$$\lim_{\xi\to-\infty} (\phi_1(\xi),\phi_2(\xi)) = (u_{1-},u_{2-}), \quad \lim_{\xi\to+\infty} (\phi_1(\xi),\phi_2(\xi)) = (u_{1+},u_{2+}).$$

To obtain the existence of traveling wave solutions, we consider the characteristic function for (2.1) with respect to the equilibrium (u_{1-},u_{2-}):

$$\Delta(\lambda,c) := \Delta_1(\lambda,c) - \Delta_2(\lambda,c),$$

where

$$\Delta_1(\lambda,c) := (\lambda^2 - c\lambda - 1)(d\lambda^2 - c\lambda - \beta), \quad \Delta_2(\lambda,c) := g_1'(0)g_2'(0)e^{-c\lambda(\tau_1+\tau_2)}.$$

Denote

$$\lambda_{11} := \frac{c + \sqrt{c^2+4}}{2}, \quad \lambda_{21} := \frac{c + \sqrt{c^2+4d\beta}}{2d}, \quad \lambda^c := \min\{\lambda_{11},\lambda_{21}\}.$$

We have the following properties for the characteristic function $\Delta(\lambda,c)$.

Lemma 2.1. *Assume that* (A1)–(A2) *hold. There exists* $c_{\min} > 0$ *such that*

- *if* $c > c_{\min}$*, then the equation* $\Delta(\lambda, c) = 0$ *has three positive real roots* $\lambda_1(c)$*,* $\lambda_2(c)$*, and* $\lambda_3(c)$ *with* $0 < \lambda_1(c) < \lambda_2(c) < \lambda^c < \lambda_3(c)$ *and*

$$\Delta(\lambda, c) \begin{cases} > 0 & \text{if } \lambda \in (\lambda_1(c), \lambda_2(c)) \cup (\lambda_3(c), +\infty), \\ < 0 & \text{if } \lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), \lambda_3(c)); \end{cases}$$

- *if* $c = c_{\min}$*, then* $0 < \lambda_1(c_{\min}) = \lambda_2(c_{\min}) < \lambda^c$*.*

Proof. We first prove the following claim.

Claim. $\Delta(\lambda, c)$ has at most three distinct positive zeros for any given $c > 0$.

Suppose to the contrary that $\Delta(\lambda, c)$ would have more than three distinct positive zeros. Using the Rolle's Theorem, there exist at least three distinct $\eta_i > 0$ such that $\frac{\partial}{\partial\lambda}\Delta(\lambda, c)|_{\lambda=\eta_i} = 0$, $i = 1, 2, 3$. Since $\Delta_1(\lambda_{11}, c) = \Delta_1(\lambda_{12}, c) = \Delta_1(\lambda_{21}, c) = \Delta_1(\lambda_{22}, c) = 0$, where $\lambda_{12} := \frac{c-\sqrt{c^2+4}}{2} < 0$, $\lambda_{22} := \frac{c-\sqrt{c^2+4d\beta}}{2d} < 0$, $\frac{\partial}{\partial\lambda}\Delta_1(\lambda, c)$ has three distinct real zeros and one of these zeros is negative, say λ_1^*, λ_2^*, λ_3^* with $\lambda_1^* < 0$. Define

$$\begin{aligned} \Delta^*(\lambda, c) &:= \frac{1}{\lambda - \lambda_1^*} \frac{\partial}{\partial\lambda}\Delta(\lambda, c) \\ &= 4d(\lambda - \lambda_2^*)(\lambda - \lambda_3^*) + \frac{c(\tau_1 + \tau_2)g_1'(0)g_2'(0)e^{-c\lambda(\tau_1+\tau_2)}}{\lambda - \lambda_1^*}, \quad \forall\, \lambda \geq 0. \end{aligned}$$

It is easy to see that $\Delta^*(\lambda, c)$ has the same number of positive zeros as $\frac{\partial}{\partial\lambda}\Delta(\lambda, c)$. A direct computation yields $\frac{\partial^2}{\partial\lambda^2}\Delta^*(\lambda, c) > 0$, $\forall\, \lambda \geq 0$. We conclude that $\Delta^*(\lambda, c)$ has at most two distinct positive zeros. Hence $\frac{\partial}{\partial\lambda}\Delta(\lambda, c)$ has at most two distinct positive zeros and then $\Delta(\lambda, c)$ has at most three distinct positive zeros.

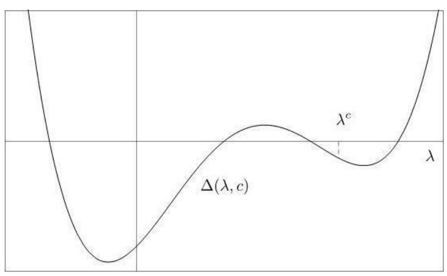

Figure 2.1: Graph of $y = \Delta(\lambda, c)$ with $c > c_{\min}$.

Clearly, $\lim_{c\to\infty} \Delta(1/\sqrt{c}, c) = +\infty$. If c is large enough, then we have $\Delta(\lambda, c)|_{\lambda=1/\sqrt{c}} > 0$ and $0 < 1/\sqrt{c} < \lambda^c$. Define

$$c_{\min} := \inf \{c > 0 : \Delta(\lambda, c) > 0 \text{ for some } \lambda \in (0, \lambda^c)\}.$$

From the assumptions (A1) and (A2), we have $\beta < g_1'(0)g_2'(0)$, and then $\Delta(\lambda, 0) < \beta - g_1'(0)g_2'(0) < 0$ for $\lambda \in (0, \lambda^c)$. Hence $c_{\min} > 0$. Assume $c > c_{\min}$. Then there exists some

$\lambda^* \in (0, \lambda^c)$ such that $\Delta(\lambda^*, c) > 0$. Since $\Delta(\infty, c) = +\infty$, $\Delta(0, c) < 0$, and $\Delta(\lambda^c, c) < 0$, we conclude that $\Delta(\lambda, c)$ has at least three distinct positive zeros and at least two of these zeros distribute in $(0, \lambda^c)$ and that at least one of these zeros distributes in $(\lambda^c, +\infty)$ for any $c > c_{\min}$ (see Figure 2.1). Then, from the Claim, we know that $\Delta(\lambda, c)$ has exactly three distinct positive zeros, two of these zeros distribute in $(0, \lambda^c)$, and one of these zeros distributes in $(\lambda^c, +\infty)$, say $\lambda_1(c) < \lambda_2(c) < \lambda_3(c)$. Clearly, $\lambda_1(c) < \lambda^* < \lambda_2(c) < \lambda^c < \lambda_3(c)$. Since $\Delta(\lambda^*, c) > 0$ and $\Delta(\infty, c) = +\infty$, we get $\Delta(\lambda, c) > 0$ for $\lambda \in (\lambda_1(c), \lambda_2(c)) \cup (\lambda_3(c), +\infty)$. Obviously, $\lambda_1(c) = \lambda_2(c)$ when $c = c_{\min}$. □

Let

$$\overline{\Phi}(\xi) = (\overline{\phi}_1(\xi), \overline{\phi}_2(\xi)) = \left(\min\left\{u_{1+}, e^{\lambda_1(c)\xi}\right\}, \min\left\{u_{2+}, \Lambda(c)e^{\lambda_1(c)\xi}\right\}\right),$$
$$\underline{\Phi}(\xi) = \left(\underline{\phi}_1(\xi), \underline{\phi}_2(\xi)\right) = \left(\max\left\{0, e^{\lambda_1(c)\xi} - pe^{\varrho\lambda_1(c)\xi}\right\}, \max\left\{0, \Lambda(c)e^{\lambda_1(c)\xi} - p\Gamma(c,\varrho)e^{\varrho\lambda_1(c)\xi}\right\}\right),$$

where $\varrho \in (1, \min\{2, \lambda_2(c)/\lambda_1(c)\})$, $p > \max\{1, \Lambda(c)/\Gamma(c,\varrho)\}$ is large enough, $\Lambda(c) = g_1'(0)e^{-\lambda_1(c)\tau_1}/[c\lambda_1(c) - d\lambda_1^2(c) + \beta]$ and

$$\frac{g_1'(0)e^{-\lambda_1(c)\varrho\tau_1}}{c(\varrho\lambda_1(c)) - d(\varrho\lambda_1(c))^2 + \beta} < \Gamma(c,\varrho) < \frac{c(\varrho\lambda_1(c)) - (\varrho\lambda_1(c))^2 + 1}{g_2'(0)e^{-\lambda_1(c)\varrho\tau_2}}.$$

It is easy to verify that $\overline{\Phi}(\xi)$ and $\underline{\Phi}(\xi)$ are a pair of upper and lower solutions of system (2.1). By using the upper-lower solutions and Schauder's fixed point theorem, Wu and Hsu [25] proved the existence of traveling wave fronts of system (1.2).

Theorem 2.2 (Existence of traveling wave fronts)**.** *Assume that* (A1)–(A2) *hold. Then for any* $c \geq c_{\min}$, *the system* (1.2) *has a traveling front* $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$, $\xi = x + ct$, *which satisfies* $\Phi(-\infty) = \boldsymbol{u}_-$, $\Phi(+\infty) = \boldsymbol{u}_+$. *Furthermore, if* $c > c_{\min}$, *then*

$$\lim_{\xi\to-\infty} \phi_1(\xi)e^{-\lambda_1(c)\xi} = 1, \quad \lim_{\xi\to-\infty} \phi_2(\xi)e^{-\lambda_1(c)\xi} = b(c),$$
$$\phi_1(\xi) \leq e^{\lambda_1(c)\xi}, \quad \phi_2(\xi) \leq b(c)e^{\lambda_1(c)\xi}, \quad \xi \in \mathbb{R}.$$

Lemma 2.3. *Assume that* (A1)–(A2) *hold. Then for a given traveling wave front* $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$, $\xi = x + ct$, *of system* (1.2) *with* $c > c_{\min}$, *there holds*

$$|\phi_1'(\xi)| \leq g_2(u_{2+}), \quad |\phi_2'(\xi)| \leq \sqrt{\frac{d}{\beta}}g_1(u_{1+}).$$

Proof. It is easy to see that

$$\phi_1(\xi) = \frac{1}{\lambda_{11} - \lambda_{12}}\left[\int_{-\infty}^{\xi} e^{\lambda_{12}(\xi-s)}g_2(\phi_2(s - c\tau_2))\,\mathrm{d}s + \int_{\xi}^{+\infty} e^{\lambda_{11}(\xi-s)}g_2(\phi_2(s - c\tau_2))\,\mathrm{d}s\right],$$

where $\lambda_{12} = (c - \sqrt{c^2+4})/2$. Then

$$\phi_1'(\xi) = \frac{1}{\lambda_{11}-\lambda_{12}} \left[\lambda_{12} \int_{-\infty}^{\xi} e^{\lambda_{12}(\xi-s)} g_2(\phi_2(s-c\tau_2))\,\mathrm{d}s + \lambda_{11} \int_{\xi}^{+\infty} e^{\lambda_{11}(\xi-s)} g_2(\phi_2(s-c\tau_2))\,\mathrm{d}s \right].$$

By the fact that $g_2(\phi_2(\xi - c\tau_2)) \le g_2(u_{2+})$ and $\lambda_{11} - \lambda_{12} \ge 2$, we get $|\phi_1'(\xi)| \le g_2(u_{2+})$.

