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Stabilization for the Wave Equation with Variable Coefficients and Balakrishnan-Taylor Damping

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Abstract. In this paper, we consider the wave equation with variable coefficients and Balakrishnan-Taylor damping and source terms. This work is devoted to prove, under suitable conditions on the initial data, the uniform decay rates of the energy without imposing any restrictive growth near zero assumption on the damping term.

1. Introduction

In this paper, we are concerned with the uniform energy decay rates of solutions for the wave equation:

$$\begin{cases} u'' - M(t)Lu + g(u') = |u|^\rho u & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma \times (0,\infty), \\ u(x,0) = u_0, \quad u'(x,0) = u_1, \end{cases} \tag{1.1}$$

where $Lu = \operatorname{div}(A\nabla u) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$ and $M(t) = \xi_1 + \xi_2 \int_\Omega A\nabla u \nabla u\,dx + \xi_3 \int_\Omega A\nabla u \nabla u'\,dx$. Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with boundary Γ. $'$ denotes the derivative with respect to time t.

When $A = I$ with Balakrishnan-Taylor damping ($\xi_3 \neq 0$), the model was initially proposed by Balakrishnan and Taylor in [1] and Bass and Zes [2]. The original motivation for studying this model seemed to solve the spillover problem, namely, to design a feedback control function that involves only finite many modes in order to achieve a high performance of the closed-loop systems, such as a robust and exponential stabilization of the system when there might be some uncertainly in values of the parameters. So far, there are some stability results for the problem having Balakrishnan-Taylor damping (see [11,13,16,17]). For instance, Tatar and Zaraï [13] proved an exponential decay result of the energy provided that the kernel decays exponentially. Recently, Ha [5] studied the

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uniform decay rates of the energy without imposing any restrictive growth assumption on the damping term and weakening the usual assumptions on the relaxation function.

When A is a general matrix without Balakrishnan-Taylor damping ($\xi_2 = \xi_3 = 0$), such a problem is called a wave equation with variable coefficients in principle. This equations arise in mathematical modeling of inhomogeneous media in solid mechanics, electromagnetic, fluid flows through porous media, and other areas of physics and engineering. For the variable coefficients problem, the main tool is Riemannian geometry method, which was introduced by Yao [15] and has been widely used in the literature (see [4,6–9,14] and a list of references therein). However, there were very few results considering the source term. For example, Boukhatem and Benabderrahmane [3] studied the uniform decay rate of the energy to the viscoelastic wave equation with variable coefficients and acoustic boundary conditions without damping term. But there is none, to our knowledge, for the variable coefficients problem having both damping and source terms.

Motivated by previous works, the goal of this paper is to study the uniform decay rate of the energy to the wave equation with variable coefficients and Balakrishnan-Taylor damping and source terms. This paper is organized as follows: In Section 2, we recall the hypotheses to prove our main result and introduce the existence and energy decay rate theorem. In Section 3, we prove under suitable conditions on the initial data, the uniform decay rates of the energy without imposing any restrictive growth near zero assumption on the damping term.

2. Preliminaries

We begin this section with introducing some notations and our main results. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 1$, with smooth boundary Γ. Throughout this paper we define the Hilbert space $\mathcal{H} = \left\{u \in H^1(\Omega) \mid Lu \in L^2(\Omega)\right\}$ with the norm $\|u\|_{\mathcal{H}} = \left(\|u\|^2_{H^1(\Omega)} + \|Lu\|_2^2\right)^{1/2}$ and $H_0^1(\Omega) = \left\{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma\right\}$. Moreover, $L^p(\Omega)$-norm is denoted by $\|\cdot\|_p$ and $(u,v) = \int_\Omega u(x)v(x)\,dx$.

(H_1) Hypotheses on ξ_1, ξ_2, ξ_3, ρ. Let $\xi_i > 0$, $i = 1,2,3$, and let ρ be a constant satisfying the following condition:

$$0 < \rho < \frac{2}{n-2} \text{ if } n \geq 3 \quad \text{and} \quad \rho > 0 \text{ if } n = 1, 2.$$

(H_2) Hypotheses on A. The matrix $A = (a_{ij}(x))$, where $a_{ij} \in C^1(\overline{\Omega})$, is symmetric and there exists a positive constant a_0 such that for all $x \in \overline{\Omega}$ and $\omega = (\omega_1, \ldots, \omega_n)$ we have

$$\sum_{i,j=1}^{n} a_{ij}(x)\omega_j\omega_i \geq a_0 \left|\omega\right|^2 . \tag{2.1}$$

(H$_3$) Hypotheses on g. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that $g(0) = 0$ and suppose that there exists a strictly increasing and odd function β of C^1 class on $[-1, 1]$ such that

$$\begin{aligned} |\beta(s)| \le |g(s)| \le \left|\beta^{-1}(s)\right| \qquad & \text{if } |s| \le 1, \\ c_1 |s| \le |g(s)| \le c_2 |s| \qquad & \text{if } |s| > 1, \end{aligned}$$

where β^{-1} denotes the inverse function of β and c_1, c_2 are positive constants.

