

## The Bressoud-Göllnitz-Gordon Theorem for Overpartitions of Even Moduli

Thomas Yao He, Allison Yi Fang Wang\* and Alice Xiao Hua Zhao

Abstract. We give an overpartition analogue of Bressoud’s combinatorial generalization of the Göllnitz-Gordon theorem for even moduli in general case. Let  $\tilde{O}_{k,i}(n)$  be the number of overpartitions of  $n$  whose parts satisfy certain difference condition and  $\tilde{P}_{k,i}(n)$  be the number of overpartitions of  $n$  whose non-overlined parts satisfy certain congruence condition. We show that  $\tilde{O}_{k,i}(n) = \tilde{P}_{k,i}(n)$  for  $1 \leq i < k$ .

### 1. Introduction

Throughout this article, we shall adopt the common notation as used in Andrews [3]. Let

$$(a)_\infty = (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i)$$

and

$$(a)_n = (a; q)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

The Rogers-Ramanujan identities [22] are the most fascinating identities in the theory of partitions. Gordon [14] made the first break-through with an infinite family of identities in 1961 by proving a combinatorial generalization of the Rogers-Ramanujan identities, which is stated as the following theorem.

**Theorem 1.1** (Rogers-Ramanujan-Gordon). *For  $k \geq i \geq 1$ . Let  $B_{k,i}(n)$  denote the number of partitions  $\lambda$  of  $n$  of the form  $\lambda = (1^{f_1}, 2^{f_2}, 3^{f_3}, \dots)$  where  $f_t(\lambda)$  ( $f_t$  for short) denotes the number of times the number  $t$  appears as a part in  $\lambda$  such that*

- (1)  $f_1(\lambda) \leq i - 1$ ;
- (2)  $f_t(\lambda) + f_{t+1}(\lambda) \leq k - 1$ .

*Let  $A_{k,i}(n)$  denote the number of partitions of  $n$  into parts  $\not\equiv 0, \pm i \pmod{2k+1}$ . Then, for all  $n \geq 0$ ,*

$$A_{k,i}(n) = B_{k,i}(n).$$

---

Received December 26, 2016; Accepted March 26, 2017.

Communicated by Sen-Peng Eu.

2010 *Mathematics Subject Classification*. 05A17, 11P84.

*Key words and phrases*. the Bressoud-Göllnitz-Gordon theorem, overpartition, Bailey pair, Göllnitz-Gordon marking.

\*Corresponding author.

In 1979, Bressoud [7] extended Rogers-Ramanujan-Gordon theorem to even moduli.

**Theorem 1.2** (Bressoud-Rogers-Ramanujan-Gordon). *For  $k > i \geq 1$ . Let  $\tilde{A}_{k,i}(n)$  denote the number of partitions of  $n$  of the form  $\lambda = (1^{f_1}, 2^{f_2}, 3^{f_3}, \dots)$  such that*

- (1)  $f_1(\lambda) \leq i - 1$ ;
- (2)  $f_t(\lambda) + f_{t+1}(\lambda) \leq k - 1$ ;
- (3) *If  $f_t(\lambda) + f_{t+1}(\lambda) = k - 1$ , then  $tf_t(\lambda) + (t + 1)f_{t+1}(\lambda) \equiv i - 1 \pmod{2}$ .*

Let  $\tilde{B}_{k,i}(n)$  denote the number of partitions of  $n$  into parts  $\not\equiv 0, \pm i \pmod{2k}$ . Then, for all  $n \geq 0$ ,

$$\tilde{A}_{k,i}(n) = \tilde{B}_{k,i}(n).$$

In [8], Bressoud found the following generating function version of Theorem 1.2.

**Theorem 1.3** (Bressoud). *For  $k > i \geq 1$ .*

$$(1.1) \quad \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}}}{(q)_{N_1 - N_2} (q)_{N_2 - N_3} \dots (q^2; q^2)_{N_{k-1}}} = \frac{(q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q)_\infty}.$$

Let  $\tilde{B}_{k,i}(m, n)$  denote the number of partitions enumerated by  $\tilde{B}_{k,i}(n)$  that have  $m$  parts. Using the notion of the Gordon marking of a partition, Kurşungöz [18, 19] obtained the generating function of  $\tilde{B}_{k,i}(m, n)$ .

**Theorem 1.4.** *For  $k \geq i \geq 1$ ,*

$$\sum_{n \geq 0} \tilde{B}_{k,i}(m, n) x^m q^n = \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1 - N_2} (q)_{N_2 - N_3} \dots (q^2; q^2)_{N_{k-1}}}.$$

Göllnitz-Gordon identities are independently introduced by Gordon [15, 16] and Göllnitz [12, 13]. In 1967, Andrews [2] found the combinatorial generalization of the Göllnitz-Gordon identities. In [8], Bressoud extended Andrews-Göllnitz-Gordon theorem to even moduli.

**Theorem 1.5** (Andrews-Bressoud-Göllnitz-Gordon). *For  $k > i \geq 1$ . Let  $\tilde{C}_{k,i}(n)$  denote the number of partitions of  $n$  of the form  $\lambda = (1^{f_1}, 2^{f_2}, 3^{f_3}, \dots)$  such that*

- (1)  $f_1(\lambda) + f_2(\lambda) \leq i - 1$ ;
- (2)  $f_{2t+1}(\lambda) \leq 1$ ;
- (3)  $f_{2t}(\lambda) + f_{2t+1}(\lambda) + f_{2t+2}(\lambda) \leq k - 1$ ;

(4) If  $f_{2t}(\lambda) + f_{2t+1}(\lambda) + f_{2t+2}(\lambda) = k - 1$ , then  $tf_{2t}(\lambda) + tf_{2t+1}(\lambda) + (t + 1)f_{2t+2}(\lambda) \equiv O_{2t+1} + i - 1 \pmod{2}$ , where  $O_{2t+1}(\lambda)$  denotes the number of odd parts in  $\lambda$  those not exceed  $2t + 1$ .

Let  $\tilde{D}_{k,i}(n)$  denote the number of partitions of  $n$  into parts  $\not\equiv 2 \pmod{4}$  and  $\not\equiv 0, \pm(2i - 1) \pmod{4k - 2}$ . Then, for all  $n \geq 0$ ,

$$\tilde{C}_{k,i}(n) = \tilde{D}_{k,i}(n).$$

Bressoud [8] obtained the following generating function version of Theorem 1.5.

**Theorem 1.6** (Bressoud). For  $k > i \geq 1$ .

$$(1.2) \quad \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})}}{(q^2; q^2)_{N_1 - N_2} (q^2; q^2)_{N_2 - N_3} \dots (q^4; q^4)_{N_{k-1}}} \\ = \frac{(q^2; q^4)_\infty (q^{4k-2}, q^{2i-1}, q^{4k-2i-1}; q^{4k-2})_\infty}{(q; q)_\infty}.$$

Recall that an overpartition  $\lambda$  of  $n$  is a partition of  $n$  in which the first occurrence of a number can be overlined. In this paper, we write an overpartition  $\lambda$  as the form  $(a^{f_a}, \bar{a}^{f_{\bar{a}}}, \dots, 1^{f_1}, \bar{1}^{f_{\bar{1}}})$  where  $f_t$  (resp.  $f_{\bar{t}}$ ) denotes the number of times the number  $t$  (resp.  $\bar{t}$ ) appears as a part (resp. overlined part) in  $\lambda$ .

Using the  $q$ -difference method, Chen, Sang and Shi [11] also obtained overpartition analogue of Theorem 1.2 in the general case.

**Theorem 1.7.** For  $k > i \geq 1$ . Let  $\tilde{E}_{k,i}(n)$  denote the number of overpartitions of  $n$  of the form  $(\bar{1}^{f_{\bar{1}}}, 1^{f_1}, \bar{2}^{f_{\bar{2}}}, 2^{f_2}, \dots)$  such that

- (1)  $f_1(\lambda) \leq i - 1$ ;
- (2)  $f_t(\lambda) + f_{\bar{t}}(\lambda) + f_{t+1}(\lambda) \leq k - 1$ ;
- (3) If  $f_t(\lambda) + f_{\bar{t}}(\lambda) + f_{t+1}(\lambda) = k - 1$  then  $tf_t(\lambda) + tf_{\bar{t}}(\lambda) + (t + 1)f_{t+1}(\lambda) \equiv V_t(\lambda) + i - 1 \pmod{2}$ , where  $V_s(\lambda)$  denotes the number of overlined parts those not exceed  $s$ .

Let  $\tilde{F}_{k,i}(n)$  denote the number of overpartitions of  $n$  whose non-overlined parts are not congruent to  $0, \pm i$  modulo  $2k - 1$ . Then, for all  $n \geq 0$ ,

$$\tilde{E}_{k,i}(n) = \tilde{F}_{k,i}(n).$$

Chen, Sang and Shi [11] also gave generating function version of Theorem 1.7.

**Theorem 1.8.** For  $k > i \geq 1$ .

$$(1.3) \quad \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-N_1})_{N_1-1} q^{N_1^2+N_2^2+\dots+N_{k-1}^2+N_{i+1}+\dots+N_{k-1}} (1+q^{N_i})}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}} \\ = \frac{(-q)_\infty (q^i, q^{2k-i-1}, q^{2k-1}; q^{2k-1})_\infty}{(q)_\infty}.$$

Let  $\tilde{E}_{k,i}(m, n)$  denote the number of overpartitions counted by  $\tilde{E}_{k,i}(n)$  with exactly  $m$  parts, Sang and Shi [23] gave a combinatorial proof of the following generating function of  $\tilde{E}_{k,i}(m, n)$ .

**Theorem 1.9.** For  $k > i \geq 1$ , we have

$$\sum_{m, n \geq 0} \tilde{E}_{k,i}(m, n) x^m q^n \\ = \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-N_1})_{N_1-1} q^{N_1^2+\dots+N_{k-1}^2+N_{i+1}+\dots+N_{k-1}} (1+q^{N_i}) x^{N_1+\dots+N_{k-1}}}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.$$

In this article, we will give an another proof of Theorem 1.8 in the next section by using Bailey’s lemma and the change of base formula and a direct combinatorial proof of Theorem 1.9 by introduce the notation of Gordon marking for an overpartition introduced by Chen, Sang and Shi [10].

The main result of this article is the following overpartition analogue of Theorem 1.5.

