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Iterative Method for a New Class of Evolution Equations with Non-instantaneous Impulses

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Abstract. In this paper, we are concerned with the existence of mild solutions for the initial value problem to a new class of abstract evolution equations with non-instantaneous impulses on ordered Banach spaces. The existence and uniqueness theorem of mild solution for the associated linear evolution equation with non-instantaneous impulses is established. With the aid of this theorem, the existence of mild solutions for nonlinear evolution equation with non-instantaneous impulses is obtained by using perturbation technique and iterative method under the situation that the corresponding solution semigroup $T(\cdot)$ and non-instantaneous impulsive function g_k are compact, $T(\cdot)$ is not compact and g_k is compact, $T(\cdot)$ and g_k are not compact, respectively. The results obtained in this paper essentially improve and extend some related conclusions on this topic. Two concrete examples to parabolic partial differential equations with non-instantaneous impulses are given to illustrate that our results are valuable.

1. Introduction

In this paper, we use the perturbation technique and iterative method in the presence of lower and upper solutions to study the existence of mild solutions for the initial value problem (IVP) to a new class of semi-linear evolution equations with non-instantaneous impulses in Banach space E

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)) & t \in \bigcup_{k=0}^{p} (s_k, t_{k+1}], \\ u(t) = g_k(u(t)) & t \in \bigcup_{k=1}^{p} (t_k, s_k], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $A\colon D(A) \subset E \to E$ is a closed linear operator, $-A$ is the infinitesimal generator of a C_0-semigroup $T(t)$ $(t \geq 0)$ in E, $0 < t_1 < t_2 < \cdots < t_p < t_{p+1} := a$, $a > 0$ is a constant,

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$p \in \mathbb{Z}^+$, $J = [0, a]$, $s_0 := 0$ and $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \ldots, p$, $f: [s_k, t_{k+1}] \times E \to E$ is a continuous nonlinear function for $k = 0, 1, \ldots, p$, $g_k \in C(E, E)$ is non-instantaneous impulsive function for all $k = 1, 2, \ldots, p$, and $u_0 \in E$.

The theory of instantaneous impulsive differential equations is an important branch of differential equation theory, which has extensive physical, chemical, biological, engineering background and realistic mathematical model, and hence has been emerging as an important area of investigation in the last few decades. For more details on differential equations with instantaneous impulses, one can see the monographs of Lakshmikantham, Bainov and Simeonov [31], Benchohra, Henderson and Ntouyas [10] and the papers of Ahmed [3], Abada, Benchohra and Hammouche [1], Barreira and Valls [9], Bonottoa et al. [11], Qian, Chen and Sun [41], Guo and Liu [26], Chang and Li [12], Li and Liu [34], Chen, Li and Zhang [17], Fan and Li [23], Liang, Liu and Xiao [35, 36], where numerous properties of their solutions are studied and detailed bibliographies are given. Differential equations with instantaneous impulses consider basically problems for which the impulses are abrupt and instantaneous. The most important feature of instantaneous impulsive differential equations in this class of equations is linked to their utility in simulating processes and phenomena subject to short time perturbations during their evolution, and the perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena when construct mathematical models.

However, one can see that the models with instantaneous impulses could not explain the certain dynamics of evolution processes in pharmacotherapy. Just as pointed out by Hernández and O'Regan in [30], when we consider the simplified situation concerning the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. Therefore, one can interpret this situation as an impulsive action which starts abruptly and stays active on a finite time interval. We call such phenomenon non-instantaneous impulses during construct mathematical models. It is reported that many models arising from realistic models can be described as partial differential equations with non-instantaneous impulses.

In the past three years, nonlinear differential equations with non-instantaneous impulses have been studied by several authors and some interesting results have been obtained, see [2,18,24,30,40,43,45,46]. In 2013, Hernández and O'Regan [30] first studied the initial value problem for a new class of abstract evolution equations with non-instantaneous impulses in Banach spaces. In the same year, Pierri, O'Regan and Rolnik [40] obtained the existence of mild solutions for a class of semi-linear abstract differential equations with non-instantaneous impulses by using the theory of analytic semigroups. Gautam and Dabas [24] studied the existence, uniqueness and continuous dependence results of mild solution for fractional functional integro-differential equations with non-instantaneous impulses by using the theory of analytic α-resolvent family and fixed point theorems in 2014. In 2015, Colao, Mugliam and Xu [18] obtained the existence of solutions for a second-order differential equation with non-instantaneous impulses and delay on an unbounded interval by establish a compactness criterion in a certain class of functions. Yu and Wang [46] investigated the existence of solutions to periodic boundary value problems for nonlinear evolution equations with non-instantaneous impulses on Banach spaces by using the theory of semigroups and fixed point methods also in 2015. In 2014, Wang and Li [43] obtained the existence of solutions for periodic boundary value problem of nonlinear ordinary differential equations with non-instantaneous impulses. In addition, fractional ordinary and partial differential equations with non-instantaneous impulses have also been studied by Wang, Zhou and Lin [45] in 2014 and Abbas and Benchohra [2] in 2015, respectively.

But so far we have not seen relevant papers that study abstract evolution equations with non-instantaneous impulses by applying the iterative method, perturbation technique and the method of lower and upper solutions. The most advantage by using the iterative method based on lower and upper solutions is that it not only provides a method to obtain the existence of extremal mild solutions, but also yields iterative sequences of lower and upper approximate solutions that converge to the minimal and maximal mild solutions between the lower and upper solutions. The iterative sequences are very useful in numerical calculation, which provide a computing rule in computer simulation. As early as 1976, Amann [4] established the lower and upper solutions theorem for operator equation in ordered Banach spaces. In 1982, Du and Lakshmikantham [21] investigated the existence of extremal solutions to initial value problem of ordinary differential equation without impulse by using the method of lower and upper solutions and the iterative method. Later, Guo and Liu [26], Li and Liu [34], Chen, Li and Zhang [17] developed the iterative method for ordinary differential equations with instantaneous impulses in Banach spaces. Recently, the iterative method has been extended to evolution equations in ordered Banach spaces, we refer to the papers by Li [32], Wang and Wang [44] and EI-Gebeily, O'Regan and Nieto [22] for evolution equations with classical initial vlaue conditions, and to the paper by Chen and Li [13], Chen, Li and Yang [16] and Chen and Li [15] for evolution equations with instantaneous impulses in Banach spaces. In this paper, inspired by the above-mentioned aspects, we use the positive operator semigroups theory, perturbation technique and iterative method to study the existence of mild solutions for IVP (1.1). In addition, by using a perturbation technique for nonlinear function f, we extend to the situation that the nonlinear function f is not monotone increasing on the ordered interval in this paper.

The theory of operator semigroups plays an important role in studying of abstract evolution equations. In a study of nonlinear evolution equations, if the corresponding nonhomogeneous problem is well-posed, then one can define a strongly continuous linear and bounded operator on $\mathbb{R}^+$, which is called solution semigroup, such that the solution for the initial value problem of the corresponding nonhomogeneous problem and also the nonlinear evolution equations can be uniquely expressed by this linear operator semigroup. For more details, one can see the monographs by Pazy [38] and Vrabie [42]. The discussion of this paper is just based on the theory of linear operator semigroups. By using the theory of linear operator semigroups, we can transform the corresponding linear evolution equation with non-instantaneous impulses into an equivalent integral equation (see Theorem 3.1). With the aid of Theorem 3.1, we can define a solution operator of IVP (1.1) and then applying perturbation technique and iterative method to discuss the existence of mild solutions for IVP (1.1) under the situation that $T(\cdot)$ and g_k are compact, $T(\cdot)$ is not compact and g_k is compact, $T(\cdot)$ and g_k are not compact, respectively.

The rest of this article is organized as follows. We provide in Section 2 some definitions, notations and necessary preliminaries about cone, partial order, measure of noncompactness and operator semigroups, which are used throughout this paper. The existence and uniqueness theorem of mild solution for the associated linear evolution equation with non-instantaneous impulses is established in Section 3. In Section 4, we obtained the existence of extremal mild solutions as well as mild solutions for IVP (1.1) under the situation that $T(t)$ $(t \geq 0)$ is a compact and positive C_0-semigroup and the non-instantaneous impulsive function g_k is compact in E for $k = 1, 2, \ldots, p$. In Section 5, we discuss the existence of extremal mild solutions for IVP (1.1) under the situation that $-A$ only generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in E and the non-instantaneous impulsive function g_k is compact in E for $k = 1, 2, \ldots, p$, in which the Gronwall-Bellman type inequalities paly an important role. The existence of extremal mild solutions for IVP (1.1) is obtained under the situation that the semigroup $T(t)$ $(t \geq 0)$ is a positive C_0-semigroup in Banach space E and the non-instantaneous impulsive function g_k is only continuous for $k = 1, 2, \ldots, p$, or the positive C_0-semigroup $T(t)$ $(t \geq 0)$ generated by $-A$ is equicontinuous in E and the non-instantaneous impulsive function g_k is Lipschitz continuous for $k = 1, 2, \ldots, p$ in Section 6. In the last section, two concrete examples to parabolic partial differential equation with non-instantaneous impulses are given to illustrate the feasibility of our abstract results.

2. Preliminaries

In this section, we recall some basic theories of the cone, partial order, measure of noncompactness and operator semigroups. Let E be a Banach space with the norm $\|\cdot\|$ and partial order "$\leq$", whose positive cone $P = \{x \in E \mid x \geq \theta\}$ is normal with normal constant N, where θ is the zero element in E. We denote by $C(J,E)$ the Banach space of all continuous functions from J into E endowed with the sup-norm $\|u\|_C = \sup_{t\in J}\|u(t)\|$ for every $u \in C(J,E)$. Then $C(J,E)$ is an ordered Banach space induced by the convex cone

$$P_C = \{u \in C(J,E) \mid u(t) \geq \theta, t \in J\},$$

and P_C is also a normal cone with the same normal constant N. Let

$$PC(J,E) = \{u\colon J \to E \mid u \text{ is continuous at } t \neq t_k, \text{ left continuous at } t = t_k \text{ and } u(t_k^+) \text{ exists for all } k = 1,2,\ldots,p\}$$

be a piecewise continuous function space. It is easy to see that $PC(J,E)$ is a Banach space endowed with the norm

$$\|u\|_{PC} = \sup_{t\in J}\|u(t)\|, \quad \forall\, u \in PC(J,E). \tag{2.1}$$

Evidently, $PC(J,E)$ is also an ordered Banach space with the partial order "$\leq$" induced by the positive cone

$$K_{PC} = \{u \in PC(J,E) \mid u(t) \geq \theta, t \in J\}.$$

K_{PC} is also normal with the same normal constant N. For $v,w \in PC(J,E)$ with $v \leq w$, we use $[v,w]$ to denote the order interval

$$\{u \in PC(J,E) \mid v \leq u \leq w\}$$

in $PC(J,E)$, and $[v(t),w(t)]$ to denote the order interval

$$\{u \in E \mid v(t) \leq u(t) \leq w(t), t \in J\}$$

in E. For more definitions and details of the cone and partial order, we refer to the monographs by Guo and Lakshmikantham [25] and Deimling [19]. We use E_1 to denote the Banach space $D(A)$ with the graph norm

$$\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|.$$

Let $J' = J\backslash\{t_1,t_2,\ldots,t_p\}$, $J'' = J\backslash\{0,t_1,t_2,\ldots,t_p\}$. An abstract function $u \in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ is called a solution of IVP (1.1) if $u(t)$ satisfies all the equalities of (1.1).