Similarly,

$$\phi_2(\xi) = \frac{1}{\lambda_{21}-\lambda_{22}} \left[\int_{-\infty}^{\xi} e^{\lambda_{22}(\xi-s)} g_1(\phi_1(s-c\tau_1))\,\mathrm{d}s + \int_{\xi}^{+\infty} e^{\lambda_{21}(\xi-s)} g_1(\phi_1(s-c\tau_1))\,\mathrm{d}s \right],$$

where $\lambda_{22} = (c - \sqrt{c^2+4d\beta})/(2d)$. Then

$$\phi_2'(\xi) = \frac{1}{\lambda_{21}-\lambda_{22}} \left[\lambda_{22} \int_{-\infty}^{\xi} e^{\lambda_{22}(\xi-s)} g_1(\phi_1(s-c\tau_1))\,\mathrm{d}s + \lambda_{21} \int_{\xi}^{+\infty} e^{\lambda_{21}(\xi-s)} g_1(\phi_1(s-c\tau_1))\,\mathrm{d}s \right].$$

Since $g_1(\phi_1(\xi - c\tau_1)) \le g_1(u_{1+})$ and $\lambda_{21} - \lambda_{22} \ge 2\sqrt{\beta/d}$, we obtain $|\phi_2'(\xi)| \le \sqrt{d/\beta}\, g_1(u_{1+})$. $\square$

Lemma 2.4. *Assume that* (A1)–(A2) *hold and* $g_1'(u_{1+}) + g_2'(u_{2+}) < \min\{2, 2\beta\}$. *Then there exists* $\xi_0 > 0$ *such that for each* $\xi \ge \xi_0$,

$$g_i'(\phi_i(\xi)) < g_i'(u_{i+}) + \epsilon, \quad g_i'(\phi_i(\xi - c\tau_i)) < g_i'(u_{i+}) + \epsilon, \quad i = 1, 2,$$

where $\epsilon < \min\{(2 - g_1'(u_{1+}) - g_2'(u_{2+}))/4, (2\beta - g_1'(u_{1+}) - g_2'(u_{2+}))/4\}$.

The proof is easy, so we omit it.

Lemma 2.5 (Boundedness)**.** *Assume that* (A1)–(A2) *hold. If* $u_{i-} \le u_{i0}(x,s) \le u_{i+}$, $(x,s) \in \mathbb{R} \times [-\tau_i, 0]$, *then the solution* $(u_1(x,t), u_2(x,t))$ *of Cauchy problem* (1.2) *and* (1.3) *satisfies*

$$u_{i-} \le u_i(x,t) \le u_{i+}, \quad (x,t) \in \mathbb{R} \times [0, +\infty).$$

Proof. We first prove $u_1(x,t) \ge 0$. For $t \in [0, \tau_2]$, then $u_2(x, t-\tau_2) = u_{20}(x, t-\tau_2)$ and $g_2(u_{20}(x, t-\tau_2)) \ge 0$. Notice that $u_1(x,t)$ satisfies the following equation

$$\frac{\partial u_1(x,t)}{\partial t} - \frac{\partial^2 u_1(x,t)}{\partial x^2} + u_1(x,t) = g_2(u_{20}(x, t-\tau_2)) \ge 0, \quad (x,t) \in \mathbb{R} \times [0, \tau_2].$$

Applying the comparison principle for parabolic equations, we have $u_1(x,t) \ge 0$ on $\mathbb{R} \times [0, \tau_2]$. Repeating this procedure to each interval $[k\tau_2, (k+1)\tau_2]$, $k = 1, 2, \ldots$, we get $u_1(x,t) \ge 0$ on $\mathbb{R} \times [0, +\infty)$.

Next we prove $u_1(x,t) \le u_{1+}$. It is easy to see that

$$\frac{\partial[u_{1+} - u_1(x,t)]}{\partial t} = \frac{\partial^2[u_{1+} - u_1(x,t)]}{\partial x^2} - [u_{1+} - u_1(x,t)] + g_2(u_{2+}) - g_2(u_2(x,t-\tau_2)).$$

As $t \in [0,\tau_2]$, then $u_{2-} \le u_2(x,t-\tau_2) = u_{20}(x,t-\tau_2) \le u_{2+}$. By the monotonicity of $g_2(u)$, we have $g_2(u_{2+}) - g_2(u_2(x,t-\tau_2)) \ge 0$. Thus $u_1(x,t)$ satisfies the following differential equation

$$\frac{\partial[u_{1+} - u_1(x,t)]}{\partial t} - \frac{\partial^2[u_{1+} - u_1(x,t)]}{\partial x^2} + [u_{1+} - u_1(x,t)] = g_2(u_{2+}) - g_2(u_2(x,t-\tau_2)) \ge 0.$$

Applying the comparison principle for parabolic equations again, we have $u_1(x,t) \le u_{1+}$ on $\mathbb{R} \times [0,\tau_2]$. Repeating this procedure to each interval $[k\tau_2,(k+1)\tau_2]$, $k = 1,2,\ldots$, we get $u_1(x,t) \le u_{1+}$ on $\mathbb{R} \times [0,+\infty)$.

Similarly, we can prove $u_{2-} \le u_2(x,t) \le u_{2+}$, $(x,t) \in \mathbb{R} \times [0,+\infty)$. □

Lemma 2.6 (Comparison principle)**.** *Assume that* (A1)–(A2) *hold. Let* $\boldsymbol{u}^+ = (u_1^+(x,t), u_2^+(x,t))$ *and* $\boldsymbol{u}^- = (u_1^-(x,t), u_2^-(x,t))$ *be the solution of system* (1.2) *with the initial data* $(u_{10}^+(x,s), u_{20}^+(x,s))$ *and* $(u_{10}^-(x,s), u_{20}^-(x,s))$ *respectively. If* $u_{i-} \le u_i^-(x,s) \le u_i^+(x,s) \le u_{i+}$, $(x,s) \in \mathbb{R} \times [-\tau_i, 0]$, *then*

$$u_{i-} \le u_i^-(x,t) \le u_i^+(x,t) \le u_{i+}, \quad (x,t) \in \mathbb{R} \times [0,+\infty).$$

Proof. From Lemma 2.5, it suffices to prove $u_i^-(x,t) \le u_i^+(x,t)$. We only prove $u_1^-(x,t) \le u_1^+(x,t)$ and the proof for $u_2^-(x,t) \le u_2^+(x,t)$ is similar. If we let $u_1(x,t) = u_1^+(x,t) - u_1^-(x,t)$ and $u_{10}(x,s) = u_{10}^+(x,s) - u_{10}^-(x,s)$, then $u_1(x,t)$ satisfies

$$\frac{\partial u_1(x,t)}{\partial t} - \frac{\partial^2 u_1(x,t)}{\partial x^2} + u_1(x,t) = g_2(u_2^+(x,t-\tau_2)) - g_2(u_2^-(x,t-\tau_2)).$$

Since $u_{2-} \le u_2^-(x,t-\tau_2) \le u_2^+(x,t-\tau_2) \le u_{2+}$ for $(x,t) \in \mathbb{R} \times [0,\tau_2]$, by the monotonicity of $g_2(u)$, we have $g_2(u_2^-(x,t-\tau_2)) \le g_2(u_2^+(x,t-\tau_2))$. Therefore $u_1(x,t)$ satisfies the following differential equation

$$\frac{\partial u_1(x,t)}{\partial t} - \frac{\partial^2 u_1(x,t)}{\partial x^2} + u_1(x,t) = g_2(u_2^+(x,t-\tau_2)) - g_2(u_2^-(x,t-\tau_2)) \ge 0.$$

Applying the comparison principle for parabolic equations, we have $u_1(x,t) \ge 0$ on $\mathbb{R} \times [0,\tau_2]$. Repeating this procedure to each interval $[k\tau_2,(k+1)\tau_2]$, $k = 1,2,\ldots$, we get $u_1(x,t) \ge 0$ on $\mathbb{R} \times [0,+\infty)$, which implies $u_1^-(x,t) \le u_1^+(x,t)$. □

Next, we present the global existence and uniqueness result of solution $(u_1(x,t), u_2(x,t))$ to Cauchy problem (1.2) and (1.3).

Theorem 2.7 (Global Existence and Uniqueness)**.** *Assume that the initial data satisfy* $u_{i-} \le u_{i0}(x,s) \le u_{i+}$ *and are continuous for any* $(x,s) \in \mathbb{R} \times [-\tau_i, 0]$*. For a given traveling wave front* $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$*,* $\xi = x + ct$*, of the system* (1.2)*, if the initial perturbation* $u_{i0}(x,s) - \phi_i(x+cs) \in C([-\tau_i, 0], H^1(\mathbb{R}))$*, then there exists a unique global solution* $(u_1(x,t), u_2(x,t))$ *of Cauchy problem* (1.2) *and* (1.3) *such that* $u_i(x,t) - \phi_i(x + ct) \in C([0,+\infty), H^1(\mathbb{R}))$*.*

Let $V_i(x,t) = u_i(x,t) - \phi_i(x+ct)$, where $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$, $\xi = x+ct$, is a given traveling wave front of system (1.2). Then Cauchy problem (1.2) and (1.3) can be rewritten as

$$(2.2) \qquad \begin{cases} \dfrac{\partial V_1(x,t)}{\partial t} = \dfrac{\partial^2 V_1(x,t)}{\partial x^2} - V_1(x,t) + G_2(x, t-\tau_2), \\ \dfrac{\partial V_2(x,t)}{\partial t} = d\dfrac{\partial^2 V_2(x,t)}{\partial x^2} - \beta V_2(x,t) + G_1(x, t-\tau_1), \end{cases}$$

with

$$(2.3) \qquad \begin{aligned} V_1(x,s) &= u_{10}(x,s) - \phi_1(x+cs) = V_{10}(x,s), \quad (x,s) \in \mathbb{R} \times [-\tau_1, 0], \\ V_2(x,s) &= u_{20}(x,s) - \phi_2(x+cs) = V_{20}(x,s), \quad (x,s) \in \mathbb{R} \times [-\tau_2, 0], \end{aligned}$$

where

$$\begin{aligned} G_1(x,t-\tau_1) &= g_1(V_1(x,t-\tau_1) + \phi_1(x+ct-c\tau_1)) - g_1(\phi_1(x+ct-c\tau_1)), \\ G_2(x,t-\tau_2) &= g_2(V_2(x,t-\tau_2) + \phi_2(x+ct-c\tau_2)) - g_2(\phi_2(x+ct-c\tau_2)). \end{aligned}$$

We give the following two lemmas on the local existence, uniqueness, extension of solutions and boundedness of solutions of (2.2) and (2.3), which will imply Theorem 2.7.

Lemma 2.8 (Local Existence and Uniqueness)**.** *For* $V_{i0}(x,s) \in C([-\tau_i, 0], H^1(\mathbb{R}))$*, there exists* $t_0 > 0$ *such that Cauchy problem* (2.2) *and* (2.3) *has a unique solution* $(V_1(x,t), V_2(x,t)) \in C([0,t_0), (H^1(\mathbb{R}))^2)$*. Furthermore, if* $[0, T_{\max})$ *is its maximal interval of existence and* $(V_1(x,t), V_2(x,t)) \in C([0,T_{\max}), (H^1(\mathbb{R}))^2)$*, then either* $T_{\max} = +\infty$ *or the solution blows up in finite time, i.e.,* $T_{\max} < +\infty$ *and*

$$\lim_{t \to T_{\max}^-} \|V_i(\,\cdot\,,t)\|_{H^1(\mathbb{R})} = +\infty.$$

It can be proved by using the standard iteration method (see [7,15,16]), so we omit it.