**Theorem 1.10.** For  $k > i \geq 1$ . Let  $\tilde{O}_{k,i}(n)$  denote the number of overpartitions of  $n$  of the form  $(a^{f_a}, \bar{a}^{f_{\bar{a}}}, \dots, 1^{f_1}, \bar{1}^{f_{\bar{1}}})$  such that

- (1)  $f_{\bar{1}}(\lambda) + f_2(\lambda) \leq i - 1$ ;
- (2)  $f_{2t}(\lambda) + f_{\bar{2t}}(\lambda) + f_{\overline{2t+1}}(\lambda) + f_{2t+2}(\lambda) \leq k - 1$ ;
- (3) If  $f_{2t+1}(\lambda) \geq 1$ , then  $f_{2t+2}(\lambda) \leq k - 2$ ;
- (4) If  $f_{2t}(\lambda) + f_{\bar{2t}}(\lambda) + f_{\overline{2t+1}}(\lambda) + f_{2t+2}(\lambda) = k - 1$  then  $tf_{2t}(\lambda) + tf_{\bar{2t}}(\lambda) + tf_{\overline{2t+1}}(\lambda) + (t + 1)f_{2t+2}(\lambda) \equiv V_{2t+1}(\lambda) + i - 1 \pmod{2}$ ;
- (5) If  $f_{2t+1}(\lambda) \geq 1$  and  $f_{2t+2}(\lambda) = k - 2$ , then  $t + (t + 1)f_{2t+2}(\lambda) \equiv V_{2t+1}(\lambda) + i - 1 \pmod{2}$ ,

where  $V_s(\lambda)$  denotes the number of overlined parts those not exceed  $s$ .

Let  $\tilde{P}_{k,i}(n)$  denote the number of overpartitions of  $n$  whose non-overlined parts are not congruent to 0,  $\pm(2i - 1)$  modulo  $4k - 4$ . Then, for all  $n \geq 0$ ,

$$\tilde{O}_{k,i}(n) = \tilde{P}_{k,i}(n).$$

It should be noted that if an overpartition  $\lambda$  counted by  $\tilde{O}_{k,i}(n)$  does not contain overlined even parts and non-overlined odd part, and we change the overlined odd parts in  $\lambda$  to non-overlined parts, then we get a partition enumerated by  $\tilde{C}_{k,i}(n)$ . Hence we say that Theorem 1.10 is the overpartition analogue of Theorem 1.5.

The corresponding generating function version of Theorem 1.10 is given as the following theorem.

**Theorem 1.11.** For  $k > i \geq 1$ ,

$$\begin{aligned}
 (1.4) \quad & \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i})}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^4; q^4)_{N_{k-1}}} \\
 &= \frac{(-q; q)_\infty (q^{2i-1}, q^{4k-3-2i}, q^{4k-4}; q^{4k-4})_\infty}{(q; q)_\infty}.
 \end{aligned}$$

We will first give an analytic proof of Theorem 1.11 in the next section by using Bailey’s lemma and the change of base formula. We then use Theorem 1.11 to derive Theorem 1.10. To be more precisely, let  $\tilde{O}_{k,i}(m, n)$  denote the number of overpartitions counted by  $\tilde{O}_{k,i}(n)$  with exactly  $m$  parts, we shall give a combinatorial proof of the following generating function of  $\tilde{O}_{k,i}(m, n)$  by introducing the Göllnitz-Gordon marking [17] of an overpartition.

**Theorem 1.12.** For  $k > i \geq 1$ , we have

$$\begin{aligned}
 (1.5) \quad & \sum_{m, n \geq 0} \tilde{O}_{k,i}(m, n) x^m q^n \\
 &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i}) x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^4; q^4)_{N_{k-1}}}.
 \end{aligned}$$

By setting  $x = 1$  in (1.5), we obtain the generating function for  $\tilde{O}_{k,i}(n)$  which is the left-hand side of (1.4). On the other hand, it is evident that the generating function of  $\tilde{P}_{k,i}(n)$  equals

$$\sum_{n \geq 0} \tilde{P}_{k,i}(n) q^n = \frac{(-q; q)_\infty (q^{2i-1}, q^{4k-3-2i}, q^{4k-4}; q^{4k-4})_\infty}{(q; q)_\infty},$$

which is the right-hand side of (1.4). Hence we are led to Theorem 1.10 by Theorem 1.11.

This paper is organized as follows. In Section 2, we show Theorems 1.8 and 1.11 by using Bailey’s lemma and the change of base formula due to Bressoud, Ismail and Stanton. In Section 3, we recall the definition and related notations of Gordon marking of an overpartition and give the outline of the proof of Theorem 1.9. In Section 4, we show

the detail of the proof of Theorem 1.9. In Section 5, we recall the definition and related notations of Göllnitz-Gordon marking of an overpartition and give an outline of the proof of Theorem 1.12. In Section 6, we show the proof of Theorem 1.12, thus we complete the proof of Theorem 1.10.

## 2. Proofs of Theorems 1.8 and 1.11

We first briefly review Bailey pairs and Bailey’s lemma. Recall that a pair of sequences  $(\alpha_n(a, q), \beta_n(a, q))$  is called a Bailey pair with parameters  $(a, q)$  if they have the following relation for all  $n \geq 0$ ,

$$\beta_n(a, q) = \sum_{r=0}^n \frac{\alpha_r(a, q)}{(q; q)_{n-r}(aq; q)_{n+r}}.$$

Bailey’s lemma was first given by Bailey [6] and was formulated by Andrews [4, 5] in the following form.

**Theorem 2.1** (Bailey’s lemma). *If  $(\alpha_n(a, q), \beta_n(a, q))$  is a Bailey pair with parameters  $(a, q)$ , then  $(\alpha'_n(a, q), \beta'_n(a, q))$  is another Bailey pair with parameters  $(a, q)$ , where*

$$\begin{aligned} \alpha'_n(a, q) &= \frac{(\rho_1; q)_n(\rho_2; q)_n}{(aq/\rho_1; q)_n(aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a, q), \\ \beta'_n(a, q) &= \sum_{j=0}^n \frac{(\rho_1; q)_j(\rho_2; q)_j(aq/\rho_1\rho_2; q)_{n-j}}{(aq/\rho_1; q)_n(aq/\rho_2; q)_n(q; q)_{n-j}} \left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j(a, q). \end{aligned}$$

Andrews first noticed that Bailey’s lemma can create a new Bailey pair from a given one. Hence the iteration of the lemma leads to a sequence of Bailey pairs called a Bailey chain. Based on this observation, Andrews [4] used Bailey’s lemma to show the Bressoud-Gordon identity (1.1) in Theorem 1.3 with  $i = 1$  and  $i = k$ . Subsequently, Agarwal, Andrews and Bressoud [1] gave an extension of the Bailey chain known as the Bailey lattice, by means of which enable us to prove the Andrews-Gordon identity and the Bressoud-Gordon identity (1.1) with  $1 \leq i \leq k$ . In [9], Bressoud, Ismail and Stanton established versions of Bailey’s lemma, known as the change of base formulas, which can be used to prove the Bressoud-Göllnitz-Gordon identity (1.2).

In this section, we will show Theorems 1.8 and 1.11 by combining Bailey’s lemma and the change of base formula. First, we need to review two limiting cases of Bailey’s lemma which are both appeared in [20, 21]. The first limiting case is obtained by letting  $\rho_1, \rho_2 \rightarrow \infty$  in Theorem 2.1.

**Lemma 2.2.** *If  $(\alpha_n(a, q), \beta_n(a, q))$  is a Bailey pair with parameters  $(a, q)$ , then  $(\alpha'_n, \beta'_n)$  is also a Bailey pair with parameters  $(a, q)$ , where*

$$\alpha'_n(a, q) = a^n q^{n^2} \alpha_n(a, q) \quad \text{and} \quad \beta'_n(a, q) = \sum_{j=0}^n \frac{a^j q^{j^2}}{(q)_{n-j}} \beta_j(a, q).$$

The second limiting case is obtained by letting  $\rho_1 \rightarrow \infty, \rho_2 = -1$  in Theorem 2.1.

**Lemma 2.3.** *If  $(\alpha_n(a, q), \beta_n(a, q))$  is a Bailey pair with parameters  $(a, q)$ , then  $(\alpha'_n, \beta'_n)$  is also a Bailey pair with parameters  $(a, q)$ , where*

$$\alpha'_n(a, q) = \frac{(-1)_n a^n q^{(n^2+n)/2}}{(-aq)_n} \alpha_n(a, q),$$

$$\beta'_n(a, q) = \frac{1}{(-aq)_n} \sum_{j=0}^n \frac{(-1)_j a^j q^{(j^2+j)/2}}{(q)_{n-j}} \beta_j(a, q).$$

The proof of Theorem 1.11 requires the following special case of change of base formula.

**Lemma 2.4.** [9, Theorem 2.5,  $B \rightarrow \infty$ ] *If  $(\alpha_n(a, q), \beta_n(a, q))$  is a Bailey pair with parameters  $(a, q)$ , then  $(\alpha'_n, \beta'_n)$  is also a Bailey pair with parameters  $(a, q)$ , where*

$$\alpha'_n(a, q) = \frac{1+a}{1+aq^{2n}} q^n \alpha_n(a^2, q^2) \quad \text{and} \quad \beta'_n(a, q) = \sum_{k=0}^n \frac{(-a; q)_{2k} q^k}{(q^2; q^2)_{n-k}} \beta_k(a^2, q^2).$$

The following proposition is useful in our proofs as well.

**Proposition 2.5.** [9, Proposition 4.1] *If  $(\alpha_n, \beta_n)$  is a Bailey pair with parameters  $(1, q)$ , and*

$$\alpha_n = \begin{cases} 1 & \text{for } n = 0, \\ (-1)^n q^{An^2} (q^{(A-1)n} + q^{-(A-1)n}) & \text{for } n > 0, \end{cases}$$

*then  $(\alpha'_n(q), \beta'_n(q))$  is a Bailey pair with parameters  $(1, q)$ , where*

$$\alpha'_n(q) = \begin{cases} 1 & \text{for } n = 0, \\ (-1)^n q^{An^2} (q^{An} + q^{-An}) & \text{for } n > 0, \end{cases} \quad \text{and} \quad \beta'_n(q) = q^n \beta_n(q).$$

*Proof of Theorem 1.8.* We begin with the following Bailey pair [24, E(4)] with parameters  $(1, q)$ , where

$$\alpha_n^{(1)}(1, q) = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{n^2} (q^{-n} + q^n) & \text{if } n \geq 1, \end{cases} \quad \text{and} \quad \beta_n^{(1)}(1, q) = \frac{q^n}{(q^2; q^2)_n}.$$

Invoking Lemma 2.2 gives the following Bailey pair

$$\alpha_n^{(2)}(1, q) = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{2n^2} (q^{-n} + q^n) & \text{if } n \geq 1, \end{cases}$$

$$\beta_n^{(2)}(1, q) = \sum_{N_{k-1}=0}^n \frac{q^{N_{k-1}^2 + N_{k-1}}}{(q)_{n-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.$$

Alternately apply Proposition 2.5 and Lemma 2.2 for  $k - i - 2$  times. The result is

$$\begin{aligned}
 \alpha_n^{(2k-2i-2)}(1, q) &= \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{(k-i)n^2} (q^{-(k-i-1)n} + q^{(k-i-1)n}) & \text{if } n \geq 1, \end{cases} \\
 \beta_n^{(2k-2i-2)}(1, q) &= \sum_{n \geq N_{i+1} \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_{i+1}^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{(q)_{n-N_{i+1}} (q)_{N_{i+1}-N_{i+2}} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}
 \tag{2.1}$$