Definition 2.1. If a function $v_0 \in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ satisfies that

$$\begin{cases} v_0'(t) + Av_0(t) \leq f(t,v_0(t)) & t \in \bigcup_{k=0}^{p}(s_k,t_{k+1}], \\ v_0(t) \leq g_k(v_0(t)) & t \in \bigcup_{k=1}^{p}(t_k,s_k], \\ v_0(0) \leq u_0, \end{cases} \tag{2.2}$$

we call it a lower solution of IVP (1.1); if all the inequalities in (2.2) are reversed, we call it an upper solution of IVP (1.1).

Let $A\colon D(A) \subset E \to E$ be a closed linear operator and let $-A$ generate a C_0-semigroup $T(t)$ $(t \geq 0)$ on E. Then there exist constants $C_1 \geq 1$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq C_1 e^{\delta t}, \quad t \geq 0. \tag{2.3}$$

Denote by $\mathcal{L}(E)$ the Banach space of all linear and bounded operators on E. By (2.3) we know that

$$C := \sup_{t \in J} \|T(t)\|_{\mathcal{L}(E)} \geq 1 \tag{2.4}$$

is a finite number.

Definition 2.2. A C_0-semigroup $T(t)$ $(t \geq 0)$ in E is said to be positive, if order inequality $T(t)x \geq \theta$ holds for each $x \geq \theta$, $x \in E$ and $t \geq 0$.

Definition 2.3. A C_0-semigroup $T(t)$ $(t \geq 0)$ in E is said to be compact, if $T(t)$ is a compact operator in E for every $t > 0$.

Definition 2.4. A C_0-semigroup $T(t)$ $(t \geq 0)$ in E is said to be equicontinuous, if $T(t)$ is continuous in the operator norm for every $t > 0$.

It is easy to see that for any constant $M \geq 0$, $-(A + MI)$ also generates a C_0-semigroup $S(t) = e^{-Mt}T(t)$ $(t \geq 0)$ in E. Therefore, $S(t)$ $(t \geq 0)$ is a positive C_0-semigroup if $T(t)$ $(t \geq 0)$ is a positive C_0-semigroup, $S(t)$ $(t \geq 0)$ is a compact semigroup if $T(t)$ $(t \geq 0)$ is a compact semigroup, $S(t)$ $(t \geq 0)$ is an equicontinuous semigroup if $T(t)$ $(t \geq 0)$ is an equicontinuous semigroup. For more details about the properties of the operator semigroups and positive C_0-semigroups, we refer to the monographs by Henry [29], Pazy [38], Banasiak and Arlotti [8], Vrabie [42] and the paper by Li [32].

Next, we recall some basic definitions and properties about Kuratowski measure of noncompactness that will be used in the proof of our main results.

Definition 2.5. [7, 19] The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on bounded set S of Banach space E is

$$\alpha(S) := \inf\left\{ \delta > 0 \;\middle|\; S = \bigcup_{i=1}^{m} S_i \text{ and } \operatorname{diam}(S_i) \leq \delta \text{ for } i = 1, 2, \ldots, m \right\}.$$

The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.6. [7,19] *Let E be a Banach space and $S, U \subset E$ be bounded. Then the following properties are satisfied:*

(i) $\alpha(S) = 0$ *if and only if* $\overline{S}$ *is compact, where* $\overline{S}$ *means the closure hull of* S*;*

(ii) $\alpha(S) = \alpha(\overline{S}) = \alpha(\operatorname{conv} S)$, *where* $\operatorname{conv} S$ *means the convex hull of* S;

(iii) $\alpha(\lambda S) = |\lambda| \, \alpha(S)$ *for any* $\lambda \in \mathbb{R}$;

(iv) $S \subset U$ *implies* $\alpha(S) \le \alpha(U)$;

(v) $\alpha(S \cup U) = \max\{\alpha(S), \alpha(U)\}$;

(vi) $\alpha(S + U) \le \alpha(S) + \alpha(U)$, *where* $S + U = \{x \mid x = y + z, y \in S, z \in U\}$;

(vii) *If the map* $Q \colon \mathcal{D}(Q) \subset E \to X$ *is Lipschitz continuous with constant* k, *then* $\alpha(Q(V)) \le k\alpha(V)$ *for any bounded subset* $V \subset \mathcal{D}(Q)$, *where* X *is another Banach space.*

In this paper, we use $\alpha(\cdot)$, $\alpha_C(\cdot)$ and $\alpha_{PC}(\cdot)$ to denote the Kuratowski measure of noncompactness on the bounded set of E, $C(J, E)$ and $PC(J, E)$, respectively. For any $D \subset C(J, E)$ and $t \in J$, set $D(t) = \{u(t) \mid u \in D\}$, then $D(t) \subset E$. If $D \subset C(J, E)$ is bounded, then $D(t)$ is bounded in E and $\alpha(D(t)) \le \alpha_C(D)$. For the details about the definition and properties of the measure of noncompactness, we refer to the monographs by Deimling [19], Banas and Goebel [7], Guo and Sun [27], Ayerbe, Domínguez and López [6].

The following lemmas about the measure of noncompactness are needed in our argument.

Lemma 2.7. [14, 33] *Let* E *be a Banach space, and let* $D \subset E$ *be bounded. Then there exists a countable set* $D_0 \subset D$, *such that*

$$\alpha(D) \le 2\alpha(D_0).$$

Lemma 2.8. [28] *Let* E *be a Banach space, and let* $D = \{u_n\} \subset PC([b_1, b_2], E)$ *be a bounded and countable set for constants* $-\infty < b_1 < b_2 < +\infty$. *Then* $\alpha(D(t))$ *is Lebesgue integrable on* $[b_1, b_2]$, *and*

$$\alpha\left(\left\{\int_{b_1}^{b_2} u_n(t)\,dt \;\middle|\; n \in \mathbb{N}\right\}\right) \le 2\int_{b_1}^{b_2} \alpha(D(t))\,dt.$$

Lemma 2.9. [7, 27] *Let* E *be a Banach space, and let* $D \subset C([b_1, b_2], E)$ *be bounded and equicontinuous for constants* $-\infty < b_1 < b_2 < +\infty$. *Then* $\alpha(D(t))$ *is continuous on* $[b_1, b_2]$, *and*

$$\alpha_C(D) = \max_{t \in [b_1, b_2]} \alpha(D(t)).$$

Definition 2.10. [39] Let E be a Banach space, and let S be a nonempty subset of E. A continuous mapping $Q \colon S \to E$ is called to be strict α-set-contraction operator if there exists a constant $0 \le \beta < 1$, such that for every bounded set $\Omega \subset S$

$$\alpha(Q(\Omega)) \le \beta\alpha(\Omega).$$

Lemma 2.11. [25] *Let P be a normal cone of the ordered Banach space E and $v_0, w_0 \in E$ with $v_0 \leq w_0$. Suppose that $Q\colon [v_0, w_0] \to E$ is a nondecreasing strict α-set-contraction operator such that $v_0 \leq Qv_0$ and $Qw_0 \leq w_0$. Then Q has a minimal fixed point $\underline{u}$ and a maximal fixed point $\overline{u}$ in $[v_0, w_0]$; moreover,*

$$v_n \to \underline{u} \quad \text{and} \quad w_n \to \overline{u} \quad \text{as } n \to \infty,$$

where $v_n = Qv_{n-1}$ and $w_n = Qw_{n-1}$ $(n = 1, 2, \dots)$ which satisfy

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq \underline{u} \leq \overline{u} \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0.$$

3. Linear evolution equation with non-instantaneous impulses

In order to study the initial value problem of nonlinear evolution equations with non-instantaneous impulses (1.1), in this section, we first consider the initial value problem of linear evolution equation (LEE) with non-instantaneous impulses

$$\begin{cases} u'(t) + Au(t) = h(t) & t \in \bigcup_{k=0}^{p} (s_k, t_{k+1}], \\ u(t) = e_k(t) & t \in \bigcup_{k=1}^{p} (t_k, s_k], \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where $h \in PC(J, E)$, $e_k\colon [t_k, s_k] \to E$ is continuous function for $k = 1, 2, \dots, p$, $u_0 \in E$.

Theorem 3.1. *Let E be a Banach space, $A\colon D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a C_0-semigroup $T(t)$ $(t \geq 0)$ in E. Then for any $h \in PC(J, E)$, $e_k \in C([t_k, s_k], E)$ for $k = 1, 2, \dots, p$ and $u_0 \in E$, LEE (3.1) has a unique mild solution $u \in PC(J, E)$ given by*

$$u(t) = \begin{cases} T(t)u_0 + \int_0^t T(t-s)h(s)\,ds & t \in [0, t_1], \\ e_k(t) & t \in (t_k, s_k],\ k = 1, 2, \dots, p, \\ T(t-s_k)e_k(s_k) + \int_{s_k}^t T(t-s)h(s)\,ds & t \in (s_k, t_{k+1}],\ k = 1, 2, \dots, p. \end{cases} \tag{3.2}$$

Proof. Assume that u satisfies (3.1). If $t \in [0, t_1]$, by [38, Corollary 4.2.11] we know that u can be uniquely expressed by

$$u(t) = T(t)u(0) + \int_0^t T(t-s)h(s)\,ds. \tag{3.3}$$

If $t \in (t_k, s_k]$, $k = 1, 2, \dots, p$, then it is easy to see that

$$u(t) = e_k(t). \tag{3.4}$$

Especially,

$$u(s_k) = e_k(s_k), \quad k = 1, 2, \ldots, p. \tag{3.5}$$

If $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, p$, by again [38, Corollary 4.2.11] and (3.5) we know that

$$\begin{aligned} u(t) &= T(t - s_k)u(s_k) + \int_{s_k}^{t} T(t - s)h(s)\, ds \\ &= T(t - s_k)e_k(s_k) + \int_{s_k}^{t} T(t - s)h(s)\, ds. \end{aligned} \tag{3.6}$$

Now it is clear that (3.3), (3.4) and (3.6) imply (3.2).

