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Hypersurfaces of Randers Spaces with Constant Mean Curvature

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Abstract. Let $(\overline{M}^{n+1}, \overline{F})$ be a complete simply connected Randers space with $\overline{F}(x,Y) = \overline{a}(x,Y) + \overline{b}(x,Y)$, where $\overline{a}(x,Y) = \sqrt{\overline{a}_{ij}(x)Y^iY^j}$ and $\overline{b}(x,Y) = \overline{b}_i(x)Y^i$ are a Riemannian metric and a 1-form on the smooth $(n+1)$-dimensional manifold $\overline{M}$ respectively. Assume the 1-form $\overline{b}$ is parallel with respect to $\overline{a}$ and the sectional curvature $\overline{K}_{\overline{M}}$ of $\overline{M}$ with respect to $\overline{a}$ satisfies $\delta(n) \leq \overline{K}_{\overline{M}} \leq 1$. In this paper, we study the compact hypersurface (M,F) of the Randers space $(\overline{M}^{n+1}, \overline{F})$ with constant mean curvature $|H|$ and prove that if the norm square S of the second fundamental form of (M,F) with respect to the Finsler metric $\overline{F}$ satisfies a certain inequality, then $S = n|H|^2$ and M is the unit sphere or equality holds. In that case, we describe all M that satisfy this equality, which generalizes the result of [8] from the Riemannian case to the Randers space.

1. Introduction

Let M be an n-dimensional smooth manifold and $\pi\colon TM \to M$ be the natural projection from the tangent bundle. Let (x,Y) be a point of TM with $x \in M$, $Y \in T_xM$ and let (x^i, Y^i) be the local coordinate on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F\colon TM \to [0,+\infty)$ satisfying the following properties:

(i) Regularity: $F(x,Y)$ is smooth in $TM \setminus 0$;

(ii) Positive homogeneity: $F(x,\lambda Y) = \lambda F(x,Y)$ for $\lambda > 0$;

(iii) Strong convexity: The fundamental quadratic form $g = g_{ij}(x,Y)\, dx^i \otimes dx^j$ is positively definite, where $g_{ij} = \frac{1}{2}\partial^2(F^2)/(\partial Y^i \partial Y^j)$.

Recent studies on Finsler manifolds have taken on a new look and Finsler manifolds can be also applied to biology and physics, etc. In these researches, people find that there is a quite important metric constructed from a Riemannian metric a and a 1-form b on the smooth manifold M. We call this metric a Randers metric which is firstly studied

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by G. Randers and be applied in studying the navigation problems, etc. In [7], Z. Shen studied the projectively flat Randers metrics and have classified projectively flat Randers metrics with constant flag curvature. In [2], D. Bao, C. Robles and Z. Shen have completed classification of strongly convex Randers metrics with constant flag curvature.

The Riemannian submanifolds are important in modern differential geometry. There has been a long history for the study of Riemannian submanifolds. Many researches have been done and improved in the field of the classification theorems for Riemannian submanifolds.

For the Randers space $(M, a+b)$, where a and b are a Riemannian metric and a 1-form on M respectively, there are many the Randers spaces $(M, a+b)$ with b parallel with respect to a, which isn't Riemannian. See the example below:

Example 1.1. Let $\overline{M}$ be a 6-dimensional real vector space with the Cartesian coordinates $x^1, x^2, x^3, x^4, x^5, x^6$. For the Euclidean metric $\overline{a} = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 + dx^4 \otimes dx^4 + dx^5 \otimes dx^5 + dx^6 \otimes dx^6$ and a 1-form $\overline{b} = \lambda dx^6$ on $\overline{M}$, where $\lambda < 1$ is some positive constant, then 1-form $\overline{b}$ is parallel with respect to the Riemannian metric $\overline{a}$. The Randers space $(\overline{M}, \overline{F})$ constructed from $\overline{a}$ and $\overline{b}$ isn't Riemannian. Let $M = S^1(1/3) \times S^2(2/3) \subset \overline{M}$ be an H-torus, where $S^1(1/3) = \{(x^1, x^2) : (x^1)^2 + (x^2)^2 = 1/3\}$ and $S^2(2/3) = \{(x^3, x^4, x^5) : (x^3)^2 + (x^4)^2 + (x^5)^2 = 2/3\}$. When the Euclidean metric $\overline{a}$ is pulled back to M, it yields a Riemannian metric a. Since when 1-form dx^6 is pulled back to M, the pull-back 1-form vanishes, the pull-back 1-form b of $\overline{b}$ to M gives $b = 0$. Therefore we have that the Finsler metric $\overline{F}$ induces a Riemannian metric $F = a$ on M, i.e., the M with respect to the induced metric $F = a$ is a Riemannian submanifold of Randers space $(\overline{M}, \overline{F})$.

Motivated by the example above, we study the submanifolds of Randers space. In this paper, by the Gauss formula of Chern connection for Finsler submanifolds, we study the hypersurfaces of Randers space $(\overline{M}^{n+1}, \overline{a}+\overline{b})$ with $\overline{b}$ parallel with respect to $\overline{a}$ and obtain the following classification theorem.

Theorem 1.2 (Main Theorem). *Let $(\overline{M}^{n+1}, \overline{F})$ be a complete simply connected Randers space constructed from a Riemannian metric $\overline{a}$ and a 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. Assume $\overline{M}^{n+1}$ is a $\delta(n)$-pinching Riemannian manifold with respect to the Riemannian metric $\overline{a}$, i.e., the sectional curvature $\overline{K}_{\overline{M}}$ of $\overline{M}$ with respect to the Riemannian metric $\overline{a}$ satisfies $\delta(n) \le \overline{K}_{\overline{M}} \le 1$. If M^n is a compact hypersurface of $(\overline{M}^{n+1}, \overline{F})$ with constant mean curvature $|H|$ and the norm square S of the second fundamental form of (M, F) with respect to the Finsler metric $\overline{F}$ satisfies $\alpha(1-\delta) \le S - n|H|^2 \le B_H$, where $\alpha = \frac{1}{12}\sqrt{n(n-1)(52n-50)}$ and B_H is the positive solution of the following equation*

$$-x - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\, x^{1/2} + n\delta - \left(\alpha + \frac{7}{2}\right)(1-\delta) = 0,$$

then either M is the unit sphere or $S-n\,|H|^2 = B_H$ and one of the following cases occurs:

(1) $H=0$ *and* $M = S^k\big(\sqrt{k/n}\big)\times S^{n-k}\big(\sqrt{(n-k)/n}\big)$ *is a minimal Clifford hypersurface,*

(2) $H \neq 0$, $n \geq 3$ *and* $M = S^1(r_1)\times S^{n-1}(r_2)$ *is an H-torus, where* $r_1^2 + r_2^2 = 1$ *and* $r_2^2 < (n-1)/2$,

(3) $H \neq 0$, $n = 2$ *and* $M = S^1(r_1)\times S^1(r_2)$ *is an H-torus, where* $r_1^2 + r_2^2 = 1$ *and* $r_1^2 \neq 1/2$.

Remark 1.3. Theorem 1.2 generalizes the result of [8] from the Riemannian case to the Randers space.

2. Preliminaries

Let (M^n, F) be an n-dimensional Finsler manifold. F inherits the *Hilbert* form, the fundamental tensor and the *Cartan* tensor as follows:

$$\omega = \frac{\partial F}{\partial Y^i}dx^i,\ g_Y = g_{ij}(x,Y)\,dx^i\otimes dx^j,\ A_Y = A_{ijk}\,dx^i\otimes dx^j\otimes dx^k,\ A_{ijk} := \frac{F\partial g_{ij}}{2\partial Y^k}.$$

It is well known that there exists uniquely the Chern connection ∇ on π^*TM with $\nabla\frac{\partial}{\partial x^i} = \omega_i^j\frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j\,dx^k$ satisfying that

$$d(dx^i) - dx^j\wedge\omega_j^i = -dx^j\wedge\omega_j^i = 0 \quad\text{and}\quad dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\frac{\delta Y^k}{F},$$

where $\delta Y^i = dY^i + N_j^i\,dx^j$, $N_j^i = \gamma_{jk}^i Y^k - \frac{1}{F}A_{jk}^i\gamma_{st}^k Y^s Y^t$ and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}.

The curvature 2-forms of the Chern connection ∇ are

$$d\omega_j^i - \omega_j^k\wedge\omega_k^i = \Omega_j^i = \frac{1}{2}R_{jkl}^i\,dx^k\wedge dx^l + \frac{1}{F}P_{jkl}^i\,dx^k\wedge\delta Y^l,$$

where R_{jkl}^i and P_{jkl}^i are the components of the hh-curvature tensor and hv-curvature tensor of the Chern connection, respectively.

Take a g-orthonormal frame $\{e_i = u_i^j\frac{\partial}{\partial x^j}\}$ with $e_n = \ell$ for each fibre of π^*TM and $\{\omega^i\}$ is its dual coframe, where $\pi\colon TM\to M$ denotes the natural projection. The collection $\{\omega^i,\omega_n^i\}$ forms an orthonormal basis for $T^*(TM\setminus\{0\})$ with respect to the Sasakitype metric $g_{ij}\,dx^i\otimes dx^j + g_{ij}\,\delta Y^i\otimes\delta Y^j$. The pull-back of the Sasaki metric from $TM\setminus\{0\}$ to the sphere bundle SM is a Riemannian metric $\widehat{g} = g_{ij}\,dx^i\otimes dx^j + \delta_{ab}\,\omega_n^a\otimes\omega_n^b$. The collection $\{e_i^H,\widehat{e}_{n+\lambda}\}$ forms an orthonormal basis on the sphere bundle SM, where $e_i^H = u_i^j\frac{\delta}{\delta x^j} = u_i^j\big(\frac{\partial}{\partial x^j} - N_j^k\frac{\partial}{\partial Y^k}\big)$ denotes the horizontal part of e_i and $\widehat{e}_{n+\lambda} = u_\lambda^j\frac{F\partial}{\partial Y^j}$, $\lambda = 1,2,\dots,n-1$. Thus the volume element dV_{SM} of SM may be defined as

$$dV_{SM} = dv\wedge\omega_n^1\wedge\cdots\wedge\omega_n^{n-1} = \Omega\,dx\wedge d\tau,$$

where $dv = \sqrt{\det(g_{ij})}\, dx$, $\Omega = \det(g_{ij}/F)$, $d\tau = \sum_i (-1)^{i-1} Y^i dY^1 \wedge \cdots \wedge \widehat{dY^i} \wedge \cdots \wedge dY^n$, $dx = dx^1 \wedge \cdots \wedge dx^n$.