Applying Proposition 2.5 to the Bailey pair (2.1) yields

$$\begin{aligned}
 \alpha_n^{(2k-2i-1)}(1, q) &= \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{(k-i)n^2} (q^{-(k-i)n} + q^{(k-i)n}) & \text{if } n \geq 1, \end{cases} \\
 \beta_n^{(2k-2i-1)}(1, q) &= q^n \sum_{n \geq N_{i+1} \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_{i+1}^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{(q)_{n-N_{i+1}} (q)_{N_{i+1}-N_{i+2}} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}
 \tag{2.2}$$

Adding these two Bailey pairs (2.1) and (2.2) together according to the definition of Bailey pair, we get a new Bailey pair relative to  $(1, q)$

$$\begin{aligned}
 \alpha_n^{(2k-2i)}(1, q) &= \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{(k-i)n^2} (q^{-(k-i)n} + q^{(k-i-1)n})(1 + q^n)/2 & \text{if } n \geq 1, \end{cases} \\
 \beta_n^{(2k-2i)}(1, q) &= \sum_{n \geq N_{i+1} \geq \dots \geq N_{k-1} \geq 0} \frac{(1 + q^n) q^{N_{i+1}^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{2(q)_{n-N_{i+1}} (q)_{N_{i+1}-N_{i+2}} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}
 \tag{2.3}$$

Then applying Lemma 2.2 to (2.3)  $i - 1$  times, we can get a Bailey pair  $(\alpha_n^{(2k-i-1)}(1, q), \beta_n^{(2k-i-1)}(1, q))$  with parameters  $(1, q)$ , where

$$\begin{aligned}
 \alpha_n^{(2k-i-1)}(1, q) &= \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{(k-1)n^2} (q^{-(k-i)n} + q^{(k-i-1)n})(1 + q^n)/2 & \text{if } n \geq 1, \end{cases} \\
 \beta_n^{(2k-i-1)}(1, q) &= \sum_{n \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_2^2 + N_3^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} (1 + q^{N_i})}{2(q)_{n-N_2} (q)_{N_2-N_3} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}
 \tag{2.4}$$

Invoking Lemma 2.3 to (2.4) with parameters  $(1, q)$  gives

$$\begin{aligned}
 \alpha_n^{(2k-i)}(1, q) &= \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{(2k-1)n^2/2} (q^{-(2k-2i-1)n/2} + q^{(2k-2i-1)n/2}) & \text{if } n \geq 1, \end{cases} \\
 \beta_n^{(2k-i)}(1, q) &= \frac{1}{(-q)_n} \sum_{n \geq N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-1)_{N_1} q^{(N_1+1)N_1/2 + N_2^2 + N_3^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} (1 + q^{N_i})}{2(q)_{n-N_1} (q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}$$



By the definition of Bailey pairs, we have

$$\begin{aligned}
 (2.5) \quad & \frac{1}{(-q)_n} \sum_{n \geq N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-1)_{N_1} q^{(N_1+1)N_1/2 + N_2^2 + N_3^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} (1 + q^{N_i})}{2(q)_{n-N_1} (q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}} \\
 &= \frac{1}{(q)_n (q)_n} + \sum_{r=1}^n \frac{(-1)^r q^{(2k-1)r^2/2} (q^{-(2k-2i-1)r/2} + q^{(2k-2i-1)r/2})}{(q)_{n-r} (q)_{n+r}}.
 \end{aligned}$$

Letting  $n \rightarrow \infty$  and multiplying both sides by  $(q^2; q^2)_\infty$  in (2.5), we obtain

$$\begin{aligned}
 (2.6) \quad & \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q)_{N_1-1} q^{(N_1+1)N_1/2 + N_2^2 + N_3^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} (1 + q^{N_i})}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}} \\
 &= \frac{(-q)_\infty}{(q)_\infty} \left( 1 + \sum_{n=1}^\infty (-1)^n q^{(2k-1)n^2/2} (q^{-(2k-2i-1)n/2} + q^{(2k-2i-1)n/2}) \right).
 \end{aligned}$$

Letting  $q \rightarrow q^{(2k-1)/2}$ ,  $z \rightarrow -q^{(2k-2i-1)/2}$  in the following Jacobi's triple product identity

$$(2.7) \quad 1 + \sum_{n=1}^\infty q^{n^2} (z^{-n} + z^n) = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty.$$

We derive that

$$\begin{aligned}
 (2.8) \quad & 1 + \sum_{n=1}^\infty (-1)^n q^{(2k-1)n^2/2} (q^{-(2k-2i-1)n/2} + q^{(2k-2i-1)n/2}) \\
 &= (q^i, q^{2k-i-1}, q^{2k-1}; q^{2k-1})_\infty.
 \end{aligned}$$

Submitting (2.8) into (2.6), and noting that

$$(-q)_{N_1-1} q^{(N_1+1)N_1/2} = (-q^{1-N_1})_{N_1-1} q^{N_1^2},$$

we obtain (1.3). Thus we complete the proof of Theorem 1.8. □

*Proof of Theorem 1.11.* Invoking Lemma 2.4 to (2.4) with parameters  $(1, q)$ , we get another Bailey pair  $(\alpha_n^{(2k-i)}(1, q), \beta_n^{(2k-i)}(1, q))$  with parameters  $(1, q)$ , where

$$\begin{aligned}
 & \alpha_n^{(2k-i)}(1, q) = \begin{cases} 1 & \text{if } n = 0, \\ (-1)^n q^{2(k-1)n^2} (q^{-(2k-2i-1)n} + q^{(2k-2i-1)n}) & \text{if } n \geq 1, \end{cases} \\
 & \beta_n^{(2k-i)}(1, q) \\
 &= \sum_{n \geq N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q; q)_{2N_1-1} q^{N_1+2(N_2^2+N_3^2+\dots+N_{k-1}^2+N_{i+1}+\dots+N_{k-1})} (1 + q^{2N_i})}{(q^2; q^2)_{n-N_1} (q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^4; q^4)_{N_{k-1}}}.
 \end{aligned}$$

By the definition of Bailey pairs, letting  $n \rightarrow \infty$  and multiplying both sides by  $(q^2; q^2)_\infty$ , we obtain

$$(2.9) \quad \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q)_{2N_1-1} q^{N_1+2(N_2^2+N_3^2+\dots+N_{k-1}^2+N_{i+1}+\dots+N_{k-1})} (1+q^{2N_i})}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^4; q^4)_{N_{k-1}}} \\ = \frac{(-q)_\infty}{(q)_\infty} \left( 1 + \sum_{n=1}^\infty (-1)^n q^{2(k-1)n^2} (q^{-(2k-2i-1)n} + q^{(2k-2i-1)n}) \right).$$

Letting  $q \rightarrow q^{2(k-1)}$ ,  $z \rightarrow -q^{(2k-2i-1)}$  in (2.7), we derive that

$$(2.10) \quad 1 + \sum_{n=1}^\infty (-1)^n q^{2(k-1)n^2} (q^{-(2k-2i-1)n} + q^{(2k-2i-1)n}) \\ = (q^{2i-1}, q^{4k-2i-3}, q^{4k-4}; q^{4k-4})_\infty.$$

Submitting (2.10) into (2.9), and noting that

$$(-q)_{2N_1-1} q^{N_1} = (-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2N_1^2},$$

we arrive at (1.4). Thus, we complete the proof of Theorem 1.11. □

### 3. Gordon marking of an overpartition

In this section, we give an outline of the proof of Theorem 1.9. Let  $\tilde{\mathbb{E}}_{k,i}(m, n)$  denote the set of overpartitions counted by  $\tilde{E}_{k,i}(m, n)$ , we further classify  $\tilde{\mathbb{E}}_{k,i}(m, n)$  by considering whether the smallest part is overlined or not. Note that the parts of an overpartition are ordered in the following order

$$(3.1) \quad \bar{1} < 1 < \bar{2} < 2 < \dots$$

Let  $\tilde{\mathbb{G}}_{k,i}(m, n)$  denote the set of overpartitions in  $\tilde{\mathbb{E}}_{k,i}(m, n)$  for which the smallest part is non-overlined, and let  $\tilde{\mathbb{H}}_{k,i}(m, n)$  denote the set of overpartitions in  $\tilde{\mathbb{E}}_{k,i}(m, n)$  with the smallest part is overlined.

Let  $\tilde{G}_{k,i}(m, n) = |\tilde{\mathbb{G}}_{k,i}(m, n)|$  and  $\tilde{H}_{k,i}(m, n) = |\tilde{\mathbb{H}}_{k,i}(m, n)|$ . Sang and Shi obtained [23] a relation between  $\tilde{G}_{k,i}(m, n)$  and  $\tilde{H}_{k,i}(m, n)$ .

**Theorem 3.1.** *For  $k > i \geq 2$ , we have*

$$(3.2) \quad \tilde{G}_{k,i}(m, n) = \tilde{H}_{k,i-1}(m, n).$$

*For  $k > i = 1$ , we have*

$$(3.3) \quad \tilde{G}_{k,1}(m, n) = \tilde{H}_{k,k-1}(m, n - m).$$



where column indicates the value of parts, and the row (counted from bottom to top) indicates the mark.

Let  $N_r$  be the number of parts in the  $r$ -th row of  $G(\lambda)$ . It is easy to find that  $N_1 \geq N_2 \geq \dots$ . For the example above, we have  $N_1 = 8, N_2 = 7, N_3 = 5$ .

Now, let  $\tilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}(n), \tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n)$  and  $\tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}(n)$  denote the sets of overpartitions  $\lambda$  in  $\tilde{\mathbb{E}}_{k, i}(m, n), \tilde{\mathbb{G}}_{k, i}(m, n)$  and  $\tilde{\mathbb{H}}_{k, i}(m, n)$  respectively that have  $N_r$   $r$ -marked parts in their Gordon marking  $G(\lambda)$  for  $1 \leq r \leq k - 1$ , where  $\sum_{r=1}^{k-1} N_r = m$ . Obviously, we have

$$(3.4) \quad \tilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}(n) = \tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n) \cup \tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}(n).$$

Let  $\tilde{G}_{N_1, \dots, N_{k-1}; i}(n) = |\tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n)|$  and  $\tilde{H}_{N_1, \dots, N_{k-1}; i}(n) = |\tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}(n)|$ . Using the same method proving Theorem 3.1, we can easily get a relation between  $\tilde{G}_{N_1, \dots, N_{k-1}; i}(n)$  and  $\tilde{H}_{N_1, \dots, N_{k-1}; i}(n)$ .

**Lemma 3.2.** *For  $k > i \geq 2$ , we have*

$$(3.5) \quad \tilde{G}_{N_1, \dots, N_{k-1}; i}(n) = \tilde{H}_{N_1, \dots, N_{k-1}; i-1}(n).$$

*For  $k > i = 1$ , we have*

$$(3.6) \quad \tilde{G}_{N_1, \dots, N_{k-1}; 1}(n) = \tilde{H}_{N_1, \dots, N_{k-1}; k-1}(n - N_1 - \dots - N_{k-1}).$$

Let  $\tilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i}(n)$  denote the set of overpartitions in  $\tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n)$  for which there do not exist overlined part.

