TAIWANESE JOURNAL OF MATHEMATICS
Vol. 21, No. 6, pp. 1213–1231, December 2017
DOI: 10.11650/tjm/8031

The IC-indices of Complete Multipartite Graphs

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Abstract. Given a connected graph G, a function f mapping the vertex set of G into the set of all integers is a coloring of G. For any subgraph H of G, we denote as $f(H)$ the sum of the values of f on the vertices of H. If for any integer $k \in \{1, 2, \ldots, f(G)\}$, there exists an induced connected subgraph H of G such that $f(H) = k$, then the coloring f is called an IC-coloring of G. The IC-index of G, written $M(G)$, is defined to be the maximum value of $f(G)$ over all possible IC-colorings f of G. In this paper, we give a lower bound on the IC-index of any complete ℓ-partite graph for all $\ell \geq 3$ and then show that, when $\ell = 3$, our lower bound also serves as an upper bound. As a consequence, the exact value of the IC-index of any tripartite graph is determined.

1. Introduction

The postage stamp problem in number theory has been extensively studied and formulated into several versions in different fields [1–6,8,9,12,13]. In this paper, we consider the version called the IC-coloring of a graph. Throughout this paper, all graphs involved are simple graphs. For the terminologies and notations in graph theory, please refer to [14]. Given a connected graph G, a function $f \colon V(G) \to \mathbb{N}$ is called a *coloring* of G. The number $f(v)$ is the *color* of the vertex v of G. For any subgraph H of G, we denote the sum $\sum_{v \in V(H)} f(v)$ as $f(H)$. A coloring f of G is referred to as an *IC-coloring* of G if, for any integer $k \in \{1, 2, \ldots, f(G)\}$, there exists an induced connected subgraph H of G such that $f(H) = k$. Every connected graph G admits a trivial IC-coloring which assigns the value 1 to every vertex of G. The problem of finding an IC-coloring with the largest value of $f(G)$ arose naturally. The *IC-index* of a graph G, denoted $M(G)$, is defined to be

$$M(G) = \max \{f(G) \mid f \text{ is an IC-coloring of } G\}.$$

An IC-coloring f satisfying $f(G) = M(G)$ is called a *maximal IC-coloring* of G.

Received October 5, 2016; Accepted March 16, 2017.
Communicated by Sen-Peng Eu.
2010 *Mathematics Subject Classification.* 05C15.
Key words and phrases. IC-coloring, IC-index, complete multipartite graph, complete tripartite graph.
Lu was supported in part by the Ministry of Science and Technology of Taiwan under Grant MOST 104-2115-M-239-001.
Shiue was supported in part by the National Science Council under Grant NSC-100-2115-M033-002.
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Determining the exact values of the IC-index of a graph is challenging. In the past decades, not much achievements have been made. In 1992, Glenn Chappel formulated the IC-coloring problem as a "subgraph sums problem" and gave an upper bound on the IC-index of an n-cycle. He showed that $M(C_n) \leq n^2 - n + 1$. Later, in 1995, Penrice [7] introduced the IC-coloring as the stamp covering and determined the exact values of $M(K_n)$ and $M(K_{1,n})$, namely, $M(K_n) = 2^n - 1$ for $n \geq 1$ and $M(K_{1,n}) = 2^n + 2$ for $n \geq 2$. In 2005, Salehi et al. [8] proved that $M(K_{2,n}) = 3 \cdot 2^n + 1$ for $n \geq 2$. Shiue and Fu [10] completely settled the problem regarding complete bipartite graphs in 2008 by showing that $M(K_{m,n}) = 3 \cdot 2^{m+n-2} - 2^{m-2} + 2$ for $2 \leq m \leq n$. In this present paper, we consider *complete multipartite graphs.* A complete multipartite graph $K_{m_1,m_2,\ldots,m_\ell}$ is a graph whose vertex set can be partitioned into ℓ partite sets $V_1, V_2, \ldots, V_\ell$, where $|V_i| = m_i$ for all $i \in \{1, 2, \ldots, \ell\}$, such that there are no edges within each V_i and any two vertices from different partite sets are adjacent. A complete multipartite graph with ℓ partite sets is called a *complete ℓ-partite graph.* We also denote as $K_{1(n),m_{n+1},m_{n+2},\ldots,m_\ell}$, $n \leq \ell$, the complete ℓ-partite graph in which there are n partite sets which are of size one and the rest $(\ell - n)$ partite sets have sizes $m_{n+1}, m_{n+2}, \ldots, m_\ell$ respectively.

In [11], we first considered complete multipartite graphs. We gave a lower bound on $M(K_{m_1,m_2,\ldots,m_\ell})$ for $1 = m_1 = \cdots = m_n < m_{n+1} \leq m_{n+2} \leq \cdots \leq m_\ell$ and showed that, when $\ell = n + 1$ and $n \geq 2$, our lower bound is the exact value of it, that is, $M(K_{1(n),m}) = 2^{m+n} - 2^m + 1$ for $m \geq 2$ and $n \geq 2$. In this present paper, we investigate the problem of the IC-indices of general complete multipartite graphs. In Section 2, we introduce some previous results which are useful in our discussion. In Section 3, we introduce our lower bounds on $M(K_{m_1,m_2,\ldots,m_\ell})$ for $2 \leq m_1 \leq m_2 \leq \cdots \leq m_\ell$ and $1 = m_1 < m_2 \leq \cdots \leq m_\ell$ by constructing suitable IC-colorings. Subsequently, we prove in Section 4 that, when $\ell = 3$, the lower bounds given in Section 3 are in fact the exact values of the IC-indices of complete tripartite graphs. Our work completely solves the problem regarding complete tripartite graphs. Finally, a concluding remark is given in Section 5.

2. Preliminaries

In dealing with the IC-index of a graph, we view the colors of all vertices as a sequence satisfying some properties. We introduce some basic counting tools from [10] to analyse the sequence of colors. For convenience, we let $[1, \ell]$ denote the set $\{1, 2, \ldots, \ell\}$. A sequence of 0 and 1 is called a *binary sequence.*

Lemma 2.1. [10] *If $a_1, a_2, \ldots, a_n$ are n positive integers which satisfy that $a_1 = 1$ and $a_i \leq a_{i+1} \leq \sum_{j=1}^{i} a_j + 1$ for all $i \in [1, n-1]$, then for each $\ell \in [1, \sum_{j=1}^{n} a_j]$ there exists a binary sequence $c_1, c_2, \ldots, c_n$ such that $\ell = \sum_{j=1}^{n} c_j a_j$.*

Lemma 2.2. [10] *If $s_0, s_1, \ldots, s_n$ is a sequence of integers, then for each $i \in [1, n]$ there exists an integer $r_i \in \mathbb{Z}$ such that $s_i = \sum_{j=0}^{i-1} s_j + r_i$ and $\sum_{j=0}^{n} s_j = 2^n s_0 + \sum_{j=1}^{n} 2^{n-j} r_j$.*

Lemma 2.3. [10] *Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. If f is an IC-coloring of G such that $f(u_i) \leq f(u_{i+1})$ for all $i \in [1, n-1]$, then $f(u_1) = 1$ and $f(u_{i+1}) \leq \sum_{j=1}^{i} f(u_j) + 1$ for all $i \in [1, n-1]$.*

Lemma 2.4. [10] *Let f be an IC-coloring of a graph G such that $f(u_i) < f(u_{i+1})$ for $i \in [1, n-1]$, where $V(G) = \{u_1, u_2, \ldots, u_n\}$. For each pair (i_1, i_2) where $1 \leq i_1 < i_2 \leq n$, if $f(u_{i_1}) = \sum_{j=1}^{i_1-1} f(u_j) + 1$ and $u_{i_1}u_{i_2} \notin E(G)$, then either $f(u_{i_2}) \leq \sum_{j=1}^{i_2-1} f(u_j) - f(u_{i_1})$ or $f(u_{i_2+1}) \leq f(u_{i_1}) + f(u_{i_2})$.*

Lemma 2.5. [10] *Let $r_1, r_2, \ldots, r_n$ be n numbers. If there are two integers i and k such that $1 \leq i < k \leq n$ and $r_i < r_k$, then*

$$\sum_{j=1}^{n} 2^{n-j} r_j < \sum_{j=1}^{n} 2^{n-j} r_j - (2^{n-i} r_i + 2^{n-k} r_k) + (2^{n-i} r_k + 2^{n-k} r_i).$$

3. Lower bounds on $M(K_{m_1,m_2,\ldots,m_\ell})$

A lower bound on the IC-index of $K_{m_1,m_2,\ldots,m_\ell}$ for $1 = m_1 = \cdots = m_n < m_{n+1} \leq m_{n+2} \leq \cdots \leq m_\ell$ has been given in [11] as $M(K_{m_1,m_2,\ldots,m_\ell}) \geq (2^{m_\ell}(2^{m_{\ell-1}}(\cdots(2^{m_{n+1}}(2^n - 1) + 1)\cdots) + 1) + 1)$. In this section, we introduce our lower bounds on $M(K_{m_1,m_2,\ldots,m_\ell})$ separately in two cases where $2 \leq m_1 \leq m_2 \leq \cdots \leq m_\ell$ and $1 = m_1 < m_2 \leq \cdots \leq m_\ell$. In what follows, G represents the complete multipartite graph $K_{m_1,m_2,\ldots,m_\ell}$ with partite sets $W_1, W_2, \ldots, W_\ell$ where $W_i = \{w_{i,j} \mid j \in [1, m_i]\}$ for all $i = 1, 2, \ldots, \ell$. For any $S \subseteq V(G)$, we denote the subgraph of G induced by S as $\langle S \rangle$.