Lemma 2.9 (Boundedness)**.** *If* $(V_1(x,t), V_2(x,t)) \in C([0,T), (H^1(\mathbb{R}))^2)$ *is a solution of Cauchy problem* (2.2) *and* (2.3)*, where* $0 < T < +\infty$*, then there exist positive constants* A *and* B*, independent of* T*, such that*

$$(2.4) \qquad \|V_i(t)\|^2_{H^1(\mathbb{R})} \le A\left(\sum_{i=1}^{2} \|V_{i0}(0)\|^2_{H^1(\mathbb{R})} + \sum_{i=1}^{2} \int_{-\tau_i}^{0} \|V_{i0}(s)\|^2_{H^1(\mathbb{R})}\, \mathrm{d}s\right) e^{Bt}$$

for all $t \in [0, T)$, *where*

$$A = \max\{1, L_{11} + L_{12}M_1, L_{21} + L_{22}M_2\}, \quad B = L_{11} + L_{21} + L_{22}(M_1 + M_2).$$

Proof. Multiplying the first and the second equations of system (2.2) by $2V_1(x,t)$ and $2V_2(x,t)$, respectively, we have

$$(V_1^2)_t - 2(V_1V_{1x})_x + 2V_{1x}^2 + 2V_1^2 = 2G_2(x, t-\tau_2)V_1(x,t), \tag{2.5}$$

$$(V_2^2)_t - 2d(V_2V_{2x})_x + 2dV_{2x}^2 + 2\beta V_2^2 = 2G_1(x, t-\tau_1)V_2(x,t). \tag{2.6}$$

Integrating (2.5) and (2.6) over $\mathbb{R} \times [0,t]$ with respect to ξ and t, respectively, we get

$$\begin{aligned}&\|V_1(t)\|_{L^2(\mathbb{R})}^2 + 2\int_0^t \|V_{1x}(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s + 2\int_0^t \|V_1(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\ &= \|V_{10}(0)\|_{L^2(\mathbb{R})}^2 + 2\int_0^t\int_{\mathbb{R}} G_2(x, s-\tau_2)V_1(x,s)\,\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\|V_2(t)\|_{L^2(\mathbb{R})}^2 + 2d\int_0^t \|V_{2x}(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s + 2\beta\int_0^t \|V_2(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\ &= \|V_{20}(0)\|_{L^2(\mathbb{R})}^2 + 2\int_0^t\int_{\mathbb{R}} G_1(x, s-\tau_1)V_2(x,s)\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{2.8}$$

By the mean-value theorem, we get

$$|G_1(x, s-\tau_1)| \le L_{11}|V_1(x, s-\tau_1)|, \quad |G_2(x, s-\tau_2)| \le L_{21}|V_2(x, s-\tau_2)|. \tag{2.9}$$

From (2.7)–(2.9) and the Cauchy inequality, we obtain

$$\begin{aligned}&\sum_{i=1}^2 \|V_i(t)\|_{L^2(\mathbb{R})}^2 + 2\int_0^t \|V_{1x}(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s + 2d\int_0^t \|V_{2x}(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\ &\quad + 2\int_0^t \|V_1(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s + 2\beta\int_0^t \|V_2(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\ &= \sum_{i=1}^2 \|V_{i0}(0)\|_{L^2(\mathbb{R})}^2 + 2\int_0^t\int_{\mathbb{R}} G_2(x, s-\tau_2)V_1(x,s)\,\mathrm{d}x\mathrm{d}s \\ &\quad + 2\int_0^t\int_{\mathbb{R}} G_1(x, s-\tau_1)V_2(x,s)\,\mathrm{d}x\mathrm{d}s \\ &\le \sum_{i=1}^2 \|V_{i0}(0)\|_{L^2(\mathbb{R})}^2 + \sum_{i=1}^2 L_{i1}\int_{-\tau_i}^0 \|V_{i0}(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\ &\quad + (L_{11} + L_{21})\sum_{i=1}^2 \int_0^t \|V_i(s)\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s.\end{aligned} \tag{2.10}$$

Similarly, by the mean-value theorem, we have

$$|G_{1x}(x, s-\tau_1)| \le L_{11}|V_{1x}(x, s-\tau_1)| + L_{12}M_1|V_1(x, s-\tau_1)|, \tag{2.11}$$

$$|G_{2x}(x, s-\tau_2)| \le L_{21}|V_{2x}(x, s-\tau_2)| + L_{22}M_2|V_2(x, s-\tau_2)|. \tag{2.12}$$

Differentiating the first and second equations of system (2.2) with respect to x and multiplying them by $2V_{1x}(x,t)$ and $2V_{2x}(x,t)$, respectively, we have

$$(V_{1x}^2)_t - 2(V_{1x}V_{1xx})_x + 2V_{1xx}^2 + 2V_{1x}^2 = 2G_{2x}(x,t-\tau_2)V_{1x}(x,t), \tag{2.13}$$

$$(V_{2x}^2)_t - 2d(V_{2x}V_{2xx})_x + 2dV_{2xx}^2 + 2\beta V_{2x}^2 = 2G_{1x}(x,t-\tau_1)V_{2x}(x,t). \tag{2.14}$$

Integrating (2.13) and (2.14) over $\mathbb{R}\times[0,t]$ with respect to x and t, from (2.11), (2.12), and the Cauchy inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^2 \|V_{ix}(t)\|_{L^2(\mathbb{R})}^2 + 2\int_0^t \|V_{1xx}(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s + 2d\int_0^t \|V_{2xx}(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s \\
&\quad + 2\int_0^t \|V_{1x}(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s + 2\beta\int_0^t \|V_{2x}(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s \\
&= \sum_{i=1}^2 \|V_{ix0}(0)\|_{L^2(\mathbb{R})}^2 + 2\int_0^t\int_{\mathbb{R}} G_{2x}(x,s-\tau_2)V_{1x}(x,s)\,\mathrm{d}x\mathrm{d}s \\
&\quad + 2\int_0^t\int_{\mathbb{R}} G_{1x}(x,s-\tau_1)V_{2x}(x,s)\,\mathrm{d}x\mathrm{d}s \\
&\le \sum_{i=1}^2 \|V_{ix0}(0)\|_{L^2(\mathbb{R})}^2 + \sum_{i=1}^2 L_{i2}M_i\int_{-\tau_i}^0 \|V_{i0}(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s + \sum_{i=1}^2 L_{i1}\int_{-\tau_i}^0 \|V_{ix0}(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s \\
&\quad + (L_{11}+L_{21}+L_{12}M_1+L_{22}M_2)\sum_{i=1}^2\int_0^t \|V_{ix}(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s \\
&\quad + \sum_{i=1}^2 L_{i2}M_i\int_0^t \|V_i(s)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s.
\end{aligned} \tag{2.15}$$

From (2.10) and (2.15), we have

$$\begin{aligned}
\sum_{i=1}^2 \|V_i(t)\|_{H^1(\mathbb{R})}^2 &\le \sum_{i=1}^2 \|V_{i0}(0)\|_{H^1(\mathbb{R})}^2 + \sum_{i=1}^2 (L_{i1}+L_{i2}M_i)\int_{-\tau_i}^0 \|V_{i0}(s)\|_{H^1(\mathbb{R})}^2\,\mathrm{d}s \\
&\quad + (L_{11}+L_{21}+L_{12}M_1+L_{22}M_2)\sum_{i=1}^2\int_0^t \|V_i(s)\|_{H^1(\mathbb{R})}^2\,\mathrm{d}s \\
&\le A\left(\sum_{i=1}^2 \|V_{i0}(0)\|_{H^1(\mathbb{R})}^2 + \sum_{i=1}^2\int_{-\tau_i}^0 \|V_{i0}(s)\|_{H^1(\mathbb{R})}^2\,\mathrm{d}s\right) \\
&\quad + B\int_0^t \sum_{i=1}^2 \|V_i(s)\|_{H^1(\mathbb{R})}^2\,\mathrm{d}s,
\end{aligned}$$

where

$$A = \max\{1, L_{11}+L_{12}M_1, L_{21}+L_{22}M_2\}, \quad B = L_{11}+L_{21}+L_{12}M_1+L_{22}M_2.$$

From the Gronwall's inequality, we obtain

$$\sum_{i=1}^{2}\|V_i(t)\|_{H^1(\mathbb{R})}^2 \le A\left(\sum_{i=1}^{2}\|V_{i0}(0)\|_{H^1(\mathbb{R})}^2 + \sum_{i=1}^{2}\int_{-\tau_i}^{0}\|V_{i0}(s)\|_{H^1(\mathbb{R})}^2\,\mathrm{d}s\right)e^{Bt},$$

i.e.,

$$\|V_i(t)\|_{H^1(\mathbb{R})}^2 \le A\left(\sum_{i=1}^{2}\|V_{i0}(0)\|_{H^1(\mathbb{R})}^2 + \sum_{i=1}^{2}\int_{-\tau_i}^{0}\|V_{i0}(s)\|_{H^1(\mathbb{R})}^2\,\mathrm{d}s\right)e^{Bt}. \qquad \square$$

Thus, Theorem 2.7 follows immediately from Lemmas 2.8 and 2.9.

In order to state our stability result, we make some technique assumptions.

(A3) $g_1''(u_1) \le 0$ for $u_1 \in [0, K_1]$, $g_2''(u_2) \le 0$ for $u_2 \in [0, K_2]$;

(A4) $g_1'(u_{1+}) + g_2'(u_{2+}) < \min\{2, 2\beta\}$;

(A5) $g_1 \in C^3([0, u_{1+}], \mathbb{R}_+)$ and $g_2 \in C^3([0, u_{2+}], \mathbb{R}_+)$.

We define a weight function as

$$w(\xi) = \begin{cases} e^{-\gamma(\xi-\xi_0)} & \text{if } \xi \le \xi_0, \\ 1 & \text{if } \xi > \xi_0, \end{cases} \tag{2.16}$$

where $\gamma = c/(d+1)$ and ξ_0 is defined as in Lemma 2.4.

Now, we present the corresponding stability theorem for Cauchy problem (1.2) and (1.3) as follows.

Theorem 2.10 (Stability). *Assume that* (A1)–(A5) *hold. For any given traveling wave front* $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$ $(\xi = x + ct)$ *of system* (1.2) *with the wave speed*

$$c > \max\left\{c_{\min}, (d+1)\sqrt{|g_1'(0) + g_2'(0) - 2\beta|}, \sqrt{\frac{(d+1)^2}{d}\,|g_1'(0) + g_2'(0) - 2|}\right\},$$

if the initial data satisfies

$$u_{1-} \le u_{10}(x,s) \le u_{1+}, \quad (x,s) \in \mathbb{R}\times[-\tau_1, 0],$$
$$u_{2-} \le u_{20}(x,s) \le u_{2+}, \quad (x,s) \in \mathbb{R}\times[-\tau_2, 0],$$

and the initial perturbations satisfy

$$u_{10}(x,s) - \phi_1(x+cs) \in C^0([-\tau_1, 0], H_w^1(\mathbb{R})), \quad (x,s) \in \mathbb{R}\times[-\tau_1, 0],$$
$$u_{20}(x,s) - \phi_2(x+cs) \in C^0([-\tau_2, 0], H_w^1(\mathbb{R})), \quad (x,s) \in \mathbb{R}\times[-\tau_2, 0],$$

where the weight function $w(\xi)$ is defined by (2.16), then the solution $(u_1(x,t), u_2(x,t))$ of Cauchy problem (1.2) and (1.3) satisfies

$$u_{i-} \le u_i(x,t) \le u_{i+}, \quad (x,t) \in \mathbb{R} \times [0,+\infty),\ i = 1,2,$$
$$u_i(x,t) - \phi_i(x+ct) \in C^0([0,+\infty), H^1_w(\mathbb{R})) \cap L^2([0,+\infty), H^2_w(\mathbb{R})), \quad i = 1,2$$

for some positive constant μ and C.