Set

$$\begin{aligned} \tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i} &= \bigcup_{n \geq 0} \tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n), \\ \tilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i} &= \bigcup_{n \geq 0} \tilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i}(n). \end{aligned}$$

We shall give a bijection for the following relation.

**Theorem 3.3.** *For  $N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0$ , we have*

$$\sum_{\lambda \in \tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\lambda)} q^{|\lambda|} = (-q^{1-N_1})_{N_1-1} \sum_{\mu \in \tilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\mu)} q^{|\mu|},$$

where  $\ell(\lambda)$  denotes the number of parts of  $\lambda$ .





which lay in  $[-N, -1]$ . To prove Theorem 3.3, we give a bijection between  $\widetilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}$  and  $R_{N_1-1} \times \widetilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i}$ . More precisely, set

$$\mathbb{RB}_{N_1, \dots, N_{k-1}; i}(n) = \left\{ (\tau, \mu) \in R_{N_1-1} \times \widetilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i} \mid |\tau| + |\mu| = n \right\},$$

we establish a bijection between  $\widetilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n)$  and  $\mathbb{RB}_{N_1, \dots, N_{k-1}; i}(n)$ .

On the one hand, for an overpartition  $\lambda \in \widetilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n)$ , we construct a pair  $(\tau, \mu) \in \mathbb{RB}_{N_1, \dots, N_{k-1}; i}(n)$ .

If there is no overlined part in  $\lambda$ , we set  $\tau = \emptyset$  and  $\mu = \lambda$ .

If there exist overlined parts in  $\lambda$ , then the overlined parts must be marked 1, set the overlined parts be  $\lambda_{j_1}^{(1)}, \lambda_{j_2}^{(1)}, \dots, \lambda_{j_s}^{(1)}$ , where  $1 \leq j_1 < j_2 < \dots < j_s < N_1$ . Let  $M_0 = 0$  and  $M_l = \sum_{t=1}^l j_t$  where  $1 \leq l \leq s$ . Set  $\lambda^0 = \lambda$ , we do the following operations for  $l$  from 1 to  $s$  successively.

For each  $l$ , we iterate the following operation for  $p$  from 1 to  $j_l$ : we do the first dilation of  $p$ -th kind of  $\lambda^{M_{l-1}+p-1}$  and denote the resulting overpartition by  $\lambda^{M_{l-1}+p}$ .

After the  $M_s$  operations above, we can get an overpartition  $\lambda^{M_s}$  which contains no overlined part, what is more,  $\lambda^{M_s} \in \widetilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i}(n + M_s)$ . Finally, set  $\mu = \lambda^{M_s}$  and  $\tau = (-j_1, -j_2, \dots, -j_s)$ , we get the desired pair  $(\tau, \mu) \in \mathbb{RB}_{N_1, \dots, N_{k-1}; i}(n)$ .

On the contrary, for a pair  $(\tau, \mu) \in \mathbb{RB}_{N_1, \dots, N_{k-1}; i}(n)$ , we want to construct an overpartition  $\lambda \in \widetilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n)$ .

If  $\tau = \emptyset$ , we just set  $\lambda = \mu$ . If  $\tau \neq \emptyset$ , then we set  $\tau = (-j_1, -j_2, \dots, -j_s)$  where  $1 \leq j_1 < j_2 < \dots < j_s < N_1$ . Let  $M_l = \sum_{t=1}^l j_t$  where  $1 \leq l \leq s$  and set  $\lambda^{M_s} = \mu$ , we do the following operations for  $l$  from  $s$  to 1 successively.

For each  $l$ , we iterate the following operation for  $p$  from 1 to  $j_l$ : we do the first reduction of  $p$ -th kind of  $\mu^{M_l-p+1}$  and denote the resulting overpartition by  $\mu^{M_l-p}$ .

After the  $M_s$  operations above, we can get an overpartition  $\mu^0$  which belongs to  $\widetilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}(n)$ . Finally, we just need to set  $\lambda = \mu^0$ . Thus we complete the proof of Theorem 3.3. □

Now we are in a position to prove Theorem 1.9. We first recall construction called Gordon marking of a partition [19].

For a partition  $\eta = (\eta_1, \eta_2, \dots, \eta_m)$  where  $1 \leq \eta_1 \leq \eta_2 \leq \dots \leq \eta_m$ . The Gordon marking of  $\eta$  is an assignment of positive integers (marks) to  $\eta$  from smallest part to largest part such that the marks are as small as possible subject to equal or consecutive parts are assigned distinct marks. For example, the Gordon marking of  $\eta = (1, 1, 2, 3, 3, 3, 4, 5, 6, 7, 7, 8)$  is

$$(1_1, 1_2, 2_3, 3_1, 3_2, 3_4, 4_3, 5_1, 6_2, 7_1, 7_3, 7_4, 8_2),$$

which can be represented by an array as follows, where column indicates the value of parts and the row (counted from bottom to top) indicates the mark:

$$\begin{bmatrix} & 3 & & 7 & \\ & 2 & 4 & 7 & \\ 1 & 3 & & 6 & 8 \\ 1 & 3 & 5 & 7 & \end{bmatrix} \begin{matrix} 4 \\ 3 \\ 2 \\ 1 \end{matrix} .$$

Let  $\mathbb{B}_{N_1, \dots, N_{k-1}; i}(n)$  denote the set of partitions  $\eta$  counted by  $B_{k,i}(m, n)$  that have  $N_r$   $r$ -marked parts in the Gordon marking of  $\eta$  for  $1 \leq r \leq k - 1$ , where  $m = \sum_{r=1}^{k-1} N_r$ .

Define

$$\mathbb{B}_{N_1, \dots, N_{k-1}; i} = \sum_{n \geq 0} \mathbb{B}_{N_1, \dots, N_{k-1}; i}(n).$$

Applying Kurşungöz’s bijection for Theorem 1.4 in [19], we see that for  $k > i \geq 1$ ,

$$(4.1) \quad \sum_{\eta \in \mathbb{B}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\eta)} q^{|\eta|} = \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1 - N_2} \cdots (q)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}.$$

Notice that ordinary partitions can be regarded as special overpartitions with no overlined part. For an ordinary partition  $\lambda$ , it is easy to check that the Gordon marking for partition of  $\lambda$  is the same as the Gordon marking for overpartition of  $\lambda$ . Thus we can get that  $\tilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i} = \mathbb{B}_{N_1, \dots, N_{k-1}; i}$ . By (4.1), we obtain that for  $k > i \geq 1$ ,

$$\begin{aligned} \sum_{\mu \in \tilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\mu)} q^{|\mu|} &= \sum_{\eta \in \mathbb{B}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\eta)} q^{|\eta|} \\ &= \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1 - N_2} \cdots (q)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned}$$

By Theorem 3.3, we get that for  $k > i \geq 1$ ,

$$\begin{aligned} \sum_{\lambda \in \tilde{\mathcal{G}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\lambda)} q^{|\lambda|} &= (-q^{1-N_1})_{N_1-1} \sum_{\mu \in \tilde{\mathbb{B}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\mu)} q^{|\mu|} \\ &= \frac{(-q^{1-N_1})_{N_1-1} q^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1 - N_2} \cdots (q)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned}$$

Given the relation between  $\tilde{\mathcal{G}}_{N_1, \dots, N_{k-1}; i}(n)$  and  $\tilde{\mathcal{H}}_{N_1, \dots, N_{k-1}; i}(n)$  as stated in Lemma 3.2, let  $\tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i} = \bigcup_{n \geq 0} \tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}(n)$ , we get the generating function for  $\tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}$ :

**Theorem 4.3.** For  $k > i \geq 1$ ,

$$(4.2) \quad \sum_{\pi \in \tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\pi)} q^{|\pi|} = \frac{(-q^{1-N_1})_{N_1-1} q^{N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1 - N_2} \cdots (q)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}.$$



*Proof.* From the relation (3.5), we deduce that for  $1 \leq i < k - 1$ ,

$$\begin{aligned}
 (4.3) \quad \sum_{\pi \in \tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\pi)} q^{|\pi|} &= \sum_{\lambda \in \tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i+1}} x^{\ell(\lambda)} q^{|\lambda|} \\
 &= \frac{(-q^{1-N_1})_{N_1-1} q^{N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}$$

For  $i = k - 1$ , from (3.6) it follows that

$$\begin{aligned}
 (4.4) \quad \sum_{\pi \in \tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; k-1}} x^{\ell(\pi)} q^{|\pi|} &= \sum_{\lambda \in \tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; 1}} x^{\ell(\lambda)} q^{|\lambda| - N_1 - \dots - N_{k-1}} \\
 &= \frac{(-q^{1-N_1})_{N_1-1} q^{N_1^2 + \dots + N_{k-1}^2} x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}$$

Observe that the above formulas (4.3) for  $1 \leq i < k - 1$  and (4.4) for  $i = k - 1$  take the same form (4.2) as in the theorem. This completes the proof.  $\square$

We are now ready to finish the proof of Theorem 1.9.

*Proof of Theorem 1.9.* By the generating functions of  $\tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}$  and  $\tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}$  and relation (3.4), we find that

$$\begin{aligned}
 (4.5) \quad &\sum_{\delta \in \tilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\delta)} q^{|\delta|} \\
 &= \sum_{\lambda \in \tilde{\mathbb{G}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\lambda)} q^{|\lambda|} + \sum_{\pi \in \tilde{\mathbb{H}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\pi)} q^{|\pi|} \\
 &= \frac{(-q^{1-N_1})_{N_1-1} q^{N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} (1 + q^{N_i}) x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 &\sum_{m, n \geq 0} \tilde{E}_{k,i}(m, n) x^m q^n \\
 &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \sum_{\delta \in \tilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\delta)} q^{|\delta|} \\
 &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-N_1})_{N_1-1} q^{N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}} (1 + q^{N_i}) x^{N_1 + \dots + N_{k-1}}}{(q)_{N_1-N_2} \cdots (q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}.
 \end{aligned}$$

This completes the proof of Theorem 1.9.  $\square$



It is not hard to see that the non-overlined odd parts and overlined even parts in  $\lambda$  have only appeared in the first row of  $GG(\lambda)$ . Furthermore, the parts in each row except for the first row are distinct. Moreover, there are only odd parts repeated in the first row. We use  $2j + 1^t$  to denote that there are  $t$  multiplies of  $2j + 1$  in the first row for  $t \geq 2$ , and  $\overline{2j + 1}^t$  to denote that there are one  $\overline{2j + 1}$  and  $t - 1$  multiplies of  $2j + 1$  in the first row.

Let  $N_r$  be the number of parts in the  $r$ -th row of  $GG(\lambda)$ . It is easy to find that  $N_1 \geq N_2 \geq \dots$ . For the example above, we have  $N_1 = 8, N_2 = 6, N_3 = 3$ .

For a part  $\lambda_j$  of an overpartition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ , we write  $\lambda_j = \overline{a_j}$  to indicate that  $\lambda_j$  is an overlined part and write  $\lambda_j = a_j$  to indicate that  $\lambda_j$  is a non-overlined part. Set  $|\lambda_j| = a_j$ .