Inversely, assume that $u \in PC(J, E)$ satisfies (3.2), then we can verify directly that the function u defined by (3.2) is a mild solution of LEE (3.1). This completes the proof of Theorem 3.1. □

By Theorem 3.1, we can easily obtain the following result.

Theorem 3.2. *Let E be a Banach space, $A\colon D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a C_0-semigroup $T(t)$ $(t \geq 0)$ in E. Then for any $h \in PC(J, E) \cap C^1(J', E)$, $u_0 \in D(A)$, $e_k \in C([t_k, s_k], E)$ and $e_k(t) \in D(A)$ for every $t \in (t_k, s_k]$, $k = 1, 2, \ldots, p$, LEE* (3.1) *has a unique classical solution $u \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ given by* (3.2).

4. $T(\cdot)$ and g_k are compact

In this section, we discuss the existence of mild solutions for IVP (1.1) under the situation that $T(t)$ $(t \geq 0)$ is a compact C_0-semigroup and the non-instantaneous impulsive function g_k is compact in E for $k = 1, 2, \ldots, p$.

Theorem 4.1. *Let E be an ordered Banach space, whose positive cone P is normal, $A\colon D(A) \subset E \to E$ be a closed linear operator and the positive C_0-semigroup $T(t)$ $(t \geq 0)$ generated by $-A$ be compact in E, $g_k \in C(E, E)$ be a compact operator for $k = 1, 2, \ldots, p$, $f \in C([s_k, t_{k+1}] \times E, E)$ for $k = 0, 1, \ldots, p$. Assume that IVP* (1.1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \leq w_0$. Assume that*

(H_f1) *there exists a constant $M > 0$ such that*

$$f(t, u_2) - f(t, u_1) \geq -M(u_2 - u_1)$$

for any $t \in [s_k, t_{k+1}]$ $(k = 0, 1, \ldots, p)$ and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$;

(H_g) *the non-instantaneous impulsive function* $g_k(u)$ *is increasing on ordered interval* $[v_0, w_0]$ *for* $k = 1, 2, \ldots, p$.

Then IVP (1.1) *has a minimal mild solution* $\underline{u}$ *and a maximal mild solution* $\overline{u}$ *between* v_0 *and* w_0*, which can be obtained by a monotone iterative procedure starting from* v_0 *and* w_0*, respectively.*

Proof. It is easy to see that IVP (1.1) is equivalent to the following initial value problem

$$\begin{cases} u'(t) + Au(t) + Mu(t) = f(t, u(t)) + Mu(t) & t \in \bigcup_{k=0}^{p} (s_k, t_{k+1}], \\ u(t) = g_k(u(t)) & t \in \bigcup_{k=1}^{p} (t_k, s_k], \\ u(0) = u_0 \end{cases}$$

for any constant $M > 0$. Therefore, define an operator $\mathcal{F}\colon [v_0, w_0] \to PC(J, E)$ by

$$(\mathcal{F}u)(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s, u(s)) + Mu(s)]\, ds & t \in [0, t_1], \\ g_k(u(t)) & t \in (t_k, s_k],\ k = 1, 2, \ldots, p, \\ S(t - s_k) g_k(u(s_k)) & \\ \quad + \int_{s_k}^t S(t-s)[f(s, u(s)) + Mu(s)]\, ds & t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, p, \end{cases} \tag{4.1}$$

where $S(t) = e^{-Mt}T(t)$ $(t \geq 0)$ is the C_0-semigroup generated by $-(A + MI)$. Then it is clear that $\mathcal{F}\colon [v_0, w_0] \to PC(J, E)$ is a continuous operator. By Theorem 3.1, the mild solution of IVP (1.1) is equivalent to the fixed point of operator $\mathcal{F}$ defined by (4.1). Since $S(t)$ $(t \geq 0)$ is a positive C_0-semigroup, combining this fact with the assumptions (H_f1) and (H_g), it is easy to prove that $\mathcal{F}$ is an increasing operator in $[v_0, w_0]$.

Next, we show that $v_0 \leq \mathcal{F}v_0$ and $\mathcal{F}w_0 \leq w_0$. Let $h(t) = v_0'(t) + Av_0(t) + Mv_0(t)$, $t \in \bigcup_{k=0}^{p} (s_k, t_{k+1}]$. From (2.2) we know that $h \in PC(J, E)$ and $h(t) \leq f(t, v_0(t)) + Mv_0(t)$ for $t \in \bigcup_{k=0}^{p} (s_k, t_{k+1}]$. Set $e_k(t) = v_0(t)$ for $t \in (t_k, s_k]$, $k = 1, 2, \ldots, p$. Again from (2.2) we know that $e_k(t) \leq g_k(v_0(t))$ for $t \in (t_k, s_k]$, $k = 1, 2, \ldots, p$. By the above facts, Theorem 3.1 and Definition 2.1, we get that

$$v_0(t) = \begin{cases} S(t)v_0(0) + \int_0^t S(t-s)h(s)\, ds & t \in [0, t_1], \\ e_k(t) & t \in (t_k, s_k],\ k = 1, 2, \ldots, p, \\ S(t - s_k) e_k(s_k) & \\ \quad + \int_{s_k}^t S(t-s)h(s)\, ds & t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, p \end{cases}$$

$$\leq \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s, v_0(s)) + Mv_0(s)]\, ds & t \in [0, t_1], \\ g_k(v_0(t)) & t \in (t_k, s_k],\ k = 1, 2, \ldots, p, \\ S(t - s_k) g_k(v_0(s_k)) & \\ \quad + \int_{s_k}^t S(t-s)[f(s, v_0(s)) + Mv_0(s)]\, ds & t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, p \end{cases} \tag{4.2}$$

$$= (\mathcal{F}v_0)(t), \quad t \in J.$$

(4.2) means that $v_0 \leq \mathcal{F}v_0$. Similarly, it can be shown that $\mathcal{F}w_0 \leq w_0$. Therefore, $\mathcal{F}\colon [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = \mathcal{F}v_{n-1}, \quad w_n = \mathcal{F}w_{n-1}, \quad n = 1, 2, \ldots. \tag{4.3}$$

Then from the monotonicity of $\mathcal{F}$ it follows that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0. \tag{4.4}$$

Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent on J. For convenience, let $B = \{v_n \mid n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$. Then $B = \mathcal{F}(B_0)$. For any $v_{n-1} \in B_0$, by the assumption (H$_f$1), we get that for every $t \in \bigcup_{k=0}^{p}[s_k, t_{k+1}]$,

$$f(t, v_0(t)) + Mv_0(t) \leq f(t, v_{n-1}(t)) + Mv_{n-1}(t) \leq f(t, w_0(t)) + Mw_0(t). \tag{4.5}$$

By the normality of the cone P and (4.5), we know that there exists a constant $M_1 > 0$ such that

$$\|f(t, v_{n-1}(t)) + Mv_{n-1}(t)\| \leq M_1, \quad t \in \bigcup_{k=0}^{p}[s_k, t_{k+1}],\ v_{n-1} \in B_0. \tag{4.6}$$

For $t \in (0, t_1]$ and $0 < \epsilon < t$, the operator

$$\begin{aligned} (\mathcal{F}_\epsilon v_{n-1})(t) &= S(t)u_0 + \int_0^{t-\epsilon} S(t-s)[f(s, v_{n-1}(s)) + Mv_{n-1}(s)]\, ds \\ &= S(\epsilon)\left\{ S(t-\epsilon)u_0 + \int_0^{t-\epsilon} S(t-s-\epsilon)[f(s, v_{n-1}(s)) + Mv_{n-1}(s)]\, ds \right\} \end{aligned} \tag{4.7}$$

is precompact in E since $S(t)$ is compact for $t > 0$. By (2.4), (4.1), (4.6) and (4.7), we get that

$$\begin{aligned} \|(\mathcal{F}v_{n-1})(t) - \mathcal{F}_\epsilon v_{n-1}(t)\| &= \int_{t-\epsilon}^{t} \|S(t-s)\| \cdot \|f(s, v_{n-1}(s)) + Mv_{n-1}(s)\|\, ds \\ &\leq CM_1\epsilon. \end{aligned} \tag{4.8}$$

(4.8) means that there exists precompact set $\{(\mathcal{F}_\epsilon v_{n-1})(t) \mid v_{n-1} \in B_0\}$ sufficiently close to the set $\{(\mathcal{F}v_{n-1})(t) \mid v_{n-1} \in B_0\}$ for every $t \in (0, t_1]$. Above discussion combined with the fact that $\{(\mathcal{F}v_{n-1})(0) = u_0 \mid v_{n-1} \in B_0\}$ is precompact in E, we know that for $t \in [0, t_1]$, $\{(\mathcal{F}v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in E.

For $t \in \bigcup_{k=1}^{p}(t_k, s_k]$, the set $\{(\mathcal{F}v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in E due to the compactness of g_k for $k = 1, 2, \ldots, p$.

For $t \in \bigcup_{k=1}^{p}(s_k, t_{k+1}]$ and $0 < \epsilon < t - s_k$, $k = 1, 2, \ldots, p$, the operator

$$\begin{aligned}
&(\mathcal{F}^{\epsilon} v_{n-1})(t) \\
&= S(t - s_k) g_k(v_{n-1}(s_k)) + \int_{s_k}^{t-\epsilon} S(t - s)[f(s, v_{n-1}(s)) + M v_{n-1}(s)]\, ds \\
&= S(t - s_k) g_k(v_{n-1}(s_k)) + S(\epsilon) \int_{s_k}^{t-\epsilon} S(t - s - \epsilon)[f(s, v_{n-1}(s)) + M v_{n-1}(s)]\, ds
\end{aligned}$$

is precompact in E since $S(t)$ is compact for $t > 0$ and g_k is compact for $k = 1, 2, \ldots, p$. Choose ϵ is small enough such that $t, t - \epsilon \in (s_k, t_{k+1}]$ for $k = 1, 2, \ldots, p$, then we know that

$$\begin{aligned}
\|(\mathcal{F} v_{n-1})(t) - \mathcal{F}^{\epsilon} v_{n-1}(t)\| &= \int_{t-\epsilon}^{t} \|S(t - s)\| \cdot \|f(s, v_{n-1}(s)) + M v_{n-1}(s)\|\, ds \\
&\le C M_1 \epsilon.
\end{aligned} \tag{4.9}$$

(4.9) means that there exists precompact set $\{(\mathcal{F}^{\epsilon} v_{n-1})(t) \mid v_{n-1} \in B_0\}$ sufficiently close to the set $\{(\mathcal{F} v_{n-1})(t) \mid v_{n-1} \in B_0\}$ for every $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, p$. Therefore, we know that for all $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, p$, the set $\{(\mathcal{F} v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in E. Hence $v_n(t)$ is precompact in E for any $t \in J$, and therefore $\{v_n(t)\}$ has a convergent subsequence. Combining this with the monotonicity (4.4), we can easily to prove that $\{v_n(t)\}$ itself is convergent, i.e., $\lim_{n\to\infty} v_n(t) = \underline{u}(t)$, $t \in J$. Similarly, we can prove that $\lim_{n\to\infty} w_n(t) = \overline{u}(t)$, $t \in J$.