Proposition 3.1. *Let $m = \sum_{i=1}^{\ell} m_i$. Then*

$$M(K_{m_1,m_2,\ldots,m_\ell}) \geq 13 \cdot 2^{m-4} + \sum_{j=3}^{\ell-1} 2^{m-(\sum_{x=3}^{j} m_x + 4)} - g(m_1, m_2),$$

where

$$g(m_1, m_2) = \begin{cases} 3 \cdot 2^{m_1-2} - 4 & \text{if } 2 \leq m_1 \leq m_2 \leq \cdots \leq m_\ell, \\ 2^{m_2-2} - 2 & \text{if } 1 = m_1 < m_2 \leq \cdots \leq m_\ell. \end{cases}$$

Proof. We prove the lower bound by constructing an IC-coloring of G. First, let us consider the case where $2 \leq m_1 \leq m_2 \leq \cdots \leq m_\ell$. Before explicitly defining the coloring f, we arrange the vertices of G into a new order $u_1, u_2, \ldots, u_m$ such that the values of f can be defined recursively. Let

$$u_i = \begin{cases} w_{2,1} & \text{if } i = 1, \\ w_{1,1} & \text{if } i = 2, \\ w_{1,2} & \text{if } i = 3, \\ w_{2,2} & \text{if } i = 4, \\ w_{j,i-\sum_{x=3}^{j-1} m_x - 4} & \text{if } i \in [\sum_{x=3}^{j-1} m_x + 5, \sum_{x=3}^{j} m_x + 4] \text{ and } j \in [3, \ell], \\ w_{2,i-\sum_{x=3}^{\ell} m_x - 2} & \text{if } i \in [\sum_{x=3}^{\ell} m_x + 5, \sum_{x=2}^{\ell} m_x + 2], \\ w_{1,i-\sum_{x=2}^{\ell} m_x} & \text{if } i \in [\sum_{x=2}^{\ell} m_x + 3, m]. \end{cases}$$

Each partite set actually contains the u_i's as follows:

$$W_1 = \{u_2, u_3\} \cup \left\{ u_i \;\middle|\; i \in \left[\sum\nolimits_{x=2}^{\ell} m_x + 3, m \right] \right\},$$
$$W_2 = \{u_1, u_4\} \cup \left\{ u_i \;\middle|\; i \in \left[\sum\nolimits_{x=3}^{\ell} m_x + 5, \sum\nolimits_{x=2}^{\ell} m_x + 2 \right] \right\}$$

and

$$W_j = \left\{ u_i \;\middle|\; i \in \left[\sum\nolimits_{x=3}^{j-1} m_x + 5, \sum\nolimits_{x=3}^{j} m_x + 4 \right] \right\} \quad \text{for } j \in [3, \ell].$$

Now, we define $f \colon V(G) \to \mathbb{N}$ recursively as $f(u_1) = 1$ and $f(u_i) = \sum_{j=1}^{i-1} f(u_j) + r_i$ for $i \in [2, m]$, where

$$r_i = \begin{cases} 1 & \text{if } i \in \{2, 3\} \cup \left\{ \sum_{x=3}^{j} m_x + 4 \mid j \in [3, \ell] \right\}, \\ 0 & \text{if } i \in [5, \sum_{x=3}^{\ell} m_x + 4] \setminus \left\{ \sum_{x=3}^{j} m_x + 4 \mid j \in [3, \ell] \right\}, \\ -1 & \text{if } i \in \{4\} \cup [\sum_{x=3}^{\ell} m_x + 5, \sum_{x=2}^{\ell} m_x + 2], \\ -4 & \text{if } i \in [\sum_{x=2}^{\ell} m_x + 3, m]. \end{cases}$$

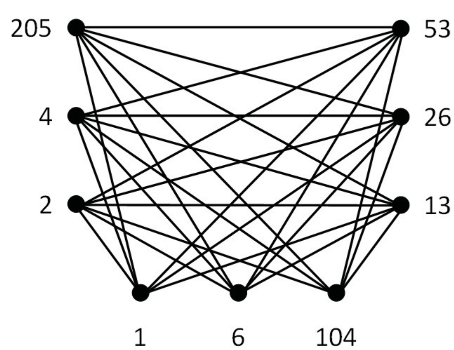

Figure 3.1: An IC-coloring of $K_{3,3,3}$

Now, considering $s_0 = 0$ and $s_i = f(u_i)$ for $i \in [1, m]$ in Lemma 2.2, we have

$$\begin{aligned}
f(G) &= \sum_{j=0}^{m} s_j = 2^m \cdot s_0 + \sum_{j=1}^{m} 2^{m-j} \cdot r_j \\
&= 2^{m-1} + 2^{m-2} + 2^{m-3} + 2^{m-4} \cdot (-1) + \sum_{j=3}^{\ell} 2^{m-(\sum_{x=3}^{j} m_x + 4)} \\
&\quad + \sum_{j=m-m_1-m_2+5}^{m-m_1+2} 2^{m-j}(-1) + \sum_{j=m-m_1+3}^{m} 2^{m-j}(-4) \\
&= 13 \cdot 2^{m-4} + \sum_{j=3}^{\ell} 2^{m-(\sum_{x=3}^{j} m_x+4)} - (2^{m_1+m_2-4} - 2^{m_1-2}) - 4(2^{m_1-2} - 1) \\
&= 13 \cdot 2^{m-4} + \sum_{j=3}^{\ell-1} 2^{m-(\sum_{x=3}^{j} m_x+4)} - 3 \cdot 2^{m_1-2} + 4.
\end{aligned}$$

Next, we will show that f is an IC-coloring of G. Given any $k \in [1, f(G)]$, we need to identify a connected subgraph H such that $f(H) = k$. Since $f(u_1) = 1$ and $f(u_i) < f(u_{i+1}) \le \sum_{j=1}^{i} f(u_j) + 1$ for all $i \in [1, m-1]$, Lemma 2.1 guarantees the existence of a binary sequence $c_1, c_2, \ldots, c_m$ such that $k = \sum_{j=1}^{m} c_j \cdot f(u_j)$. Let $S = \{u_j \mid c_j = 1, j \in [1, m]\}$. Then $f(\langle S \rangle) = k$. It suffices to consider the situation where $\langle S \rangle$ is disconnected, that is, $S \subseteq W_j$ for some $j \in [1, \ell]$ and $|S| \ge 2$. There are five possible cases.

Case 1*:* $\{u_2, u_3\} \subseteq S \subseteq W_1$.

Observe that $f(u_2) + f(u_3) = f(u_4)$. Let $S_1 = (S \setminus \{u_2, u_3\}) \cup \{u_4\}$. Then the subgraph $H = \langle S_1 \rangle$ is connected and $f(H) = k - f(u_2) - f(u_3) + f(u_4) = k$.

Case 2*:* $S \subseteq W_1$ *and* $\{u_2, u_3\} \nsubseteq S$.

In this case, $\{w_{1,j} \mid j \ge 3\} \cap S \ne \emptyset$, that is, there is some $u_j \in S$ where $j \in [\sum_{x=2}^{\ell} m_x + 3, m]$. Let $t = \min\left\{j \mid u_j \in S \text{ and } \sum_{x=2}^{\ell} m_x + 3 \le j \le m\right\}$. Then we have $f(u_t) = \sum_{j=1}^{t-1} f(u_j) - 4$ from the definition of f.

(1) If $u_2 \in S$, then $f(u_t) + f(u_2) = \sum_{j=1}^{t-1} f(u_j) - 2 = \sum_{j=1}^{t-1} f(u_j) - f(u_2) = f(u_1) + \sum_{j=3}^{t-1} f(u_j)$. By letting $S_1 = (S \setminus \{u_2, u_t\}) \cup \{u_1\} \cup \{u_3, u_4, \ldots, u_{t-1}\}$, we have a connected subgraph $H = \langle S_1 \rangle$ satisfying $f(H) = k$.