We recall the cluster defined on the Göllnitz-Gordon marking of an overpartition defined in [17].

**Definition 5.2.** For an overpartition  $\lambda$ , we denote the 1-marked parts in the Göllnitz-Gordon marking  $GG(\lambda)$  of  $\lambda$  by

$$\lambda_{N_1}^{(1)} \leq \lambda_{N_1-1}^{(1)} \leq \dots \leq \lambda_1^{(1)}$$

also in the following order

$$\overline{1} < 1 < \overline{2} < 2 < \dots$$

We proceed to decompose  $\lambda$  into  $N_1$  clusters according to 1-marked parts of  $\lambda$  in the above order. In other words, we aim to define  $N_1$ -cluster corresponding to  $\lambda_{N_1}^{(1)}$ ,  $(N_1 - 1)$ -cluster corresponding to  $\lambda_{N_1-1}^{(1)}, \dots$ , 1-cluster corresponding to  $\lambda_1^{(1)}$  consecutively. Denote  $j$ -cluster by  $\alpha^{(j)}$ , then

$$\lambda = \left\{ \alpha^{(N_1)}, \alpha^{(N_1-1)}, \dots, \alpha^{(1)} \right\}.$$

We suppose that  $(N_1 + 1)$ -cluster is  $\emptyset$ . The  $j$ -cluster  $\alpha^{(j)}$  corresponding to  $\lambda_j^{(1)}$  is defined by considering the following three cases:

- (1) If  $|\lambda_j^{(1)}| = |\lambda_{j-1}^{(1)}|$ , then  $\alpha^{(j)}$  has only one part equal to  $\lambda_j^{(1)}$ .
- (2) If  $\lambda_j^{(1)}$  is odd and  $|\lambda_{j-1}^{(1)}| - |\lambda_j^{(1)}| = 1$ , then  $\alpha^{(j)}$  has only one part equal to  $\lambda_j^{(1)}$ .
- (3) Otherwise,  $\alpha^{(j)}$  is a maximal length sub-overpartition  $\alpha_1^{(j)} \leq \alpha_2^{(j)} \leq \dots \leq \alpha_s^{(j)}$  where  $\alpha_1^{(j)} = \lambda_j^{(1)}$  and for  $2 \leq b \leq s$ ,  $\alpha_b^{(j)}$  is a  $b$ -marked part of  $\lambda$  and not in  $(j + 1)$ -cluster satisfying the following conditions:
  - (i) If  $\alpha_{b-1}^{(j)}$  is odd, then  $|\alpha_b^{(j)}| - |\alpha_{b-1}^{(j)}| = 1$ .
  - (ii) If  $|\alpha_{b-1}^{(j)}|$  is even and there does not exist  $(b - 1)$ -marked part with size  $|\alpha_{b-1}^{(j)}| + 2$  in  $\lambda$ , then  $0 \leq |\alpha_b^{(j)}| - |\alpha_{b-1}^{(j)}| \leq 2$ .

- (iii) If  $\alpha_{b-1}^{(j)}$  is even and there exists a  $(b-1)$ -marked part with size  $|\alpha_{b-1}^{(j)}| + 2$  in  $\lambda$ , then  $0 \leq |\alpha_b^{(j)}| - |\alpha_{b-1}^{(j)}| \leq 1$ .

For the example in (5.1), the overpartition  $\lambda$  has seven clusters, namely

$$GG(\lambda) = \left[ \begin{array}{ccccccc} \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 4 \\ \hline \end{array} & \begin{array}{|c|} \hline \bar{7} \\ \hline \end{array} & \begin{array}{|c|} \hline 9^2 \\ \hline \end{array} & \begin{array}{|c|} \hline 12 \\ \hline \end{array} & \begin{array}{|c|} \hline 14 \\ \hline 14 \\ \hline 14 \\ \hline \end{array} & \begin{array}{|c|} \hline \bar{17} \\ \hline \end{array} \\ \hline 1 & 3 & \bar{6} & \bar{6} & \bar{11} & \bar{14} & \bar{16} \\ \hline 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

From the definition of cluster, it is easy to see that any overpartition  $\lambda$  along with its Göllnitz-Gordon marking has a unique decomposition into non-overlapping clusters. The following proposition is useful, here we omit the proof for which you can see [17].

**Proposition 5.3.** *For an overpartition  $\lambda$ , if the Göllnitz-Gordon marking  $GG(\lambda)$  is decomposed as follows according to its clusters*

$$GG(\lambda) = \left\{ \alpha^{(N_1)}, \alpha^{(N_1-1)}, \dots, \alpha^{(1)} \right\},$$

then for  $1 \leq t \leq N_1$ , there exists at most one odd part in the  $t$ -cluster  $\alpha^{(t)}$ . Moreover, if there exists an odd part in  $\alpha^{(t)}$ , say  $\alpha_d^{(t)}$ , then  $|\alpha_d^{(t)}| - |\alpha_1^{(t)}| \leq 1$ .

The following proposition is also important.

**Proposition 5.4.** *For an overpartition  $\lambda$ , if the Göllnitz-Gordon marking  $GG(\lambda)$  is decomposed as follows according to its clusters*

$$GG(\lambda) = \left\{ \alpha^{(N_1)}, \alpha^{(N_1-1)}, \dots, \alpha^{(1)} \right\}$$

for  $1 \leq j < N_1$ , we set

$$\alpha(j) = \left\{ r \mid |\alpha_r^{(j)}| - |\alpha_r^{(j+1)}| \leq 2 \text{ with restrict inequality if } \alpha_r^{(j)} \text{ or } \alpha_r^{(j+1)} \text{ is odd} \right\},$$

then  $\alpha(j)$  has at most one element.

*Proof.* Assume that  $\alpha(j)$  has at least two elements.

For  $r \in \alpha(j)$ , we first prove that  $\alpha_r^{(j)}$  and  $\alpha_r^{(j+1)}$  are both even. Otherwise,  $\alpha_r^{(j)}$  or  $\alpha_r^{(j+1)}$  is odd, by the definition of Göllnitz-Gordon marking, we must have  $r = 1$ . Now if  $\alpha_1^{(j+1)}$  or  $\alpha_1^{(j)}$  is odd, since  $|\alpha_1^{(j)}| - |\alpha_1^{(j+1)}| < 2$ , by the definition of cluster, we get that  $\alpha^{(j+1)}$  or  $\alpha^{(j)}$  contains only one part, which contracts to the fact that  $\alpha(j)$  has at least two elements. So  $\alpha_r^{(j)}$  and  $\alpha_r^{(j+1)}$  are both even.

Set  $b$  and  $c$  be the smallest element and the second smallest element in  $\alpha(j)$ , then  $c > b$ . Set  $\alpha_c^{(j)} = 2t + 2$ , then  $\alpha_c^{(j+1)} = 2t$ . By the definition of Göllnitz-Gordon marking,

we know that there is a 1-marked  $2t + 1$  or  $\overline{2t + 2}$  and  $c$  is the smallest mark of  $\overline{2t}$ ,  $2t$  and  $\overline{2t + 1}$ . So there exist 2-marked, 3-marked,  $\dots$ ,  $(c - 1)$ -marked  $2t + 2$  in  $GG(\lambda)$ . By the definition of cluster, it is easy to see that  $\alpha_1^{(j)} = 2t + 1$  or  $\overline{2t + 2}$  and  $\alpha_r^{(j)} = 2t + 2$  for  $2 \leq r \leq c$ . Since  $b < c$ , so we have  $\alpha_b^{(j)} = \overline{2t + 2}$  or  $2t + 2$ . But there is no  $b$ -marked  $\overline{2t}$ ,  $2t$  and  $\overline{2t + 1}$ , which implies that  $|\alpha_b^{(j)}| \leq 2t - 2$ . So  $|\alpha_b^{(j)}| - |\alpha_b^{(j+1)}| \geq 4$ , which contradicts to the choice of  $b$ .

Thus, the assumption is false, namely,  $\alpha(j)$  has at most one element. □

Let  $\widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n)$  denote the set of overpartitions  $\nu$  in  $\widetilde{\mathcal{O}}_{k, i}(m, n)$  that have  $N_r$   $r$ -marked parts in their Göllnitz-Gordon marking  $GG(\nu)$  for  $1 \leq r \leq k - 1$ , where  $\sum_{r=1}^{k-1} N_r = m$ .

For an overpartition  $\nu$ , assume that the Göllnitz-Gordon marking representation of  $\nu$  is decomposed as follows according to the clusters

$$GG(\nu) = \{ \beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)} \}.$$

If there is an odd part in  $\beta^{(p)}$  but no odd part in  $\beta^{(p-1)}$  for  $1 < p \leq N_1$  or there is an odd part in  $\beta^{(p)}$  for  $p = 1$ , we define the second dilation of  $p$ -th kind as follows.

**The second dilation of  $p$ -th kind:**

We set the odd part in  $\beta^{(p)}$  be  $\beta_r^{(p)}$ . If  $p = 1$ , we just change  $\beta_r^{(1)}$  to an  $r$ -marked non-overlined (resp. overlined) even part with size  $|\beta_r^{(1)}| + 1$  if  $\beta_r^{(1)}$  is overlined (resp. non-overlined). If  $1 < p \leq N_1$ , we define

$$\beta(p - 1) = \left\{ j \mid |\alpha_j^{(p-1)}| - |\alpha_j^{(p)}| \leq 2 \text{ with restrict inequality if } \alpha_j^{(p-1)} \text{ or } \alpha_j^{(p)} \text{ is odd} \right\},$$

we consider following two cases:

*Case 1.*  $\beta(p - 1)$  is empty, we first change  $\beta_r^{(p)}$  to an  $r$ -marked non-overlined (resp. overlined) even part with size  $|\beta_r^{(p)}| + 1$  if  $\beta_r^{(p)}$  is overlined (resp. non-overlined). Then we choose the part with the smallest size but the largest mark in  $\beta^{(p-1)}$ , set  $b$ -marked  $\beta_b^{(p-1)}$ , then we change  $\beta_b^{(p-1)}$  to an  $b$ -marked non-overlined (resp. overlined) odd part with size  $|\beta_b^{(p-1)}| + 1$  if  $\beta_b^{(p-1)}$  is overlined (resp. non-overlined).

*Case 2.*  $\beta(p - 1) = \{b\}$ , we first change  $\beta_r^{(p)}$  to an  $r$ -marked non-overlined (resp. overlined) even part with size  $|\beta_r^{(p)}| + 1$  if  $\beta_r^{(p)}$  is overlined (resp. non-overlined). Then we change  $\beta_b^{(p-1)}$  to an  $b$ -marked non-overlined (resp. overlined) odd part with size  $|\beta_b^{(p-1)}| + 1$  if  $\beta_b^{(p-1)}$  is overlined (resp. non-overlined).