Obviously, $\{v_n(t)\} \subset PC(J, E)$, and $\underline{u}(t)$ is bounded integrable when t belongs to $[0, t_1]$, $(t_k, s_k]$ and $(s_k, t_{k+1}]$ respectively for $k = 1, 2, \ldots, p$. For any $t \in J$, we know from (4.1) that

$$\begin{aligned}
v_n(t) &= (\mathcal{F} v_{n-1})(t) \\
&= \begin{cases}
S(t) u_0 + \int_0^t S(t - s)[f(s, v_{n-1}(s)) + M v_{n-1}(s)]\, ds & t \in [0, t_1], \\
g_k(v_{n-1}(t)) & t \in (t_k, s_k],\ k = 1, 2, \ldots, p, \\
S(t - s_k) g_k(v_{n-1}(s_k)) & \\
\quad + \int_{s_k}^t S(t - s)[f(s, v_{n-1}(s)) + M v_{n-1}(s)]\, ds & t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, p.
\end{cases}
\end{aligned}$$

Letting $n \to \infty$ in the above equality, by the Lebesgue dominated convergence theorem, we know that $\underline{u}(t) = (\mathcal{F}\underline{u})(t)$ and $\underline{u}(t) \in PC(J, E)$. Similarly, we know that $\overline{u}(t) = (\mathcal{F}\overline{u})(t)$ and $\overline{u}(t) \in PC(J, E)$. Combining this fact with monotonicity (4.4), we see that $v_0(t) \le \underline{u}(t) \le \overline{u}(t) \le w_0(t)$, $t \in J$.

Next, we show that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of the operator $\mathcal{F}$ in $[v_0, w_0]$, respectively. In fact, for any $u \in [v_0, w_0]$ and $\mathcal{F}u = u$, we have $v_0 \le u \le w_0$, and

$$v_1 = \mathcal{F} v_0 \le \mathcal{F} u = u \le \mathcal{F} w_0 = w_1.$$

Continuing such a progress, we know $v_n \le u \le w_n$. Letting $n \to \infty$, we get that $\underline{u} \le u \le \overline{u}$. Therefore, $\underline{u}$ and $\overline{u}$ are minimal and maximal mild solutions of IVP (1.1) in $[v_0, w_0]$, and $\underline{u}$ and $\overline{u}$ can be obtained by the iterative scheme (4.3) starting from v_0 and w_0, respectively. This completes the proof of Theorem 4.1. □

By the proof of Theorem 4.1, we can easily obtain the following result.

Corollary 4.2. *Let E be an ordered Banach space, whose positive cone P is normal, $A\colon D(A) \subset E \to E$ be a closed linear operator and the positive C_0-semigroup $T(t)$ $(t \ge 0)$ generated by $-A$ be compact in E, $f \in C([s_k, t_{k+1}] \times E, E)$ for $k = 0, 1, \dots, p$, $g_k \in C(E, E)$ map a monotonic set into a precompact set for $k = 1, 2, \dots, p$. Assume that IVP* (1.1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \le w_0$, and the assumptions* $(\mathrm{H}_f 1)$ *and* (H_g) *hold, then IVP* (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

We can also prove the following existence result.

Theorem 4.3. *Let E be an ordered Banach space, whose positive cone P is normal, $A\colon D(A) \subset E \to E$ be a closed linear operator and the positive C_0-semigroup $T(t)$ $(t \ge 0)$ generated by $-A$ be compact in E, $g_k \in C(E, E)$ be a compact operator for $k = 1, 2, \dots, p$, $f \in C([s_k, t_{k+1}] \times E, E)$ for $k = 0, 1, \dots, p$. Assume that IVP* (1.1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \le w_0$, and the assumptions* $(\mathrm{H}_f 1)$ *and* (H_g) *hold, then IVP* (1.1) *has at least one mild solution in ordered interval $[v_0, w_0]$.*

Proof. By the proof of Theorem 4.1, we know that $\mathcal{F}$ defined by (4.1) is a continuous mapping from $[v_0, w_0]$ to $[v_0, w_0]$. Therefore, to be able to apply Schauder's fixed point theorem to obtain a fixed point and hence a mild solution, we need to prove that $\mathcal{F}$ is a completely continuous operator.

For this purpose, let

$$\Pi_1 = \{(\mathcal{F}u)(\cdot) \mid \cdot \in [0, t_1], u \in [v_0, w_0]\}.$$

We first prove that Π_1 is precompact in $C([0, t_1], E)$. For $t \in [0, t_1]$, the set $\{S(t)u_0 \mid u_0 \in E\}$ is precompact in E since the semigroup $T(t)$ $(t \ge 0)$ is compact and therefore $S(t)$ $(t \ge 0)$ is also compact for every $t > 0$. For $t \in (0, t_1]$ and $0 < \epsilon < t$, the set

$$\begin{aligned} &\left\{ \int_0^{t-\epsilon} S(t-s)[f(s, u(s)) + Mu(s)]\, ds \;\middle|\; u \in [v_0, w_0] \right\} \\ &= \left\{ S(\epsilon) \int_0^{t-\epsilon} S(t-s-\epsilon)[f(s, u(s)) + Mu(s)]\, ds \;\middle|\; u \in [v_0, w_0] \right\} \end{aligned} \tag{4.10}$$

is precompact in E since $S(t)$ is compact for $t > 0$. Furthermore, by the continuity of the nonlinear function f, we know that for every $u \in [v_0, w_0]$,

$$\begin{aligned}(4.11) \qquad &\int_0^{t-\epsilon} S(t-s)[f(s,u(s)) + Mu(s)]\,ds \\ &\to \int_0^t S(t-s)[f(s,u(s)) + Mu(s)]\,ds \quad \text{as } \epsilon \to 0.\end{aligned}$$

By (4.10), (4.11) and total boundedness, we get that the set

$$\left\{ \int_0^t S(t-s)[f(s,u(s)) + Mu(s)]\,ds \;\middle|\; u \in [v_0, w_0] \right\}$$

is precompact in E. Therefore, for each $t \in [0, t_1]$, $\Pi_1(t)$ is precompact in E.

Next, we show the equicontinuity of Π_1. For any $u \in [v_0, w_0]$ and $t \in [0, t_1]$, by the assumption ($\mathrm{H}_f 1$), we know that

$$f(t, v_0(t)) + Mv_0(t) \le f(t, u(t)) + Mu(t) \le f(t, w_0(t)) + Mw_0(t).$$

By the normality of the cone P, there exists $M_2 > 0$ such that

$$\|f(t,u(t)) + Mu(t)\| \le M_2, \quad t \in [0, t_1],\ u \in [v_0, w_0].$$

For $0 \le t' < t'' \le t_1$ and any $u \in [v_0, w_0]$, we have

$$\begin{aligned}&\big\|(\mathcal{F}u)(t'') - (\mathcal{F}u)(t')\big\| \\ &= \bigg\| S(t'')u_0 - S(t')u_0 + \int_0^{t'} [S(t''-s) - S(t'-s)] \cdot [f(s,u(s)) + Mu(s)]\,ds \\ &\quad + \int_{t'}^{t''} S(t''-s)[f(s,u(s)) + Mu(s)]\,ds \bigg\| \\ &\le C\,\big\|S(t''-t')u_0 - u_0\big\| + M_2 \int_0^{t'} \big\|S(t''-s) - S(t'-s)\big\|\,ds + CM_2(t''-t') \\ &\le C\,\big\|S(t''-t')u_0 - u_0\big\| + M_2 \int_0^{t'} \big\|S(t''-t'+s) - S(s)\big\|\,ds + CM_2(t''-t').\end{aligned}$$

Since the semigroup $S(t)$ $(t \ge 0)$ is strongly continuous for $t \ge 0$ and is continuous in the uniform operator topology for $t > 0$, then it is easy to see that $\|(\mathcal{F}u)(t'') - (\mathcal{F}u)(t')\|$ tends to zero independently of $u \in [v_0, w_0]$ as $t'' - t' \to 0$, which means that the functions in Π_1 are equicontinuous. Therefore, by the Arzela-Ascoli theorem one can easily to justify that Π_1 is precompact in $C([0, t_1], E)$.

Secondly, we prove that

$$\Pi_2 = \{(\mathcal{F}u)(\cdot) \mid \cdot \in [s_1, t_2], u \in [v_0, w_0]\}$$

is precompact in $C([s_1,t_2],E)$. By the assumptions we know that the semigroup $S(t)$ $(t \ge 0)$ is compact for every $t > 0$ and g_k is compact for $k = 1,2,\ldots,p$. Therefore, for each $t \in [s_1,t_2]$, the set

$$\{S(t-s_1)g_1(u(s_1)) \mid u \in [v_0,w_0]\}$$

is precompact in E. Using a completely similar method to the one we used to prove the precompactness of the set

$$\left\{\int_0^t S(t-s)[f(s,u(s)) + Mu(s)]\,ds \,\middle|\, u \in [v_0,w_0]\right\}$$

in E for $t \in (0,t_1]$, we can prove for each $t \in (s_1,t_2]$, the set

$$\left\{\int_{s_1}^t S(t-s)[f(s,u(s)) + Mu(s)]\,ds \,\middle|\, u \in [v_0,w_0]\right\}$$

is precompact in E. Therefore, for each $t \in [s_1,t_2]$, $\Pi_2(t)$ is precompact in E.

In the following, we prove the equicontinuity of Π_2. For $s_1 \le t' < t'' \le t_2$ and any $u \in [v_0,w_0]$, we get that

$$\begin{aligned} &\left\|S(t''-s_1)g_1(u(s_1)) - S(t'-s_1)g_1(u(s_1))\right\| \\ &= \left\|S(t'-s_1)[S(t''-t') - S(0)]g_1(u(s_1))\right\| \\ &\le C\left\|[S(t''-t') - S(0)]g_1(u(s_1))\right\|. \end{aligned} \tag{4.12}$$

By the compactness of g_1, the strong continuity of the operator $S(t)$ for $t > 0$ and (4.12), we know that the functions in

$$\{S(\,\cdot\,-s_1)g_1(u(s_1)) \mid \cdot \in [s_1,t_2], u \in [v_0,w_0]\}$$

are equicontinuous. Using a completely similar method to the one we used to prove the equicontinuity of the functions in the set

$$\left\{\int_0^{\cdot} S(\,\cdot\,-s)[f(s,u(s)) + Mu(s)]\,ds \,\middle|\, \cdot \in [0,t_1], u \in [v_0,w_0]\right\}$$

one can easily to prove that the functions in

$$\left\{\int_{s_1}^{\cdot} S(\,\cdot\,-s)[f(s,u(s)) + Mu(s)]\,ds \,\middle|\, \cdot \in [s_1,t_2], u \in [v_0,w_0]\right\}$$

are equicontinuous. Thus we have proved that the functions in Π_2 are equicontinuous. Therefore, by the Arzela-Ascoli theorem one can easily to justify that Π_2 is precompact in $C([s_1,t_2],E)$. As the cases for interval $[s_k,t_{k+1}]$, $k = 2,3,\ldots,p$, the proofs are the same.