The volume form dV_M of an n-dimensional Finsler manifold (M, F) can be defined by

$$dV_M = \sigma(x)\, dx, \quad \sigma(x) = \frac{1}{C_{n-1}} \int_{S_x M} \Omega\, d\tau,$$

where $S_x M = \{Y \in T_x M : F(Y) = 1\}$ is the fibre of SM at point x and C_{n-1} denotes the volume of the unit Euclidean sphere S^{n-1} .

$\varphi\colon (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ is called an isometric immersion from a Finsler manifold to a Finsler manifold if $F(Y) = \overline{F}(\varphi_*(Y))$. We have that (see, [6])

$$\begin{aligned} g_Y(U, V) &= \overline{g}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V)), \\ A_Y(U, V, W) &= \overline{A}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V), \varphi_*(W)), \end{aligned} \tag{2.1}$$

where $Y, U, V, W \in TM$, $\overline{g}$ and $\overline{A}$ are the fundamental tensor and the *Cartan* tensor of $\overline{M}$, respectively.

It can be seen from (2.1) that $\varphi^*(\overline{\omega}) = \omega$, where $\overline{\omega}$ is the *Hilbert* form of $\overline{M}$.

In the following we simplify A_Y and g_Y to A and g, respectively. Any vector field $U \in \Gamma(TM)$ will be identified with the corresponding vector field $d\varphi(U) \in \Gamma(T\overline{M})$. We will use the following convention:

$$1 \le i, j, \ldots \le n; \qquad n+1 \le \alpha, \beta, \ldots \le n+p;$$
$$1 \le \lambda, \mu, \ldots \le n-1; \qquad 1 \le A, B, \ldots \le n+p.$$

Let $\varphi\colon (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion. Take a $\overline{g}$-orthonormal frame form $\{e_A\}$ for each fibre of $\pi^* T\overline{M}$ and $\{\omega^A\}$ is its local dual coframe, such that $\{e_i\}$ is a frame field for each fibre of $\pi^* TM$ and ω^n is the *Hilbert* form, where $\pi\colon TM \to M$ denotes the natural projection. Let θ^A_B and ω^i_j denote the Chern connection 1-form of $\overline{F}$ and F, respectively, i.e., $\overline{\nabla} e_A = \theta^B_A e_B$ and $\nabla e_i = \omega^j_i e_j$, where $\overline{\nabla}$ and ∇ are the Chern connection of $\overline{M}$ and M, respectively. We obtain that $A(e_i, e_j, e_n) = \overline{A}(e_A, e_B, e_n) = 0$, where $e_n = \frac{Y^i}{F} \frac{\partial}{\partial x^i}$ is the natural dual of the Hilbert form ω^n.

The collection $\{e^H_i, \widehat{e}_{n+\lambda}\}$ forms an orthonormal basis on the projectivized tangent bundle PTM and $\{\omega^i, \omega^\lambda_n\}$ is its local dual coframe, where $e^H_i = u^j_i \frac{\delta}{\delta x^j} = u^j_i \big(\frac{\partial}{\partial x^j} - N^k_j \frac{\partial}{\partial Y^k}\big)$ denotes the horizontal part of e_i, $\widehat{e}_{n+\lambda} = u^j_\lambda \frac{\delta}{\delta Y^j} = u^j_\lambda \frac{F\partial}{\partial Y^j}$, $\omega^i = v^i_j\, dx^j$ and $\omega^\lambda_n = v^\lambda_j \delta Y^j$. we have $\omega^\alpha = 0$ on the projectivized tangent bundle PTM. By the structure equations of

$\overline{M}$, we have that $\theta_j^\alpha \wedge \omega^j = 0$, which implies that $\theta_j^\alpha = h_{ij}^\alpha \omega^i$, $h_{ij}^\alpha = h_{ji}^\alpha$. Let

$$\begin{aligned}\overline{\nabla} V = \nabla V &+ \omega^i \otimes B(V, e_i) \\ &+ \sum_i \Big\{ \overline{A}(V, e_j, B(e_i, e_n)) - \overline{A}(e_j, e_i, B(V, e_n)) \\ &\quad - \overline{A}(V, e_i, B(e_j, e_n)) - \sum_\lambda \overline{A}(e_j, V, e_\lambda)\overline{A}(e_\lambda, e_i, B(e_n, e_n)) \\ &\quad + \sum_\lambda \overline{A}(e_j, e_i, e_\lambda)\overline{A}(e_\lambda, V, B(e_n, e_n)) \\ &\quad + \sum_\lambda \overline{A}(V, e_i, e_\lambda)\overline{A}(e_\lambda, e_j, B(e_n, e_n)) \Big\} \omega^j \otimes e_i,\end{aligned} \tag{2.2}$$

where $V = v^i e_i \in \Gamma(\pi^* TM)$, $B(V, e_i) = \theta_i^\alpha(V) e_\alpha = v^j h_{ij}^\alpha e_\alpha$.

If the $\overline{\nabla}$ is the Chern connection of $\overline{M}$, then we obtain in [3] that the ∇ is the induced Chern connection of M.

We obtain from (2.2) that (see, [3])

$$\omega_i^j = \theta_i^j - \Psi_{jik}\omega^k,$$

where

$$\Psi_{jik} = h_{jn}^\alpha \overline{A}_{ki\alpha} - h_{kn}^\alpha \overline{A}_{ji\alpha} - h_{in}^\alpha \overline{A}_{kj\alpha} - h_{nn}^\alpha \overline{A}_{iks}\overline{A}_{sj\alpha} + h_{nn}^\alpha \overline{A}_{ijs}\overline{A}_{sk\alpha} + h_{nn}^\alpha \overline{A}_{jks}\overline{A}_{si\alpha}.$$

In particular,

$$\omega_i^n = \theta_i^n - h_{nn}^\alpha \overline{A}_{ki\alpha}\omega^k.$$

Using the almost $\overline{g}$-compatibility, we have

$$\theta_\alpha^j = \left(-h_{ij}^\alpha - 2h_{ni}^\beta \overline{A}_{j\alpha\beta} + 2h_{nn}^\beta \overline{A}_{j\lambda\alpha}\overline{A}_{i\lambda\beta} \right) \omega^i - 2\overline{A}_{j\alpha\lambda}\omega_n^\lambda.$$

In particular, $\theta_\alpha^n = -h_{ni}^\alpha \omega^i$.

We quote the following propositions.

Proposition 2.1. (Gauss equations, [3]) *Let $\varphi\colon (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold, then we have that*

$$\begin{aligned} P_{ik\lambda}^j &= \overline{P}_{ik\lambda}^j + \Psi_{jik;\lambda} - 2\Psi_{sik}A_{js\lambda} - 2h_{ik}^\alpha \overline{A}_{j\lambda\alpha}, \\ R_{ikl}^j &= \overline{R}_{ikl}^j - h_{ik}^\alpha h_{jl}^\alpha + h_{il}^\alpha h_{jk}^\alpha + \Psi_{jik|l} - \Psi_{jil|k} + \Psi_{sik}\Psi_{jsl} - \Psi_{sil}\Psi_{jsk} \\ &\quad - 2h_{ik}^\alpha h_{nl}^\beta \overline{A}_{j\alpha\beta} + 2h_{il}^\alpha h_{nk}^\beta \overline{A}_{j\alpha\beta} + 2h_{ik}^\alpha h_{nn}^\beta \overline{A}_{js\alpha}\overline{A}_{ls\beta} - 2h_{il}^\alpha h_{nn}^\beta \overline{A}_{js\alpha}\overline{A}_{ks\beta} \\ &\quad - h_{nn}^\alpha \overline{A}_{sl\alpha}\overline{P}_{iks}^j + h_{nn}^\alpha \overline{A}_{sk\alpha}\overline{P}_{ils}^j + h_{nl}^\alpha \overline{P}_{ik\alpha}^j - h_{nk}^\alpha \overline{P}_{il\alpha}^j, \end{aligned}$$

where " ; " and " | " denote the vertical and the horizontal covariant differentials with respect to the Chern connection ∇ respectively.