Furthermore, the solution $(u_1(x,t), u_2(x,t))$ converges to the traveling wave front $(\phi_1(x+ct), \phi_2(x+ct))$ exponentially in time t, i.e.,

$$\sup_{x\in\mathbb{R}} |u_i(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0,\ i = 1,2. \tag{2.17}$$

3. A priori estimates

In this section, we are going to establish the a priori estimates, by means of the weighted-energy method and the comparison principle.

Let the initial data $(u_{10}(x,s), u_{20}(x,s))$ satisfy

$$u_{1-} \le u_{10}(x,s) \le u_{1+}, \qquad (x,s) \in \mathbb{R} \times [-\tau_1, 0],$$
$$u_{2-} \le u_{20}(x,s) \le u_{2+}, \qquad (x,s) \in \mathbb{R} \times [-\tau_2, 0].$$

Define

$$\begin{aligned}
U^+_{10}(x,s) &= \max\{u_{10}(x,s), \phi_1(x+cs)\}, \qquad (x,s) \in \mathbb{R} \times [-\tau_1, 0],\\
U^-_{10}(x,s) &= \min\{u_{10}(x,s), \phi_1(x+cs)\}, \qquad (x,s) \in \mathbb{R} \times [-\tau_1, 0],\\
U^+_{20}(x,s) &= \max\{u_{20}(x,s), \phi_2(x+cs)\}, \qquad (x,s) \in \mathbb{R} \times [-\tau_2, 0],\\
U^-_{20}(x,s) &= \min\{u_{20}(x,s), \phi_2(x+cs)\}, \qquad (x,s) \in \mathbb{R} \times [-\tau_2, 0].
\end{aligned}$$

Obviously,

$$\begin{aligned}
&u_{1-} \le U^-_{10}(x,s) \le u_{10}(x,s) \le U^+_{10}(x,s) \le u_{1+}, \qquad (x,s) \in \mathbb{R} \times [-\tau_1, 0],\\
&u_{1-} \le U^-_{10}(x,s) \le \phi_1(x+cs) \le U^+_{10}(x,s) \le u_{1+}, \qquad (x,s) \in \mathbb{R} \times [-\tau_1, 0], \qquad (3.1)\\
&u_{2-} \le U^-_{20}(x,s) \le u_{20}(x,s) \le U^+_{20}(x,s) \le u_{2+}, \qquad (x,s) \in \mathbb{R} \times [-\tau_2, 0],\\
&u_{2-} \le U^-_{20}(x,s) \le \phi_2(x+cs) \le U^+_{20}(x,s) \le u_{2+}, \qquad (x,s) \in \mathbb{R} \times [-\tau_2, 0]. \qquad (3.2)
\end{aligned}$$

Let $(U^+_1(x,t), U^+_2(x,t))$ and $(U^-_1(x,t), U^-_2(x,t))$ be the solutions of (1.2) and (1.3) with the initial data $(U^+_{10}(x,s), U^+_{20}(x,s))$ and $(U^-_{10}(x,s), U^-_{20}(x,s))$, i.e.,

$$\begin{cases}
\dfrac{\partial U^\pm_1(x,t)}{\partial t} = \dfrac{\partial^2 U^\pm_1(x,t)}{\partial x^2} - U^\pm_1(x,t) + g_2(U^\pm_2(x,t-\tau_2)),\\
\dfrac{\partial U^\pm_2(x,t)}{\partial t} = d\dfrac{\partial^2 U^\pm_2(x,t)}{\partial x^2} - \beta U^\pm_2(x,t) + g_1(U^\pm_1(x,t-\tau_1)),\\
U^\pm_1(x,s) = U^\pm_{10}(x,s), \quad (x,s) \in \mathbb{R} \times [-\tau_1, 0],\\
U^\pm_2(x,s) = U^\pm_{20}(x,s), \quad (x,s) \in \mathbb{R} \times [-\tau_2, 0].
\end{cases}$$

It follows from the comparison principle that

$$u_{1-} \le U_1^-(x,t) \le u_1(x,t) \le U_1^+(x,t) \le u_{1+}, \qquad (x,t) \in \mathbb{R}\times[0,\infty),$$

$$u_{1-} \le U_1^-(x,t) \le \phi_1(x+ct) \le U_1^+(x,t) \le u_{1+}, \qquad (x,t) \in \mathbb{R}\times[0,\infty), \tag{3.3}$$

$$u_{2-} \le U_2^-(x,t) \le u_2(x,t) \le U_2^+(x,t) \le u_{2+}, \qquad (x,t) \in \mathbb{R}\times[0,\infty),$$

$$u_{2-} \le U_2^-(x,t) \le \phi_2(x+ct) \le U_2^+(x,t) \le u_{2+}, \qquad (x,t) \in \mathbb{R}\times[0,\infty). \tag{3.4}$$

Let $\xi = x + ct$ and

$$\begin{aligned} V_i(\xi,t) &= U_i^+(x,t) - \phi_i(\xi), && i=1,2,\\ V_{10}(\xi,s) &= U_{10}^+(x,s) - \phi_1(x+cs), && (x,s)\in\mathbb{R}\times[-\tau_1,0],\\ V_{20}(\xi,s) &= U_{20}^+(x,s) - \phi_2(x+cs), && (x,s)\in\mathbb{R}\times[-\tau_2,0]. \end{aligned}$$

Then by (3.1), (3.2), (3.3) and (3.4), we have

$$V_i(\xi,t) \ge 0, \quad V_{i0}(\xi,s) \ge 0, \quad i=1,2.$$

First of all, we are going to derive the a priori estimates for $V_1(\xi,t)$ and $V_2(\xi,t)$ in the weighted Sobolev space $H_w^1(\mathbb{R})$.

Lemma 3.1. *Assume that* (A1)–(A3) *hold. Then for any* $c > c_{\min}$, *it holds that*

$$\begin{aligned} &\sum_{i=1}^2 e^{2\mu t}\|V_i(t)\|^2_{L^2_w(\mathbb{R})} + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s + d\int_0^t e^{2\mu s}\|V_{2\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\\ &+\int_0^t e^{2\mu s}\int_{\mathbb{R}} B^{(1)}_{\mu,w}(\xi)w(\xi)V_1^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s + \int_0^t e^{2\mu s}\int_{\mathbb{R}} B^{(2)}_{\mu,w}(\xi)w(\xi)V_2^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\ \le{}& \sum_{i=1}^2 \|V_{i0}(0)\|^2_{L^2_w(\mathbb{R})} + \sum_{i=1}^2\int_{-\tau_i}^0\int_{\mathbb{R}} e^{2\mu(s+\tau_i)}w(\xi+c\tau_i)g_i'(\phi_i(\xi))V_{i0}^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s, \end{aligned}$$

where

$$\begin{aligned} B^{(1)}_{\mu,w}(\xi) ={}& -2\mu - c\left(\frac{w'}{w}\right) - \left(\frac{w'}{w}\right)^2 + 2\\ &- g_2'(\phi_2(\xi - c\tau_2)) - e^{2\mu\tau_1}\frac{w(\xi+c\tau_1)}{w(\xi)}g_1'(\phi_1(\xi)), \end{aligned} \tag{3.5}$$

$$\begin{aligned} B^{(2)}_{\mu,w}(\xi) ={}& -2\mu - c\left(\frac{w'}{w}\right) - d\left(\frac{w'}{w}\right)^2 + 2\beta\\ &- g_1'(\phi_1(\xi - c\tau_1)) - e^{2\mu\tau_2}\frac{w(\xi+c\tau_2)}{w(\xi)}g_2'(\phi_2(\xi)), \end{aligned} \tag{3.6}$$

the weight function $w(\xi)$ *is defined by* (2.16), *and* μ *is arbitrarily given positive constant at this moment and defined later.*

Proof. It is easy to verify that $V_i(\xi,t)$, $i=1,2$ satisfy

$$\begin{cases} \dfrac{\partial V_1(\xi,t)}{\partial t}+c\dfrac{\partial V_1(\xi,t)}{\partial \xi}-\dfrac{\partial^2 V_1(\xi,t)}{\partial \xi^2}+V_1(\xi,t)-g_2'(\phi_2(\xi-c\tau_2))V_2(\xi-c\tau_2,t-\tau_2) \\ =P(\xi-c\tau_2,t-\tau_2), \\ \dfrac{\partial V_2(\xi,t)}{\partial t}+c\dfrac{\partial V_2(\xi,t)}{\partial \xi}-d\dfrac{\partial^2 V_2(\xi,t)}{\partial \xi^2}+\beta V_2(\xi,t)-g_1'(\phi_1(\xi-c\tau_1))V_1(\xi-c\tau_1,t-\tau_1) \\ =Q(\xi-c\tau_1,t-\tau_1), \\ V_1(\xi,s)=V_{10}(\xi,s), \quad (\xi,s)\in\mathbb{R}\times[-\tau_1,0], \\ V_2(\xi,s)=V_{20}(\xi,s), \quad (\xi,s)\in\mathbb{R}\times[-\tau_2,0], \end{cases} \tag{3.7}$$

where

$$\begin{aligned} P(\xi-c\tau_2,t-\tau_2) &= g_2(V_2(\xi-c\tau_2,t-\tau_2)+\phi_2(\xi-c\tau_2)) \\ &\quad - g_2(\phi_2(\xi-c\tau_2))-g_2'(\phi_2(\xi-c\tau_2))V_2(\xi-c\tau_2,t-\tau_2), \\ Q(\xi-c\tau_1,t-\tau_1) &= g_1(V_1(\xi-c\tau_1,t-\tau_1)+\phi_1(\xi-c\tau_1)) \\ &\quad - g_1(\phi_1(\xi-c\tau_1))-g_1'(\phi_1(\xi-c\tau_1))V_1(\xi-c\tau_1,t-\tau_1). \end{aligned}$$

Multiplying the first equation of (3.7) by $e^{2\mu t}w(\xi)V_1(\xi,t)$, where $\mu>0$ is a small constant to be specified later, we obtain

$$\begin{aligned} &\left\{\frac{1}{2}e^{2\mu t}wV_1^2\right\}_t+e^{2\mu t}\left\{\frac{1}{2}cwV_1^2-wV_1V_{1\xi}\right\}_\xi+\left\{-\mu-\frac{1}{2}c\left(\frac{w'}{w}\right)+1\right\}e^{2\mu t}wV_1^2 \\ &\quad+e^{2\mu t}w'V_1V_{1\xi}+e^{2\mu t}wV_{1\xi}^2-e^{2\mu t}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1(\xi,t)V_2(\xi-c\tau_2,t-\tau_2) \\ &= e^{2\mu t}w(\xi)V_1(\xi,t)P(\xi-c\tau_2,t-\tau_2). \end{aligned} \tag{3.8}$$

By the Cauchy inequality $|xy|\le \epsilon x^2+y^2/(4\epsilon)$, we obtain

$$\left|e^{2\mu t}w'V_1V_{1\xi}\right|\le \epsilon e^{2\mu t}wV_{1\xi}^2+\frac{1}{4\epsilon}\left(\frac{w'}{w}\right)^2 wV_1^2e^{2\mu t}.$$