**Example 5.5.** Set  $GG(\nu) = \{ \beta^{(7)}, \beta^{(6)}, \dots, \beta^{(1)} \}$  be the following overpartition

$$GG(\nu) = \left[ \begin{array}{cccccc} \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

There is an odd part 1-marked 9 in  $\beta^{(3)}$  but no odd part in  $\beta^{(2)}$ , so we can do the second dilation of 3-th kind. Since  $|\beta_2^{(2)}| - |\beta_2^{(3)}| = |12| - |10| = 2$ , we have  $\beta(2) = 2$ . So it belongs to Case 2. Now we first change 1-marked 9 to a 1-marked  $\overline{10}$ , then we change 2-marked 12 to a 2-marked  $\overline{13}$ . Thus we get the following overpartition

$$GG(\nu') = \left[ \begin{array}{cccccc} \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

For the resulting overpartition  $GG(\nu') = \{\gamma^{(7)}, \gamma^{(6)}, \dots, \gamma^{(1)}\}$  illustrated above, since there is an odd part 2-marked  $\overline{13}$  in  $\gamma^{(2)}$  but no odd part in  $\gamma^{(1)}$ , we can do the second dilation of 2-nd kind. It is easy to check that  $\gamma(1)$  is empty. Such a condition satisfies Case 1. So we first change 2-marked  $\overline{13}$  to a 2-marked 14. Then we choose the part with the smallest size but the largest mark in  $\gamma(1)$ , namely 2-marked 16. Finally, we change 2-marked 16 to a 2-marked  $\overline{17}$  to get the following overpartition

$$GG(\nu'') = \left[ \begin{array}{cccccc} \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

There exists an odd part 2-marked  $\overline{17}$  in the 1-cluster of  $GG(\nu'')$  above. Thus we can do the second dilation of 1-st kind. We just change the 2-marked  $\overline{17}$  to a 2-marked 18, then we get the following overpartition

$$GG(\nu''') = \left[ \begin{array}{cccccc} \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \\ \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} \end{array} \right] \begin{array}{l} 3 \\ 2 \\ 1 \end{array}$$

For an overpartition  $\nu$ , assume that the Göllnitz-Gordon marking representation of  $\nu$  is decomposed as follows according to the clusters

$$GG(\nu) = \{\beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)}\}.$$

If there is an odd part in  $\beta^{(p-1)}$  but no odd part in  $\beta^{(p)}$  for  $1 < p \leq N_1$  or there is no odd part in  $\beta^{(p)}$  for  $p = 1$ , we define the second reduction of  $p$ -th kind as follows.

**The second reduction of  $p$ -th kind:**

If  $p = 1$ , we choose the part with the largest size but the smallest mark in  $\beta^{(p)}$ , set  $b$ -marked  $\beta_b^{(p)}$ , then we change  $\beta_b^{(p)}$  to an  $b$ -marked non-overlined (resp. overlined) odd part with size  $|\beta_b^{(p)}| - 1$  if  $\beta_b^{(p)}$  is overlined (resp. non-overlined).

If  $1 < p \leq N_1$ , set the odd part in  $\beta^{(p-1)}$  be  $\beta_r^{(p-1)}$  and define

$$\beta(p-1) = \left\{ j \mid |\alpha_j^{(p-1)}| - |\alpha_j^{(p)}| \leq 2 \text{ with restrict inequality if } \alpha_j^{(p-1)} \text{ or } \alpha_j^{(p)} \text{ is odd} \right\},$$

we consider following two cases:

*Case 1.*  $\beta(p-1)$  is empty, we first change  $\beta_r^{(p-1)}$  to an  $r$ -marked non-overlined (resp. overlined) even part with size  $|\beta_r^{(p-1)}| - 1$  if  $\beta_r^{(p-1)}$  is overlined (resp. non-overlined). Then we choose the part with the smallest size but the largest mark in  $\beta^{(p)}$ , set  $b$ -marked  $\beta_b^{(p)}$ , then we change  $\beta_b^{(p)}$  to an  $b$ -marked non-overlined (resp. overlined) odd part with size  $|\beta_b^{(p)}| - 1$  if  $\beta_b^{(p)}$  is overlined (resp. non-overlined).

*Case 2.*  $\beta(p-1) = \{b\}$ , we first change  $\beta_r^{(p-1)}$  to an  $r$ -marked non-overlined (resp. overlined) even part with size  $|\beta_r^{(p-1)}| - 1$  if  $\beta_r^{(p-1)}$  is overlined (resp. non-overlined). Then we change  $\beta_b^{(p)}$  to an  $b$ -marked non-overlined (resp. overlined) odd part with size  $|\beta_b^{(p)}| - 1$  if  $\beta_b^{(p)}$  is overlined (resp. non-overlined).

It is easy to check that the second dilation of  $p$ -th kind and the second reduction of  $p$ -th kind are inverse mapping for each other. Now we want show that the second dilation of  $p$ -th kind and the second reduction of  $p$ -th kind both preserve conditions (1)–(5) in Theorem 1.10. Such operations preserve conditions (1)–(3) are easy to check. We first show that the second dilation preserves conditions (4) and (5) in Theorem 1.10.

**Proposition 5.6.** *For an overpartition  $\nu \in \tilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n)$ , if the Göllnitz-Gordon marking representation of  $\nu$  is decomposed as follows according to the clusters*

$$GG(\nu) = \left\{ \beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)} \right\}.$$

*If there is an odd part  $\beta_r^{(p)}$  in  $\beta^{(p)}$  but no odd part in  $\beta^{(p-1)}$  for  $1 < p \leq N_1$  or there is an odd part in  $\beta^{(p)}$  for  $p = 1$ . Denote the overpartition after doing the second dilation of  $p$ -th kind of  $\nu$  by  $\nu'$ , we shall show that if*

$$(5.2) \quad f_{2l}(\nu') + f_{\overline{2l}}(\nu') + f_{\overline{2l+1}}(\nu') + f_{2l+2}(\nu') = k - 1$$

*for some  $l$ , then*

$$(5.3) \quad lf_{2l}(\nu') + lf_{\overline{2l}}(\nu') + lf_{\overline{2l+1}}(\nu') + (l + 1)f_{2l+2}(\nu') \equiv V_{2l+1}(\nu') + i - 1 \pmod{2},$$

and if

$$(5.4) \quad f_{2l+1}(\nu') \geq 1 \quad \text{and} \quad f_{2l+2}(\nu) = k - 2$$

for some  $l$ , then

$$(5.5) \quad l + (l + 1)f_{2l+2}(\nu') \equiv V_{2l+1}(\nu') + i - 1 \pmod{2}.$$

*Proof.* Assume that  $|\beta_r^{(p)}| = 2t + 1$ .

If  $p = 1$ , we change  $r$ -marked  $\beta_r^{(1)}$  to an  $r$ -marked even part  $2t + 2$  (resp.  $\overline{2t + 2}$ ) if  $\beta_r^{(1)}$  is  $\overline{2t + 1}$  (resp.  $2t + 1$ ). For integer  $l$  satisfying (5.2) or (5.4), if  $l \neq t$ , it is easy to check that (5.3) or (5.5) is established.

If  $\beta_r^{(1)}$  is  $\overline{2t + 1}$ , it is easy to get  $f_{2t}(\nu') = f_{2t}(\nu)$ ,  $f_{\overline{2t}}(\nu') = f_{\overline{2t}}(\nu)$ ,  $f_{\overline{2t+1}}(\nu') = f_{\overline{2t+1}}(\nu) - 1$ ,  $f_{2t+2}(\nu') = f_{2t+2}(\nu) + 1$  and  $V_{2t+1}(\nu') = V_{2t+1}(\nu) - 1$ . So if (5.2) is established for  $l = t$ , we also have

$$f_{2t}(\nu) + f_{\overline{2t}}(\nu) + f_{\overline{2t+1}}(\nu) + f_{2t+2}(\nu) = k - 1.$$

Thus,

$$\begin{aligned} & t f_{2t}(\nu') + t f_{\overline{2t}}(\nu') + t f_{\overline{2t+1}}(\nu') + (t + 1) f_{2t+2}(\nu') \\ &= t f_{2t}(\nu) + t f_{\overline{2t}}(\nu) + t (f_{\overline{2t+1}}(\nu) - 1) + (t + 1) (f_{2t+2}(\nu) + 1) \\ &\equiv V_{2t+1}(\nu) + i - 1 - t + (t + 1) \pmod{2} \\ &\equiv V_{2t+1}(\nu') + i - 1 \pmod{2}. \end{aligned}$$

So (5.3) is established.

If  $\beta_r^{(1)}$  is  $2t + 1$ , it is easy to get  $f_{2t}(\nu') = f_{2t}(\nu)$ ,  $f_{\overline{2t}}(\nu') = f_{\overline{2t}}(\nu)$ ,  $f_{\overline{2t+1}}(\nu') = f_{\overline{2t+1}}(\nu)$ ,  $f_{2t+2}(\nu') = f_{2t+2}(\nu)$  and  $V_{2t+1}(\nu') = V_{2t+1}(\nu)$ . So if (5.2) is established for  $l = t$ , we also have

$$f_{2t}(\nu) + f_{\overline{2t}}(\nu) + f_{\overline{2t+1}}(\nu) + f_{2t+2}(\nu) = k - 1.$$

Thus,

$$\begin{aligned} & t f_{2t}(\nu') + t f_{\overline{2t}}(\nu') + t f_{\overline{2t+1}}(\nu') + (t + 1) f_{2t+2}(\nu') \\ &= t f_{2t}(\nu) + t f_{\overline{2t}}(\nu) + t (f_{\overline{2t+1}}(\nu)) + (t + 1) (f_{2t+2}(\nu)) \\ &\equiv V_{2t+1}(\nu) + i - 1 \pmod{2} \\ &\equiv V_{2t+1}(\nu') + i - 1 \pmod{2}. \end{aligned}$$

So (5.3) is established.

Assume that (5.4) set up for  $l = t$ , then  $f_{2t+1}(\nu') \geq 1$ . By the definition of cluster, we must have  $\beta_r^{(1)} = 2t + 1$ . So,  $f_{2t+2}(\nu') = f_{2t+2}(\nu) = k - 2$  and  $V_{2t+1}(\nu') = V_{2t+1}(\nu)$ . Thus,

$$\begin{aligned} t + (t + 1) f_{2l+2}(\nu') &= t + (t + 1) f_{2l+2}(\nu) \equiv V_{2l+1}(\nu) + i - 1 \\ &\equiv V_{2l+1}(\nu') + i - 1 \pmod{2}, \end{aligned}$$



which is (5.5) for  $l = t$ .

Now we consider the case  $1 < p \leq N_1$ . Assume that the part we choose to change in  $\beta^{(p-1)}$  is  $\beta_b^{(p-1)}$ . Set  $|\beta_b^{(p-1)}| = 2s + 2$ , we assert that  $t \leq s$ . By Proposition 5.3, we know that  $0 \leq |\beta_r^{(p)}| - |\beta_1^{(p)}| \leq 1$ , namely  $2t \leq |\beta_1^{(p)}| \leq 2t + 1$ . Since  $\beta_1^{(p-1)}$  is even, by the definition of Göllnitz-Gordon marking, we have  $|\beta_1^{(p-1)}| - |\beta_1^{(p)}| \geq 1$  with restrict inequality if  $\beta_1^{(p)}$  is even. Thus we have  $|\beta_1^{(p-1)}| \geq 2t + 2$ . By the definition of cluster, we know that  $2s + 2 = |\beta_b^{(p-1)}| \geq |\beta_1^{(p-1)}| \geq 2t + 2$ , which leads to our assertion.