By using the hypothesis (H$_2$), we verify that the bilinear form $a(\cdot\,,\cdot)\colon H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a(u(t), v(t)) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u(t)}{\partial x_j} \frac{\partial v(t)}{\partial x_i}\, dx = \int_\Omega A\nabla u(t) \nabla v(t)\, dx$$

is symmetric and continuous. On the other hand, from (2.1) for $\omega = \nabla u$, we get

$$a(u(t), u(t)) \ge a_0 \int_\Omega \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 dx = a_0 \left\| \nabla u(t) \right\|_2^2 . \tag{2.2}$$

Now, we state the local existence theorem which can be complete arguing as [14–16].

Theorem 2.1. *Suppose that* (H$_1$)–(H$_3$) *hold. Then given* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$*, there exist* $T > 0$ *and a unique solution* u *of the problem* (1.1) *such that*

$$u \in C(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega)).$$

In order to study the global existence and decay of a local solution for problem (1.1) given by Theorem 2.1, we will find a stable region. First of all, we set the constant

$$0 < K_0 := \sup_{u \in H_0^1(\Omega), u \neq 0} \left(\frac{\|u\|_{\rho+2}}{[a(u,u)]^{1/2}} \right) < +\infty \tag{2.3}$$

and the functional

$$J(u) = \frac{\xi_1}{2} a(u,u) - \frac{1}{\rho+2} \left\| u \right\|_{\rho+2}^{\rho+2}, \quad u \in H_0^1(\Omega). \tag{2.4}$$

And we define the function

$$j(\lambda) = \frac{\xi_1}{2} \lambda^2 - \frac{1}{\rho+2} K_0^{\rho+2} \lambda^{\rho+2}, \quad \lambda > 0,$$

then

$$\lambda_0 = \left(\frac{\xi_1}{K_0^{\rho+2}} \right)^{1/\rho}$$

is the absolute maximum point of j and

$$d_0 := j(\lambda_0) = \frac{\xi_1}{2}\lambda_0^2 - \frac{1}{\rho+2}K_0^{\rho+2}\lambda_0^\rho\lambda_0^2 = \frac{\xi_1\rho}{2(\rho+2)}\lambda_0^2.$$

Moreover, since $\lambda_0 > 0$, we have

$$\xi_1 - K_0^{\rho+2}\lambda_0^\rho = 0.$$

The energy associated to the problem (1.1) is given by

$$E(u(t)) = E(t) = \frac{1}{2}\left\|u'(t)\right\|_2^2 + \frac{1}{2}\left(\xi_1 + \frac{\xi_2}{2}a(u(t),u(t))\right)a(u(t),u(t)) - \frac{1}{\rho+2}\left\|u(t)\right\|_{\rho+2}^{\rho+2}.$$

Then

$$E'(t) = -\frac{\xi_3}{4}\left(\frac{d}{dt}a(u(t),u(t))\right)^2 - (g(u'(t),u'(t)) \le 0,$$

it follows that $E(t)$ is a nonincreasing positive function.

By (2.3) and (2.4), we have

$$\begin{aligned} E(t) \ge J(u(t)) &\ge \frac{\xi_1}{2}a(u(t),u(t)) - \frac{1}{\rho+2}K_0^{\rho+2}\left[a(u(t),u(t))\right]^{(\rho+2)/2} \\ &= j([a(u(t),u(t))]^{1/2}). \end{aligned} \tag{2.5}$$

Now, if one considers

$$a(u(t),u(t)) < \lambda_0^2, \tag{2.6}$$

then from (2.5), we arrive at

$$E(t) \ge J(u(t)) > a(u(t),u(t))\left(\frac{\xi_1}{2} - \frac{1}{\rho+2}K_0^{\rho+2}\lambda_0^\rho\right) = \frac{\xi_1\rho}{2(\rho+2)}a(u(t),u(t)) \tag{2.7}$$

and, consequently,

$$J(t) \ge 0 \ (J(t) = 0 \text{ iff } u = 0) \quad \text{and} \quad a(u(t),u(t)) \le \frac{2(\rho+2)}{\xi_1\rho}E(t). \tag{2.8}$$

Moreover, if we define the functional I by

$$I(u(t)) = \xi_1 a(u(t),u(t)) - \|u\|_{\rho+2}^{\rho+2},$$

then from the relationship $I(u(t)) = (\rho+2)J(u(t)) - \frac{\xi_1\rho}{2}a(u(t),u(t))$ and the strict inequality (2.7), we obtain

$$I(u(t)) > 0 \quad \text{for all } t \ge 0. \tag{2.9}$$

Lemma 2.2. *Suppose that*

$$E(0) < d_0 \quad \textit{and} \quad a(u_0, u_0) < \lambda_0^2. \tag{2.10}$$

Then (2.6) *is satisfied, that is,* $a(u(t), u(t)) < \lambda_0^2$ *for all* $t \geq 0$.