For interval $[t_k,s_k]$, $k = 1,2,\ldots,p$, by the compactness of g_k for $k = 1,2,\ldots,p$, we know that

$$\{g_k(u(\cdot)) \mid \cdot \in [t_k,s_k], u \in [v_0,w_0], k = 1,2,\ldots,p\}$$

is precompact in $C([t_k, s_k], E)$ for $k = 1, 2, \dots, p$. Hence, we have proved that $\mathcal{F}$ is a compact operator, and therefore a completely continuous operator. Hence, the famous Schauder's fixed point theorem implies that $\mathcal{F}$ has at least one fixed point, which gives rise to a mild solution of IVP (1.1). This completes the proof of Theorem 4.3. $\square$

Remark 4.4. In Theorem 4.1, we obtained the existence of two mild solutions (minimal mild solution and a maximal mild solution) by using the iterative method. In Theorem 4.3, we obtained the existence of at least one mild solution by utilizing Schauder's fixed point theorem. The results as well as the proof method in these two theorems are all different.

5. $T(\cdot)$ is not compact, g_k is compact

In this section, we discuss the existence of minimal and maximal mild solutions for IVP (1.1) under the situation that $-A$ only generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in E and the non-instantaneous impulsive function g_k is compact in E for $k = 1, 2, \dots, p$.

Theorem 5.1. *Let E be an ordered Banach space, whose positive cone P is normal, $A\colon D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in E, $g_k \in C(E, E)$ be a compact operator for $k = 1, 2, \dots, p$, $f \in C([s_k, t_{k+1}] \times E, E)$ for $k = 0, 1, \dots, p$. Assume that IVP* (1.1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \leq w_0$. If the assumptions* (H_f1), (H_g) *and the following assumption*

(H_f2) *there exists a constant $L > 0$ such that*

$$\alpha(\{f(t, u_n)\}) \leq L\alpha(\{u_n\}), \quad \forall\, t \in \bigcup_{k=0}^{p} (s_k, t_{k+1}],$$

where $\{u_n\} \subset [v_0(t), w_0(t)]$ is countable and increasing or decreasing monotonic set,

hold, then IVP (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. By the proof of Theorem 4.1, we know that $\mathcal{F}$ defined by (4.1) maps $[v_0, w_0]$ to $[v_0, w_0]$ is continuous and monotone increasing. Moreover, the sequences $\{v_n\}$ and $\{w_n\}$ defined by (4.3) satisfying (4.4). Next, we use a different method with which used in Theorem 4.1 to prove that $\{v_n\}$ and $\{w_n\}$ are convergent on J. For convenience, let $B = \{v_n \mid n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$. Then $B = \mathcal{F}(B_0)$. From $B_0 = B \cup \{v_0\}$ it follows that $\alpha(B_0(t)) = \alpha(B(t))$ for $t \in J$. Let $\varphi(t) = \alpha(B(t))$, here we prove interval by interval that $\varphi(t) \equiv 0$ on J.

For $t \in [0, t_1]$, by (4.1), (4.3) and Lemma 2.8, we get that

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(\mathcal{F}(B_0)(t)) \\
&= \alpha\left(\left\{S(t)u_0 + \int_0^t S(t-s)[f(s, v_{n-1}(s)) + Mv_{n-1}(s)]\, ds\right\}\right) \\
&\le 2C \int_0^t \alpha(\{f(s, v_{n-1}(s)) + Mv_{n-1}(s)\})\, ds \\
&\le 2C \int_0^t [L(\alpha(B_0(s)) + M\alpha(B_0(s))]\, ds \\
&\le 2C(L+M) \int_0^t \varphi(s)\, ds.
\end{aligned} \tag{5.1}$$

(5.1) and Bellman-Gronwall's inequality implies that $\varphi(t) \equiv 0$ on $[0, t_1]$. For $t \in (t_1, s_1]$, from the compactness of g_1 one can easily to get that $\varphi(t) = \alpha(\{g_1(v_{n-1}(t))\}) \equiv 0$ on $(t_1, s_1]$. For $t \in (s_1, t_2]$, by (4.1), (4.3), Lemma 2.8 and compactness of the function g_1, we know that

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(\mathcal{F}(B_0)(t)) \\
&= \alpha\left(\left\{S(t-s_1)g_1(v_{n-1}(s_1)) + \int_{s_1}^t S(t-s)[f(s, v_{n-1}(s)) + Mv_{n-1}(s)]\, ds\right\}\right) \\
&\le 2C \int_0^t \alpha(\{f(s, v_{n-1}(s)) + Mv_{n-1}(s)\})\, ds \\
&\le 2C \int_0^t [L(\alpha(B_0(s)) + M\alpha(B_0(s))]\, ds \\
&\le 2C(L+M) \int_0^t \varphi(s)\, ds.
\end{aligned} \tag{5.2}$$

(5.2) and Bellman-Gronwall's inequality implies that $\varphi(t) \equiv 0$ on $(s_1, t_2]$. Continuing such a process interval by interval for $k = 2, 3, \ldots, p$, we can prove that $\varphi(t) \equiv 0$ for every $t \in J$. Therefore, $v_n(t)$ is precompact in E for each $t \in J$, and therefore $\{v_n(t)\}$ has a convergent subsequence. Combining this with the monotonicity (4.4), we can easily to prove that $\{v_n(t)\}$ itself is convergent, i.e., $\lim_{n\to\infty} v_n(t) = \underline{u}(t)$, $t \in J$. Similarly, we can prove that $\lim_{n\to\infty} w_n(t) = \overline{u}(t)$, $t \in J$. Using a completely similar method to the one we used to prove Theorem 4.1 we can easily to prove that $\underline{u}$ and $\overline{u}$ are minimal and maximal mild solutions of IVP (1.1) in $[v_0, w_0]$, and $\underline{u}$ and $\overline{u}$ can be obtained by the iterative scheme (4.3) starting from v_0 and w_0, respectively. This completes the proof of Theorem 5.1. □

By the proof of Theorem 5.1, we can easily obtain the following result.

Corollary 5.2. *Let E be an ordered Banach space, whose positive cone P is normal, $A\colon D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \ge 0)$ in E, $f \in C([s_k, t_{k+1}] \times E, E)$ for $k = 0, 1, \ldots, p$, $g_k \in C(E, E)$ map a* *monotonic set into a precompact set for* $k = 1, 2, \ldots, p$. *Assume that IVP* (1.1) *has a lower solution* $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ *and an upper solution* $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ *with* $v_0 \le w_0$, *and the assumptions* ($\mathrm{H}_f 1$), ($\mathrm{H}_f 2$) *and* (H_g) *hold, then IVP* (1.1) *has a minimal mild solution* $\underline{u}$ *and a maximal mild solution* $\overline{u}$ *between* v_0 *and* w_0, *which can be obtained by a monotone iterative procedure starting from* v_0 *and* w_0, *respectively.*

6. $T(\cdot)$ and g_k are not compact

In this section, we first discuss the existence of extremal mild solutions for IVP (1.1) under the situation that the semigroup $T(t)$ $(t \ge 0)$ generated by $-A$ is a positive C_0-semigroup in Banach space E and the non-instantaneous impulsive function g_k is only continuous for $k = 1, 2, \ldots, p$.

Theorem 6.1. *Let E be an ordered and weakly sequentially complete Banach space, whose positive cone P is normal, $A\colon D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \ge 0)$ in E, $g_k \in C(E, E)$ for $k = 1, 2, \ldots, p$ and $f \in C([s_k, t_{k+1}] \times E, E)$ for $k = 0, 1, \ldots, p$. Assume that IVP* (1.1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \le w_0$, and the assumptions* ($\mathrm{H}_f 1$) *and* (H_g) *hold, then IVP* (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. From the proof of Theorem 4.1, we know that the operator $\mathcal{F}$ defined by (4.1) maps $[v_0, w_0]$ to $[v_0, w_0]$ is continuous and monotone increasing. Furthermore, if the conditions ($\mathrm{H}_f 1$) and (H_g) are satisfied, then the sequences $\{v_n\}$ and $\{w_n\}$ defined by (4.3) satisfying (4.4). Therefore, for any $t \in J$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are monotone and order-bounded sequences in E. Noticing that E is a weakly sequentially complete Banach space, by Theorem 2.2 in [20], we know that $\{v_n(t)\}$ and $\{w_n(t)\}$ are precompact in E. Combining this fact with the monotonicity (4.4), it follows that $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent in E. Similar with the proof of Theorem 4.1, we know that IVP (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between v_0 and w_0, which can be obtained by the iterative scheme (4.3) starting from v_0 and w_0, respectively. This completes the proof of Theorem 6.1. □

Remark 6.2. In the application of differential equations, such as the Hilbert space, reflexive space and L_1 space, these spaces are all weakly sequentially complete spaces. Therefore, it is interesting to discuss the existence of solutions for differential equations in weakly sequentially complete space.

Similarly, in a general ordered Banach space, whose positive cone is regular, we have the following result.

Theorem 6.3. *Let E be an ordered Banach space, whose positive cone P is regular, $A\colon D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in E, $g_k \in C(E,E)$ for $k = 1,2,\ldots,p$ and $f \in C([s_k,t_{k+1}] \times E, E)$ for $k = 0,1,\ldots,p$. Assume that IVP* (1.1) *has a lower solution $v_0 \in PC(J,E) \cap C^1(J'',E) \cap C(J',E_1)$ and an upper solution $w_0 \in PC(J,E) \cap C^1(J'',E) \cap C(J',E_1)$ with $v_0 \leq w_0$, and the assumptions* (H_f1) *and* (H_g) *hold, then IVP* (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Next, we discuss the existence of extremal mild solutions for IVP (1.1) under the situation that the positive C_0-semigroup $T(t)$ $(t \geq 0)$ generated by $-A$ is equicontinuous and the non-instantaneous impulsive function g_k is Lipschitz continuous in E for $k = 1,2,\ldots,p$.