(2) If $u_2 \notin S$, then $f(u_t) = \sum_{j=1}^{t-1} f(u_j) - f(u_3) = f(u_1) + f(u_2) + \sum_{j=4}^{t-1} f(u_j)$. The subgraph H induced by $(S \setminus \{u_t\}) \cup \{u_1, u_2\} \cup \{u_4, u_5, \ldots, u_{t-1}\}$ is connected and satisfies $f(H) = k$.

Case 3*:* $S \subseteq W_2$.

(1) If $u_4 \in S$, then the subgraph induced by $(S \setminus \{u_4\}) \cup \{u_2, u_3\}$ is the desired connected subgraph because $f(u_4) = f(u_2) + f(u_3)$.

(2) If $u_4 \notin S$, then $\{w_{2,j} \mid j \ge 3\} \cap S \ne \emptyset$. There is some u_j in S where $j \in [\sum_{x=3}^{\ell} m_x + 5, \sum_{x=2}^{\ell} m_x + 2]$. Let $t = \min\left\{j \mid u_j \in S \text{ and } \sum_{x=3}^{\ell} m_x + 5 \le j \le \sum_{x=2}^{\ell} m_x + 2\right\}$. Then,

from the definition of f, we have $f(u_t) = \sum_{j=1}^{t-1} f(u_j) - 1 = \sum_{j=1}^{t-1} f(u_j) - f(u_1) = \sum_{j=2}^{t-1} f(u_j)$. By letting H be the subgraph induced by $(S \setminus \{u_t\}) \cup \{u_2, u_3, \ldots, u_{t-1}\}$, we have $f(H) = k$ and H is connected.

Case 4: $S \subseteq W_i$ for some $i \in [3, \ell]$.

Let $t = \min\left\{j \mid u_j \in S \text{ and } j \in [\sum_{x=3}^{i-1} m_x + 5, \sum_{x=3}^{i} m_x + 4]\right\}$. Since $|S| \geq 2$, $t < \sum_{x=3}^{i} m_x + 4$. It follows that $f(u_t) = \sum_{j=1}^{t-1} f(u_j)$. Now, let $S_1 = (S \setminus \{u_t\}) \cup \{u_1, u_2, \ldots, u_{t-1}\}$, the subgraph induced by S_1 is desired.

For the second case where $1 = m_1 < m_2 \leq \cdots \leq m_\ell$, the result can be shown similarly. Let

$$u_i = \begin{cases} w_{2,1} & \text{if } i = 1, \\ w_{2,2} & \text{if } i = 2, \\ w_{3,1} & \text{if } i = 3, \\ w_{1,1} & \text{if } i = 4, \\ w_{3,i-3} & \text{if } i \in [5, m_3 + 3], \\ w_{j,i-\sum_{x=3}^{j-1} m_x - 3} & \text{if } i \in [\sum_{x=3}^{j-1} m_x + 4, \sum_{x=3}^{j} m_x + 3] \text{ and } j \in [4, \ell], \\ w_{2,i-\sum_{x=3}^{\ell} m_x - 1} & \text{if } i \in [\sum_{x=3}^{\ell} m_x + 4, m]. \end{cases}$$

Then we have

$$\begin{aligned} W_1 &= \{u_4\}, \\ W_2 &= \{u_1, u_2\} \cup \left\{u_j \;\middle|\; j \in \left[\sum\nolimits_{x=3}^{\ell} m_x + 4, m\right]\right\}, \\ W_3 &= \{u_3\} \cup \{u_j \mid j \in [5, m_3 + 3]\} \end{aligned}$$

and

$$W_j = \left\{u_i \;\middle|\; i \in \left[\sum\nolimits_{x=3}^{j-1} m_x + 4, \sum\nolimits_{x=3}^{j} m_x + 3\right]\right\} \quad \text{for } j \in [4, \ell].$$

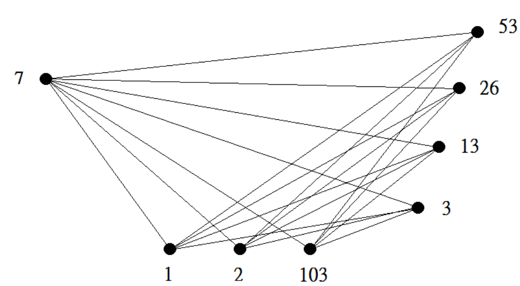

Figure 3.2: An IC-coloring of $K_{1,3,4}$

Now, we define $f\colon V(G) \to \mathbb{N}$ recursively as $f(u_1) = 1$ and $f(u_i) = \sum_{j=1}^{i-1} f(u_j) + r_i$ for $i \in [2, m]$, where

$$r_i = \begin{cases} 1 & \text{if } i \in \{2,4\} \cup \left\{\sum_{x=3}^{j} m_x + 3 \mid j \in [3,\ell]\right\}, \\ 0 & \text{if } i = 3 \text{ or } i \in [5, \sum_{x=3}^{\ell} m_x + 3] \setminus \left\{\sum_{x=3}^{j} m_x + 3 \mid j \in [3,\ell]\right\}, \\ -2 & \text{if } i \in [\sum_{x=3}^{\ell} m_x + 4, m]. \end{cases}$$

The value of $f(G)$ can be determined using Lemma 2.2 as follows:

$$\begin{aligned} f(G) &= \sum_{j=1}^{m} 2^{m-j} \cdot r_j \\ &= 2^{m-1} + 2^{m-2} + 2^{m-4} + \sum_{j=3}^{\ell} 2^{m-(\sum_{x=3}^{j} m_x + 3)} + \sum_{j=m-m_2+3}^{m} 2^{m-j}(-2) \\ &= 13 \cdot 2^{m-4} + \sum_{j=3}^{\ell} 2^{m-(\sum_{x=3}^{j} m_x + 3)} - 2(2^{m_2-2} - 1) \\ &= 13 \cdot 2^{m-4} + \sum_{j=3}^{\ell-1} 2^{m-(\sum_{x=3}^{j} m_x + 3)} - 2^{m_2-2} + 2. \end{aligned}$$

Next, given $k \in [1, f(G)]$, Lemma 2.1 implies that there exists a binary sequence $c_1, c_2, \ldots, c_m$ such that $k = \sum_{j=1}^{m} c_j \cdot f(u_j)$. Let $S = \{u_j \mid c_j = 1, j \in [1,m]\}$. Then $f(\langle S \rangle) = k$. The subgraph $\langle S \rangle$ is disconnected only when the following three cases occur. We construct a connected subgraph H with $f(H) = k$ in each case.

Case 1*: $S \subseteq W_2$.*

If $\{u_1, u_2\} \subseteq S$, then the subgraph H induced by $(S \setminus \{u_1, u_2\}) \cup \{u_3\}$ is connected and $f(H) = k$ because $u_3 \in W_3$ and $f(u_1) + f(u_2) = f(u_3)$. If $\{u_1, u_2\} \nsubseteq S$, we let $t = \min\left\{j \mid u_j \in S \text{ and } j \in [\sum_{x=3}^{\ell} m_x + 4, m]\right\}$, then $f(u_t) = \sum_{j=1}^{t-1} f(u_j) - 2$. First, observe that $f(u_t) + f(u_1) = \sum_{j=1}^{t-1} f(u_j) - 1 = \sum_{j=1}^{t-1} f(u_j) - f(u_1) = \sum_{j=2}^{t-1} f(u_j)$. If $u_1 \in S$ and $u_2 \notin S$, then the subgraph induced by $(S \setminus \{u_1, u_t\}) \cup \{u_2, u_3, \ldots, u_{t-1}\}$ is the desired one. Second, note that $f(u_t) = \sum_{j=1}^{t-1} f(u_j) - f(u_2) = f(u_1) + \sum_{j=3}^{t-1} f(u_j)$. If $u_1 \notin S$, then the subgraph H induced by $(S \setminus \{u_t\}) \cup \{u_1\} \cup \{u_3, u_4, \ldots, u_{t-1}\}$ is connected and satisfies $f(H) = k$.

Case 2*: $S \subseteq W_3$.*

If $u_3 \in S$, then the subgraph induced by $(S \setminus \{u_3\}) \cup \{u_1, u_2\}$ certainly satisfies our requirement. If $u_3 \notin S$, then we let $t = \min\{j \mid u_j \in S, 5 \le j \le m_3 + 3\}$. Since $|S| \ge 2$, $t < m_3 + 3$ and $f(u_t) = \sum_{j=1}^{t-1} f(u_j)$. The subgraph H induced by $(S \setminus \{u_t\}) \cup \{u_1, u_2, \ldots, u_{t-1}\}$ is desired.