For integer  $l$  satisfying (5.2) or (5.4), if  $l \neq t$  and  $s + 1$ , it is easy to check that (5.3) or (5.5) is established.

For integer  $l$  satisfying (5.2) or (5.4), if  $l = t$  or  $s + 1$ . If  $s > t$ , similar to the proof of  $p = 1$ , it is easy to check that (5.3) or (5.5) is established. We just consider the case  $s = t$  and  $l = t$  or  $t + 1$ .

For  $s = t$ , we have  $2t + 2 = |\beta_b^{(p-1)}| \geq |\beta_1^{(p-1)}| \geq 2t + 2$ , so  $|\beta_1^{(p-1)}| = 2t + 2$ . Since  $2t \leq |\beta_1^{(p)}| \leq 2t + 1$ , we know that  $\beta_1^{(p-1)} = \overline{2t + 2}$  and  $\beta(p - 1) = 1$ . So the second dilation of  $p$ -th kind only has following two cases:

*Case 1.*  $\beta_r^{(p)}$  is  $2t + 1$ , we first change  $r$ -marked  $2t + 1$  to an  $r$ -marked  $\overline{2t + 2}$ , and then we change the part  $\beta_1^{(p-1)} = \overline{2t + 2}$  to a 1-marked  $2t + 3$ . We get that the occurrences of  $2t, \overline{2t + 1}, 2t + 2, \overline{2t + 2}, \overline{2t + 3}$  and  $2t + 4$  in  $\nu'$  are the same as those in  $\nu$ . Notice that  $V_{2t+1}(\nu') = V_{2t+1}(\nu)$  and  $V_{2t+3}(\nu') = V_{2t+3}(\nu)$ , it is obvious that if (5.2) or (5.4) holds for  $l = t$  or  $t + 1$ , then (5.3) or (5.5) also holds for  $l = t$  or  $t + 1$ .

*Case 2.*  $\beta_r^{(p)}$  is  $\overline{2t + 1}$ , we first change  $r$ -marked  $\overline{2t + 1}$  to an  $r$ -marked  $2t + 2$ , and then we change the part  $\beta_1^{(p-1)} = \overline{2t + 2}$  to a 1-marked  $2t + 3$ . In such case, we must have  $f_{2t+1}(\nu) = 0$ , otherwise we must have  $\beta_1^{(p-1)} = 2t + 1$ . But  $\beta_1^{(p-1)} = \overline{2t + 2}$ , which leads to a contradiction. So we have  $f_{2t+1}(\nu) = 0$ . Now it is easy to check that  $f_{\overline{2t}}(\nu') = f_{\overline{2t}}(\nu)$ ,  $f_{2t}(\nu') = f_{2t}(\nu)$ ,  $f_{\overline{2t+1}}(\nu') = f_{\overline{2t+1}}(\nu) - 1$ ,  $f_{2t+1}(\nu') = f_{2t+1}(\nu) = 0$ ,  $f_{2t+2}(\nu') = f_{2t+2}(\nu) + 1$ ,  $f_{\overline{2t+2}}(\nu') = f_{\overline{2t+2}}(\nu) - 1$ ,  $f_{\overline{2t+3}}(\nu') = f_{\overline{2t+3}}(\nu)$ , and  $f_{2t+4}(\nu') = f_{2t+4}(\nu)$ . What is more, we have  $V_{2t+1}(\nu') = V_{2t+1}(\nu) - 1$  and  $V_{2t+3}(\nu') = V_{2t+3}(\nu) - 2$ . Next we show that (5.2) and (5.4) lead to (5.3) and (5.5) respectively for  $l = t$  or  $t + 1$ .

Since  $f_{2t+1}(\nu') = 0$ , so (5.4) can not hold for  $l = t$ . If (5.2) holds for  $l = t$ , then we also have

$$f_{2t}(\nu) + f_{\overline{2t}}(\nu) + f_{\overline{2t+1}}(\nu) + f_{2t+2}(\nu) = k - 1.$$

Thus,

$$\begin{aligned} & t f_{2t}(\nu') + t f_{\overline{2t}}(\nu') + t f_{\overline{2t+1}}(\nu') + (t + 1) f_{2t+2}(\nu') \\ &= t f_{2t}(\nu) + t f_{\overline{2t}}(\nu) + t (f_{\overline{2t+1}}(\nu) - 1) + (t + 1) (f_{2t+2}(\nu) + 1) \\ &\equiv V_{2t+1}(\nu) + i - 1 + 1 \pmod{2} \\ &\equiv V_{2t+1}(\nu') + i - 1 \pmod{2}. \end{aligned}$$

So (5.3) is established for  $l = t$ .

If (5.4) holds for  $l = t + 1$ . Since  $f_{2t+4}(\nu) = f_{2t+4}(\nu') = k - 2$  and  $f_{\overline{2t+2}}(\nu) = 1$ , we have  $f_{2t+2}(\nu) = f_{\overline{2t+3}}(\nu) = 0$  and

$$f_{2t+2}(\nu) + f_{\overline{2t+2}}(\nu) + f_{\overline{2t+3}}(\nu) + f_{2t+4}(\nu) = k - 1.$$

So we can easily get

$$\begin{aligned} & (t + 1) + (t + 2)f_{\overline{2t+4}}(\nu') \\ &= (t + 1)f_{\overline{2t+2}}(\nu) + (t + 2)f_{\overline{2t+4}}(\nu) \\ &= (t + 1)f_{2t+2}(\nu) + (t + 1)f_{\overline{2t+2}}(\nu) + (t + 1)f_{\overline{2t+3}}(\nu) + (t + 2)f_{2t+4}(\nu) \\ &\equiv V_{2t+3}(\nu) + i - 1 \pmod{2} \\ &\equiv V_{2t+3}(\nu') + i - 1 \pmod{2}, \end{aligned}$$

which leads to (5.5) for  $l = t + 1$ .

Finally if (5.2) holds for  $l = t + 1$ , we must have

$$f_{2t+2}(\nu) + f_{\overline{2t+2}}(\nu) + f_{\overline{2t+3}}(\nu) + f_{2t+4}(\nu) = k - 1.$$

Thus,

$$\begin{aligned} & (t + 1)f_{2t+2}(\nu') + (t + 1)f_{\overline{2t+2}}(\nu') + (t + 1)f_{\overline{2t+3}}(\nu') + (t + 2)f_{2t+4}(\nu') \\ &= (t + 1)(f_{2t+2}(\nu) + 1) + (t + 1)(f_{\overline{2t+2}}(\nu) - 1) + (t + 1)f_{\overline{2t+3}}(\nu) + (t + 2)f_{2t+4}(\nu) \\ &\equiv V_{2t+3}(\nu) + i - 1 \pmod{2} \\ &\equiv V_{2t+3}(\nu') + i - 1 \pmod{2}. \end{aligned}$$

So (5.3) is established for  $l = t + 1$ . Now we complete the proof. □

If the Göllnitz-Gordon markings of an overpartition  $\lambda$  are not exceed  $k - 1$ , and  $\lambda$  satisfies condition (5.2) (or (5.4)) but does not satisfies condition (5.3) (or (5.5)) for some  $l$ . Assume that we can do the second dilation of  $\lambda$  and denote the resulting overpartition by  $\nu$ . Using the same method above, one can easily get that  $\nu$  satisfies condition (5.2) (or (5.4)) but does not satisfies condition (5.3) (or (5.5)) for some  $t$ , we omit the details here.

Notice that the second dilation of  $p$ -th kind and the second reduction of  $p$ -th kind are inverse mapping for each other, so the second reduction of  $p$ -th kind also preserves conditions (4) and (5) in Theorem 1.10.

Thus, we have the following proposition.

**Proposition 5.7.** *For an overpartition  $\nu \in \widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n)$  and it satisfies the condition of the second dilation of  $p$ -th kind (resp. the second reduction of  $p$ -th kind), if we do the such*

operation for  $\nu$  and denote the resulting overpartition by  $\nu'$ , then  $\nu' \in \widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n+2)$  (resp.  $\nu' \in \widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n-2)$ ) for  $1 < p \leq N_1$  or  $\nu' \in \widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n+1)$  (resp.  $\nu' \in \widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n-1)$ ) for  $p = 1$ .

At the end of this section, we will show the outline proof of Theorem 1.12. To prove Theorem 1.12, it suffices to show the following generating function

$$(5.6) \quad \sum_{\nu \in \widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\nu)} q^{|\nu|} = \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i}) x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^4; q^4)_{N_{k-1}}}.$$

Let  $\widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}(n)$  denote the set of overpartitions in  $\widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n)$  without odd parts. Set

$$\widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i} = \bigcup_{n \geq 0} \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}(n),$$

we will show the following relation in the next section by constructing a bijection in terms of the second dilation and the second reduction.

**Theorem 5.8.** *For  $N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0$ , we have*

$$\sum_{\nu \in \widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\nu)} q^{|\nu|} = (-q^{1-2N_1}; q^2)_{N_1} \sum_{\omega \in \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\omega)} q^{|\omega|},$$

which leads a proof to (5.6).

### 6. Proof of Theorem 1.12

We first give a proof of Theorem 5.8.

*Proof of Theorem 5.8.* Let  $Q_N$  denote the set of partitions with distinct negative odd parts which lay in  $[1 - 2N, -1]$ . We give a bijection between  $\widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}$  and  $Q_{N_1} \times \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}$ . More precisely, set

$$\mathbb{QW}_{N_1, \dots, N_{k-1}; i}(n) = \left\{ (\xi, \omega) \in Q_{N_1} \times \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i} \mid |\xi| + |\omega| = n \right\},$$

we establish a bijection between  $\widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n)$  and  $\mathbb{QW}_{N_1, \dots, N_{k-1}; i}(n)$ .

On the one hand, for an overpartition  $\nu \in \widetilde{\mathcal{O}}_{N_1, \dots, N_{k-1}; i}(n)$ , we construct a pair  $(\xi, \omega) \in \mathbb{QW}_{N_1, \dots, N_{k-1}; i}(n)$ .

If there is no odd part in  $\nu$ , we set  $\xi = \emptyset$  and  $\omega = \nu$ . If there exist odd parts in  $\nu$ , we consider the Göllnitz-Gordon marking  $GG(\nu) = \{\beta^{(N_1)}, \beta^{(N_1-1)}, \dots, \beta^{(1)}\}$ . Then by

Proposition 5.3, the odd parts must in different clusters of  $GG(\nu)$ , set such clusters be  $\beta^{(j_1)}, \beta^{(j_2)}, \dots, \beta^{(j_s)}$ , where  $1 \leq j_1 < j_2 < \dots < j_s \leq N_1$ . Let  $M_0 = 0$  and  $M_l = \sum_{t=1}^l j_t$  where  $1 \leq l \leq s$ . Set  $\nu^0 = \nu$ , we do the following operations for  $l$  from 1 to  $s$  successively.