Theorem 6.4. *Let E be an ordered Banach space, whose positive cone P is normal, $A\colon D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive and equicontinuous C_0-semigroup $T(t)$ $(t \geq 0)$ in E, $g_k \in C(E,E)$ for $k = 1,2,\ldots,p$ and $f \in C([s_k,t_{k+1}] \times E, E)$ for $k = 0,1,\ldots,p$. Assume that IVP* (1.1) *has a lower solution $v_0 \in PC(J,E) \cap C^1(J'',E) \cap C(J',E_1)$ and an upper solution $w_0 \in PC(J,E) \cap C^1(J'',E) \cap C(J',E_1)$ with $v_0 \leq w_0$. If the assumptions* (H_f1), (H_g) *and the following assumption*

(H_{fg}) *there exist positive constants $\overline{L}$ and L_k $(k = 1,2,\ldots,p)$ satisfying $4aC(\overline{L} + M) + C\sum_{k=1}^{p} L_k < 1$ such that*

$$\|g_k(u(t)) - g_k(v(t))\| \leq L_k \|u(t) - v(t)\|, \quad t \in \bigcup_{k=1}^{p} (t_k, s_k],\ u(t), v(t) \in E$$

and

$$\alpha(\{f(t,u_n)\}) \leq \overline{L}\alpha(\{u_n\}), \quad \forall\, t \in \bigcup_{k=0}^{p} (s_k, t_{k+1}],$$

where $\{u_n\} \subset [v_0(t), w_0(t)]$ is countable and increasing or decreasing monotonic set,

hold, then IVP (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ in $[v_0, w_0]$; moreover,*

$$v_n(t) \to \underline{u}(t), \quad w_n(t) \to \overline{u}(t), \quad (n \to +\infty) \quad \text{uniformly for } t \in J,$$

where $v_n(t) = (\mathcal{F}v_{n-1})(t)$ and $w_n(t) = (\mathcal{F}w_{n-1})(t)$, which satisfy

$$\begin{aligned} v_0(t) &\leq v_1(t) \leq \cdots \leq v_n(t) \leq \cdots \leq \underline{u}(t) \leq \overline{u}(t) \leq \cdots \\ &\leq w_n(t) \leq \cdots \leq w_1(t) \leq w_0(t), \qquad \forall\, t \in J. \end{aligned}$$

Proof. From the proof of Theorem 4.1, we know that $\mathcal{F}$ defined by (4.1) maps $[v_0, w_0]$ to $[v_0, w_0]$ is continuous and monotone increasing. Next, we prove that the operator $\mathcal{F}\colon [v_0, w_0] \to [v_0, w_0]$ is strict α-set-contraction. For this purpose, we denote by

$$(\mathcal{F}_1 u)(t) = \begin{cases} g_k(u(t)) & t \in (t_k, s_k],\ k = 1, 2, \ldots, p, \\ S(t - s_k) g_k(u(s_k)) & t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, p \end{cases} \tag{6.1}$$

and

$$(\mathcal{F}_2 u)(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s, u(s)) + Mu(s)]\, ds & t \in [0, t_1], \\ \int_{s_k}^t S(t-s)[f(s, u(s)) + Mu(s)]\, ds & t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, p. \end{cases}$$

Then it is easy to see that

$$(\mathcal{F}u)(t) = (\mathcal{F}_1 u)(t) + (\mathcal{F}_2 u)(t).$$

In what follows, we prove that the operator $\mathcal{F}_1\colon [v_0, w_0] \to [v_0, w_0]$ is Lipschitz continuous. For $t \in (t_k, s_k]$, $k = 1, 2, \ldots, p$, and $u, v \in [v_0, w_0]$, by (6.1) and the assumption (H_{fg}), we get that

$$\|(\mathcal{F}_1 u)(t) - (\mathcal{F}_1 v)(t)\| \le L_k \|u(t) - v(t)\| \le L_k \|u - v\|_{PC}. \tag{6.2}$$

For $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, p$ and $u, v \in [v_0, w_0]$, by (6.1), (2.4) and the assumption (H_{fg}), we know that

$$\|(\mathcal{F}_1 u)(t) - (\mathcal{F}_1 v)(t)\| \le CL_k \|u(s_k) - v(s_k)\| \le CL_k \|u - v\|_{PC}. \tag{6.3}$$

From (6.2), (6.3), (2.1) and (2.4), we get that

$$\|\mathcal{F}_1 u - \mathcal{F}_1 v\|_{PC} \le C \sum_{k=1}^{p} L_k \|u - v\|_{PC}. \tag{6.4}$$

Therefore, by Lemma 2.6(vii) and (6.4) we know that for any bounded set $D \subset [v_0, w_0]$,

$$\alpha(\mathcal{F}_1(D))_{PC} \le C \sum_{k=1}^{p} L_k \alpha(D)_{PC}. \tag{6.5}$$

Next, we make estimate for the measure of noncompactness to operator $\mathcal{F}_2$. By the proof of Theorem 4.3, one can easily to prove that the operator $\mathcal{F}_2\colon [v_0, w_0] \to [v_0, w_0]$ is equicontinuous. Therefore, for any bounded set $D \subset [v_0, w_0]$, $\mathcal{F}_2(D)$ is bounded and equicontinuous. Hence, by Lemma 2.7, there exists a countable set $D_0 = \{u_n\} \subset D$, such that

$$\alpha_{PC}(\mathcal{F}_2(D)) \le 2\alpha_{PC}(\mathcal{F}_2(D_0)). \tag{6.6}$$

For every $t \in [0, t_1]$, by assumption (H_{fg}) and Lemma 2.8, we get that

$$\begin{aligned}
\alpha(\mathcal{F}_2(D_0)(t)) &= \alpha\left(S(t)u_0 + \int_0^t S(t-s)[f(s,u_n(s)) + Mu_n(s)]\,ds\right) \\
&\le 2\int_0^t \|S(t-s)\|\,\alpha(f(s,D_0(s)) + MD_0(s))\,ds \\
&\le 2C\int_0^t (\overline{L}+M)\alpha(D_0(s))\,ds \\
&\le 2aC(\overline{L}+M)\alpha_{PC}(D).
\end{aligned} \tag{6.7}$$

For every $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, p$, by the assumption (H_{fg}) and Lemma 2.8, we know that

$$\begin{aligned}
\alpha(\mathcal{F}_2(D_0)(t)) &= \alpha\left(\int_{s_k}^t S(t-s)[f(s,u_n(s)) + Mu_n(s)]\,ds\right) \\
&\le 2C\int_{s_k}^t \alpha(f(s,D_0(s)) + MD_0(s))\,ds \\
&\le 2C\int_{s_k}^t (\overline{L}+M)\alpha(D_0(s))\,ds \\
&\le 2aC(\overline{L}+M)\alpha_{PC}(D).
\end{aligned} \tag{6.8}$$

Therefore, from (6.7) and (6.8) we know that for every $t \in \bigcup_{k=0}^p (s_k, t_{k+1}]$,

$$\alpha(\mathcal{F}_2(D_0)(t)) \le 2aC(\overline{L}+M)\alpha_{PC}(D). \tag{6.9}$$

We modify the value of u_n at t_k via $u_n(t_k) = u_n(t_k^+)$ for $k = 1, 2, \ldots, p$ and $n \in \mathbb{N}$, thus the set $\{u_n\}$ is continuous on interval $[s_k, t_{k+1}]$ for $k = 0, 1, \ldots, p$ and $n \in \mathbb{N}$. Since $\mathcal{F}_2(D_0)$ is equicontinuous, by Lemma 2.9, we know that

$$\alpha_{PC}(\mathcal{F}_2(D_0)) = \max_{\substack{t\in[s_k,t_{k+1}] \\ k=0,1,\ldots,p}} \alpha(\mathcal{F}_2(D_0)(t)). \tag{6.10}$$

Combining (6.10) with (6.9) and (6.6) we get that

$$\alpha_{PC}(\mathcal{F}_2(D)) \le 4aC(\overline{L}+M)\alpha(D)_{PC}. \tag{6.11}$$

Therefore, from (6.5), (6.11), Lemma 2.6(vi) and the assumption (H_{fg}) we get that

$$\alpha(\mathcal{F}(D))_{PC} \le \alpha(\mathcal{F}_1(D))_{PC} + \alpha(\mathcal{F}_2(D))_{PC} \le \gamma\alpha(D)_{PC},$$

where

$$\gamma = 4aC(\overline{L}+M) + C\sum_{k=1}^p L_k < 1.$$

Hence, by Definition 2.10 we know that $\mathcal{F}\colon [v_0, w_0] \to [v_0, w_0]$ is a strict α-set-contraction operator. Therefore, our conclusion follows from Lemma 2.11. This completes the proof of Theorem 6.4. $\square$

Remark 6.5. Analytic semigroup and differentiable semigroup are equicontinuous semigroup [38, 42]. In the application of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroups are analytic semigroups. Therefore, Theorem 6.4 in this paper are convenient to applications.

7. Applications

In this section, we will give two examples to indicate how our abstract results can be applied to concrete problems.

Example 7.1. Let $N \ge 1$ be a integer, $\Omega \subset \mathbb{R}^N$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Consider the following initial boundary value problem of parabolic partial differential equation with non-instantaneous impulses

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) - \Delta u(x,t) = f(x,t,u(x,t)) & x \in \Omega,\ t \in (0,1] \cup (2,3], \\ u(x,t) = \int_1^t K(s) \ln(1 + |u(x,s)|)\, ds & x \in \Omega,\ t \in (1,2], \\ u(x,t) = 0 & x \in \partial\Omega,\ t \in [0,3], \\ u(x,0) = \varphi(x) & x \in \Omega, \end{cases} \tag{7.1}$$

where Δ is the Laplace operator, $f\colon \Omega \times (0,1] \cup (2,3] \times \mathbb{R} \to \mathbb{R}$ is continuous, $K(\cdot) \in L((1,2], \mathbb{R}^+)$, $\varphi \in C(\overline{\Omega}, \mathbb{R})$.

Let $E = C(\overline{\Omega}, \mathbb{R})$, $P = \big\{u \in C(\overline{\Omega}, \mathbb{R}) \mid u(x) \ge 0, \forall\, x \in \Omega\big\}$. Then E is a Banach space and P is a normal cone of E. Consider the operator $A\colon D(A) \subset E \to E$ defined by

$$Au = -\Delta u$$

with the domain

$$D(A) = \left\{ u \in \bigcap_{q \ge 1} W^{2,q}(\Omega) \,\middle|\, u, \Delta u \in E, \frac{\partial u}{\partial n} = 0 \right\},$$

where n is the outer unit normal on $\partial\Omega$. By [37, Corollary 3.1.24] we know that $-A$ generates a positive and compact C_0-semigroup $T(t)$ $(t \ge 0)$ in E. Let $a = t_2 = 3$, $s_0 = 0$, $t_1 = 1$, $s_1 = 2$, $u(t) = u(\cdot\,, t)$, $f(t, u(t)) = f(\cdot\,, t, u(\cdot\,, t))$, $g_1(u(t)) = \int_1^t K(s) \ln(1 + |u(\cdot\,, s)|)\, ds$, $u_0 = \varphi(\cdot)$, then the parabolic partial differential equation with non-instantaneous impulses (7.1) can be rewritten into the abstract form of IVP (1.1) in $C(\overline{\Omega}, \mathbb{R})$

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)) & t \in (0,1] \cup (2,3], \\ u(t) = g_1(u(t)) & t \in (1,2], \\ u(0) = u_0. \end{cases}$$

From [35] we know that $g_1\colon E \to E$ is a compact operator.