Let $\epsilon=1/2$. Then (3.8) reduces to

$$\begin{aligned} &\left\{\frac{1}{2}e^{2\mu t}wV_1^2\right\}_t+e^{2\mu t}\left\{\frac{1}{2}cwV_1^2-wV_1V_{1\xi}\right\}_\xi \\ &\quad+\left\{-\mu-\frac{1}{2}c\left(\frac{w'}{w}\right)-\frac{1}{2}\left(\frac{w'}{w}\right)^2+1\right\}e^{2\mu t}wV_1^2 \\ &\quad+\frac{1}{2}e^{2\mu t}wV_{1\xi}^2-e^{2\mu t}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1(\xi,t)V_2(\xi-c\tau_2,t-\tau_2) \\ &\le e^{2\mu t}w(\xi)V_1(\xi,t)P(\xi-c\tau_2,t-\tau_2). \end{aligned} \tag{3.9}$$

Integrating (3.9) over $\mathbb{R}\times[0,t]$ with respect to ξ and t, respectively, we have

$$
\begin{aligned}
&e^{2\mu t}\|V_1(t)\|^2_{L^2_w(\mathbb{R})}+\int_0^t\int_{\mathbb{R}}\left\{-2\mu-c\left(\frac{w'}{w}\right)-\left(\frac{w'}{w}\right)^2+2\right\}e^{2\mu s}w(\xi)V_1^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
&+\int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\\
&-2\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1(\xi,s)V_2(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s\\
\le{}&\|V_{10}(0)\|^2_{L^2_w(\mathbb{R})}+2\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)V_1(\xi,s)P(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s.
\end{aligned}\tag{3.10}
$$

Using the Cauchy inequality again, we have

$$
\begin{aligned}
&\left|e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1(\xi,s)V_2(\xi-c\tau_2,s-\tau_2)\right|\\
&\le e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))\left[\frac{V_1^2(\xi,s)}{2}+\frac{V_2^2(\xi-c\tau_2,s-\tau_2)}{2}\right].
\end{aligned}
$$

Thus, the fourth term on the left hand side of (3.10) is reduced to

$$
\begin{aligned}
&2\left|\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1(\xi,s)V_2(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s\right|\\
&\le 2\int_0^t\int_{\mathbb{R}}\left|e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1(\xi,s)V_2(\xi-c\tau_2,s-\tau_2)\right|\mathrm{d}\xi\mathrm{d}s\\
&\le\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_2^2(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s\\
&=\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
&\quad+\int_{-\tau_2}^{t-\tau_2}\int_{\mathbb{R}}e^{2\mu(s+\tau_2)}w(\xi+c\tau_2)g_2'(\phi_2(\xi))V_2^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
&\le\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_1^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
&\quad+\int_{-\tau_2}^0\int_{\mathbb{R}}e^{2\mu(s+\tau_2)}w(\xi+c\tau_2)g_2'(\phi_2(\xi))V_{20}^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}}e^{2\mu(s+\tau_2)}w(\xi+c\tau_2)g_2'(\phi_2(\xi))V_2^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s.
\end{aligned}
$$

Then, (3.10) is reduced to

$$
\begin{aligned}
&e^{2\mu t}\|V_1(t)\|^2_{L^2_w(\mathbb{R})}+\int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\\
&-\int_0^t\int_{\mathbb{R}}e^{2\mu(s+\tau_2)}w(\xi+c\tau_2)g_2'(\phi_2(\xi))V_2^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}}\left\{-2\mu-c\left(\frac{w'}{w}\right)-\left(\frac{w'}{w}\right)^2+2-g_2'(\phi_2(\xi-c\tau_2))\right\}e^{2\mu s}w(\xi)V_1^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&\le \|V_{10}(0)\|^2_{L^2_w(\mathbb{R})}+\int_{-\tau_2}^0\int_{\mathbb{R}}e^{2\mu(s+\tau_2)}w(\xi+c\tau_2)g_2'(\phi_2(\xi))V_{20}^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&+2\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)V_1(\xi,s)P(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s.
\end{aligned} \tag{3.11}
$$

Multiplying the second equation of (3.7) by $e^{2\mu t}w(\xi)V_2(\xi,t)$ and integrating it over $\mathbb{R}\times[0,t]$ with respect to ξ and t, respectively, we can similarly obtain

$$
\begin{aligned}
&e^{2\mu t}\|V_2(t)\|^2_{L^2_w(\mathbb{R})}+d\int_0^t e^{2\mu s}\|V_{2\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s \\
&-\int_0^t\int_{\mathbb{R}}e^{2\mu(s+\tau_1)}w(\xi+c\tau_1)g_1'(\phi_1(\xi))V_1^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&+\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)V_2^2(\xi,s)\left\{-2\mu-c\left(\frac{w'}{w}\right)-d\left(\frac{w'}{w}\right)^2+2\beta-g_1'(\phi_1(\xi-c\tau_1))\right\}\mathrm{d}\xi\mathrm{d}s \\
&\le \|V_{20}(0)\|^2_{L^2_w(\mathbb{R})}+\int_{-\tau_1}^0\int_{\mathbb{R}}e^{2\mu(s+\tau_1)}w(\xi+c\tau_1)g_1'(\phi_1(\xi))V_{10}^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&+2\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)V_2(\xi,s)Q(\xi-c\tau_1,s-\tau_1)\,\mathrm{d}\xi\mathrm{d}s.
\end{aligned} \tag{3.12}
$$

From (3.11) and (3.12), we get

$$
\begin{aligned}
&\sum_{i=1}^2 e^{2\mu t}\|V_i(t)\|^2_{L^2_w(\mathbb{R})}+\int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s+d\int_0^t e^{2\mu s}\|V_{2\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s \\
&+\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)V_1^2(\xi,s) \\
&\quad\times\left\{-2\mu-c\left(\frac{w'}{w}\right)-\left(\frac{w'}{w}\right)^2+2-g_2'(\phi_2(\xi-c\tau_2))-e^{2\mu\tau_1}\frac{w(\xi+c\tau_1)}{w(\xi)}g_1'(\phi_1(\xi))\right\}\mathrm{d}\xi\mathrm{d}s \\
&+\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)V_2^2(\xi,s) \\
&\quad\times\left\{-2\mu-c\left(\frac{w'}{w}\right)-d\left(\frac{w'}{w}\right)^2+2\beta-g_1'(\phi_1(\xi-c\tau_1))-e^{2\mu\tau_2}\frac{w(\xi+c\tau_2)}{w(\xi)}g_2'(\phi_2(\xi))\right\}\mathrm{d}\xi\mathrm{d}s \\
&\le\sum_{i=1}^2\|V_{i0}(0)\|^2_{L^2_w(\mathbb{R})}+\sum_{i=1}^2\int_{-\tau_i}^0\int_{\mathbb{R}}e^{2\mu(s+\tau_i)}w(\xi+c\tau_i)g_i'(\phi_i(\xi))V_{i0}^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&+2\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)V_1(\xi,s)P(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s \\
&+2\int_0^t\int_{\mathbb{R}}e^{2\mu s}w(\xi)V_2(\xi,s)Q(\xi-c\tau_1,s-\tau_1)\,\mathrm{d}\xi\mathrm{d}s.
\end{aligned} \tag{3.13}
$$

For the nonlinearity $P(\xi - c\tau_2, t - \tau_2)$, using Taylor's formula, we have

$$\begin{aligned} P(\xi - c\tau_2, t - \tau_2) &= g_2(V_2(\xi - c\tau_2, t - \tau_2) + \phi_2(\xi - c\tau_2)) \\ &\quad - g_2(\phi_2(\xi - c\tau_2)) - g_2'(\phi_2(\xi - c\tau_2))V_2(\xi - c\tau_2, t - \tau_2) \\ &= \frac{g_2''(\widetilde{\phi}_2)}{2!} V_2^2(\xi - c\tau_2, t - \tau_2), \end{aligned}$$

where $\widetilde{\phi}_2 \in [\phi_2(\xi - c\tau_2), V_2(\xi - c\tau_2, t - \tau_2) + \phi_2(\xi - c\tau_2)]$. Since $V_2(\xi - c\tau_2, t - \tau_2) + \phi_2(\xi - c\tau_2) \in [u_{2-}, u_{2+}]$, from (A2), we get $g_2''(\widetilde{\phi}_2) \le 0$, i.e., $P(\xi - c\tau_2, t - \tau_2) \le 0$. Since $V_1(\xi, t) \ge 0$, we have

$$2\int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) V_1(\xi, s) P(\xi - c\tau_2, s - \tau_2)\, \mathrm{d}\xi \mathrm{d}s \le 0.$$

Similarly, we get

$$2\int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) V_2(\xi, s) Q(\xi - c\tau_1, s - \tau_1)\, \mathrm{d}\xi \mathrm{d}s \le 0.$$

Thus (3.13) is reduced to

$$\begin{aligned} &\sum_{i=1}^2 e^{2\mu t} \|V_i(t)\|_{L_w^2(\mathbb{R})}^2 + \int_0^t e^{2\mu s} \|V_{1\xi}(s)\|_{L_w^2(\mathbb{R})}^2 \mathrm{d}s + d\int_0^t e^{2\mu s} \|V_{2\xi}(s)\|_{L_w^2(\mathbb{R})}^2 \mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}} B_{\mu,w}^{(1)}(\xi) e^{2\mu s} w(\xi) V_1^2(\xi, s)\, \mathrm{d}\xi \mathrm{d}s + \int_0^t \int_{\mathbb{R}} B_{\mu,w}^{(2)}(\xi) e^{2\mu s} w(\xi) V_2^2(\xi, s)\, \mathrm{d}\xi \mathrm{d}s \\ \le &\sum_{i=1}^2 \|V_{i0}(0)\|_{L_w^2(\mathbb{R})}^2 + \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{\mathbb{R}} e^{2\mu(s+\tau_i)} w(\xi + c\tau_i) g_i'(\phi_i(\xi)) V_{i0}^2(\xi, s)\, \mathrm{d}\xi \mathrm{d}s, \end{aligned}$$

where $B_{\mu,w}^{(1)}(\xi)$ and $B_{\mu,w}^{(2)}(\xi)$ are given in (3.5) and (3.6). □

Let

$$\begin{aligned} C_1(\mu) &= \frac{dc^2}{(d+1)^2} + 2 - 2\mu - g_2'(0) - g_1'(0) e^{2\mu\tau_1}, \\ C_2(\mu) &= \frac{2 - g_1'(\mu_{1+}) - g_2'(\mu_{2+})}{2} - 2\mu - g_1'(0)(e^{2\mu\tau_1} - 1), \\ C_3(\mu) &= \frac{c^2}{(d+1)^2} + 2\beta - 2\mu - g_1'(0) - g_2'(0) e^{2\mu\tau_2}, \end{aligned}$$

and

$$C_4(\mu) = \frac{2\beta - g_1'(\mu_{1+}) - g_2'(\mu_{2+})}{2} - 2\mu - g_2'(0)(e^{2\mu\tau_2} - 1).$$

Lemma 3.2. *Assume that* (A4) *holds. Then for any*

$$c > \max\left\{ c_{\min}, (d+1)\sqrt{|g_1'(0) + g_2'(0) - 2\beta|}, \sqrt{\frac{(d+1)^2}{d} |g_1'(0) + g_2'(0) - 2|} \right\},$$

there exists a unique root $\mu_i \in (0, \infty)$ for the equation $C_i(\mu) = 0$, $i = 1, 2, 3, 4$ respectively.