Proposition 2.2. (Codazzi equations, [3]) *Let $\varphi\colon (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold, then we have that*

$$\begin{aligned} h^{\alpha}_{ij;\lambda} &= -\overline{P}^{\alpha}_{ij\lambda}, \\ h^{\alpha}_{ij|k} - h^{\alpha}_{ik|j} &= -\overline{R}^{\alpha}_{ijk} + h^{\beta}_{nj}\overline{P}^{\alpha}_{ik\beta} - h^{\beta}_{nk}\overline{P}^{\alpha}_{ij\beta} \\ &\quad - h^{\alpha}_{lk}\Psi_{lij} + h^{\alpha}_{lj}\Psi_{lik} - h^{\beta}_{nn}\overline{A}_{lj\beta}\overline{P}^{\alpha}_{ikl} + h^{\beta}_{nn}\overline{A}_{lk\beta}\overline{P}^{\alpha}_{ijl}. \end{aligned}$$

Proposition 2.3. [3] *An isometric immersion $\varphi\colon (M, F) \to (\overline{M}, \overline{F})$ is minimal if and only if*

$$\int_{SM} \langle V, B(e_n, e_n)\rangle \, dV_{SM} = 0,$$

or

$$\int_{SM} \langle V, nH\rangle \, dV_{SM} = 0,$$

for any vector $V \in \Gamma(TM)^{\perp}$, where $B = h^{\alpha}_{ij}\, e_\alpha \otimes \omega^i \otimes \omega^j$,

$$\begin{aligned} H = \frac{1}{n}\sum_i \Big\{ & B(e_i, e_i) + \sum_\alpha \big[2\overline{C}(e_\alpha, e_i, \overline{\nabla}_{e_i^H}(Fe_n)) \\ & + (\overline{\nabla}_{Fe_n^H}\overline{C})(e_i, e_i, e_\alpha) + 2\overline{C}(\overline{\nabla}_{Fe_n^H} e_i, e_i, e_\alpha)\big] e_\alpha \Big\}, \end{aligned} \tag{2.3}$$

and $\overline{C} = \frac{\overline{A}}{\overline{F}}$, $e_i^H = u_i^j\big(\frac{\partial}{\partial x^j} - N_j^k \frac{\partial}{\partial Y^k}\big)$ denotes the horizontal part of $e_i = u_i^j \frac{\partial}{\partial x^j}$.

Definition 2.4. H is called the *mean curvature vector* and the length $|H|$ of it is called the *mean curvature*.

Let $(\overline{M}^{n+p}, \overline{F})$ be a Randers space with $\overline{F} = \overline{a} + \overline{b}$, where $\overline{a} = \sqrt{\overline{a}_{AB}\overline{Y}^A\overline{Y}^B}$ is a Riemannian metric and $\overline{b} = \overline{b}_A \, d\overline{x}^A$ is a 1-form.

In [1], we know that

$$\overline{\ell}_A = \widetilde{\ell}_A + \overline{\beta}_A,$$

where $\overline{\ell}_A = \frac{\overline{g}_{AB}\overline{Y}^B}{\overline{F}}$ and $\widetilde{\ell}_A = \frac{\overline{a}_{AB}\overline{Y}^B}{\overline{a}}$.

Proposition 2.5. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and 1-form $\overline{b}$. If M^n is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$, then $\overline{A}_{jln+1} = \overline{A}_{n+1n+1n+1} = 0$ and $\Psi_{ijk} = 0$.*

Proof. For the Randers space $(\overline{M}^{n+1}, \overline{F})$, in [1] we have that

$$\overline{A}\left(\frac{\partial}{\partial \overline{x}^A}, \frac{\partial}{\partial \overline{x}^B}, \frac{\partial}{\partial \overline{x}^C}\right) = \frac{1}{2}\left[\eta_{AB}\left(\overline{b}_C - \frac{\overline{b}}{\overline{a}}\widetilde{\ell}_C\right) + \eta_{BC}\left(\overline{b}_A - \frac{\overline{b}}{\overline{a}}\widetilde{\ell}_A\right) + \eta_{CA}\left(\overline{b}_B - \frac{\overline{b}}{\overline{a}}\widetilde{\ell}_B\right)\right], \tag{2.4}$$

where $\eta_{AB} = \overline{g}_{AB} - \overline{\ell}_A\overline{\ell}_B$.

For the 1-form $\bar{b} = \bar{b}_i \, dx^i$ on M, define a 1-form $\bar{b}^*$ on the projectivized tangent bundle PTM as $\bar{b}^* = (\bar{b}_i \circ \pi) \, dx^i$. By the fact that 1-form $\overline{\beta} = \overline{\beta}_i \, dx^i$ is globally defined on M, we have that 1-form $\bar{b}^*$ is globally defined on PTM.

Since the collection $\{e_i^H, \widehat{e}_{n+\lambda}\}$ forms an orthonormal basis on the projectivized tangent bundle PTM and $\{\omega^i, \omega_n^\lambda\}$ is its dual coframe, where $\omega^i = v_j^i \, dx_j$ and $\omega_n^\lambda = v_i^\lambda \frac{\delta Y^i}{F} = v_i^\lambda \left(\frac{dY^i - N_j^i \, dx^j}{F} \right)$, so 1-form $\bar{b}^*$ can be written as $\bar{b}^* = \bar{b}_i \, dx^i = \overline{\beta}_i \omega^i$ on PTM. Then $\bar{b}^*(e_{n+1}) = \bar{b}_i \omega^i(e_{n+1}) = 0$, which implies $u_{n+1}^i \bar{b}_i = 0$, where $e_{n+1} = u_{n+1}^i \frac{\partial}{\partial x^i}$ is the unit normal vector with respect to the Finsler metric $\overline{F}$. This, together with $u_{n+1}^i \overline{\ell}_i = \overline{g}(e_{n+1}, e_n) = 0$ and the fact $\overline{\ell}_i = \widetilde{\ell}_i + \bar{b}_i$, implies that $u_{n+1}^i \widetilde{\ell}_i = \frac{\overline{F}}{\overline{a}} \overline{a}(e_{n+1}, e_n) = 0$. Then by $u_A^C u_B^D \eta_{CD} = \delta_{AB} - \delta_{AC}\delta_{BD}$ and (2.4), we obtain that

$$\overline{A}_{n+1ij} = u_{n+1}^k u_i^l u_j^s \overline{A} \left(\frac{\partial}{\partial \overline{x}^l}, \frac{\partial}{\partial \overline{x}^k}, \frac{\partial}{\partial \overline{x}^s} \right) = 0,$$

and

$$\overline{A}_{n+1n+1n+1} = u_{n+1}^k u_{n+1}^l u_{n+1}^s \overline{A} \left(\frac{\partial}{\partial \overline{x}^l}, \frac{\partial}{\partial \overline{x}^k}, \frac{\partial}{\partial \overline{x}^s} \right) = 0.$$

This completes the proof of Proposition 2.5. □

Let $\overline{\gamma}_{ijk}$ and $\widetilde{\gamma}_{ijk}$ are the formal Christoffel symbols of the Finsler metric $\overline{F}$ and the Riemannian metric $\overline{a}$ respectively. In [1], we have that

$$\overline{\gamma}_{ijk} = \frac{\overline{F}}{\overline{a}} \widetilde{\gamma}_{ijk} - \frac{\overline{a}}{2\overline{F}} \eta_{jk} \left(\bar{b}_{l,x^i} \widetilde{\ell}^l - \frac{\bar{b}}{\overline{a}} \widetilde{\gamma}_{nni} \right) + \frac{1}{2} (\clubsuit_{ijk} + \clubsuit_{ikj}), \tag{2.5}$$

where

$$\begin{aligned} \clubsuit_{ijk} = {} & \frac{\overline{a}}{\overline{F}} \eta_{ij} \left(\bar{b}_{l,x^k} \widetilde{\ell}^l - \frac{\bar{b}}{\overline{a}} \widetilde{\gamma}_{nnk} \right) + \overline{\ell}_i \bar{b}_{j,x^k} - \overline{\ell}_j \bar{b}_{k,i} + \overline{\ell}_k \bar{b}_{i,x^j} \\ & + \overline{\xi}_i (\widetilde{\gamma}_{jnk} + \widetilde{\gamma}_{njk}) - \overline{\xi}_j (\widetilde{\gamma}_{kni} + \widetilde{\gamma}_{nki}) + \overline{\xi}_k (\widetilde{\gamma}_{inj} + \widetilde{\gamma}_{nij}) \\ & - \overline{\xi}_i \widetilde{\ell}_j \widetilde{\gamma}_{nnk} + \overline{\xi}_j \widetilde{\ell}_k \widetilde{\gamma}_{nni} - \overline{\xi}_k \widetilde{\ell}_i \widetilde{\gamma}_{nnj} \end{aligned} \tag{2.6}$$

and $\overline{\xi}_i = \overline{\ell}_i - \frac{\overline{F}}{\overline{a}} \widetilde{\ell}_i$, $\bar{b}_{i,x^j} = \frac{\partial \bar{b}_i}{\partial x^j}$, $\widetilde{\ell}^i = \frac{\overline{Y}^i}{\overline{a}}$, $\widetilde{\gamma}_{njk} = \widetilde{\ell}^i \widetilde{\gamma}_{ijk}$, $\widetilde{\gamma}_{jnk} = \widetilde{\ell}^i \widetilde{\gamma}_{jik}$.

Proposition 2.6. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and 1-form $\bar{b}$, where $\bar{b}$ is parallel with respect to $\overline{a}$. If M^n is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$, then*

$$\begin{aligned} & \left\{ \overline{\Gamma}_{jk}^l u_n^j u_\lambda^k \frac{\partial}{\partial x^l} - \widetilde{\gamma}_{jk}^l u_n^j u_\lambda^k \frac{\partial}{\partial x^l} \right\}^N \\ & = \left\{ \overline{A}_{n+1n+1\lambda} \widetilde{\gamma}_{inj} u_n^j u_{n+1}^i - \overline{A}_{n+1n+1\lambda} \overline{\Gamma}_{jk}^s \widetilde{\ell}^k u_n^j u_{n+1}^i \overline{a}_{si} \right\} e_{n+1}, \end{aligned}$$

where X^N denotes the normal component of X.