Case 3*: $S \subseteq W_i$ for some $i \in [4, \ell]$.*

Let $t = \min\left\{j \mid u_j \in S, j \in [\sum_{x=3}^{i-1} m_x + 4, \sum_{x=3}^{i} m_x + 3]\right\}$. Then $t < \sum_{x=3}^{i} m_x + 3$ and $f(u_t) = \sum_{j=1}^{t-1} f(u_j)$. The subgraph H induced by $(S \setminus \{u_t\}) \cup \{u_1, u_2, \ldots, u_{t-1}\}$ is what we need. □

4. The exact value of $M(K_{m_1,m_2,m_3})$

In this section, we prove that the lower bound on $M(K_{m_1,m_2,m_3})$ given in the previous section also serves as an upper bound on it. To be precise, we shall show that $M(K_{m_1,m_2,m_3})$ is upper-bounded by $13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4$ for the case where $2 \le m_1 \le m_2 \le m_3$, and by $13 \cdot 2^{m-4} - 2^{m_2-2} + 2$ for the case where $1 = m_1 < m_2 \le m_3$. In what follows, for the given IC-coloring f, we always assume that $\{u_1, u_2, \ldots, u_m\}$ is the vertex set of G such that $f(u_i) \le f(u_{i+1})$ for all $i \in [1, m-1]$. For brevity, we let $f_0 = 0$ and denote the sum $\sum_{i=1}^{j} f(u_i)$ as f_j for $j \in [1, m]$. The following properties are essential for a maximal IC-coloring of G.

Lemma 4.1. *If f is a maximal IC-coloring of G, then $f_j < 2^{j-i}(f_i + 1)$ for each pair (i, j) with $1 \le i \le j \le m$.*

Proof. It suffices to consider the case $i < j$. For the given pair (i, j), let us consider the sequence $s_0 = f_i$ and $s_k = f(u_{i+k})$ for $k \in [1, j-i]$. Since $s_k \le \sum_{\ell=0}^{k-1} s_\ell + 1$ by Lemma 2.3, we obtain from Lemma 2.2 that

$$\begin{aligned} f_j = f_i + \sum_{k=1}^{j-i} f(u_{i+k}) &\le 2^{j-i} f_i + \sum_{k=1}^{j-i} 2^{j-i-k} \cdot 1 \\ &= 2^{j-i} f_i + (2^{j-i} - 1) < 2^{j-i}(f_i + 1). \end{aligned}$$ □

Lemma 4.2. *If f is a maximal IC-coloring of G, then all colors of the vertices of G are distinct.*

Proof. Suppose that there exist two distinct vertices u_i and u_{i+1} such that $f(u_i) = f(u_{i+1})$. Then we have $f_{i-1} \le 2^{(i-1)-1}(f_1 + 1) - 1 = 2^{i-1} - 1$ from Lemma 4.1 and $f(u_{i+1}) = f(u_i) \le f_{i-1} + 1$ from Lemma 2.3. Thus, $f_{i+1} = f(u_{i+1}) + f(u_i) + f_{i-1} \le 3 \cdot f_{i-1} + 2 \le 3 \cdot 2^{i-1} - 1$.

Now, Lemma 4.1 implies that

$$\begin{aligned} f(G) &< 2^{m-(i+1)}(f_{i+1} + 1) \le 3 \cdot 2^{m-2} \\ &= 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} - 2^{m_1-2}(2^{m_2+m_3-2} - 3) \qquad \text{(4.1a)} \\ &= 13 \cdot 2^{m-4} - 2^{m_2-2} - 2^{m_2-2}(2^{m_1+m_3-2} - 1) \qquad \text{(4.1b)} \\ &= 2^m - 2^{m-2}. \end{aligned}$$

The value in (4.1a) is less than $13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4$ when $2 \le m_1 \le m_2 \le m_3$ and the one in (4.1b) is smaller than $13 \cdot 2^{m-4} - 2^{m_2-2} + 2$ when $1 = m_1 < m_2 \le m_3$. These lead to a contradiction to Proposition 3.1. We have the result. □

Lemma 4.3. *If f is a maximal IC-coloring of K_{m_1,m_2,m_3}, where $2 \le m_1 \le m_2 \le m_3$ or $1 = m_1 < m_2 \le m_3$, then $f_j > 51 \cdot 2^{j-6} - 1$ for any $j \in [1, m]$.*

Proof. Suppose that $f_j \le 51 \cdot 2^{j-6} - 1$ for some $j \in [1, m]$. Then from Lemma 4.1 we have

$$
\begin{aligned}
f(G) &< 2^{m-j} \cdot (f_j + 1) \le 51 \cdot 2^{m-6} \\
&= 13 \cdot 2^{m-4} - 2^{m-6} && \text{(4.2a)} \\
&= 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} - 2^{m_1-2}(2^{m_2+m_3-4} - 3) && \text{(4.2b)} \\
&= 13 \cdot 2^{m-4} - 2^{m_2-2} - 2^{m_2-2}(2^{m_1+m_3-4} - 1). && \text{(4.2c)}
\end{aligned}
$$

The value in (4.2b) is less than $13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4$ when $2 \le m_1 \le m_2 \le m_3$ and the upper bound in (4.2c) is smaller than $13 \cdot 2^{m-4} - 2^{m_2-2} + 2$ when $1 = m_1 < m_2 \le m_3$, contradicting to Proposition 3.1. The result follows. □

Lemma 4.4. *Suppose that f is a maximal IC-coloring of K_{m_1,m_2,m_3}, where $2 \le m_1 \le m_2 \le m_3$ or $1 = m_1 < m_2 \le m_3$, and $f_k \le 14 \cdot 2^{k-4} - 1$ for some $k \ge 4$. If $f(u_j) = f_{j-1} + 1$ for some $j \in [1, m]$ and $u_j u_{j+\ell} \notin E(G)$ for some $\ell \in [k + 1 - j, m - j]$, then $f(u_{j+\ell}) \le f_{j+\ell-1} - f(u_j)$.*

Proof. First, we derive upper bounds on f_{j-1} as follows. By Lemma 4.1, if $j \ge k+1$, then $f_{j-1} \le 2^{(j-1)-k}(f_k + 1) - 1 \le 14 \cdot 2^{j-5} - 1 \le 7 \cdot 2^{j+\ell-5} - 1$. Otherwise, $j \le k$, we have $f_{j-1} \le f_{k-1} \le 2^{(k-1)-1}(f_1+1)-1 = 2^{j+\ell-5} \cdot 2^{(k-1)-(j+\ell-5)} - 1 = 2^{j+\ell-5} \cdot 2^{3-(j+\ell-k-1))} - 1$. Since $j + \ell \ge k + 1$, these two bounds can be combined into $f_{j-1} \le 8 \cdot 2^{j+\ell-5} - 1$. Now, suppose that $f(u_{j+\ell}) > f_{j+\ell-1} - f(u_j)$. Then $f(u_{j+\ell+1}) \le f(u_{j+\ell}) + f(u_j)$ by Lemma 2.4. This implies that

$$
\begin{aligned}
f_{j+\ell+1} &= f(u_{j+\ell+1}) + f_{j+\ell} \\
&\le f(u_{j+\ell}) + f(u_j) + f(u_{j+\ell}) + f_{j+\ell-1} \\
&\le (f_{j+\ell-1} + 1) + (f_{j-1} + 1) + (f_{j+\ell-1} + 1) + f_{j+\ell-1} \\
&= 3(f_{j+\ell-1} + 1) + f_{j-1} \\
&< 3 \cdot [2^{(j+\ell-1)-k} \cdot (f_k + 1)] + 8 \cdot 2^{j+\ell-5} - 1 \\
&\le 50 \cdot 2^{j+\ell-5} - 1.
\end{aligned}
$$

Lemma 4.1 then enables us to find a bound on $f(G)$:

$$f(G) < 2^{m-(j+\ell+1)} \cdot (f_{j+\ell+1} + 1) \le 2^{m-(j+\ell+1)} \cdot (50 \cdot 2^{j+\ell-5}) = 13 \cdot 2^{m-4} - 2^{m-5}.$$

This value is smaller than (4.2a) and we have a contradiction. The result follows. □

Lemma 4.5. *If f is a maximal IC-coloring of K_{m_1,m_2,m_3}, where $2 \le m_1 \le m_2 \le m_3$ or $1 = m_1 < m_2 \le m_3$, then $f_4 \ge 12$. Furthermore, given $i \in [1, m-1]$, let $r_\ell = f(u_{i+\ell}) - f_{i+\ell-1}$ for all $\ell \in [1, m-i]$. Then $f(G) = 2^{m-i} f_i + \sum_{\ell=1}^{m-i} 2^{m-i-\ell} r_\ell$.*

Proof. The first result is a direct consequence from Lemma 4.3 which gives that $f_4 > 51 \cdot 2^{4-6} - 1 > 11$. To prove the second result, for given $i \in [1, m-1]$, we let $s_0 = f_i$ and $s_\ell = f(u_{i+\ell})$ for all $\ell \in [1, m-i]$. Then Lemma 2.2 gives

$$f_{i+j} = f_i + \sum_{\ell=1}^{j} f(u_{i+\ell}) = 2^j f_i + \sum_{\ell=1}^{j} 2^{j-\ell} r_\ell. \tag{4.3}$$

Therefore, the result follows by letting $j = m - i$ in (4.3). □

Now, we are in a position to show our upper bounds on $M(G)$.