Proof. We first prove that the equation $C_1(\mu) = 0$ has a unique root in $(0, \infty)$. Let $H_1(\mu) = dc^2/(d+1)^2 + 2 - g_2'(0) - 2\mu$ and $H_2(\mu) = g_1'(0)e^{2\mu\tau_1}$. Since $H_1(0) = dc^2/(d+1)^2 + 2 - g_2'(0) > g_1'(0) = H_2(0)$, sketching the graphs of $H_1(\mu)$ and $H_2(\mu)$ (see Figure 3.1a), it is easy to verify that there exists only one root μ_1 in $(0, \infty)$.

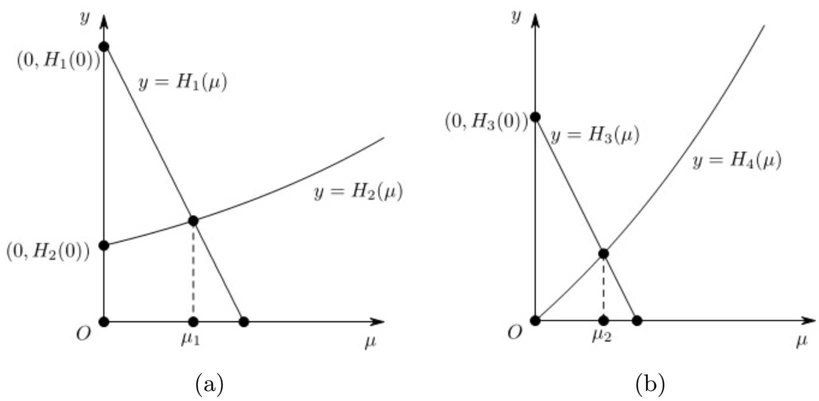

Figure 3.1: (a) Graphs of $y = H_1(\mu)$ and $y = H_2(\mu)$. (b) Graphs of $y = H_3(\mu)$ and $y = H_4(\mu)$.

Next, we prove that the equation $C_2(\mu) = 0$ has a unique root in $(0, \infty)$. Let $H_3(\mu) = (2 - g_1'(\mu_{1+}) - g_2'(\mu_{2+}))/2 - 2\mu$ and $H_4(\mu) = g_1'(0)(e^{2\mu\tau_1} - 1)$. By the condition (A4), we get $H_3(0) = (2 - g_1'(\mu_{1+}) - g_2'(\mu_{2+}))/2 > 0 = H_4(0)$. It can be verified by sketching the graphs of $H_3(\mu)$ and $H_4(\mu)$ (see Figure 3.1b) that there exists a unique root μ_2 in $(0, \infty)$. Using the similar method above, we can prove that the equation $C_i(\mu) = 0$ $(i = 3, 4)$ has a unique root in $(0, \infty)$. □

Lemma 3.3. *Assume that* (A1)–(A4) *hold. Let* $w(\xi)$ *be the weight function defined as in* (2.16). *Then for any*

$$c > \max\left\{c_{\min}, (d+1)\sqrt{|g_1'(0) + g_2'(0) - 2\beta|}, \sqrt{\frac{(d+1)^2}{d}\,|g_1'(0) + g_2'(0) - 2|}\right\},$$

we have

$$B_{\mu,w}^{(1)}(\xi) > C_0^{(1)}(\mu) = \min\{C_1(\mu), C_2(\mu)\} \tag{3.14}$$

and

$$B_{\mu,w}^{(2)}(\xi) > C_0^{(2)}(\mu) = \min\{C_3(\mu), C_4(\mu)\} \tag{3.15}$$

for all $\xi \in \mathbb{R}$ *and* $0 < \mu < \mu_0 = \min\{\mu_1, \mu_2, \mu_3, \mu_4\}$, *where* $\mu_i > 0$ *is the unique solution to the equation* $C_i(\mu) = 0$, $i = 1, 2, 3, 4$.

Proof. We only prove (3.14), because another is similar. From (A3), we have $0 \le g_1'(\phi_1(\xi)), g_1'(\phi_1(\xi - c\tau_1)) \le g_1'(0)$ and $0 \le g_2'(\phi_2(\xi)), g_2'(\phi_2(\xi - c\tau_2)) \le g_2'(0)$.

Case 1: $\xi \le \xi_0$. From (2.16), we have $w(\xi) = e^{-\gamma(\xi-\xi_0)}$. Since $w(\xi)$ is non-increasing, we obtain

$$\begin{aligned} B_{\mu,w}^{(1)}(\xi) &= -2\mu + c\gamma - \gamma^2 + 2 - g_2'(\phi_2(\xi - c\tau_2)) - e^{2\mu\tau_1}\frac{w(\xi + c\tau_1)}{w(\xi)} g_1'(\phi_1(\xi)) \\ &\ge -2\mu + c\gamma - \gamma^2 + 2 - g_2'(0) - e^{2\mu\tau_1} g_1'(0) \\ &= \frac{dc^2}{(d+1)^2} + 2 - 2\mu - g_2'(0) - g_1'(0)e^{2\mu\tau_1} \\ &= C_1(\mu) > 0. \end{aligned}$$

Case 2: $\xi > \xi_0$. In this case, $w(\xi) = w(\xi + c\tau_1) = 1$ and $w'(\xi) = 0$. Thus, by Lemma 2.4, we have

$$\begin{aligned} B_{\mu,w}^{(1)}(\xi) &= -2\mu + 2 - g_2'(\phi_2(\xi - c\tau_2)) - e^{2\mu\tau_1}\frac{w(\xi + c\tau_1)}{w(\xi)} g_1'(\phi_1(\xi)) \\ &\ge -2\mu + 2 - g_2'(\phi_2(\xi - c\tau_2)) - g_1'(\phi_1(\xi)) - (e^{2\mu\tau_1} - 1) g_1'(\phi_1(\xi)) \\ &\ge -2\mu + 2 - g_2'(u_{2+}) - g_1'(u_{1+}) - 2\epsilon - (e^{2\mu\tau_1} - 1) g_1'(0) \\ &\ge \frac{2 - g_1'(\mu_{1+}) - g_2'(\mu_{2+})}{2} - 2\mu - g_1'(0)(e^{2\mu\tau_1} - 1) \\ &= C_2(\mu) > 0. \end{aligned}$$

If we choose $C_0^{(1)}(\mu) = \min\{C_1(\mu), C_2(\mu)\}$, then (3.14) holds. □

Applying Lemma 3.1 and dropping the positive term

$$\int_0^t \int_{\mathbb{R}} B_{\mu,w}^{(1)}(\xi) e^{2\mu s} w(\xi) V_1^2(\xi, s)\, \mathrm{d}\xi \mathrm{d}s, \quad \int_0^t \int_{\mathbb{R}} B_{\mu,w}^{(2)}(\xi) e^{2\mu s} w(\xi) V_2^2(\xi, s)\, \mathrm{d}\xi \mathrm{d}s$$

and

$$\int_0^t e^{2\mu s} \|V_{1\xi}(s)\|_{L_w^2(\mathbb{R})}^2 \,\mathrm{d}s, \quad d\int_0^t e^{2\mu s} \|V_{2\xi}(s)\|_{L_w^2(\mathbb{R})}^2 \,\mathrm{d}s,$$

we immediately obtain that

$$\sum_{i=1}^{2} e^{2\mu t} \|V_i(t)\|_{L_w^2(\mathbb{R})}^2 \le C_5 \left(\sum_{i=1}^{2} \|V_{i0}(0)\|_{L_w^2(\mathbb{R})}^2 + \sum_{i=1}^{2} \int_{-\tau_i}^{0} \|V_{i0}(s)\|_{L_w^2(\mathbb{R})}^2 \,\mathrm{d}s \right),$$

where $C_5 = \max\{1, e^{2\mu\tau_1} L_{11}, e^{2\mu\tau_2} L_{21}\}$. Thus, we obtain the first energy estimates as follows.

Lemma 3.4. *Assume that* (A1)–(A4) *hold. Let* $w(\xi)$ *be the weight function defined as in* (2.16)*. Then for any*

$$c > \max\left\{ c_{\min}, (d+1)\sqrt{|g_1'(0) + g_2'(0) - 2\beta|}, \sqrt{\frac{(d+1)^2}{d} |g_1'(0) + g_2'(0) - 2|} \right\},$$

it holds that

$$\begin{aligned}&\|V_i(t)\|^2_{L^2_w(\mathbb{R})}\\&\leq C_5 e^{-2\mu t}\left(\sum_{i=1}^{2}\|V_{i0}(0)\|^2_{L^2_w(\mathbb{R})}+\sum_{i=1}^{2}\int_{-\tau_i}^{0}\|V_{i0}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\right),\quad t\geq 0,\ i=1,2,\end{aligned}$$

where $C_5=\max\left\{1,e^{2\mu\tau_1}L_{11},e^{2\mu\tau_2}L_{21}\right\}$ *and* $0<\mu<\mu_0=\min\{\mu_1,\mu_2,\mu_3,\mu_4\}$.

Next, we derive the estimates for $V_{1\xi}(\xi,t)$ and $V_{2\xi}(\xi,t)$.

Lemma 3.5. *Assume that* (A1)–(A5) *hold. Let* $w(\xi)$ *be the weight function defined as in* (2.16). *Then for any*

$$c>\max\left\{c_{\min},(d+1)\sqrt{|g_1'(0)+g_2'(0)-2\beta|},\sqrt{\frac{(d+1)^2}{d}\,|g_1'(0)+g_2'(0)-2|}\right\},$$

it holds that

$$\begin{aligned}&\|V_{i\xi}(t)\|^2_{L^2_w(\mathbb{R})}\\&\leq C_7 e^{-2\mu t}\left(\sum_{i=1}^{2}\|V_{i0}(0)\|^2_{H^1_w(\mathbb{R})}+\sum_{i=1}^{2}\int_{-\tau_i}^{0}\|V_{i0}(s)\|^2_{H^1_w(\mathbb{R})}\,\mathrm{d}s\right),\quad t\geq 0,\ i=1,2,\end{aligned}$$

where $C_7=\max\left\{C_6\sum_{i=1}^2\left(3L_{i2}M_i+2L_{i2}u_{i+}+\frac{3}{2}L_{i3}M_iu_{i+}\right)e^{2\mu\tau_i},C_5+L_{12}u_{1+},C_5+L_{22}u_{2+}\right\}$, $C_6=C_5\max\left\{1,1/d,1/C_5,1/C_0^{(1)}(\mu),1/C_0^{(2)}(\mu)\right\}$ *and* $0<\mu<\mu_0=\min\{\mu_1,\mu_2,\mu_3,\mu_4\}$.