Proof. Using the fact $\eta_{ij}u^i_{n+1}u^j_n = \eta_{ij}u^i_l u^j_n = 0$ and $\eta_{ik}u^i_\mu u^k_\lambda = \delta_{\lambda\mu}$, we have that

$$\left\{\left[\frac{\overline{a}}{\overline{F}}\left(\overline{b}_{s,x^k}\widetilde{\ell}^s - \frac{\overline{b}}{\overline{a}}\widetilde{\gamma}_{nnk}\right)\eta_{ij} + \frac{\overline{a}}{\overline{F}}\left(\overline{b}_{s,x^j}\widetilde{\ell}^s - \frac{\overline{b}}{\overline{a}}\widetilde{\gamma}_{nnj}\right)\eta_{ik}\right]u^j_n u^k_\lambda \overline{g}^{il}\frac{\partial}{\partial x^l}\right\}^N = 0. \tag{2.7}$$

When $\overline{b}$ is parallel with respect to $\overline{a}$, we have $\overline{b}_{i,x^j} - \overline{b}_{j,x^i} = 0$. Since $\overline{\ell}_k u^k_\lambda = 0$ and $\overline{\ell}_j u^j_n = 1$, we obtain that

$$\left\{\left[\overline{\ell}_i\overline{b}_{j,x^k} - \overline{\ell}_j\overline{b}_{k,i} + \overline{\ell}_k\overline{b}_{i,x^j} + \overline{\ell}_i\overline{b}_{k,x^j} - \overline{\ell}_k\overline{b}_{j,i} + \overline{\ell}_j\overline{b}_{i,x^k}\right]u^j_n u^k_\lambda \overline{g}^{il}\frac{\partial}{\partial x^l}\right\}^N = 0. \tag{2.8}$$

From $u^j_n\overline{\xi}_j = 0$ and $u^k_\lambda\overline{\xi}_k = -\frac{\overline{F}}{\overline{a}}\widetilde{\ell}_k u^k_\lambda$, we obtain that

$$\begin{aligned}
&\Big\{\big[\overline{\xi}_i(\widetilde{\gamma}_{jnk} + \widetilde{\gamma}_{njk}) - \overline{\xi}_j(\widetilde{\gamma}_{kni} + \widetilde{\gamma}_{nki}) + \overline{\xi}_k(\widetilde{\gamma}_{inj} + \widetilde{\gamma}_{nij}) \\
&\quad - \overline{\xi}_i(\widetilde{\gamma}_{knj} + \widetilde{\gamma}_{nkj}) - \overline{\xi}_k(\widetilde{\gamma}_{jni} + \widetilde{\gamma}_{nji}) + \overline{\xi}_j(\widetilde{\gamma}_{ink} + \widetilde{\gamma}_{nik})\big]u^j_n u^k_\lambda \overline{g}^{il}\frac{\partial}{\partial x^l}\Big\}^N \\
&= -\frac{\overline{F}}{\overline{a}}\widetilde{\ell}_k u^k_\lambda(\widetilde{\gamma}_{inj} - \widetilde{\gamma}_{jni})u^j_n u^i_{n+1}e_{n+1}.
\end{aligned} \tag{2.9}$$

Using $u^i_{n+1}\widetilde{\ell}_i = 0$, $\widetilde{\ell}_j u^j_n = \frac{\overline{a}}{\overline{F}}$ and $\widetilde{\gamma}_{nni} = \frac{\overline{F}}{\overline{a}}\widetilde{\gamma}_{jni}u^j_n$, we see that

$$\begin{aligned}
&\left\{\left[-\overline{\xi}_i\widetilde{\ell}_j\widetilde{\gamma}_{nnk} + \overline{\xi}_j\widetilde{\ell}_k\widetilde{\gamma}_{nni} - \overline{\xi}_k\widetilde{\ell}_i\widetilde{\gamma}_{nnj} - \overline{\xi}_i\widetilde{\ell}_k\widetilde{\gamma}_{nnj} + \overline{\xi}_k\widetilde{\ell}_j\widetilde{\gamma}_{nni} - \overline{\xi}_j\widetilde{\ell}_i\widetilde{\gamma}_{nnk}\right]u^j_n u^k_\lambda \overline{g}^{il}\frac{\partial}{\partial x^l}\right\}^N \\
&= -\frac{\overline{F}}{\overline{a}}\widetilde{\ell}_k u^k_\lambda\widetilde{\gamma}_{jni}u^j_n u^i_{n+1}e_{n+1}.
\end{aligned} \tag{2.10}$$

Substituting (2.7)–(2.10) into (2.6) implies that

$$\left\{\frac{1}{2}(\clubsuit_{ijk} + \clubsuit_{ikj})\overline{g}^{il}u^j_n u^k_\lambda\frac{\partial}{\partial x^l}\right\}^N = -\frac{\overline{a}}{2\overline{F}}\widetilde{\ell}_k u^k_\lambda\widetilde{\gamma}_{inj}u^j_n u^i_{n+1}e_{n+1}. \tag{2.11}$$

Since $\overline{g}_{ij} = \frac{\overline{F}}{\overline{a}}(\overline{a}_{ij} - \widetilde{\ell}_i\widetilde{\ell}_j) + \ell_i\ell_j$ and $\widetilde{\ell}_i = \ell_i - \overline{b}_i$, we obtain that

$$\begin{aligned}
&\frac{\overline{F}}{\overline{a}}\widetilde{\gamma}_{ijk}u^j_n u^k_\lambda\overline{g}^{il}\frac{\partial}{\partial x^l} \\
&= \widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda\overline{g}_{is}u^i_{n+1}e_{n+1} + (\widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda\ell_s - \widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda\overline{b}_s)e_n + \frac{\overline{F}}{\overline{a}}\widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda u^i_\mu\overline{a}_{si}e_\mu \\
&= \left\langle\widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda\frac{\partial}{\partial x^s}, e_{n+1}\right\rangle_{\overline{g}}e_{n+1} + \left(\left\langle\widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda\frac{\partial}{\partial x^s}, e_n\right\rangle_{\overline{g}}e_n - \widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda\overline{b}_s\right)e_n \\
&\quad + \widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda u^i_\mu\left\{\overline{g}_{si} - \ell_s\ell_i + \frac{\overline{F}}{\overline{a}}\widetilde{\ell}_s\widetilde{\ell}_i\right\}e_\mu \\
&= \widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda\frac{\partial}{\partial x^s} - \widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda\overline{b}_s e_n + \frac{\overline{F}}{\overline{a}}\widetilde{\gamma}^s_{jk}u^j_n u^k_\lambda u^i_\mu\widetilde{\ell}_s\widetilde{\ell}_i e_\mu.
\end{aligned} \tag{2.12}$$

Using $\eta_{jk}u_n^j u_\lambda^k = 0$, we have that $-\frac{1}{2}\big(\overline{b}_{s,x^i}\widetilde{\ell}^s - \frac{\overline{b}}{\overline{a}}\widetilde{\gamma}_{nni}\big)\eta_{jk}u_n^j u_\lambda^k \overline{g}^{il}\frac{\partial}{\partial x^l} = 0$. It follows from $\eta_{ij}u_{n+1}^i u_\lambda^j = 0$ and (2.3) that $\overline{A}_{n+1n+1\lambda} = \frac{1}{2}\frac{\overline{F}}{\overline{a}}u_\lambda^k \overline{b}_k$. Substituting this, (2.11) and (2.12) into (2.5), we obtain that

$$\left\{\overline{\gamma}^l_{jk}u_n^j u_\lambda^k \frac{\partial}{\partial x^l} - \widetilde{\gamma}^l_{jk}u_n^j u_\lambda^k \frac{\partial}{\partial x^l}\right\}^N = \overline{A}_{n+1n+1\lambda}\widetilde{\gamma}_{inj}u_n^j u_{n+1}^i e_{n+1}. \tag{2.13}$$

On the other hand, we have that

$$\begin{aligned}\left\{\left[\overline{\Gamma}^l_{jk} - \overline{\gamma}^l_{jk}\right]u_n^j u_\lambda^k \frac{\partial}{\partial x^l}\right\}^N &= \Bigg\{ -\overline{g}^{li}\Bigg[\overline{A}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^s}\right)\frac{\overline{N}^s_k}{\overline{F}} - \overline{A}\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^s}\right)\frac{\overline{N}^s_i}{\overline{F}} \\ &\quad + \overline{A}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^s}\right)\frac{\overline{N}^s_j}{\overline{F}}\Bigg]u_n^j u_\lambda^k \frac{\partial}{\partial x^l}\Bigg\}^N \\ &= -\overline{A}_{n+1n+1\lambda}\overline{\Gamma}^s_{jk}\widetilde{\ell}^k u_n^j u_{n+1}^i \overline{a}_{si}e_{n+1}.\end{aligned} \tag{2.14}$$

We get Proposition 2.6 immediately from (2.13) and (2.14). □

Similarly, we can obtain the following

Proposition 2.7. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. If M^n is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$, then*

$$\left\{\overline{\Gamma}^l_{jk}u_n^j u_n^k \frac{\partial}{\partial x^l} - \widetilde{\gamma}^l_{jk}u_n^j u_n^k \frac{\partial}{\partial x^l}\right\}^N = -\frac{1}{2}\left(\overline{b}_{s,x^i}\widetilde{\ell}^i - \frac{\overline{b}}{\overline{a}}\widetilde{\gamma}_{nni}\right)u_{n+1}^i e_{n+1}.$$

A direct calculation gives

$$d\left(\frac{\overline{b}}{\overline{a}}\right) = \left(\overline{b}_{s,x^i}\widetilde{\ell}^i - \frac{\overline{b}}{\overline{a}}\widetilde{\gamma}_{nni}\right)dx^i + \left(\frac{b_i\overline{a} - \overline{b}\widetilde{\ell}_i}{\overline{a}^2}\right)d\overline{Y}^i. \tag{2.15}$$

Since $d(\frac{\overline{b}}{\overline{a}})(e_{n+1}) = 0$ on the projectivized tangent bundle PTM, we see that $(\overline{b}_{s,x^i}\widetilde{\ell}^i - \frac{\overline{b}}{\overline{a}}\widetilde{\gamma}_{nni})u_{n+1}^i = 0$ from (2.15). Then we get that $\left\{\overline{\Gamma}^l_{jk}u_n^j u_n^k \frac{\partial}{\partial x^l} - \widetilde{\gamma}^l_{jk}u_n^j u_n^k \frac{\partial}{\partial x^l}\right\}^N = 0$ from Proposition 2.7. Substituting this into Proposition 2.6 yields that $\{\overline{\Gamma}^l_{jk}u_n^j u_\lambda^k \frac{\partial}{\partial x^l} - \widetilde{\gamma}^l_{jk}u_n^j u_\lambda^k \frac{\partial}{\partial x^l}\}^N = 0$. we obtain immediately