Proposition 4.6. *If f is a maximal IC-coloring of K_{m_1,m_2,m_3}, then*

$$f(K_{m_1,m_2,m_3}) \le \begin{cases} 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4 & \text{if } 2 \le m_1 \le m_2 \le m_3, \\ 13 \cdot 2^{m-4} - 2^{m_2-2} + 2 & \text{if } 1 = m_1 < m_2 \le m_3, \end{cases}$$

where $m = m_1 + m_2 + m_3$.

Proof. Since $f(u_2) = 2 < f(u_3) \le f_2 + 1 = 4$, the value of $f(u_3)$ is either 3 or 4. We split the long proof into two parts depending on the value of $f(u_3)$. In the first part, let us assume that $f(u_3) = 4$. Note that Lemma 4.3 gives $f_5 > 51 \cdot 2^{5-6} - 1$ and then $25 \le f_5 = f_4 + f(u_5) \le f_4 + (f_4 + 1) = 2f(u_4) + 15$. This implies that $f(u_4) \ge 5$. On the other hand, we see from Lemma 2.3 that $f(u_4) \le f_3 + 1 = 8$. Hence, $5 \le f(u_4) \le 8$. We discuss the problem for each possible value of $f(u_4)$. Since $f(u_3) = 4$, u_1u_2 must be an edge of G for otherwise there would be no connected subgraph H satisfying $f(H) = 3$. Therefore in each of the following three cases, $u_1u_2 \in E(G)$ is true.

Case 1*:* $f(u_4) = 8$.

Since $f(u_3) > f_2 - f(u_1)$ and $f(u_4) > f(u_1) + f(u_3)$, we have $u_1u_3 \in E(G)$ by Lemma 2.4. Similarly, u_2u_3 is an edge of G as well. Hence, the subgraph induced by $\{u_1, u_2, u_3\}$ is isomorphic to K_3. First, let us consider the situation where $u_1u_4 \notin E(G)$ or $u_2u_4 \notin E(G)$. Then $f(u_5) \le f(u_2) + f(u_4) = 10$ by Lemma 2.4. In addition, $f_5 > 51 \cdot 2^{5-6} - 1$ from Lemma 4.3 implies that $f(u_5) = f_5 - f_4 \ge 25 - 15 = 10$. Therefore, $f(u_5) = 10$ and $f_5 = 25$. If $m = 5$, then $(m_1, m_2, m_3) = (1, 2, 2)$ and $f(G) = f_5 = 25 < 27 = 13 \cdot 2^{m-4} - 2^{m_2-2} + 2$. Now, suppose that $m \ge 6$. The fact $f_6 > 51 \cdot 2^{6-6} - 1$ implies that $f(u_6) = f_6 - f_5 \ge 51 - 25 = 26$. Hence, we see that $f(u_6) = f_5 + 1 = 26$. Since $\langle\{u_1, u_2, u_3\}\rangle \cong K_3$, there is some $t \in \{1, 2, 3\}$ such that $u_6u_t \notin E(G)$. Therefore, Lemma 2.4 gives that $f(u_7) \le f(u_6) + f(u_t) \le f(u_6) + f(u_3) = 30$. However, this leads to $f_7 = f(u_7) + f(u_6) + f_5 \le 81 < 51 \cdot 2^{7-6} - 1$, which contradicts to Lemma 4.3. We therefore conclude that in this situation where $u_1u_4 \notin E(G)$ or $u_2u_4 \notin E(G)$, "$m \ge 6$" is impossible to be true.

Next, we consider the situation where $u_1u_4 \in E(G)$ and $u_2u_4 \in E(G)$. Then u_3 and u_4 must be in the same partite set of G. We therefore have that $\langle\{u_1,u_2,u_3,u_4\}\rangle \cong K_{1,1,2}$ and then $f(u_5) \leq f(u_4)+f(u_3) = 12$ by Lemma 2.4. Since $f_5 = f_4+f(u_5) \leq 15+12 = 27$, $f(G) \leq 13\cdot 2^{m-4} - 2^{m_2-2}+2$ holds when $m=5$. When $m \geq 6$, $f_6 > 51\cdot 2^{6-6}-1$ implies that $f(u_6) = f_6 - f_5 > 50-27 = 23 > f(u_4)+f(u_5)$. We then have $u_4u_5 \in E(G)$ by Lemma 2.4 because $f_4 - f(u_4) < 8 = f(u_4) < f(u_5)$ and $f(u_4) = f_3+1$. Hence, $\langle\{u_1,u_2,u_3,u_4,u_5\}\rangle \cong K_{1,2,2}$.

Now, Lemma 4.5 enables us to obtain that $f(G) = 2^{m-5}f_5 + \sum_{\ell=1}^{m-5} 2^{m-5-\ell}r_\ell \leq 27\cdot 2^{m-5} + \sum_{\ell=1}^{m-5} 2^{m-5-\ell}r_\ell$. From the previous discussion, we know that $\langle\{u_1,u_2,u_4\}\rangle \cong K_3$. Let us denote the partite set containing u_j as V_{u_j}, $j=1,2,4$. Then $f(u_j) = f_{j-1}+1$ and $\{|V_{u_j}| \mid j=1,2,4\} = \{m_1,m_2,m_3\}$. Since $f_5 \leq 27 = 14\cdot 2^{5-4}-1$, by Lemma 4.4, we know that $f(u_{5+\ell}) \leq f_{5+\ell-1} - f(u_j)$ whenever $u_{5+\ell} \in V_{u_j}$ for $j=1,2,4$. Therefore, $r_\ell \leq -f(u_1) = -1$ whenever $u_{5+\ell} \in V_{u_1}$; $r_\ell \leq -f(u_2) = -2$ whenever $u_{5+\ell} \in V_{u_2}$ and $r_\ell \leq -f(u_4) = -8$ whenever $u_{5+\ell} \in V_{u_4}$. With the fact $\langle\{u_1,u_2,u_3,u_4,u_5\}\rangle \cong K_{1,2,2}$ in mind, we are ready for the discussion about the upper bounds on $f(G)$.

In the case where $2 \leq m_1 \leq m_2 \leq m_3$, the sum $\sum_{\ell=1}^{m-5} 2^{m-5-\ell}r_\ell$ is maximized when $(|V_{u_1}|,|V_{u_2}|,|V_{u_4}|) = (m_3,m_2,m_1)$ and $r_\ell = -1$ for all $\ell = 1,2,\ldots,m_3-1$; $r_\ell = -2$ for all $\ell = m_3, m_3+1,\ldots,m_3+m_2-3$ and $r_\ell = -8$ for all $\ell = m_3+m_2-2,\ldots,m-5$. Therefore,

$$\begin{aligned}
f(G) &\leq 2^{m-5}\cdot 27 + \sum_{j=1}^{m_3-1} 2^{m-5-j}\cdot(-1) + \sum_{j=m_3}^{m_3+m_2-3} 2^{m-5-j}\cdot(-2) \\
&\quad + \sum_{j=m_3+m_2-2}^{m-5} 2^{m-5-j}\cdot(-8) \\
&\leq 27\cdot 2^{m-5} - (2^{m-5}-2^{m_1+m_2-4}) - 2(2^{m_1+m_2-4}-2^{m_1-2}) - 8(2^{m_1-2}-1) \\
&= 13\cdot 2^{m-4} - 2^{m_1+m_2-4} - 3\cdot 2^{m_1-1} + 8 \\
&= 13\cdot 2^{m-4} - 3\cdot 2^{m_1-2} + 4 - (2^{m_1+m_2-4} + 3\cdot 2^{m_1-2} - 4) \\
&\leq 13\cdot 2^{m-4} - 3\cdot 2^{m_1-2} + 4.
\end{aligned}$$

In the case where $1 = m_1 < m_2 \leq m_3$, the sum $\sum_{\ell=1}^{m-5} 2^{m-5-\ell}r_\ell$ is maximized when $(|V_{u_1}|,|V_{u_2}|,|V_{u_4}|) = (m_3,1,m_2)$ and $r_\ell = -1$ for all $\ell = 1,2,\ldots,m_3-2$; $r_\ell = -8$ for all $\ell = m_3-1, m_3,\ldots,m_3+m_2-4 = m-5$. Hence,

$$\begin{aligned}
f(G) &\leq 2^{m-5}\cdot 27 + \sum_{j=1}^{m_3-2} 2^{m-5-j}\cdot(-1) + \sum_{j=m_3-1}^{m-5} 2^{m-5-j}\cdot(-8) \\
&\leq 27\cdot 2^{m-5} - (2^{m-5}-2^{m_2-2}) - 8(2^{m_2-2}-1) \\
&\leq 13\cdot 2^{m-4} - 2^{m_2-2} + 2.
\end{aligned}$$

Case 2*:* $f(u_4) = 7$.