Proof. Differentiating the first and second equations of (3.7) with respect to ξ and multiplying them by $e^{2\mu t}w(\xi)V_{1\xi}(\xi,t)$ and $e^{2\mu t}w(\xi)V_{2\xi}(\xi,t)$ respectively, we get

$$\begin{aligned}&\left\{\frac{1}{2}e^{2\mu t}wV_{1\xi}^2\right\}_t+e^{2\mu t}\left\{\frac{1}{2}cwV_{1\xi}^2-wV_{1\xi}V_{1\xi\xi}\right\}_\xi+\left\{-\mu-\frac{1}{2}c\left(\frac{w'}{w}\right)+1\right\}e^{2\mu t}wV_{1\xi}^2\\&\quad+e^{2\mu t}w'V_{1\xi}V_{1\xi\xi}+e^{2\mu t}wV_{1\xi\xi}^2-e^{2\mu t}w(\xi)g_2'(\phi_2(\xi-c\tau_2))V_{1\xi}(\xi,t)V_{2\xi}(\xi-c\tau_2,t-\tau_2)\\&=e^{2\mu t}w(\xi)g_2''(\phi_2(\xi-c\tau_2))\phi_2'(\xi-c\tau_2)V_{1\xi}(\xi,t)V_2(\xi-c\tau_2,t-\tau_2)\\&\quad+e^{2\mu t}w(\xi)V_{1\xi}(\xi,t)P_\xi(\xi-c\tau_2,t-\tau_2)\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}&\left\{\frac{1}{2}e^{2\mu t}wV_{2\xi}^2\right\}_t+e^{2\mu t}\left\{\frac{1}{2}cwV_{2\xi}^2-dwV_{2\xi}V_{2\xi\xi}\right\}_\xi+\left\{-\mu-\frac{1}{2}c\left(\frac{w'}{w}\right)+\beta\right\}e^{2\mu t}wV_{2\xi}^2\\&\quad+de^{2\mu t}w'V_{2\xi}V_{2\xi\xi}+de^{2\mu t}wV_{2\xi\xi}^2-e^{2\mu t}w(\xi)g_1'(\phi_1(\xi-c\tau_1))V_{2\xi}(\xi,t)V_{1\xi}(\xi-c\tau_1,t-\tau_1)\\&=e^{2\mu t}w(\xi)g_1''(\phi_1(\xi-c\tau_1))\phi_1'(\xi-c\tau_1)V_{2\xi}(\xi,t)V_1(\xi-c\tau_1,t-\tau_1)\\&\quad+e^{2\mu t}w(\xi)V_{2\xi}(\xi,t)Q_\xi(\xi-c\tau_1,t-\tau_1).\end{aligned}\tag{3.17}$$

Integrating (3.16) and (3.17) over $\mathbb{R}\times[0,t]$ with respect to ξ and t, respectively, we have

$$
\begin{aligned}
&\sum_{i=1}^{2} e^{2\mu t}\|V_{i\xi}(t)\|^2_{L^2_w(\mathbb{R})}+\int_0^t e^{2\mu s}\|V_{1\xi\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s+d\int_0^t e^{2\mu s}\|V_{2\xi\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\\
&+\sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}} B^{(i)}_{\mu,w}(\xi)e^{2\mu s}w(\xi)V^2_{i\xi}(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
\le{}& C_5\left(\sum_{i=1}^{2}\|V_{i\xi0}(0)\|^2_{L^2_w(\mathbb{R})}+\sum_{i=1}^{2}\int_{-\tau_i}^{0}\|V_{i\xi0}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\right)\\
&+2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)V_{1\xi}(\xi,s)P_\xi(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s\\
&+2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)V_{2\xi}(\xi,s)Q_\xi(\xi-c\tau_1,s-\tau_1)\,\mathrm{d}\xi\mathrm{d}s\\
&+2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)g_2''(\phi_2(\xi-c\tau_2))\phi_2'(\xi-c\tau_2)V_{1\xi}(\xi,s)V_2(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s\\
&+2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)g_1''(\phi_1(\xi-c\tau_1))\phi_1'(\xi-c\tau_1)V_{2\xi}(\xi,s)V_1(\xi-c\tau_1,s-\tau_1)\,\mathrm{d}\xi\mathrm{d}s.
\end{aligned}
\tag{3.18}
$$

Let

$$\delta_1=\sum_{i=1}^{2}\|V_{i0}(0)\|^2_{L^2_w(\mathbb{R})}+\sum_{i=1}^{2}\int_{-\tau_i}^{0}\|V_{i0}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s.$$

Then, we have

$$
\begin{aligned}
&e^{2\mu t}\|V_1(t)\|^2_{L^2_w(\mathbb{R})}\le C_6\delta_1, && e^{2\mu t}\|V_2(t)\|^2_{L^2_w(\mathbb{R})}\le C_6\delta_1,\\
&\int_0^t e^{2\mu s}\|V_1(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\le C_6\delta_1, && \int_0^t e^{2\mu s}\|V_2(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\le C_6\delta_1,\\
&\int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\le C_6\delta_1, && \int_0^t e^{2\mu s}\|V_{2\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\le C_6\delta_1,
\end{aligned}
$$

where $C_6=C_5\max\left\{1,1/d,1/C_5,1/C_0^{(1)}(\mu),1/C_0^{(2)}(\mu)\right\}$. By the Cauchy inequality, we get

$$
\begin{aligned}
&2\left|\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)g_2''(\phi_2(\xi-c\tau_2))\phi_2'(\xi-c\tau_2)V_{1\xi}(\xi,s)V_2(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s\right|\\
\le{}&\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\left|g_2''(\phi_2(\xi-c\tau_2))\phi_2'(\xi-c\tau_2)\right|\left[V^2_{1\xi}(\xi,s)+V^2_2(\xi-c\tau_2,s-\tau_2)\right]\mathrm{d}\xi\mathrm{d}s\\
\le{}& L_{22}M_2\left[\int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)V_2^2(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s\right]\\
\le{}& L_{22}M_2\left[\int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s+\int_0^t e^{2\mu(s+\tau_2)}\|V_2(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\right.\\
&\left.\qquad+\int_{-\tau_2}^{0} e^{2\mu(s+\tau_2)}\|V_{20}(s)\|^2_{L^2_w(\mathbb{R})}\,\mathrm{d}s\right]\\
\le{}& 3C_6L_{22}M_2\delta_1 e^{2\mu\tau_2}.
\end{aligned}
\tag{3.19}
$$

Similarly, we have

$$
\begin{aligned}
&2\left|\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)g_1''(\phi_1(\xi-c\tau_1))\phi_1'(\xi-c\tau_1)V_{2\xi}(\xi,s)V_1(\xi-c\tau_1,s-\tau_1)\,\mathrm{d}\xi\mathrm{d}s\right| \\
&\leq 3C_6L_{12}M_1\delta_1e^{2\mu\tau_1}.
\end{aligned} \tag{3.20}
$$

For the nonlinearity $P_\xi(\xi-c\tau_2,s-\tau_2)$, using Taylor's formula, we have

$$
\begin{aligned}
P_\xi(\xi-c\tau_2,s-\tau_2) &= g_2''(\widetilde{\phi}_2)V_2(\xi-c\tau_2,s-\tau_2)V_{2\xi}(\xi-c\tau_2,s-\tau_2) \\
&\quad + \frac{1}{2}g_2'''(\overline{\phi}_2)\phi_2'(\xi-c\tau_2)V_2^2(\xi-c\tau_2,s-\tau_2),
\end{aligned}
$$

where $\overline{\phi}_2,\widetilde{\phi}_2 \in [\phi_2(\xi-c\tau_2),V_2(\xi-c\tau_2,t-\tau_2)+\phi_2(\xi-c\tau_2)]$ and $u_{2-}\leq \overline{\phi}_2,\widetilde{\phi}_2\leq u_{2+}$. Changing variables $\xi-c\tau_2\to\xi$, $s-\tau_2\to s$ and using the Cauchy inequality, we get

$$
\begin{aligned}
&2\left|\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)V_{1\xi}(\xi,s)P_\xi(\xi-c\tau_2,s-\tau_2)\,\mathrm{d}\xi\mathrm{d}s\right| \\
&\leq 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\left|V_{1\xi}(\xi,s)\right|\left[\left|g_2''(\widetilde{\phi}_2)V_2(\xi-c\tau_2,s-\tau_2)V_{2\xi}(\xi-c\tau_2,s-\tau_2)\right|\right. \\
&\qquad\left. + \frac{1}{2}\left|g_2'''(\overline{\phi}_2)\phi_2'(\xi-c\tau_2)\right|V_2^2(\xi-c\tau_2,s-\tau_2)\right]\mathrm{d}\xi\mathrm{d}s \\
&\leq L_{22}u_{2+}\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\left[V_{1\xi}^2(\xi,s)+V_{2\xi}^2(\xi-c\tau_2,s-\tau_2)\right]\mathrm{d}\xi\mathrm{d}s \\
&\quad + \frac{1}{2}L_{23}u_{2+}M_2\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)\left[V_{1\xi}^2(\xi,s)+V_2^2(\xi-c\tau_2,s-\tau_2)\right]\mathrm{d}\xi\mathrm{d}s \\
&\leq \left(2L_{22}+\frac{3}{2}L_{23}M_2\right)C_6\delta_1u_{2+}e^{2\mu\tau_2} + L_{22}u_{2+}e^{2\mu\tau_2}\int_{-\tau_2}^0 e^{2\mu s}\left\|V_{2\xi0}(s)\right\|_{L_w^2(\mathbb{R})}^2\mathrm{d}s.
\end{aligned} \tag{3.21}
$$

Similarly, we have

$$
\begin{aligned}
&2\left|\int_0^t\int_{\mathbb{R}} e^{2\mu s}w(\xi)V_{2\xi}(\xi,s)Q_\xi(\xi-c\tau_1,s-\tau_1)\,\mathrm{d}\xi\mathrm{d}s\right| \\
&\leq \left(2L_{12}+\frac{3}{2}L_{13}M_1\right)C_6\delta_1u_{1+}e^{2\mu\tau_1} + L_{12}u_{1+}e^{2\mu\tau_1}\int_{-\tau_1}^0 e^{2\mu s}\left\|V_{1\xi0}(s)\right\|_{L_w^2(\mathbb{R})}^2\mathrm{d}s.
\end{aligned} \tag{3.22}
$$

From (3.18)–(3.22), we obtain

$$
\begin{aligned}
\sum_{i=1}^2 e^{2\mu t}\left\|V_{i\xi}(t)\right\|_{L_w^2(\mathbb{R})}^2 &\leq C_5\left(\sum_{i=1}^2\left\|V_{i\xi0}(0)\right\|_{L_w^2(\mathbb{R})}^2 + \sum_{i=1}^2\int_{-\tau_i}^0\left\|V_{i\xi0}(s)\right\|_{L_w^2(\mathbb{R})}^2\mathrm{d}s\right) \\
&\quad + \sum_{i=1}^2 L_{i2}u_{i+}e^{2\mu\tau_i}\int_{-\tau_i}^0 e^{2\mu s}\left\|V_{i\xi0}(s)\right\|_{L_w^2(\mathbb{R})}^2\mathrm{d}s \\
&\quad + C_6\delta_1\sum_{i=1}^2\left(3L_{i2}M_i+2L_{i2}u_{i+}+\frac{3}{2}L_{i3}M_iu_{i+}\right)e^{2\mu\tau_i} \\
&\leq C_7\left(\sum_{i=1}^2\left\|V_{i0}(0)\right\|_{H_w^1(\mathbb{R})}^2 + \sum_{i=1}^2\int_{-\tau_i}^0\left\|V_{i0}(s)\right\|_{H_w^1(\mathbb{R})}^2\mathrm{d}s\right),
\end{aligned}
$$

where $C_7 = \max\left\{C_6 \sum_{i=1}^{2}\left(3L_{i2}M_i + 2L_{i2}u_{i+} + \frac{3}{2}L_{i3}M_iu_{i+}\right)e^{2\mu\tau_i}, C_5 + L_{12}u_{1+}, C_5 + L_{22}u_{2+}\right\}$, which implies

$$\|V_{i\xi}(t)\|^2_{L^2_w(\mathbb{R})} \le C_7e^{-2\mu t}\left(\sum_{i=1}^{2}\|V_{i0}(0)\|^2_{H^1_w(\mathbb{R})} + \sum_{i=1}^{2}\int_{-\tau_i}^{0}\|V_{i0}(s)\|^2_{H^1_w(\mathbb{R})}\,\mathrm{d}s\right), \quad i = 1, 2. \quad \square$$

From Lemmas 3.4 and 3.5, we obtain the following a prior estimates.