Proposition 2.8. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. If M^n is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$, then $\left\{\overline{\nabla}e_n - \overline{\nabla}^{\overline{\alpha}}e_n\right\}^N = 0$.*

Proposition 2.9. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. If M^n is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$, then $h^{n+1}_{ij}\overline{A}_{n+1n+1\lambda} = 0$ and $h^{n+1}_{ij}\theta^{n+1}_{n+1} = 0$.*

Proof. Since $u^i_{n+1}\overline{\ell}_i = \langle e_{n+1}, e_n\rangle_{\overline{g}} = 0$ and $u^i_{n+1}\widetilde{\ell}_i = \langle e_{n+1}, e_n\rangle_{\overline{a}} = 0$, where $\overline{\ell}_i = \frac{\overline{g}_{ij}\overline{Y}^j}{\overline{F}}$, $\widetilde{\ell}_i = \frac{\overline{a}_{ij}\overline{Y}^j}{\overline{a}}$ and $e_{n+1} = u^i_{n+1}\frac{\partial}{\partial x^i}$ is the unit normal vector with respect to the Finsler metric $\overline{F}$, together with the fact $\overline{g}_{ij} = \frac{\overline{F}}{\overline{a}}(\overline{a}_{ij} - \widetilde{\ell}_i\widetilde{\ell}_j) + \overline{\ell}_i\overline{\ell}_j$, we obtain that $\langle e_{n+1}, e_{n+1}\rangle_{\overline{a}} = \frac{\overline{a}}{\overline{F}}$. This implies that $\widetilde{e}_{n+1} = \sqrt{\frac{\overline{F}}{\overline{a}}}e_{n+1}$ is the unit normal vector with respect to the Riemannian metric $\overline{a}$. Then from Proposition 2.8 we get that

$$\theta^{n+1}_n = \left\langle \overline{\nabla} e_n, e_{n+1}\right\rangle_{\overline{g}} = \frac{\overline{F}}{\overline{a}}\left\langle \overline{\nabla} e_n, e_{n+1}\right\rangle_{\overline{a}} = \sqrt{\frac{\overline{a}}{\overline{F}}}\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_n, \widetilde{e}_{n+1}\right\rangle_{\overline{a}}, \tag{2.16}$$

where $\widetilde{e}_n = \widetilde{\ell}^i\frac{\partial}{\partial x^i} = \frac{\overline{Y}^i}{\overline{a}}\frac{\partial}{\partial x^i}$.

Similarly, we have that

$$\begin{aligned}\theta^{n+1}_{n+1} &= \left\langle \overline{\nabla}^{\overline{a}} e_{n+1}, e_{n+1}\right\rangle_{\overline{g}} + \Phi_i\omega^i \\ &= \frac{\overline{F}}{\overline{a}}\left\langle \overline{\nabla}^{\overline{a}}\left(\sqrt{\frac{\overline{\alpha}}{\overline{F}}}\widetilde{e}_{n+1}\right), \sqrt{\frac{\overline{a}}{\overline{F}}}\widetilde{e}_{n+1}\right\rangle_{\overline{a}} + \Phi_i\omega^i \\ &= \sqrt{\frac{\overline{F}}{\overline{a}}}\, d\sqrt{\frac{\overline{a}}{\overline{F}}} + \Phi_i\omega^i,\end{aligned}\tag{2.17}$$

where $\Phi_l = \left\{\frac{\overline{a}}{\overline{F}}(\overline{b}_{s,x^j}\widetilde{\ell}^j - \frac{\overline{b}}{\overline{a}}\widetilde{\gamma}_{nnj}) - \overline{A}(e_{n+1}, e_{n+1}, \frac{1}{\overline{F}}\overline{N}^s_j\frac{\partial}{\partial x^s})\right\} u^j_l$ and

$$\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_{n+1}, \widetilde{e}_n\right\rangle_{\overline{a}} = -\left\langle \widetilde{e}_{n+1}, \overline{\nabla}^{\overline{a}}\widetilde{e}_n\right\rangle_{\overline{a}} = -\sqrt{\frac{\overline{F}}{\overline{a}}}\theta^{n+1}_n. \tag{2.18}$$

Let $\widetilde{\theta}^a_b$ denotes the Levi-Civita connection 1-form with respect to the Riemannian metric $\overline{a}$, i.e., $\overline{\nabla}^{\overline{a}}\frac{\partial}{\partial x^a} = \widetilde{\theta}^b_a\frac{\partial}{\partial x^b} = \widetilde{\gamma}^b_{ac}\, dx^c \otimes \frac{\partial}{\partial x^b}$ and $\widetilde{e}_{n+1} = \widetilde{u}^i_{n+1}\frac{\partial}{\partial x^i}$. Now exterior differentiate the right-hand side of (2.16), we obtain that from (2.17) and (2.18)

$$\begin{aligned}& d\left(\sqrt{\frac{\overline{a}}{\overline{F}}}\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_n, \widetilde{e}_{n+1}\right\rangle_{\overline{a}}\right) \\ &= d\left(\sqrt{\frac{\overline{a}}{\overline{F}}}\right)\wedge\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_n, \widetilde{e}_{n+1}\right\rangle_{\overline{a}} + \sqrt{\frac{\overline{a}}{\overline{F}}}d\left\{d(\widetilde{\ell}^i)\widetilde{u}^j_{n+1}\overline{a}_{ij} + \widetilde{\ell}^i\widetilde{u}^j_{n+1}\overline{a}_{jl}\widetilde{\theta}^l_i\right\} \\ &= d\left(\sqrt{\frac{\overline{a}}{\overline{F}}}\right)\wedge\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_n, \widetilde{e}_{n+1}\right\rangle_{\overline{a}} + \sqrt{\frac{\overline{a}}{\overline{F}}}\Big\{\overline{a}_{ij}\, d(\widetilde{u}^j_{n+1})\wedge d(\widetilde{\ell}^i) + \overline{a}_{lj}\widetilde{\ell}^i\, d(\widetilde{u}^j_{n+1})\wedge\widetilde{\theta}^l_i \\ &\qquad + \overline{a}_{ki}\widetilde{u}^j_{n+1}\,\widetilde{\theta}^k_j\wedge d(\widetilde{\ell}^i) + \overline{a}_{kl}\widetilde{\ell}^i\widetilde{u}^j_{n+1}\,\widetilde{\theta}^k_j\wedge\widetilde{\theta}^l_i\Big\} + \frac{1}{2}\overline{K}_{n+1nkl}\,\omega^k\wedge\omega^l \\ &= d\left(\sqrt{\frac{\overline{a}}{\overline{F}}}\right)\wedge\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_n, \widetilde{e}_{n+1}\right\rangle_{\overline{a}} + \sqrt{\frac{\overline{a}}{\overline{F}}}\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_{n+1}, \frac{\partial}{\partial x^l}\right\rangle_{\overline{a}}\wedge dx^l(\overline{\nabla}^{\overline{a}}\widetilde{e}_n) + \frac{1}{2}\overline{K}_{n+1nkl}\,\omega^k\wedge\omega^l\end{aligned}$$

(2.23)

$$= \theta^{n+1}_{n+1}\wedge\theta^{n+1}_n - \sqrt{\frac{\overline{a}}{\overline{F}}}\Phi_l\,\omega^l\wedge\theta^{n+1}_n + \sqrt{\frac{\overline{a}}{\overline{F}}}\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_{n+1}, e_l\right\rangle_{\overline{a}}\wedge\omega^l(\overline{\nabla}^{\overline{a}}\widetilde{e}_n) + \frac{1}{2}\overline{K}_{n+1nkl}\,\omega^k\wedge\omega^l$$

$$
\begin{aligned}
&= \theta_{n+1}^{n+1} \wedge \theta_n^{n+1} - \sqrt{\frac{\overline{a}}{\overline{F}}}\Phi_l\, \omega^l \wedge \theta_n^{n+1} \\
&\quad + \sqrt{\frac{\overline{a}}{\overline{F}}}\left\{\left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_{n+1}, e_n\right\rangle_{\overline{a}} \wedge \omega^n \left[d\left(\frac{\overline{F}}{\overline{a}}\right)e_n + \frac{\overline{F}}{\overline{a}}\overline{\nabla}^{\overline{a}}e_n\right] + \left\langle \overline{\nabla}^{\overline{a}}\widetilde{e}_{n+1}, e_\lambda\right\rangle_{\overline{a}} \wedge \omega^\lambda \left(\frac{\overline{F}}{\overline{a}}\overline{\nabla} e_n\right)\right\} \\
&\quad + \frac{1}{2}\overline{K}_{n+1nkl}\,\omega^k \wedge \omega^l \\
&= \left\{\theta_{n+1}^{n+1} \wedge \theta_n^{n+1} + \sqrt{\frac{\overline{a}}{\overline{F}}}\left[-\sqrt{\frac{\overline{F}}{\overline{a}}}\,\theta_n^{n+1} \wedge d\left(\frac{\overline{F}}{\overline{a}}\right) - \left\langle \widetilde{e}_{n+1}, \overline{\nabla}^{\overline{a}}e_\lambda\right\rangle_{\overline{a}} \wedge \left(\frac{\overline{F}}{\overline{a}}\theta_n^\lambda\right)\right]\right\} \\
&\qquad \left[\mathrm{mod}\,(\omega^k \wedge \omega^l)\right] \\
&= \left\{\theta_{n+1}^{n+1} \wedge \theta_n^{n+1} + 2\frac{\overline{F}}{\overline{a}}\,\theta_n^{n+1} \wedge \theta_{n+1}^{n+1} - \theta_\lambda^{n+1} \wedge \theta_n^\lambda\right\}\left[\mathrm{mod}\,(\omega^k \wedge \omega^l)\right],
\end{aligned}
$$

where the $\overline{K}^A_{BCD}$ is the curvature tensor of $\overline{M}$ with respect to the Riemannian metric $\overline{a}$.