For the same reason stated in the previous case, we also have $\langle\{u_1, u_2, u_3\}\rangle \cong K_3$. Now, Lemma 4.3 gives that $f_5 > 51 \cdot 2^{5-6} - 1$. This implies $f(u_5) = f_5 - f_4 \geq 11$ which is greater than $f(u_4) + f(u_i)$ for $i = 1, 2$. We then have $u_iu_4 \in E(G)$ from Lemma 2.4 because $f(u_4) = 7 > f_3 - f(u_i)$, $i = 1, 2$. It follows that u_3 and u_4 must be in the same partite set of G. Since $f_3 - f(u_3) < f(u_4)$, we have $f(u_5) \leq f(u_3) + f(u_4) = 11$ by Lemma 2.4. This implies that $f_5 \leq 25$. If $m = 5$, then $(m_1, m_2, m_3) = (1, 2, 2)$ and $f(G) = f_5 \leq 25 \leq 13 \cdot 2^{m-4} - 2^{m_2-2} + 2$. If $m \geq 6$, then the fact $f_6 > 51 \cdot 2^{6-6} - 1$ from Lemma 4.3 gives $f(u_6) = f_6 - f_5 > 50 - 25 = 25$. Hence, $f(u_6) = 26$ and then $f_5 = 25$ because $f(u_6) \leq f_5 + 1$. Now, since $u_6u_t \notin E(G)$ for some $t \in \{1, 2, 3\}$, we have $f(u_7) \leq f(u_6) + f(u_t) \leq f(u_6) + f(u_3) = 30$. However, this leads to $f_7 = f_5 + f(u_6) + f(u_7) \leq 81 < 51 \cdot 2^{7-6} - 1$, which contradicts to Lemma 4.3. We therefore conclude that "$m \geq 6$" is impossible to occur in this case.

Case 3*:* $f(u_4) = 6$.

Since $f(u_4) > f(u_3) + f(u_1)$ and $f(u_3) > f_2 - f(u_1)$, Lemma 2.4 implies that $u_3u_1 \in E(G)$. First, consider the situation where $u_2u_3 \in E(G)$. Then $\langle\{u_1, u_2, u_3\}\rangle \cong K_3$ and $u_5u_t \notin E(G)$ for some $t \in \{1, 2, 3\}$. Suppose that $f(u_5) \geq 13$. Then $f(u_5) > f_4 - f(u_1) \geq f_4 - f(u_t)$. By Lemma 2.4, we have $f(u_6) \leq f(u_5) + f(u_t) \leq f(u_5) + f(u_3) \leq (f_4 + 1) + f(u_3) \leq 18$. However, this implies that $f_6 = f_4 + f(u_5) + f(u_6) \leq 45 < 51 \cdot 2^{6-6} - 1$, which is a contradiction to Lemma 4.3. Hence we have $f(u_5) \leq 12$ and $f_5 \leq 25$. Suppose again that $m \geq 6$. Since $f_6 > 51 \cdot 2^{6-6} - 1$, one can see that $f(u_6) = f_6 - f_5 > 25 \geq f_5$. This implies that $f(u_6) = f_5 + 1 \leq 26$. Observe that if $u_6u_s \notin E(G)$ for some $s \in \{1, 2, 3\}$, we have $f(u_7) \leq f(u_6) + f(u_s) \leq f(u_6) + f(u_3) = 30$ by Lemma 2.4. However, this leads to $f_7 = f_5 + f(u_6) + f(u_7) \leq 81 < 51 \cdot 2^{7-6} - 1$. This contradiction enables us to conclude that "$m \geq 6$" is impossible to occur in the situation where $u_2u_3 \in E(G)$. Therefore we have $m = 5$ and $f(G) = f_5 \leq 25 < 27 = 13 \cdot 2^{m-4} - 2^{m_2-2} + 2$. The result holds in this situation.

Next, assume that $u_2u_3 \notin E(G)$. Since $f_5 > 51 \cdot 2^{5-6} - 1$, $f(u_5) = f_5 - f_4 \geq 25 - 13 = 12$ which is greater than $f(u_3) + f(u_4)$. We see that u_3 and u_4 must be adjacent in G by Lemma 2.4 because $f(u_4) > f_3 - f(u_3)$. It follows that $\langle\{u_1, u_2, u_3, u_4\}\rangle \cong K_{1,1,2}$ or $K_{2,2}$. Now, let the partite set containing u_j be V_{u_j}, $j = 1, 3$. Then u_4 belongs to V_{u_1} or the other partite set, written V_0. Observe that $f_4 = 13 = 14 \cdot 2^{4-4} - 1$ and $f(u_j) = f_{j-1} + 1$ for $j = 1, 3$, one can see from Lemma 4.4 that $f(u_t) \leq f_{t-1} - f(u_j)$, if $u_t \in V_{u_j}$, $j = 1, 3$ and $t \in [5, m]$. Therefore, $f(u_{4+\ell}) \leq f_{4+\ell-1} - f(u_1)$ whenever $u_{4+\ell} \in V_{u_1}$ and $f(u_{4+\ell}) \leq f_{4+\ell-1} - f(u_3)$ whenever $u_{4+\ell} \in V_{u_3}$. Let $i = 4$ in Lemma 4.5, we obtain $f(G) = 2^{m-4} f_4 + \sum_{\ell=1}^{m-4} 2^{m-4-\ell} r_\ell$, where the r_ℓ, defined as $f(u_{4+\ell}) - f_{4+\ell-1}$, does not exceed $-f(u_1)$ if $u_{4+\ell} \in V_{u_1}$ and it is not greater than $-f(u_3)$ if $u_{4+\ell} \in V_{u_3}$. Let $S_1 = \{u_k \in V_0 \mid f(u_k) = f_{k-1} + 1\}$. Then $r_\ell \leq 0$

whenever $u_{4+\ell} \in V_0 \setminus S_1$. If $S_1 \neq \emptyset$, we denote as i_1 the minimum element in $\{k \mid u_k \in S_1\}$, then $i_1 \geq 5$ and Lemma 4.4 implies that $f(u_{i_1+\ell}) \leq f_{i_1+\ell-1} - f(u_{i_1})$ whenever $u_{i_1+\ell} \in V_0$. Therefore, $S_1 = \{u_{i_1}\}$. Let $S_2 = \{u_k \in V_0 \mid f(u_k) \leq f_{k-1} - f(u_{i_1})\}$ and $|S_2| = y$. Then $r_\ell \leq -f(u_{i_1})$ whenever $u_{4+\ell} \in S_2$. With these observations in mind, we can investigate the value of $f(G)$ in the following situations.

Subcase 3.1*:* $2 \leq m_1 \leq m_2 \leq m_3$.

First, if $S_1 = \emptyset$, then $r_\ell \leq 0$ whenever $u_{4+\ell} \in V_0$. The sum $\sum_{\ell=1}^{m-4} 2^{m-4-\ell} r_\ell$ is maximized when $u_4 \in V_{u_1}$, $(|V_{u_1}|, |V_{u_3}|, |V_0|) = (m_2, m_1, m_3)$ and $r_\ell = 0$ for all $\ell = 1, 2, \ldots, m_3$; $r_\ell = -f(u_1) = -1$ for all $\ell = m_3+1, m_3+1, \ldots, m_3+m_2-2$ and $r_\ell = -f(u_3) = -4$ for all $\ell = m_3+m_2-1, \ldots, m-4$. Therefore, we have

$$
\begin{aligned}
f(G) &\leq 2^{m-4} f_4 + \sum_{j=1}^{m_3} 2^{m-4-j} \cdot 0 + \sum_{j=m_3+1}^{m_3+m_2-2} 2^{m-4-j} \cdot (-1) \\
&\quad + \sum_{j=m_3+m_2-1}^{m-4} 2^{m-4-j} \cdot (-4) \\
&= 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4 - 2^{m_1+m_2-4} \\
&\leq 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4.
\end{aligned}
$$

Second, if $S_1 \neq \emptyset$, then $r_\ell \leq -f(u_{i_1})$ whenever $u_{4+\ell} \in S_2$. We split the discussion into two parts:

(1) In the situation where $\langle \{u_1, u_2, u_3, u_4\} \rangle \cong K_{1,1,2}$, since $u_1 u_4 \in E(G)$ and $u_4 \in V_0$, $f(G)$ is maximized when $(|V_{u_1}|, |V_{u_3}|, |V_0|) = (m_2, m_1, m_3)$ and

$$
\begin{aligned}
f(G) &\leq 2^{m-4} f_4 + \sum_{j=1}^{m_3-y-2} 2^{m-4-j} \cdot 0 + 2^{m-4-(m_3-y-1)} \\
&\quad + \sum_{j=m_3-y}^{m_3+m_2-y-2} 2^{m-4-j} \cdot (-1) + \sum_{j=m_3+m_2-y-1}^{m-y-4} 2^{m-4-j} \cdot (-4) \\
&\quad + \sum_{j=m-y-3}^{m-4} 2^{m-4-j} \cdot [-f(u_{i_1})] \\
&= 13 \cdot 2^{m-4} + 2^{m_1+m_2+y-3} - 2^{m_1+y-2}(2^{m_2-1} - 1) \\
&\quad - 4 \cdot 2^y (2^{m_1-2} - 1) - f(u_{i_1})(2^y - 1) \\
&= 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1+y-2} + 4 - [f(u_{i_1}) - 4] \cdot (2^y - 1).
\end{aligned}
\tag{4.4}
$$

This implies that $f(G) \leq 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4$ because $f(u_{i_1}) \geq 4$.