Lemma 3.6. *Assume that* (A1)–(A5) *hold. Let* $w(\xi)$ *be the weight function defined as in* (2.16)*. Then for any*

$$c > \max\left\{c_{\min}, (d+1)\sqrt{|g_1'(0) + g_2'(0) - 2\beta|}, \sqrt{\frac{(d+1)^2}{d}\,|g_1'(0) + g_2'(0) - 2|}\right\},$$

it holds that

$$\begin{aligned}&\|V_i(t)\|^2_{H^1_w(\mathbb{R})}\\&\le C_8e^{-2\mu t}\left(\sum_{i=1}^{2}\|V_{i0}(0)\|^2_{H^1_w(\mathbb{R})} + \sum_{i=1}^{2}\int_{-\tau_i}^{0}\|V_{i0}(s)\|^2_{H^1_w(\mathbb{R})}\,\mathrm{d}s\right), \quad t \ge 0,\ i = 1, 2,\end{aligned}$$

where $C_8 = C_5 + C_7$ *and* $0 < \mu < \mu_0 = \min\{\mu_1, \mu_2, \mu_3, \mu_4\}$.

4. Asymptotic stability

This section is devoted to the proof of the asymptotic stability (2.17).

Using the Sobolev's embedding theorem $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ and the embedding inequality $H^1_w(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ due to $w(\xi) \ge 1$, we can immediately get

$$\sup_{\xi\in\mathbb{R}}|V_i(\xi, t)| \le C_9\|V_i(t)\|_{H^1(\mathbb{R})} \le C_{10}\|V_i(t)\|_{H^1_w(\mathbb{R})}, \quad i = 1, 2.$$

Thanks to Lemma 3.6, we obtain

$$\sup_{x\in\mathbb{R}}\left|U_i^+(x, t) - \phi_i(x + ct)\right| = \sup_{\xi\in\mathbb{R}}|V_i(\xi, t)| \le Ce^{-\mu t}, \quad t \ge 0,\ i = 1, 2, \tag{4.1}$$

where $0 < \mu < \mu_0 = \min\{\mu_1, \mu_2, \mu_3, \mu_4\}$. Let $\xi = x + ct$ and

$$\begin{aligned}V_i(\xi, t) &= \phi_i(\xi) - U_i^-(x, t), && i = 1, 2,\\V_{10}(\xi, s) &= \phi_1(x + cs) - U_{10}^-(x, s), && (x, s) \in \mathbb{R} \times [-\tau_1, 0],\\V_{20}(\xi, s) &= \phi_2(x + cs) - U_{20}^-(x, s), && (x, s) \in \mathbb{R} \times [-\tau_2, 0].\end{aligned}$$

By using the same method as in Lemma 3.6, we get

$$\sup_{x\in\mathbb{R}}\left|U_i^-(x, t) - \phi_i(x + ct)\right| = \sup_{\xi\in\mathbb{R}}|V_i(\xi, t)| \le Ce^{-\mu t}, \quad t \ge 0,\ i = 1, 2, \tag{4.2}$$

where $0 < \mu < \mu_0 = \min\{\mu_1, \mu_2, \mu_3, \mu_4\}$. From (4.1) and (4.2), by the squeezing argument, we immediately obtain

$$\sup_{x\in\mathbb{R}} |u_i(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0,\ i = 1, 2,$$

where $0 < \mu < \mu_0 = \min\{\mu_1, \mu_2, \mu_3, \mu_4\}$.

5. Applications

In this section, we apply our main result Theorem 2.10 to some monostable evolution equations and obtain the global exponential stability of traveling wave fronts.

Example 5.1. We consider the following monostable reaction-diffusion system with delay

$$\begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &= \frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha u_2(x, t-\tau_2), \\ \frac{\partial u_2(x,t)}{\partial t} &= d\frac{\partial^2 u_2(x,t)}{\partial x^2} - \beta u_2(x,t) + \frac{pu_1(x,t-\tau_1)}{1+qu_1(x,t-\tau_1)}, \end{aligned} \tag{5.1}$$

where $p, q > 0$.

Assume that $\beta < \alpha p$. It is clear that (5.1) has a trivial equilibrium $\boldsymbol{u}_- = (0, 0)$ and a unique positive equilibrium $\boldsymbol{u}_+ = \left(\frac{\alpha p-\beta}{\beta q}, \frac{\alpha p-\beta}{\alpha\beta q}\right)$. Moreover, $g_1(u) = \frac{pu}{1+qu}$ is non-decreasing on $[0, \infty)$, and

$$g_1'(u) = \frac{p}{(1+qu)^2} \le g_1'(0) = p, \quad g_1''(u) = \frac{-2pq}{(1+qu)^3} \le 0.$$

Therefore, (A1)–(A3) hold. We have the following result.

Theorem 5.2. *Assume that* $\alpha + \frac{\beta^2}{\alpha^2 p} < \min\{2, 2\beta\}$. *For any given traveling wave front* $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$ $(\xi = x + ct)$ *of* (5.1) *with the wave speed*

$$c > \max\left\{c_{\min}, (d+1)\sqrt{|p+\alpha-2\beta|}, \sqrt{\frac{(d+1)^2}{d}|p+\alpha-2|}\right\},$$

if the initial data satisfies

$$\begin{aligned} 0 \le u_{10}(x,s) &\le \frac{\alpha p-\beta}{\beta q}, \quad (x,s) \in \mathbb{R}\times[-\tau_1, 0], \\ 0 \le u_{20}(x,s) &\le \frac{\alpha p-\beta}{\alpha\beta q}, \quad (x,s) \in \mathbb{R}\times[-\tau_2, 0], \end{aligned}$$

and the initial perturbations satisfy

$$\begin{aligned} u_{10}(x,s) - \phi_1(x+cs) &\in C^0([-\tau_1, 0], H^1_w(\mathbb{R})), \quad (x,s) \in \mathbb{R}\times[-\tau_1, 0], \\ u_{20}(x,s) - \phi_2(x+cs) &\in C^0([-\tau_2, 0], H^1_w(\mathbb{R})), \quad (x,s) \in \mathbb{R}\times[-\tau_2, 0], \end{aligned}$$

where $w(\xi)$ *is the weight function defined as in* (2.16), *then the solution* $(u_1(x,t),u_2(x,t))$ *of Cauchy problem* (5.1) *and* (1.3) *satisfies*

$$0 \le u_1(x,t) \le \frac{\alpha p-\beta}{\beta q}, \quad 0 \le u_2(x,t) \le \frac{\alpha p-\beta}{\alpha\beta q}, \quad (x,t)\in\mathbb{R}\times[0,+\infty),$$
$$u_i(x,t)-\phi_i(x+ct)\in C^0([0,+\infty),H^1_w(\mathbb{R}))\cap L^2([0,+\infty),H^2_w(\mathbb{R})), \quad i=1,2$$

for some positive constants μ *and* C.

Furthermore, the solution $(u_1(x,t),u_2(x,t))$ *converges to the traveling wave front* $(\phi_1(x+ct),\phi_2(x+ct))$ *exponentially in time* t, *i.e.,*

$$\sup_{x\in\mathbb{R}}|u_i(x,t)-\phi_i(x+ct)| \le Ce^{-\mu t}, \quad t\ge 0,\ i=1,2.$$

Example 5.3. We consider the following reaction-diffusion system with delay

$$\tag{5.2} \begin{aligned} \frac{\partial u_1(x,t)}{\partial t} &= \frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha u_2(x,t-\tau_2),\\ \frac{\partial u_2(x,t)}{\partial t} &= d\frac{\partial^2 u_2(x,t)}{\partial x^2} - \beta u_2(x,t) + pu_1(x,t-\tau_1)e^{-qu_1(x,t-\tau_1)}, \end{aligned}$$

where $\alpha,p,q>0$.

Assume that $1<\alpha p/\beta\le e$. It is clear that (5.2) has a trivial equilibrium $\boldsymbol{u}_-=(0,0)$ and a unique positive equilibrium $\boldsymbol{u}_+=\left(\frac{1}{q}\ln\frac{\alpha p}{\beta},\frac{1}{\alpha q}\ln\frac{\alpha p}{\beta}\right)$. Moreover, $g_1(u)=pue^{-qu}$ is non-decreasing on $[0,1/q]$ and

$$g_1'(u)=pe^{-qu}(1-qu)\le g_1'(0)=p, \quad g_1''(u)=-pqe^{-qu}(2-qu)\le 0.$$

Thus, it is easy to see that (A1)–(A3) hold. We have the following result.

Theorem 5.4. *Assume that* $1<\alpha p/\beta\le e$, $\alpha+\frac{\beta}{\alpha}(1-\ln\frac{\alpha p}{\beta})<\min\{2,2\beta\}$. *For any given traveling wave front* $\Phi(\xi)=(\phi_1(\xi),\phi_2(\xi))$ $(\xi=x+ct)$ *of* (5.2) *with the wave speed*

$$c>\max\left\{c_{\min},(d+1)\sqrt{|p+\alpha-2\beta|},\sqrt{\frac{(d+1)^2}{d}|p+\alpha-2|}\right\},$$

if the initial data satisfies

$$0\le u_{10}(x,s)\le\frac{1}{q}\ln\frac{\alpha p}{\beta}, \qquad (x,s)\in\mathbb{R}\times[-\tau_1,0],$$
$$0\le u_{20}(x,s)\le\frac{1}{\alpha q}\ln\frac{\alpha p}{\beta}, \quad (x,s)\in\mathbb{R}\times[-\tau_2,0],$$

and the initial perturbations satisfy

$$u_{10}(x,s)-\phi_1(x+cs)\in C^0([-\tau_1,0],H^1_w(\mathbb{R})), \quad (x,s)\in\mathbb{R}\times[-\tau_1,0],$$
$$u_{20}(x,s)-\phi_2(x+cs)\in C^0([-\tau_2,0],H^1_w(\mathbb{R})), \quad (x,s)\in\mathbb{R}\times[-\tau_2,0],$$

where $w(\xi)$ *is the weight function defined as in* (2.16)*, then the solution* $(u_1(x,t), u_2(x,t))$ *of Cauchy problem* (5.2) *and* (1.3) *satisfies*

$$0 \le u_1(x,t) \le \frac{1}{q}\ln\frac{\alpha p}{\beta}, \quad 0 \le u_1(x,t) \le \frac{1}{\alpha q}\ln\frac{\alpha p}{\beta}, \quad (x,t) \in \mathbb{R}\times[0,+\infty),$$
$$u_i(x,t) - \phi_i(x+ct) \in C^0([0,+\infty), H_w^1(\mathbb{R})) \cap L^2([0,+\infty), H_w^2(\mathbb{R})), \quad i = 1,2$$

for some positive constants μ *and* C*.*

Furthermore, the solution $(u_1(x,t), u_2(x,t))$ *converges to the traveling wave front* $(\phi_1(x+ct), \phi_2(x+ct))$ *exponentially in time* t*, i.e.,*

$$\sup_{x\in\mathbb{R}} |u_i(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0,\ i = 1,2.$$

Acknowledgments

The authors thank the anonymous referees for their valuable comments and suggestions. This work is supported by the National Natural Science Foundation of China, RFDP and the Fundamental Research Funds for the Central University.

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