On the other hand, we have that

$$
\begin{aligned}
d\theta_n^{n+1} &= -\theta_{n+1}^{n+1} \wedge \theta_n^{n+1} - \theta_\lambda^{n+1} \wedge \theta_n^\lambda + \frac{1}{2}\overline{R}_{n+1nkl}\,\omega^k \wedge \omega^l \\
&\quad + \overline{P}^{n+1}_{nk\lambda}\,\omega^k \wedge \omega_n^\lambda + \overline{P}^{n+1}_{nkn+1}\,\omega^k \wedge \omega_n^{n+1}.
\end{aligned} \tag{2.24}
$$

Note $\theta_{n+1}^{n+1} = -\overline{A}_{n+1n+1\lambda}\omega_n^\lambda$ from Proposition 2.5. Now substituting (2.23) and (2.24) into (2.16) yields that

$$
2\frac{\overline{b}}{\overline{a}}h_{nk}^{n+1}\overline{A}_{n+1n+1\lambda} = -\overline{P}^{n+1}_{nk\lambda}. \tag{2.25}
$$

Set $k = n$ in (2.25), we have that $\frac{\overline{b}}{\overline{a}}h_{nn}^{n+1}\overline{A}_{n+1n+1\lambda} = 0$.

(1) In case $\overline{b} \neq 0$. Obviously, $h_{nn}^{n+1}\overline{A}_{n+1n+1\lambda} = 0$.

(2) In case $\overline{b} = 0$. We have $\overline{g}_{AB} = \overline{a}_{AB}$ and $\overline{A}_{ABC} = \frac{\overline{F}}{2}\frac{\partial \overline{g}_{AB}}{\partial \overline{Y}^C} = 0$.

Then these two statements imply that $h_{nn}^{n+1}\overline{A}_{n+1n+1\lambda} = 0$. When $h_{nn}^{n+1} = 0$, exterior differentiate this, we get that $h_{nn|j}^{n+1}\omega^j + h_{nn;\lambda}^{n+1}\omega_n^\lambda + h_{n\lambda}^{n+1}\omega_n^\lambda = 0$. On the other hand, when $\overline{b}$ is parallel with respect to $\overline{a}$, we have $\overline{P}^A_{BC\lambda} = 0$, hence we have that $h_{nn;\lambda}^{n+1} = 0$ from the first formula of Proposition 2.2. The above two formulas yield $h_{n\lambda}^{n+1}$ and hence $h_{in}^{n+1}\overline{A}_{n+1n+1\lambda} = 0$. It is easy to see that $h_{ni}^{n+1} = 0$, $\forall\, i$ or $\overline{A}_{n+1n+1\lambda} = 0$, $\forall\, \lambda$. When $h_{ni}^{n+1} = 0$, $\forall\, i$, we have that $h_{in|j}^{n+1}\omega^j + h_{in;\lambda}^{n+1}\omega_n^\lambda + h_{i\lambda}^{n+1}\omega_n^\lambda = 0$. On the other hand, we have that $\overline{P}^{n+1}_{nk\lambda} = 0$ from (2.24), which together with the first formula of Proposition 2.2 yields that $h_{in;\lambda}^{n+1} = 0$. Then we have $h_{i\lambda}^{n+1} = 0$ and obtain Proposition 2.9 immediately. □

Proposition 2.10. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and a 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. If M^n is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$ with constant mean curvature $|H|$, then $\sum_i h_{ii}^{n+1} = n\,|H|$ is constant.*

Proof. It follows from Propositions 2.5 and 2.9 that

$$\begin{aligned} 0 &= d\left(h^{n+1}_{ij}\overline{A}_{\lambda n+1n+1}\right) \\ &= \left(h^{n+1}_{ij|k}\overline{A}_{\lambda n+1n+1} + h^{n+1}_{ij}\overline{A}_{\lambda n+1n+1|k} + h^{n+1}_{ij}\overline{A}_{\lambda n+1n+1;n+1}h^{n+1}_{nk}\right)\omega^k \\ &\quad + \left(h^{n+1}_{ij;\mu}\overline{A}_{\lambda n+1n+1} + h^{n+1}_{ij}\overline{A}_{\lambda n+1n+1;\mu}\right)\omega^\mu_n, \end{aligned}$$

which gives that

$$h^{n+1}_{ij}\overline{A}_{in+1n+1;\lambda} = 0.$$

It follows from (2.25) and $\overline{A}_{ABC;D} = \overline{A}_{ABD;C}$ that

$$\begin{aligned} (\overline{\nabla}_{Fe^H_n}\overline{C})(e_i, e_i, e_{n+1}) &= \overline{C}_{iin+1;\lambda}\theta^\lambda_n(Fe^H_n) + \overline{C}_{iin+1;n+1}\theta^{n+1}_n(Fe^H_n) \\ &= F\overline{C}_{iin+1;n+1}h^{n+1}_{nn} \\ &= 0. \end{aligned} \tag{2.26}$$

Since we are assuming that $|H|$ is constant, it follows that $\sum_i h^{n+1}_{ii} = n\,|H|$ from (2.3) and (2.26). □

Proposition 2.11. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and a 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. If M^n is a hypersurface of Randers space $(\overline{M}^{n+1}, \overline{F})$, then*

$$\begin{aligned} h^{n+1}_{ij;\lambda;\mu} &= h^{n+1}_{ij;\mu;\lambda}, \\ h^{n+1}_{ij|k;\lambda} &= h^{n+1}_{ij;\lambda|k} - h^{n+1}_{\mu j}P^\mu_{ik\lambda} - h^{n+1}_{i\mu}P^\mu_{jk\lambda} - h^{n+1}_{ij;\mu}P^\mu_{nk\lambda}, \\ h^{n+1}_{ij|k|l} - h^{n+1}_{ij|k|l} &= h^{n+1}_{sj}R^s_{ikl} + h^{n+1}_{is}R^s_{jkl} + h^{n+1}_{ij;\lambda}R^\lambda_{nkl}. \end{aligned}$$

Proof. It follows from Proposition 2.9 that

$$h^{n+1}_{ij|k}\omega^k + h^{n+1}_{ij;\lambda}\omega^\lambda_n = dh^{n+1}_{ij} - h^{n+1}_{kj}\omega^k_i - h^{n+1}_{ik}\omega^k_j. \tag{2.27}$$

Exterior differentiate the left-hand side of (2.27), we obtain that

$$\begin{aligned} & dh^{n+1}_{ij|k}\wedge\omega^k + h^{n+1}_{ij|k}\,d\omega^k + dh^{n+1}_{ij;\lambda}\wedge\omega^\lambda_n + h^{n+1}_{ij;\lambda}\,d\omega^\lambda_n \\ &= \left\{h^{n+1}_{ij|k|l}\omega^l + h^{n+1}_{ij|k;l}\omega^l_n + h^{n+1}_{lj|k}\omega^l_i + h^{n+1}_{il|k}\omega^l_j + h^{n+1}_{ij|l}\omega^l_k\right\}\wedge\omega^k + h^{n+1}_{ij|k}\left\{-\omega^k_l\wedge\omega^l\right\} \\ &\quad + \left\{h^{n+1}_{ij;\lambda|l}\omega^l + h^{n+1}_{ij;\lambda;\mu}\omega^\mu_n + h^{n+1}_{lj;\lambda}\omega^l_i + h^{n+1}_{il;\lambda}\omega^l_j + h^{n+1}_{ij;\mu}\omega^\mu_\lambda\right\}\wedge\omega^\lambda_n \\ &\quad + h^{n+1}_{ij;\lambda}\left\{-\omega^\lambda_\mu\wedge\omega^\mu_n + \frac{1}{2}R^\lambda_{nls}\,\omega^l\wedge\omega^s + P^\lambda_{nl\mu}\,\omega^l\wedge\omega^\mu_n\right\} \\ &= \left\{-h^{n+1}_{ij|k|l} + \frac{1}{2}h_{ij;\lambda}R^\lambda_{nkl}\right\}\omega^k\wedge\omega^l - h^{n+1}_{ij;\lambda;\mu}\,\omega^\lambda_n\wedge\omega^\mu_n \\ &\quad + \left\{-h^{n+1}_{ij|k;\mu} + h^{n+1}_{ij;\mu|k} + h^{n+1}_{ij;\lambda}P^\lambda_{nkl}\right\}\omega^k\wedge\omega^\mu_n \\ &\quad + h^{n+1}_{lj|k}\,\omega^l_i\wedge\omega^k + h^{n+1}_{il|k}\,\omega^l_j\wedge\omega^k + h^{n+1}_{lj;\lambda}\,\omega^l_i\wedge\omega^\lambda_n + h^{n+1}_{il;\lambda}\,\omega^l_j\wedge\omega^\lambda_n. \end{aligned} \tag{2.28}$$