(2) In the situation where $\langle \{u_1, u_2, u_3, u_4\} \rangle \cong K_{2,2}$, we have $u_1 u_4 \notin E(G)$ and $u_4 \in V_{u_1}$.

By the similar argument, we obtain

$$f(G) \le 2^{m-4} f_4 + \sum_{j=1}^{m_3-y-1} 2^{m-4-j} \cdot 0 + 2^{m-4-(m_3-y)} + \sum_{j=m_3-y+1}^{m_2+m_3-y-2} 2^{m-4-j} \cdot (-1)$$
$$+ \sum_{j=m_2+m_3-y-1}^{m_-y-4} 2^{m-4-j} \cdot (-4) + \sum_{j=m-y-3}^{m-4} 2^{m-4-j} \cdot [-f(u_{i_1})].$$

This upper bound is less than the one in (4.4). Our result holds in this situation.

Subcase 3.2*:* $1 = m_1 < m_2 \le m_3$.

First, if $\langle \{u_1, u_2, u_3, u_4\} \rangle \cong K_{2,2}$, then we have $u_4 \in V_{u_1}$. Now, $f(G)$ is maximized when $(|V_{u_1}|, |V_{u_3}|, |V_0|) = (m_3, m_2, 1)$ and

$$\begin{aligned} f(G) &\le 2^{m-4} f_4 + 2^{m-4-1} \cdot 1 + \sum_{j=2}^{m_3-1} 2^{m-4-j} \cdot (-1) \\ &\quad + \sum_{j=m_3}^{m-4} 2^{m-4-j} \cdot (-4) \\ &= 13 \cdot 2^{m-4} - 3 \cdot 2^{m_2-2} + 4 \\ &= 13 \cdot 2^{m-4} - 2^{m_2-2} + 2 - 2(2^{m_2-2} - 1) \\ &\le 13 \cdot 2^{m-4} - 2^{m_2-2} + 2. \end{aligned} \tag{4.5}$$

Second, if $\langle \{u_1, u_2, u_3, u_4\} \rangle \cong K_{1,1,2}$, then $u_4 \in V_0$. When $|V_0| = 1$, the argument is very similar to the previous one and we have

$$f(G) \le 2^{m-4} f_4 + \sum_{j=1}^{m_3-1} 2^{m-4-j} \cdot (-1) + \sum_{j=m_3}^{m-4} 2^{m-4-j} \cdot (-4).$$

This value is smaller than the one in (4.5) and our result holds.

When $|V_{u_1}| = 1$, we have to consider whether S_1 is empty or not.

(1) If $S_1 = \emptyset$, then $r_\ell \le 0$ whenever $u_{4+\ell} \in V_0$. $f(G)$ is maximized when $(|V_{u_1}|, |V_{u_3}|, |V_0|) = (1, m_2, m_3)$ and

$$\begin{aligned} f(G) &\le 2^{m-4} f_4 + \sum_{j=1}^{m_3-1} 2^{m-4-j} \cdot 0 + \sum_{j=m_3}^{m_2+m_3-3} 2^{m-4-j} \cdot (-4) \\ &= 13 \cdot 2^{m-4} - 4 \cdot 2^{m_2-2} + 4 \\ &\le 13 \cdot 2^{m-4} - 2^{m_2-2} + 2. \end{aligned}$$

(2) If $S_1 \ne \emptyset$, then $r_\ell \le -f(u_{i_1})$ whenever $u_{4+\ell} \in S_2$. We can see from Lemma 4.5

that

$$\begin{aligned}
f(G) &\le 2^{m-4} f_4 + \sum_{j=1}^{m_3-y-2} 2^{m-4-j} \cdot 0 + 2^{m-4-(m_3-y-1)} \\
&\quad + \sum_{j=m_3-y}^{m_2+m_3-y-3} 2^{m-4-j} \cdot (-4) + \sum_{j=m_2+m_3-y-2}^{m_2+m_3-3} 2^{m-4-j} \cdot [-f(u_{i_1})] \\
&= 13 \cdot 2^{m-4} + 2^{m_2+y-2} - 4(2^{m_2+y-2} - 2^y) - f(u_{i_1})(2^y - 1) \\
&= 13 \cdot 2^{m-4} - 3 \cdot 2^{m_2+y-2} + 4 - [f(u_{i_1}) - 4] \cdot (2^y - 1) \\
&= 13 \cdot 2^{m-4} - 2^{m_2+y-2} + 2 - (2 \cdot 2^{m_2+y-2} - 2) - [f(u_{i_1}) - 4](2^y - 1) \\
&\le 13 \cdot 2^{m-4} - 2^{m_2-2} + 2.
\end{aligned}$$

Case 4*:* $f(u_4) = 5$.

Since $f_5 > 51 \cdot 2^{5-6} - 1$, $f(u_5) = f_5 - f_4 \ge 13$. Thus, $f(u_5) = 13 = f_4 + 1$ and $f_5 = 25$. The fact $f_6 > 51 \cdot 2^{6-6} - 1$ implies that $f(u_6) = f_6 - f_5 \ge 51 - 25 = 26$. Then $f(u_6) = 26$ and $f_6 = 51$. Since $f(u_6) > f(u_5) + f(u_t)$ and $f(u_5) > f_4 - f(u_t)$ for $t = 1, 2$, Lemma 2.4 guarantees that $u_5u_t \in E(G)$ for $t \in \{1, 2\}$. Hence, $\langle\{u_1, u_2, u_5\}\rangle \cong K_3$. Suppose that $m \ge 6$. Since $u_6u_t \notin E(G)$ for some $t \in \{1, 2, 5\}$ and $f(u_6) > f_5 - f(u_t)$, we obtain from Lemma 2.4 that $f(u_7) \le f(u_6) + f(u_t) \le f(u_6) + f(u_5) = 39$. However, this implies that $f_7 = f_6 + f(u_7) \le 90 < 51 \cdot 2^{7-6} - 1$, contradicting to Lemma 4.3. Therefore, the only possible situation in this case is "$m = 5$" and then $f(G) = f_5 = 25 < 27 = 13 \cdot 2^{m-4} - 2^{m_2-2} + 2$.

We have verified that our upper bounds are valid when $f(u_3) = 4$.

For the second part of this proof, we assume that $f(u_3) = 3$. Note that $f_3 + 1 \ge f(u_4) = f_4 - f_3 > (51 \cdot 2^{4-6} - 1) - 6$, that is, $7 \ge f(u_4) \ge 6$.

Case 1*:* $f(u_4) = 7$.

In this case, $f(u_5) = f_5 - f_4 > (51 \cdot 2^{5-6} - 1) - 13$, that is, $f(u_5) \ge 12$. First, let us consider the situation where $u_1u_2 \in E(G)$. Since $f(u_5) > f(u_4) + f(u_t)$ and $f(u_4) > f_3 - f(u_t)$ for $t = 1, 2$, Lemma 2.4 implies that $u_4u_t \in E(G)$. Hence, $\langle\{u_1, u_2, u_4\}\rangle \cong K_3$ and then there is some $t \in \{1, 2, 4\}$ such that $u_5u_t \notin E(G)$. Observe that $f(u_6) = f_6 - f_5 > (51 \cdot 2^{6-6} - 1) - f_5 \ge 50 - (f_4 + (f_4 + 1)) = 23$. Since $f(u_6) > 21 \ge f(u_4) + f(u_5)$, we obtain from Lemma 2.4 that $f(u_5) \le f_4 - f(u_t) \le f_4 - f(u_1) = 12$. Therefore, $f(u_5) = 12$ and $f_5 = 25$. Suppose that $m \ge 6$. Then $f(u_6) = f_6 - f_5 > (51 \cdot 2^{6-6} - 1) - 25$. Hence, $f(u_6) = f_5 + 1 = 26$. Now, we have $u_6u_t \notin E(G)$ for some $t \in \{1, 2, 4\}$ and $f(u_6) > f_5 - f(u_t)$. Lemma 2.4 enables us to obtain $f(u_7) \le f(u_6) + f(u_t) \le f(u_6) + f(u_4) = 33$. However, this leads to $f_7 = f_6 + f(u_7) \le 84 < 51 \cdot 2^{7-6} - 1$, giving a contradiction. Therefore, "$m \ge 6$" is impossible to occur when $u_1u_2 \in E(G)$ and then $f(G) = f_5 = 25 < 27 = 13 \cdot 2^{m-4} - 2^{m_2-2} + 2$.