Theorem 7.2. *Let λ_1 be the first eigenvalue of operator $-\Delta$ under zero boundary conditions and $\varphi_1(x)$ be the corresponding positive eigenvector. If $f(x,t,0) \geq 0$, $f(x,t,\varphi_1(x)) \leq \lambda_1\varphi_1(x)$ and the partial derivative of $f(x,t,u)$ on u is continuous on any bounded domain, then the parabolic partial differential equation with non-instantaneous impulses* (7.1) *has minimal and maximal mild solutions between* 0 *and* φ_1, *which can be obtained by a monotone iterative procedure starting from* 0 *and* φ_1, *respectively.*

Proof. It is easy to see that $v_0(t) \equiv 0$ and $w_0(t) = \varphi_1(x)$ are lower and upper solutions of the parabolic partial differential equation with non-instantaneous impulses (7.1), respectively. From the above assumptions on nonlinear term f and the definitions of non-instantaneous impulsive function g_1, we can easily verify that the assumptions ($\mathrm{H}_f 1$) and (H_g) are satisfied. Therefore, our conclusion follows from Theorem 4.1. This completes the proof of Theorem 7.2. □

Remark 7.3. Theorem 7.2 give a new method, which called eigenvalue method, to seeking the lower and upper solutions for the concrete parabolic partial differential equations.

Example 7.4. Consider the following parabolic partial differential equation with non-instantaneous impulses of the form

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + \mathcal{A}u(x,t) = f(x,t,u(x,t)) & x \in \Omega,\ t \in \bigcup_{k=0}^{p}(s_k, t_{k+1}], \\ u(x,t) = g_k(u(x,t)) & x \in \Omega,\ t \in \bigcup_{k=1}^{p}(t_k, s_k], \\ \mathcal{B}u(x,t) = 0 & x \in \partial\Omega,\ t \in J, \\ u(x,0) = \phi(x) & x \in \Omega, \end{cases} \tag{7.2}$$

where $J = [0,a]$, $a > 0$ is a constant, $p \in \mathbb{Z}^+$, $0 < t_1 < t_2 < \cdots < t_p < t_{p+1} := a$, $s_0 := 0$ and $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \ldots, p$, $J' = J \setminus \{t_1, t_2, \ldots, t_p\}$, integer $N \geq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, whose boundary $\partial\Omega$ is an $(N-1)$-dimensional $C^{2+\mu}$-manifold for some $0 < \mu < 1$,

$$\mathcal{A}u := -\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u$$

is a uniformly elliptic differential operator on $\overline{\Omega}$ with the coefficients $a_{ij} \in C^{1+\mu}(\overline{\Omega})$ $(i,j = 1,2,\ldots,N)$ and $a_0 \in C^{\mu}(\overline{\Omega})$ for some $\mu \in (0,1)$, $a_0(x) \geq 0$ on $\overline{\Omega}$. That is, $[a_{ij}(x)]_{N\times N}$ is a positive definite symmetric matrix for every $x \in \overline{\Omega}$ and there exists a constant $\mu_0 > 0$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\eta_i\eta_j \geq \mu_0 |\eta|^2, \quad \forall\, \eta = (\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^N,\ x \in \overline{\Omega};$$

$$\mathcal{B}u := \delta\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\cos(\nu, x_i)\frac{\partial u}{\partial x_j} + (1-\delta)u$$

is a boundary operator on $\partial\Omega$, where ν is an outer unit normal on $\partial\Omega$, $\delta = 0$ or 1; $\phi \in L^\kappa(\Omega)$ with $\kappa > N+2$.

Let $E = L^\kappa(\Omega)$ with $\kappa > N+2$, $\overline{P} = \{u \in L^\kappa(\Omega) \mid u(x) \geq 0 \text{ a.e. } x \in \Omega\}$. Then E is a Banach space equipped with the L^κ-norm $\|\cdot\|_\kappa$ and $\overline{P}$ is a regular cone of E. Consider the operator $A\colon D(A) \subset E \to E$ defined by

$$D(A) = \left\{u \in W^{2,\kappa}(\Omega) \mid \mathcal{B}u = 0\right\}, \quad Au = \mathcal{A}u.$$

It is well known from [5] that $-A$ generates a positive and analytic C_0-semigroup $T(t)$ $(t \geq 0)$ in E.

Set $u(t) = u(\cdot\,,t)$, $f(t,u(t)) = f(\cdot\,,t,u(\cdot\,,t))$, $g_k(u(t)) = g_k(u(\cdot\,,t))$ for $k = 1,2,\ldots,p$, $u_0 = \phi(\cdot)$. Then the parabolic partial differential equation with non-instantaneous impulses (7.2) can be rewritten into the abstract form of IVP 1.1 in $L^\kappa(\Omega)$ as follows:

$$\begin{cases} u'(t) + Au(t) = f(t,u(t)) & t \in \bigcup_{k=0}^{p}(s_k,t_{k+1}], \\ u(t) = g_k(u(t)) & t \in \bigcup_{k=1}^{p}(t_k,s_k], \\ u(0) = u_0. \end{cases} \tag{7.3}$$

In order to obtain the existence of solutions for the parabolic partial differential equation with non-instantaneous impulses (7.2), we should suppose that f and g_k $(k = 1,2,\ldots,p)$ satisfy the following assumptions:

(A1) There exist $M > 0$, $h \in PC(\Omega \times J) \cap C^{0,1}(\overline{\Omega} \times J')$, $h(x,t) \geq 0$, $\phi \in W^{2,\kappa}(\Omega)$, $\mathcal{B}\phi = 0$, $\phi(x) \geq 0$, $e_k(t) \in W^{2,\kappa}(\Omega)$, $\mathcal{B}e_k(t) = 0$ and $e_k(x,t) \geq 0$ for every $t \in \bigcup_{k=1}^{p}(t_k,s_k]$, such that for any $u \in PC(\Omega \times J)$ satisfying $u(x,t) \geq 0$,

$$\begin{aligned} &f(x,t,-u) \geq -Mu - h(x,t), && f(x,t,u) \leq Mu + h(x,t), && x \in \Omega,\ t \in \bigcup_{k=0}^{p}(s_k,t_{k+1}], \\ &-e_k(x,t) \leq g_k(-u(x,t)), && g_k(u(x,t)) \leq e_k(x,t), && x \in \Omega,\ t \in \bigcup_{k=1}^{p}(t_k,s_k]; \end{aligned}$$

(A2) The partial derivative $f'_u(x,t,u)$ is continuous on any bounded domain;

(A3) For $u_1(x,t)$ and $u_2(x,t)$ in any bounded and ordered interval of $PC(\Omega \times J)$ with $u_1(x,t) \leq u_2(x,t)$, such that for any $x \in \Omega$ and $t \in \bigcup_{k=1}^{p}(t_k,s_k]$,

$$g_k(u_1(x,t)) \leq g_k(u_2(x,t)), \quad k = 1,2,\ldots,p.$$

Theorem 7.5. *If the assumptions* (A1)–(A3) *are satisfied, then the parabolic partial differential equation with non-instantaneous impulses* (7.2) *has minimal and maximal mild solutions, which can be obtained by a monotone iterative procedure.*

Proof. We first consider the following linear parabolic partial differential equation with non-instantaneous impulses

$$
\begin{cases} \frac{\partial}{\partial t}u(x,t) + \mathcal{A}u(x,t) - Mu(x,t) = h(x,t) & x \in \Omega,\ t \in \bigcup_{k=0}^{p}(s_k, t_{k+1}], \\ u(x,t) = e_k(x,t) & x \in \Omega,\ t \in \bigcup_{k=1}^{p}(t_k, s_k], \\ \mathcal{B}u(x,t) = 0 & x \in \partial\Omega,\ t \in J, \\ u(x,0) = \phi(x) & x \in \Omega, \end{cases} \tag{7.4}
$$

where $M > 0$ is a constant will be given later. From the above discussion, linear parabolic partial differential equation with non-instantaneous impulses (7.4) can be transformed into the following linear initial value problem (LIVP) of evolution equation with non-instantaneous impulses of the form

$$
\begin{cases} u'(t) + Au(t) - Mu(t) = h(t) & t \in \bigcup_{k=0}^{p}(s_k, t_{k+1}], \\ u(t) = e_k(t) & t \in \bigcup_{k=1}^{p}(t_k, s_k], \\ u(0) = u_0 \end{cases} \tag{7.5}
$$

in the space $L^{\kappa}(\Omega)$, where $h(t) = h(\cdot\,, t)$, $e_k(t) = e_k(\cdot\,, t)$ for $k = 1, 2, \ldots, p$. Since $-(A - MI)$ generates a positive C_0-semigroup $S(t) = e^{Mt}T(t)$ $(t \geq 0)$ on E, by Theorem 3.2 we know that LIVP (7.5) has a unique positive classical solution $\overline{u} \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$. Denote by $v_0 = -\overline{u}$ and $w_0 = \overline{u}$, then from the assumption (A1) we know that

$$
\begin{cases} v_0'(t) + Av_0(t) = Mv_0(t) - h(t) \leq f(t, v_0(t)) & t \in \bigcup_{k=0}^{p}(s_k, t_{k+1}], \\ v_0(t) = -e_k(t) \leq g_k(v_0(t)) & t \in \bigcup_{k=1}^{p}(t_k, s_k], \\ v_0(0) = -u_0 \leq u_0 \end{cases}
$$

and

$$
\begin{cases} w_0'(t) + Aw_0(t) = Mw_0(t) + h(t) \geq f(t, w_0(t)) & t \in \bigcup_{k=0}^{p}(s_k, t_{k+1}], \\ w_0(t) = e_k(t) \geq g_k(w_0(t)) & t \in \bigcup_{k=1}^{p}(t_k, s_k], \\ w_0(0) = u_0 \geq u_0, \end{cases}
$$

which means that v_0 and w_0 are lower solution and upper solution of IVP (7.3) respectively, and $v_0 \leq w_0$. From the assumption (A2), we know that there exists a constant $M > 0$, such that for any $u(x,t) \in [-\overline{u}(x,t), \overline{u}(x,t)]$,

$$
\left|f_u'(x, t, u(x,t))\right| \leq M, \quad x \in \Omega,\ t \in \bigcup_{k=0}^{p}(s_k, t_{k+1}]. \tag{7.6}
$$

From (7.6) we get that for any $-\overline{u}(x,t) \leq u_1(x,t) \leq u_2(x,t) \leq \overline{u}(x,t)$ there exists $\xi(x,t) \in (u_1(x,t), u_2(x,t))$ such that

$$
\begin{aligned} |f(x,t,u_2(x,t)) - f(x,t,u_1(x,t))| &= \left|f_u'(x,t,\xi(x,t))(u_2(x,t) - u_1(x,t))\right| \\ &\leq M(u_2(x,t) - u_1(x,t)). \end{aligned} \tag{7.7}
$$

(7.7) implies that the assumption (H_f1) is satisfied. From the assumption (A3) it is easy to verify that the assumption (H_g) is satisfied. Therefore, our conclusion follows from Theorem 6.3. This completes the proof of Theorem 7.5. □

Remark 7.6. In the applications, we only need to verify the nonlinear term f and the non-instantaneous impulsive function g_k ($k = 1, 2, \ldots, p$) satisfy some monotonicity condition, which are more weak and easy to be verified than the growth condition which assumed in [18, 30, 40, 46].