Exterior differentiate the right-hand side of (2.27), we obtain that

$$
\begin{aligned}
&- dh^{n+1}_{kj}\wedge\omega^k_i - h^{n+1}_{kj}\,d\omega^k_i - dh^{n+1}_{ik}\wedge\omega^k_j - h^{n+1}_{ik}\,d\omega^k_j\\
=&-\left\{h^{n+1}_{kj|l}\omega^l + h^{n+1}_{kj;\lambda}\omega^\lambda_n + h^{n+1}_{lj}\omega^l_k + h^{n+1}_{kl}\omega^l_j\right\}\wedge\omega^k_i\\
&- h^{n+1}_{kj}\left\{-\omega^k_l\wedge\omega^l_i + \frac{1}{2}R^k_{ils}\,\omega^l\wedge\omega^s + P^k_{il\lambda}\,\omega^l\wedge\omega^\lambda_n\right\}\\
&-\left\{h^{n+1}_{ik|l}\omega^l + h^{n+1}_{ik;\lambda}\omega^\lambda_n + h^{n+1}_{lk}\omega^l_i + h^{n+1}_{il}\omega^l_k\right\}\wedge\omega^k_j\\
&- h^{n+1}_{ik}\left\{-\omega^k_l\wedge\omega^l_j + \frac{1}{2}R^k_{jls}\,\omega^l\wedge\omega^s + P^k_{jl\lambda}\,\omega^l\wedge\omega^\lambda_n\right\}\\
=&\left\{-\frac{1}{2}h^{n+1}_{sj}R^s_{ikl} - \frac{1}{2}h^{n+1}_{is}R^k_{jkl}\right\}\omega^k\wedge\omega^l + \left\{-h^{n+1}_{sj}P^s_{ik\lambda} - h^{n+1}_{sj}P^k_{jk\lambda}\right\}\omega^k\wedge\omega^\lambda_n\\
&- h^{n+1}_{kj|l}\,\omega^l\wedge\omega^k_i - h^{n+1}_{kj;\lambda}\,\omega^\lambda_n\wedge\omega^k_i - h^{n+1}_{ik|l}\,\omega^l\wedge\omega^k_j - h^{n+1}_{ik;\lambda}\,\omega^\lambda_n\wedge\omega^k_j.
\end{aligned}
\tag{2.29}
$$

It can be seen from (2.28) and (2.29) that

$$
\begin{aligned}
&h^{n+1}_{ij;\lambda;\mu}\,\omega^\lambda_n\wedge\omega^\mu_n + \left\{h^{n+1}_{ij|k|l} - \frac{1}{2}h^{n+1}_{sj}R^s_{ikl} - \frac{1}{2}h^{n+1}_{is}R^s_{jk\lambda} - \frac{1}{2}h^{n+1}_{ij;\lambda}R^\lambda_{nkl}\right\}\omega^k\wedge\omega^l\\
&+\left\{h^{n+1}_{ij|k;\lambda} - h^{n+1}_{ij;\lambda|k} + h^{n+1}_{ij;\mu}P^\mu_{nkkl} + h^{n+1}_{is}P^s_{jk\lambda} + h^{n+1}_{sj}P^s_{ik\lambda}\right\}\omega^k\wedge\omega^\lambda_n = 0.
\end{aligned}
\tag{2.30}
$$

We obtain Proposition 2.11 immediately from (2.30). □

3. Main theorem

Let $(\overline{M}^{n+1},\overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and a 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$ and (M^n,F) be a hypersurface of $(\overline{M}^{n+1},\overline{F})$ with constant mean curvature. By Proposition 2.10, we have that

$$
\begin{aligned}
\sum_i h_{ii|j}\omega^j + \sum_i h_{ii;j}\omega^j_n &= d\sum_i h_{ii} - \sum_{ij} h_{ji}\omega^j_i - \sum_{ij} h_{ij}\omega^j_i\\
&= -\sum_{ij} h_{ij}(\omega^j_i + \omega^i_j)\\
&= 2\sum_{ij} h_{ij}A_{ijk}\omega^k_n,
\end{aligned}
\tag{3.1}
$$

where $h_{ij} = h^{n+1}_{ij}$.

It follows from (3.1) that

$$
\sum_i h_{ii|j} = 0 \quad\text{and}\quad \sum_i h_{ii;j} = 2\sum_{ik} h_{ik}A_{ikj}. \tag{3.2}
$$

Exterior differentiate the first formula of (3.2), we obtain that

$$\begin{aligned}\sum_i h_{ii|j|k}\omega^k + \sum_i h_{ii|j;k}\omega_n^k &= d\sum_i h_{ii|j} - \sum_{ik} h_{ki|j}\omega_i^k - \sum_{ik} h_{ik|j}\omega_i^k - \sum_{ik} h_{ii|k}\omega_j^k \\ &= -\sum_{ik} h_{ik|j}(\omega_i^k + \omega_k^i) \\ &= 2\sum_{ik} h_{ik|l}A_{ilk}\omega_n^l,\end{aligned}$$

so we have that

$$\sum_i h_{ii|j|k} = 0 \quad \text{and} \quad \sum_i h_{ii|j;k} = 2\sum_{il} h_{il|j}A_{ilk}.$$

On the other hand, when $\bar{b}$ is parallel with respect to $\bar{a}$, we have $\overline{P}^a_{bc\lambda} = 0$, hence Proposition 2.1, together with Proposition 2.5, implies that

$$P^n_{ij\lambda} = 0 \quad \text{and} \quad R^j_{ikl} = \overline{R}^j_{ikl} - h^\alpha_{ik}h^\alpha_{jl} + h^\alpha_{il}h^\alpha_{jk}, \tag{3.3}$$

and Proposition 2.2 together with Proposition 2.5 implies that

$$h_{ij;k} = 0 \quad \text{and} \quad h_{ij|k} = h_{ik|j} - \overline{R}^{n+1}_{ijk}. \tag{3.4}$$

The pull-back of the Sasaki metric $g_{ij}\,dx^i \otimes dx^j + g_{ij}\,\delta Y^i \otimes \delta Y^j$ from $TM \setminus \{0\}$ to the sphere bundle SM is a Riemannian metric $\widehat{g} = g_{ij}\,dx^i \otimes dx^j + \delta_{ab}\,\omega_n^a \otimes \omega_n^b$. We need the following lemma.

Lemma 3.1. [5] *For* $X = \sum_i x_i\omega^i \in \Gamma(\pi^*T^*M)$, $\operatorname{div}_{\widehat{g}} X = \sum_i x_{i|i} + \sum_{\mu,\lambda} x_\mu P^n_{\lambda\lambda\mu}$.

Let $S = \sum_{ij}(h_{ij})^2$ be the norm square of the second fundamental form of (M, F) and $\omega = dS = S_{|i}\omega^i + S_{;i}\omega_n^i$, then ω is a global section on π^*T^*M. By the first formula of (3.4), i.e., $S_{;i} = 0$, the first formula of (3.3) and Lemma 3.1, we have that

$$\begin{aligned}\operatorname{div}_{\widehat{g}}\omega &= \operatorname{div}\left[2\sum_{i,j,k} h_{ij}h_{ij|k}\omega^k\right] = 2\left[\sum_{i,j,k} h_{ij}h_{ij|k}\right]_{|k} \\ &= 2\sum_{i,j,k} h^2_{ij|k} + 2\sum_{i,j,k} h_{ij}h_{ij|k|k}.\end{aligned} \tag{3.5}$$

It can be seen from (3.5), (3.3), (3.4) and Proposition 2.11 that

$$\begin{aligned}\operatorname{div}_{\widehat{g}}\omega &= 2\sum_{i,j,k} h^2_{ij|k} - 2\sum_{i,j,k} h_{ij}\left\{\overline{R}^{n+1}_{kik|j} + \overline{R}^{n+1}_{ijk|k}\right\} + 2\sum_{i,j,k,s} h_{ij}\left\{h_{si}R^s_{kjk} + h_{ks}R^s_{ijk}\right\} \\ &= 2\sum_{i,j,k} h^2_{ij|k} - 2\sum_{i,j,k} h_{ij}\left\{\overline{R}^{n+1}_{kik|j} + \overline{R}^{n+1}_{ijk|k}\right\} \\ &\quad + 2\sum_{i,j,k,s} h_{ij}\left\{h_{si}\overline{R}^s_{kjk} + h_{ks}\overline{R}^s_{ijk}\right\} + \sum_{i,j,s} 2n\,|H|\,h_{ij}h_{si}h_{sj} - 2S^2.\end{aligned} \tag{3.6}$$

Let $b_{ij} = h_{ij} - |H|\,\delta_{ij}$. Then (3.6) becomes that

$$\begin{aligned}\operatorname{div}_{\widehat{g}}\omega &= 2\sum_{i,j,k} b^2_{ij|k} - 2\sum_{i,j,k} b_{ij}\left\{\overline{R}^{n+1}_{kik|j} + \overline{R}^{n+1}_{ijk|k}\right\}\\ &\quad + 2\sum_{i,j,k,s} b_{ij}\left\{b_{si}\overline{R}^{s}_{kjk} + b_{ks}\overline{R}^{s}_{ijk}\right\} + \sum_{i,j,s} 2n\,|H|\,b_{ij}b_{si}b_{sj}\\ &\quad - 2(S - n\,|H|^2)^2 + 2n\,|H|^2\,(S - n\,|H|^2).\end{aligned} \tag{3.7}$$

Proposition 3.2. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and a 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. If the sectional curvature $\overline{K}_{\overline{M}}$ of $\overline{M}$ with respect to the Riemannian metric $\overline{a}$ satisfies $\delta \le \overline{K}_{\overline{M}} \le 1$, then*

(1) $\left|\overline{R}^{A}_{CBC}\right| \le \frac{1}{2}(1-\delta)$, *for $A \ne B$,*

(2) $\left|\overline{R}^{A}_{BCD}\right| \le \frac{2}{3}(1-\delta)$, *for A, B, C, D distinct with each other.*

Proof. Let $\overline{K}^{A}_{BCD}$ be the curvature tensor of $\overline{M}$ with respect to the Riemannian metric $\overline{a}$. If the sectional curvature $\overline{K}_{\overline{M}}$ of $\overline{M}$ with respect to the Riemannian metric $\overline{a}$ satisfies $\delta \le \overline{K}_{\overline{M}} \le 1$, then

(1) $\left|\overline{K}^{A}_{CBC}\right| \le \frac{1}{2}(1-\delta)$, for $A \ne B$,

(2) $\left|\overline{K}^{A}_{BCD}\right| \le \frac{2}{3}(1-\delta)$, for A, B, C, D distinct with each other.