Next, let us consider the situation where $u_1u_2 \notin E(G)$. Since $f(u_4) > f_3 - f(u_1) > f(u_3) + f(u_1)$, we know from Lemma 2.4 that both u_1u_4 and u_1u_3 are edges of G because $f(u_5) \geq 12 > f(u_4)+f(u_1)$ and $f(u_3) > f_2 - f(u_1)$. Therefore, we have $\langle\{u_1,u_2,u_3,u_4\}\rangle \cong K_{1,1,2}$ or $K_{2,2}$. Let us denote the partite set containing u_j as V_{u_j}, $j = 2,4$, and the other one as V_0. By Lemma 4.4, we have $f(u_{4+\ell}) \leq f_{4+\ell-1} - f(u_2) = f_{4+\ell-1} - 2$ if $u_{4+\ell} \in V_{u_2}$ and $f(u_{4+\ell}) \leq f_{4+\ell-1} - f(u_4) = f_{4+\ell-1} - 7$ if $u_{4+\ell} \in V_{u_4}$. Let $S_1 = \{u_k \in V_0 \mid f(u_k) = f_{k-1} + 1\}$. Then $r_\ell \leq 0$ whenever $u_{4+\ell} \in V_0 \setminus S_1$. If $S_1 \neq \emptyset$, we denote as i_1 the minimum element in $\{k \mid u_k \in S_1\}$, then, for the same reason as we stated in Case 3 in the proof of Proposition 4.6, we have $S_1 = \{u_{i_1}\}$. Let $S_2 = \{u_k \in V_0 \mid f(u_k) \leq f_{k-1} - f(u_{i_1})\}$ and $|S_2| = y$. Then $r_\ell \leq -f(u_{i_1})$ whenever $u_{4+\ell} \in S_2$.

Note that in the following three situations where "$2 \leq m_1 \leq m_2 \leq m_3$", "$1 = m_1 < m_2 \leq m_3$ and $\langle\{u_1,u_2,u_3,u_4\}\rangle \cong K_{2,2}$" and "$1 = m_1 < m_2 \leq m_3$, $\langle\{u_1,u_2,u_3,u_4\}\rangle \cong K_{1,1,2}$ and $|V_0| = 1$", the argument is almost the same as we used in Case 3 in the proof of Proposition 4.6. The upper bounds on $f(G)$ here can be obtained by replacing -1 with -2 and replacing -4 with -7 in the places of r_ℓ's in the expressions of the upper bounds in that proof. Since each resulting upper bound is less than the original one, our results still hold here.

The remaining situation in Case 1 is when "$1 = m_1 < m_2 \leq m_3$, $\langle\{u_1,u_2,u_3,u_4\}\rangle \cong K_{1,1,2}$ and $|V_{u_4}| = 1$". If $S_1 = \emptyset$, then $r_\ell \leq 0$ whenever $u_{4+\ell} \in V_0$. Lemma 4.5 guarantees that

$$\begin{aligned} f(G) &\leq 2^{m-4} f_4 + \sum_{j=1}^{m_3-1} 2^{m-4-j} \cdot 0 + \sum_{j=m_3}^{m-4} 2^{m-4-j} \cdot (-2) \\ &= 13 \cdot 2^{m-4} - 2 \cdot 2^{m_2-2} + 2 \\ &\leq 13 \cdot 2^{m-4} - 2^{m_2-2} + 2. \end{aligned}$$

If $S_1 \neq \emptyset$, then $r_\ell \leq -f(u_{i_1})$ whenever $u_{4+\ell} \in S_2$. We have

$$\begin{aligned} f(G) \leq{}& 2^{m-4} f_4 + \sum_{j=1}^{m_3-y-2} 2^{m-4-j} \cdot 0 + 2^{m-4-(m_3-y-1)} \\ &+ \sum_{j=m_3-y}^{m_2+m_3-y-3} 2^{m-4-j} \cdot (-2) + \sum_{j=m_2+m_3-y-2}^{m-4} 2^{m-4-j} \cdot [-f(u_{i_1})] \\ ={}& 13 \cdot 2^{m-4} + 2^{m_2+y-2} - 2(2^{m_2+y-2} - 2^y) - f(u_{i_1})(2^y - 1) \\ ={}& 13 \cdot 2^{m-4} - 2^{m_2+y-2} + 2 - (f(u_{i_1}) - 2)(2^y - 1) \\ \leq{}& 13 \cdot 2^{m-4} - 2^{m_2-2} + 2. \end{aligned}$$

Case 2: $f(u_4) = 6$.

Since $f(u_5) = f_5 - f_4 > (51 \cdot 2^{5-6} - 1) - 12$, we have $f(u_5) = f_4 + 1 = 13$ and $f_5 = 25$. In addition, $f(u_6) = f_6 - f_5 > (51 \cdot 2^{6-6} - 1) - 25 = 25$, which means $f(u_6) > f(u_5) + f(u_i)$ for $i = 1, 2$. By Lemma 2.4, we know that $u_i u_5 \in E(G)$, $i = 1, 2$, because $f(u_5) > f_4 - f(u_i)$. Hence $\langle \{u_1, u_2, u_5\} \rangle \cong K_3$ or $K_{1,2}$. If $m = 5$, then $(m_1, m_2, m_3) = (1, 2, 2)$ and $f(G) = f_5 = 25 < 27 = 13 \cdot 2^{m-4} - 2^{m_2-2} + 2$. If $m \geq 6$, then, as we just showed, $f(u_6) = f_5 + 1 = 26$ and $f_6 = 51$. Now, $f(u_7) = f_7 - f_6 > (51 \cdot 2^{7-6} - 1) - 51 = 50$, which means $f(u_7) > f(u_2) + f(u_6)$. We then have $u_2 u_6 \in E(G)$ by Lemma 2.4 because $f(u_6) > f_5 - f(u_2)$. Similarly, we also have $u_5 u_6 \in E(G)$. Therefore, $\langle \{u_2, u_5, u_6\} \rangle \cong K_3$. If $m = 6$, then $(m_1, m_2, m_3) = (1, 2, 3)$ or $(2, 2, 2)$. In the former case, $f(G) = f_6 = 51 < 53 = 13 \cdot 2^{m-4} - 2^{m_2-2} + 2$. In the latter case, $f(G) < 53 = 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4$. Both results are true. Now, suppose that $m \geq 7$. Then $f(u_7) \geq 51$ as we just showed. Since $u_7 u_t \notin E(G)$ for some $t \in \{2, 5, 6\}$ and $f(u_7) > f_6 - f(u_t)$, we see from Lemma 2.4 that $f(u_8) \leq f(u_7) + f(u_t) \leq f(u_7) + f(u_6) \leq (f_6 + 1) + f(u_6) = 78$. However, this leads to $f_8 = f_6 + f(u_7) + f(u_8) \leq 181 < 51 \cdot 2^{8-6} - 1$. We obtain a contradiction and therefore "$m \geq 7$" is impossible to occur in Case 2.

We conclude that our upper bounds are also valid when $f(u_3) = 3$. The proof is completed. $\square$

Combining Propositions 3.1, 4.6 and the result obtained in [11] we have the following result.

Theorem 4.7.

$$M(K_{m_1,m_2,m_3}) = \begin{cases} 13 \cdot 2^{m-4} - 3 \cdot 2^{m_1-2} + 4 & \text{if } 2 \leq m_1 \leq m_2 \leq m_3, \\ 13 \cdot 2^{m-4} - 2^{m_2-2} + 2 & \text{if } 1 = m_1 < m_2 \leq m_3, \\ 2^m - 2^{m_3} + 1 & \text{if } 1 = m_1 = m_2 < m_3, \end{cases}$$

where $m = m_1 + m_2 + m_3$.

5. Conclusion

In this paper, we have provided a lower bound on $M(K_{m_1,m_2,\ldots,m_\ell})$ for two cases and then proved that, when $\ell = 3$, our lower bound also serves as an upper bound on $M(K_{m_1,m_2,m_3})$ in each case. The IC-colorings we have constructed are indeed qualified maximal IC-colorings. The problem of finding the IC-index of any complete tripartite graph is completely settled. As the derivation of $M(K_{m_1,m_2,\ldots,m_\ell})$ becomes more and more involved when the value of ℓ becomes larger, a structural approach is required for future study of this problem. By analyzing the discussion in this paper, we are inspired to develop such an approach to deal with the problem for larger ℓ.

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