Remark 7.7. Theorem 7.5 give another method to seeking the lower and upper solutions for the concrete parabolic partial differential equations.

References

[1] N. Abada, M. Benchohra and H. Hammouche, *Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions*, J. Differential Equations **246** (2009), no. 10, 3834–3863.
https://doi.org/10.1016/j.jde.2009.03.004

[2] S. Abbas and M. Benchohra, *Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses*, Appl. Math. Comput. **257** (2015), 190–198. https://doi.org/10.1016/j.amc.2014.06.073

[3] N. U. Ahmed, *Existence of optimal controls for a general class of impulsive systems on Banach spaces*, SIAM J. Control Optim. **42** (2003), no. 2, 669–685.
https://doi.org/10.1137/s0363012901391299

[4] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), no. 4, 620–709. https://doi.org/10.1137/1018114

[5] ———, *Parabolic evolution equations and nonlinear boundary conditions*, J. Differential Equations **72** (1988), no. 2, 201–269.
https://doi.org/10.1016/0022-0396(88)90156-8

[6] J. M. Ayerbe Toledano, T. Domínguez Benavides and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory: Advances and Applications **99**, Birkhäuser Verlag, Basel, 1997.
https://doi.org/10.1007/978-3-0348-8920-9

[7] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics **60**, Marcel Pekker, New York, 1980.

[8] J. Banasiak and L. Arlotti, *Perturbations of Positive Semigroups with Applications*, Springer Monographs in Mathematics, Springer-Verlag, London, 2006. https://doi.org/10.1007/1-84628-153-9

[9] L. Barreira and C. Valls, *Lyapunov regularity of impulsive differential equations*, J. Differential Equations **249** (2010), no. 7, 1596–1619. https://doi.org/10.1016/j.jde.2010.07.016

[10] M. Benchohra, J. Henderson and S. Ntouyas, *Impulsive Differential Equations and Inclusions*, Contemporary Mathematics and Its Applications **2**, Hindawi Publ. Corp., New York, 2006. https://doi.org/10.1155/9789775945501

[11] E. M. Bonotto, M. C. Bortolan, A. N. Carvalho and R. Czaja, *Global attractors for impulsive dynamical systems–a precompact approach*, J. Differential Equations **259** (2015), no. 7, 2602–2625. https://doi.org/10.1016/j.jde.2015.03.033

[12] Y.-K. Chang and W.-T. Li, *Existence results for second order impulsive functional differential inclusions*, J. Math. Anal. Appl. **301** (2005), no. 2, 477–490. https://doi.org/10.1016/j.jmaa.2004.07.041

[13] P. Chen and Y. Li, *Mixed monotone iterative technique for a class of semilinear impulsive evolution equations in Banach spaces*, Nonlinear Anal. **74** (2011), no. 11, 3578–3588. https://doi.org/10.1016/j.na.2011.02.041

[14] ______, *Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions*, Results Math. **63** (2013), no. 3-4, 731–744. https://doi.org/10.1007/s00025-012-0230-5

[15] ______, *Monotone iterative method for abstract impulsive integro-differential equations with nonlocal conditions in Banach spaces*, Appl. Math. **59** (2014), no. 1, 99–120. https://doi.org/10.1007/s10492-014-0044-8

[16] P. Chen, Y. Li and H. Yang, *Perturbation method for nonlocal impulsive evolution equations*, Nonlinear Anal. Hybrid Syst. **8** (2013), 22–30. https://doi.org/10.1016/j.nahs.2012.08.002

[17] P. Chen, Y. Li and X. Zhang, *Double perturbations for impulsive differential equations in Banach spaces*, Taiwanese J. Math. **20** (2016), no. 5, 1065–1077. https://doi.org/10.11650/tjm.20.2016.5762

[18] V. Colao, L. Muglia and H.-K. Xu, *Existence of solutions for a second-order differential equation with non-instantaneous impulses and delay*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 3, 697–716. https://doi.org/10.1007/s10231-015-0484-0

[19] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985. https://doi.org/10.1007/978-3-662-00547-7

[20] Y. Du, *Fixed points of increasing operators in ordered Banach spaces and applications*, Appl. Anal. **38** (1990), no. 1-2, 1–20. https://doi.org/10.1080/00036819008839957

[21] S. W. Du and V. Lakshmikantham, *Monotone iterative technique for differential equtions in a Banach space*, J. Math. Anal. Appl. **87** (1982), no. 2, 454–459. https://doi.org/10.1016/0022-247x(82)90134-2

[22] M. A. EI-Gebeily, D. O'Regan and J. J. Nieto, *A monotone iterative technique for stationary and time dependent problems in Banach spaces*, J. Comput. Appl. Math. **233** (2010), no. 9, 2395–2404. https://doi.org/10.1016/j.cam.2009.10.024

[23] Z. Fan and G. Li, *Existence results for semilinear differential equations with nonlocal and impulsive conditions*, J. Funct. Anal. **258** (2010), no. 5, 1709–1727. https://doi.org/10.1016/j.jfa.2009.10.023

[24] G. R. Gautam and J. Dabas, *Mild solution for fractional functional integro-differential equation with not instantaneous impulse*, Malaya J. Mat. **2** (2014), no. 3, 428–437.

[25] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Notes and Reports in Mathematics in Science and Engineering **5**, Academic Press, Boston, MA, 1988. https://doi.org/10.1016/c2013-0-10750-7

[26] D. Guo and X. Liu, *Extremal solutions of nonlinear impulsive integrodifferential equations in Banach spaces*, J. Math. Anal. Appl. **177** (1993), no. 2, 538–552. https://doi.org/10.1006/jmaa.1993.1276

[27] D. Guo and J. Sun, *Ordinary Differential Equations in Abstract Spaces*, Shandong Science and Technology, Jinan, 1989.

[28] H.-P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. **7** (1983), no. 12, 1351–1371. https://doi.org/10.1016/0362-546x(83)90006-8

[29] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, New York, 1981. https://doi.org/10.1007/bfb0089647

[30] E. Hernández and D. O'Regan, *On a new class of abstract impulsive differential equations*, Proc. Amer. Math. Soc. **141** (2013), no. 5, 1641–1649. https://doi.org/10.1090/s0002-9939-2012-11613-2

[31] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics **6**, World Scientific, Teaneck, NJ, 1989. https://doi.org/10.1142/0906

[32] Y. Li, *Positive solutions of abstract semilinear evolution equations and their applications*, Acta Math. Sinica (Chin. Ser.) **39** (1996), no. 5, 666–672.

[33] ______, *Existence of solutions to initial value problems for abstract semilinear evolution equations*, Acta Math. Sinica (Chin. Ser.) **48** (2005), no. 6, 1089–1094.

[34] Y. Li and Z. Liu, *Monotone iterative technique for addressing impulsive integro-differential equations in Banach spaces*, Nonlinear Anal. **66** (2007), no. 1, 83–92. https://doi.org/10.1016/j.na.2005.11.013

[35] J. Liang, J. H. Liu and T.-J. Xiao, *Nonlocal impulsive problems for integrodifferential equations*, Dyn. COntin. Discrete Impuls. Syst. Ser. A Math. Anal. **16** (2009), Differential Equations and Dynamical Systems, suppl. S1, 157–161.

[36] ______, *Periodic solutions of delay impulsive differential equations*, Nonlinear Anal. **74** (2011), no. 17, 6835–6842. https://doi.org/10.1016/j.na.2011.07.008

[37] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and their Applications **16**, Birkhäuser, Verlag, Basel, 1995. https://doi.org/10.1007/978-3-0348-9234-6

[38] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences **44**, Springer-Verlag, New York, 1983. https://doi.org/10.1007/978-1-4612-5561-1

[39] W. V. Petryshyn, *Structure of the fixed points sets of k-set-contractions*, Arch. Rational. Mech. Anal. **40** (1971), 312–328. https://doi.org/10.1007/bf00252680

[40] M. Pierri, D. O'Regan and V. Rolnik, *Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses*, Appl. Math. Comput. **219** (2013), no. 12, 6743–6749. https://doi.org/10.1016/j.amc.2012.12.084

[41] D. Qian, L. Chen and X. Sun, *Periodic solutions of superlinear impulsive differential equations: A geometric approach*, J. Differential Equations **258** (2015), no. 9, 3088–3106. https://doi.org/10.1016/j.jde.2015.01.003

[42] I. I. Vrabie, *C_0-semigroups and Applications*, North-Holland Mathematics Studies **191**, North-Holland, Amsterdam, 2003. https://doi.org/10.1016/s0304-0208(03)x8032-9

[43] J. Wang and X. Li, *Periodic BVP for integer/fractional order nonlinear differential equations with non-instantaneous impulses*, J. Appl. Math. Comput. **46** (2014), no. 1-2, 321–334. https://doi.org/10.1007/s12190-013-0751-4

[44] L. Wang and Z. Wang, *Monotone iterative techniques for parameterized BVPs of abstract semilinear evolution equations*, Comput. Math. Appl. **46** (2003), no. 8-9, 1229–1243. https://doi.org/10.1016/s0898-1221(03)90214-8

[45] J. Wang, Y. Zhou and Z. Lin, *On a new class of impulsive fractional differential equations*, Appl. Math. Comput. **242** (2014), 649–657. https://doi.org/10.1016/j.amc.2014.06.002

[46] X. Yu and J. Wang, *Periodic boundary value problems for nonlinear impulsive evolution equations on Banach spaces*, Commun. Nonlinear Sci. Numer. Simul. **22** (2015), no. 1-3, 980–989. https://doi.org/10.1016/j.cnsns.2014.10.010

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