For each  $l$ , we iterate the following operation for  $p$  from 1 to  $j_l$ : we do the second dilation of  $p$ -th kind of  $\nu^{M_{l-1}+p-1}$  and denote the resulting overpartition by  $\nu^{M_{l-1}+p}$ .

After the  $M_s$  operations above, we can get an overpartition  $\nu^{M_s}$  which contains no odd part, what is more,  $\nu^{M_s} \in \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}(n + 2M_s - s)$ . Finally, set  $\omega = \nu^{M_s}$  and  $\xi = (1 - 2j_1, 1 - 2j_2, \dots, 1 - 2j_s)$ , we get a desired pair  $(\xi, \omega) \in \mathbb{QW}_{N_1, \dots, N_{k-1}; i}(n)$ .

On the contrary, for a pair  $(\xi, \omega) \in \mathbb{QW}_{N_1, \dots, N_{k-1}; i}(n)$ , we want to construct an overpartition  $\nu \in \widetilde{\mathbb{O}}_{N_1, \dots, N_{k-1}; i}(n)$ .

If  $\xi = \emptyset$ , we just set  $\nu = \omega$ . If  $\xi \neq \emptyset$ , then we set  $\xi = (1 - 2j_1, 1 - 2j_2, \dots, 1 - 2j_s)$  where  $1 \leq j_1 < j_2 < \dots < j_s \leq N_1$ . Let  $M_l = \sum_{t=1}^l j_t$  where  $1 \leq l \leq s$  and set  $\omega^{M_s} = \omega$ , we do the following operations for  $l$  from  $s$  to 1 successively.

For each  $l$ , we iterate the following operation for  $p$  from 1 to  $j_l$ : we do the second reduction of  $p$ -th kind of  $\omega^{M_l-p+1}$  and denote the resulting overpartition by  $\omega^{M_l-p}$ .

After the  $M_s$  operations above, we can get an overpartition  $\omega^0$  which belongs to  $\widetilde{\mathbb{O}}_{N_1, \dots, N_{k-1}; i}(n)$ . Finally, we just need to set  $\nu = \omega^0$ . Thus we complete the proof of Theorem 5.8. □

Now we are in a position to prove Theorem 1.12.

*Proof of Theorem 1.12.* Using (4.5), we shall prove that the generating function of overpartitions in  $\widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}$  can be stated as follows.

**Lemma 6.1.** *For  $k > i \geq 1$ , we have*

$$\sum_{\omega \in \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\omega)} q^{|\omega|} = \frac{(-q^{2-2N_1}; q^2)_{N_1-1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^4; q^4)_{N_{k-1}}}.$$

*Proof.* We will establish a bijection between  $\widetilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}(n)$  and  $\widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}(2n)$ . For an overpartition  $\delta = (\delta_1, \delta_2, \dots, \delta_\ell) \in \widetilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}(n)$ , where  $\delta_1 \leq \delta_2 \leq \dots \leq \delta_\ell$ . Double each part in  $\delta$  and denote the resulting overpartition by  $\omega$ , then  $\omega = (2\delta_1, 2\delta_2, \dots, 2\delta_\ell)$ . It is easy to see that  $|\omega| = 2|\delta| = 2n$  and  $f_{\overline{1}}(\omega) + f_2(\omega) = f_2(\omega) = f_1(\delta) \leq i - 1$ . Since all parts in  $\omega$  are even, so  $f_{2t+1}(\omega) = f_{\overline{2t+1}}(\omega) = 0$  and  $V_{2t+1}(\omega) = V_{2t}(\omega) = V_t(\delta)$ . If  $f_{2t}(\omega) + f_{\overline{2t}}(\omega) + f_{\overline{2t+1}}(\omega) + f_{2t+2}(\omega) = k - 1$ , so  $f_t(\delta) + f_{\overline{t}}(\delta) + f_{t+1}(\delta) = f_{2t}(\omega) + f_{\overline{2t}}(\omega) + f_{2t+2}(\omega) = k - 1$ . Notice that  $\delta \in \widetilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}(n)$ , we have

$$\begin{aligned} t f_{2t}(\omega) + t f_{\overline{2t}}(\omega) + t f_{\overline{2t+1}}(\omega) + (t + 1) f_{2t+2}(\omega) &= t f_t(\delta) + t f_{\overline{t}}(\delta) + (t + 1) f_{t+1}(\delta) \\ &\equiv V_t(\delta) + i - 1 \pmod{2} \\ &\equiv V_{2t+1}(\omega) + i - 1 \pmod{2}. \end{aligned}$$

Next, we consider the Gordon marking of  $\delta$  and the Göllnitz-Gordon marking of  $\omega$ . Recall the Gordon marking of  $\delta$  and the Göllnitz-Gordon marking of  $\omega$ , it can be checked that the mark  $\omega_j = 2\delta_j$  in the Göllnitz-Gordon marking of  $\omega$  are the same as the mark of  $\delta_j$  in the Gordon marking of  $\delta$  where  $1 \leq j \leq \ell$ . Hence we show that  $\omega \in \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}(2n)$  and the process is inversive. Thus we have constructed a bijection between  $\widetilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}(n)$  and  $\widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}(2n)$ . Finally by (4.5), we have

$$\begin{aligned} \sum_{\omega \in \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\omega)} q^{|\omega|} &= \sum_{\delta \in \widetilde{\mathbb{E}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\delta)} q^{2|\delta|} \\ &= \frac{(-q^{2-2N_1}; q^2)_{N_1-1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^4; q^4)_{N_{k-1}}}. \end{aligned}$$

Thus we complete the proof of Lemma 6.1. □

Applying Lemma 6.1 in Theorem 5.8, we obtain

$$\begin{aligned} &\sum_{\nu \in \widetilde{\mathbb{O}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\nu)} q^{|\nu|} \\ &= (-q^{1-2N_1}; q^2)_{N_1} \sum_{\omega \in \widetilde{\mathbb{W}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\omega)} q^{|\omega|} \\ &= \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i}) x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^4; q^4)_{N_{k-1}}}, \end{aligned}$$

which is (5.6). Hence we establish the following generating function of  $\widetilde{\mathcal{O}}_{k,i}(m, n)$ :

$$\begin{aligned} &\sum_{m, n \geq 0} \widetilde{\mathcal{O}}_{k,i}(m, n) x^m q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \sum_{\nu \in \widetilde{\mathbb{O}}_{N_1, \dots, N_{k-1}; i}} x^{\ell(\nu)} q^{|\nu|} \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i}) x^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1-N_2} \cdots (q^2; q^2)_{N_{k-2}-N_{k-1}} (q^4; q^4)_{N_{k-1}}}. \end{aligned}$$

This completes the proof of Theorem 1.12. □

### References

[1] A. K. Agarwal, G. E. Andrews and D. M. Bressoud, *The Bailey lattice*, J. Indian Math. Soc. (N.S.) **51** (1987), 57–73.

- [2] G. E. Andrews, *A generalization of the Göllnitz-Gordon partition theorems*, Proc. Amer. Math. Soc. **18** (1967), 945–952.
- [3] ———, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications **2**, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.
- [4] ———, *Multiple series Rogers-Ramanujan type identities*, Pacific J. Math. **114** (1984), no. 2, 267–283.
- [5] ———, *q-Series: Their development and application in analysis, number theory, combinatorics, physics, and computer algebra*, CBMS Regional Conference Series in Mathematics **66**, American Mathematical Society, Providence, RI, 1986.
- [6] W. N. Bailey, *Identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) **50** (1948), 1–10.
- [7] D. M. Bressoud, *A generalization of the Rogers-Ramanujan identities for all moduli*, J. Combin. Theory Ser. A **27** (1979), no. 1, 64–68.
- [8] ———, *Analytic and combinatorial generalizations of the Rogers-Ramanujan identities*, Mem. Amer. Math. Soc. **24** (1980), no. 227, 54 pp.
- [9] D. M. Bressoud, M. E. H. Ismail and D. Stanton, *Change of base in Bailey pairs*, Ramanujan J. **4** (2000), no. 4, 435–453.
- [10] W. Y. C. Chen, D. D. M. Sang and D. Y. H. Shi, *The Rogers-Ramanujan-Gordon theorem for overpartitions*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 6, 1371–1393.
- [11] ———, *An overpartition analogue of Bressoud’s theorem of Rogers-Ramanujan-Gordon type*, Ramanujan J. **36** (2015), no. 1-2, 69–80.
- [12] H. Göllnitz, *Einfache Partitionen*, Diplomarbeit W. S. Göttingen, 1960, 65 pp.
- [13] ———, *Partitionen mit Differenzenbedingungen*, J. Reine Angew. Math. **225** (1967), 154–190.
- [14] B. Gordon, *A combinatorial generalization of the Rogers-Ramanujan identities*, Amer. J. Math **83** (1961), 393–399.
- [15] ———, *Some Ramanujan-like continued fractions*, Abstracts of Short Communications. Int. 1962 Congr. of Math., Stockholm, 1962, 29–30.
- [16] ———, *Some continued fractions of the Rogers-Ramanujan type*, Duke Math. J. **32** (1965), 741–748.

- [17] T. Y. He, K. Q. Ji, A. Y. F. Wang and A. X. H. Zhao, *The Andrews-Göllnitz-Gordon theorem for overpartitions*, arXiv:1612.04960.
- [18] K. Kurşungöz, *Parity considerations in Andrews-Gordon identities, and the  $k$ -marked Durfee symbols*, Thesis (Ph.D.)-The Pennsylvania State University, 2009.
- [19] ———, *Parity considerations in Andrews-Gordon identities*, European J. Combin. **31** (2010), no. 3, 976–1000.
- [20] P. Paule, *On identities of the Rogers-Ramanujan type*, J. Math. Anal. Appl. **107** (1985), no. 1, 255–284.
- [21] ———, *A note on Bailey's lemma*, J. Combin. Theory Ser. A **44** (1987), no. 1, 164–167.
- [22] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **S1-25** (1894), no. 1, 318–343.
- [23] D. D. M. Sang and D. Y. H. Shi, *An Andrews-Gordon type identity for overpartitions*, Ramanujan J. **37** (2015), no. 3, 653–679.
- [24] L. J. Slater, *A new proof of Rogers's transformations of infinite series*, Proc. London Math. Soc. (2) **53** (1951), 460–475.

Thomas Yao He and Allison Yi Fang Wang

Center for Applied Mathematics, Tianjin University, Tianjin 300072, P. R. China

*E-mail address:* heyao@tju.edu.cn, wangiifang@tju.edu.cn

Alice Xiao Hua Zhao

Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071,

P. R. China

*E-mail address:* zhaoxiaohua@mail.nankai.edu.cn