On the other hand, when 1-form $\overline{b}$ is parallel with respect to $\overline{a}$, we have that $\overline{\Gamma}^{A}_{BC} = \widetilde{\gamma}^{A}_{BC}$, which implies that $\overline{R}^{A}_{BCD} = \overline{K}^{A}_{BCD}$. This proves Proposition 3.2. □

Proposition 3.3. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and a 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. Assume the sectional curvature $\overline{K}_{\overline{M}}$ of $\overline{M}$ with respect to the Riemannian metric $\overline{a}$ satisfies $\delta \le \overline{K}_{\overline{M}} \le 1$. If (M^n, F) is a hypersurface of $(\overline{M}^{n+1}, \overline{F})$ with constant mean curvature, then*

$$\sum_{i,j,k}(b_{ij}b_{si}\overline{R}^{s}_{kjk} + b_{ij}b_{ks}\overline{R}^{s}_{ijk}) \ge n\delta(S - n\,|H|^2) - \frac{7}{2}(1-\delta)(S - n\,|H|^2).$$

Proof. It follows from Proposition 3.2 that

$$\begin{aligned}\sum_{i,j,k}(b_{ij}b_{si}\overline{R}^{s}_{kjk} + b_{ij}b_{ks}\overline{R}^{s}_{ijk}) &= b^2_{ij}\overline{R}^{j}_{kjk} + b^2_{ij}\overline{R}^{j}_{iji} - b_{ii}b_{jj}\overline{R}^{j}_{iji}\\ &\quad + \sum_{s\ne j} b_{ij}b_{si}\overline{R}^{s}_{kjk} + 2\sum_{i\ne k} b_{ij}b_{jk}\overline{R}^{j}_{ijk} + 2\sum_{s\ne k} b_{ii}h_{sk}\overline{R}^{s}_{iik}\\ &= \sum_{i\ne j} b^2_{ij}\overline{R}^{j}_{kjk} + b^2_{ij}\overline{R}^{j}_{iji} + \frac{1}{2}(b_{ii} - b_{jj})^2\overline{R}^{j}_{iji}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s\neq j} b_{ij}b_{si}\overline{R}^{s}_{kjk} + 2\sum_{i\neq k} b_{ij}b_{jk}\overline{R}^{j}_{ijk} + 2\sum_{s\neq k} b_{ii}b_{sk}\overline{R}^{s}_{iik} \\
&\geq n\delta(S - n|H|^2) - \frac{7}{2}(1-\delta)(S - n|H|^2). \qquad \square
\end{aligned}
$$

Proposition 3.4. *Let $(\overline{M}^{n+1}, \overline{F})$ be a Randers space constructed from a Riemannian metric $\overline{a}$ and a 1-form $\overline{b}$, where $\overline{b}$ is parallel with respect to $\overline{a}$. Assume the sectional curvature $\overline{K}_{\overline{M}}$ of $\overline{M}$ with respect to the Riemannian metric $\overline{a}$ satisfies $\delta \leq \overline{K}_{\overline{M}} \leq 1$. If M^n is a compact hypersurface of $(\overline{M}^{n+1}, \overline{F})$ with constant mean curvature, then*

$$
\int_{SM} \sum_{i,j,k} \left[b^2_{ij|k} - \left(b_{ij}\overline{R}^{n+1}_{kik|j} + b_{ij}\overline{R}^{n+1}_{ijk|k} \right) \right] dV_{SM} \geq -\int_{SM} \frac{1}{72} n(n-1)(26n-25)(1-\delta)^2 \, dV_{SM}.
$$

Proof. Let $X = \sum_{i,j}\left(b_{ik}\overline{R}^{n+1}_{jij} + b_{ij}\overline{R}^{n+1}_{ijk}\right)\omega^k$. It follows from Lemma 3.1 and the second formula of (3.4) that

$$
\begin{aligned}
&\sum_{i,j,k} b^2_{ij|k} - \sum_{i,j,k}\left(b_{ik}\overline{R}^{n+1}_{jij|k} + b_{ij}\overline{R}^{n+1}_{ijk|k}\right) \\
&= \sum_{i,j,k} b^2_{ij|k} + \sum_{i,j,k}\left(b_{ik|k}\overline{R}^{n+1}_{jij} + b_{ij|k}\overline{R}^{n+1}_{ijk}\right) - \operatorname{div}_{\widehat{g}} X \\
&= \sum_{i,j,k} b^2_{ij|k} + \sum_{i,j,k} b_{ij|k}\overline{R}^{n+1}_{ijk} - \sum_{i}\left(\sum_{j}\overline{R}^{n+1}_{jij}\right)^2 - \operatorname{div}_{\widehat{g}} X \qquad (3.8)\\
&\geq -\frac{1}{4}\sum_{i,j,k}\left(\overline{R}^{n+1}_{ijk}\right)^2 - \frac{1}{4}n(n-1)^2(1-\delta)^2 - \operatorname{div}_{\widehat{g}} X \\
&\geq -\frac{1}{72}n(n-1)(26n-25)(1-\delta)^2 - \operatorname{div}_{\widehat{g}} X.
\end{aligned}
$$

Integrating (3.8) yields Proposition 3.4. $\square$

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Using the fact $\sum_{i,j,s} n|H|\, b_{ij}b_{is}b_{sj} \geq -\frac{n-2}{\sqrt{n(n-1)}} n|H|(s - n|H|^2)^{3/2}$, substituting Propositions 3.3 and 3.4 into (3.7), we have that

$$
\begin{aligned}
0 \geq \int_{SM} \Big\{ &- (S - n|H|^2)^2 + n\delta(S - n|H|^2) - \frac{7}{2}(1-\delta)(S - n|H|^2) \\
&- \frac{n-2}{\sqrt{n(n-1)}} n|H|(S - n|H|^2)^{3/2} - \frac{1}{72}n(n-1)(26n-25)(1-\delta)^2 \Big\} dV_{SM}. \qquad (3.9)
\end{aligned}
$$

From our assumption condition $S - n|H|^2 \leq B_H$, we can obtain that

$$
-(S - n|H|^2) + n\delta - \frac{7}{2}(1-\delta) - \frac{n-2}{\sqrt{n(n-1)}} n|H|(S - n|H|^2)^{1/2} - \alpha(1-\delta) \geq 0,
$$

which together with $S - n|H|^2 \geq \alpha(1-\delta)$ yields that

$$
\begin{aligned}
&-(S-n|H|^2)^2 + n\delta(S-n|H|^2) - \frac{7}{2}(1-\delta)(S-n|H|^2) \\
&-\frac{n-2}{\sqrt{n(n-1)}}n|H|(S-n|H|^2)^{3/2} - \frac{1}{72}n(n-1)(26n-25)(1-\delta)^2 \geq 0.
\end{aligned}
\tag{3.10}
$$

It follows from (3.9) and (3.10) that all inequalities in (3.8) are actually equalities. Then we get that $b_{ij|k} = \overline{R}^{n+1}_{ijk}$, $\forall\, i,j,k$ and $\left|\overline{R}^{n+1}_{jij}\right| = \frac{1}{2}(1-\delta)$, $\forall\, i \neq j$. It can be seen from $b_{ij|k} = \overline{R}^{n+1}_{ijk}$, $\forall\, i,j,k$ that $\overline{R}^{n+1}_{jij} = \overline{R}^{n+1}_{ijj} = 0$. Then it is easy to see that $\delta = 1$. This, together with the parallel 1-form $\bar{b}$, implies that $\overline{M}$ is a Berwald manifold with constant flag curvature $\overline{K} = 1$, then $\overline{M}$ is a complete simply connected Riemannian manifold with constant curvature 1. Hence we obtain Theorem 1.2 immediately. □

Using the same way as the proof of Theorem 1.2, we can also obtain the following

Theorem 3.5. *Let $(\overline{M}^{n+1}, \overline{F})$ be a complete simply connected Randers space constructed from a Riemannian metric $\overline{a}$ and a 1-form $\bar{b}$, where $\bar{b}$ is parallel with respect to $\overline{a}$. Assume the sectional curvature $\overline{K}_{\overline{M}}$ of $\overline{M}$ with respect to the Riemannian metric $\overline{a}$ satisfies $\delta(n) \leq \overline{K}_{\overline{M}} \leq 1$. If M^n is a compact hypersurface of $(\overline{M}^{n+1}, \overline{F})$ with constant mean curvature $|H|$ and the norm square S of the second fundamental form of (M, F) with respect to the Finsler metric $\overline{F}$ satisfies $B_1 \leq \sqrt{S - n|H|^2} \leq B_2$, where B_1, B_2 $(0 \leq B_1 < B_2)$ are the solutions of the following equation*

$$
-x^4 - \frac{n-2}{\sqrt{n(n-1)}}n|H|x^3 + \left[n\delta - \frac{7}{2}(1-\delta)\right]x^2 - \frac{1}{72}n(n-1)(26n-25)(1-\delta)^2 = 0,
$$

then either M is the unit sphere or $S - n|H|^2 = B_2$ and one of the following cases occurs:

(1) *$H = 0$ and $M = S^k\big(\sqrt{k/n}\big) \times S^{n-k}\big(\sqrt{(n-k)/n}\big)$ is a minimal Clifford hypersurface,*

(2) *$H \neq 0$, $n \geq 3$ and $M = S^1(r_1) \times S^{n-1}(r_2)$ is an H-torus, where $r_1^2 + r_2^2 = 1$ and $r_2^2 < (n-1)/2$,*

(3) *$H \neq 0$, $n = 2$ and $M = S^1(r_1) \times S^1(r_2)$ is an H-torus, where $r_1^2 + r_2^2 = 1$ and $r_1^2 \neq 1/2$.*

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