Proof. It is easy to verify that j is increasing for $0 < \lambda < \lambda_0$, decreasing for $\lambda > \lambda_0$, $j(\lambda_0) = d_0$, $j(\lambda) \to -\infty$ as $\lambda \to +\infty$. Then since $d_0 > E(u_0) \geq j([a(u_0, u_0)]^{1/2}) \geq j(0) = 0$, there exist $\lambda_0' < \lambda_0 < \widetilde{\lambda}_0$, which verify

$$j(\lambda_0') = j(\widetilde{\lambda}_0) = E(u_0). \tag{2.11}$$

Considering that $E(t)$ is nonincreasing, we have

$$E(u(t)) \leq E(u_0) \quad \text{for all } t \geq 0. \tag{2.12}$$

From (2.5) and (2.11), we deduce that

$$j([a(u_0, u_0)]^{1/2}) \leq E(u_0) = j(\lambda_0'). \tag{2.13}$$

Since $[a(u_0, u_0)]^{1/2} < \lambda_0$, $\lambda_0' < \lambda_0$ and j is increasing in $[0, \lambda_0)$, from (2.13) it holds that

$$[a(u_0, u_0)]^{1/2} \leq \lambda_0'. \tag{2.14}$$

Next, we will prove that

$$[a(u(t), u(t))]^{1/2} \leq \lambda_0' \quad \text{for all } t \geq 0. \tag{2.15}$$

In fact, we will argue by contradiction. Suppose that (2.15) does not hold. Then there exists time t^* which verifies

$$[a(u(t^*), u(t^*))]^{1/2} > \lambda_0'. \tag{2.16}$$

If $[a(u(t^*), u(t^*))]^{1/2} < \lambda_0$, from (2.5), (2.11) and (2.16) we can write

$$E(u(t^*)) \geq j([a(u(t^*), u(t^*))]^{1/2}) > j(\lambda_0') = E(u(0)),$$

which contradicts (2.12).

If $[a(u(t^*), u(t^*))]^{1/2} \geq \lambda_0$, then we have, in view of (2.14), that there exists $\overline{\lambda}_0$ which verifies

$$[a(u_0), u_0)]^{1/2} \leq \lambda_0' < \overline{\lambda}_0 < \lambda_0 \leq [a(u(t^*), u(t^*))]^{1/2}. \tag{2.17}$$

Consequently, from the continuity of the function $[a(u(\cdot), u(\cdot))]^{1/2}$, there exists time $\bar{t}$ verifying

$$[a(u(\bar{t}), u(\bar{t}))]^{1/2} = \overline{\lambda}_0. \tag{2.18}$$

Then from (2.5), (2.11), (2.17) and (2.18), we get

$$E(u(\bar{t})) \geq j([a(u(\bar{t}), u(\bar{t}))]^{1/2}) = j(\overline{\lambda}_0) > j(\lambda_0') = E(u_0),$$

which also contradicts (2.12). This completes the proof of Lemma 2.2. □

Theorem 2.3. *Let $u(t)$ be the solution of* (1.1). *If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies* (2.10)*, then the solution $u(t)$ is global.*

Proof. It suffices to show that $\|u'(t)\|_2^2 + a(u(t), u(t))$ is bounded independent of t. By virtue of (2.7) and (2.9), we get

$$J(t) = \frac{\xi_1 \rho}{2(\rho+2)} a(u(t), u(t)) + \frac{1}{\rho+2} I(t) > \frac{\xi_1 \rho}{2(\rho+2)} a(u(t), u(t)).$$

Hence,

$$\frac{1}{2}\left\|u'(t)\right\|_2^2 + \frac{\rho}{2(\rho+2)} \xi_1 a(u(t), u(t)) < \frac{1}{2}\left\|u'(t)\right\|_2^2 + J(t) \leq E(t) \leq E(0).$$

Therefore, there exists a positive constant C depending only on ρ and ξ_1 such that

$$\left\|u'(t)\right\|_2^2 + a(u(t), u(t)) \leq CE(0).$$

□

Now we are in the position to state the energy decay rates result.

Theorem 2.4. *Assume that hypotheses* (H_1)–(H_3) *and* (2.10) *hold. If the function $G(s) := \beta(s)/s$ is nondecreasing on $(0,1)$ and $G(0) = 0$, then we have*

$$E(t) \leq C_1 \left(\beta^{-1}\left(\frac{1}{t}\right)\right)^2,$$

where C_1 is a positive constant that depends only on $E(0)$.

To end this section, we recall a technical lemma which will play an essential role when establishing the asymptotic behavior.

Lemma 2.5. [10] *Let $E\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \to +\infty \quad \textit{as } t \to +\infty.$$

Assume that there exists $\sigma > 0$, $\sigma' \geq 0$ and $C > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,dt \leq CE^{1+\sigma}(S) + \frac{C}{(1+\phi(S))^{\sigma'}} E^{\sigma}(0)E(S), \quad 0 \leq S < +\infty.$$

Then, there exists $C > 0$ such that

$$E(t) \leq E(0)\frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}} \quad \textit{for all } t > 0.$$

3. Asymptotic stability

In this section we prove the uniform decay rates of equation (1.1). In the following section, the symbol C indicates positive constants, which may be different.

Let us now multiply equation (1.1) by $E(t)\phi'(t)u$, $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a concave nondecreasing function of class C^2, such that $\phi(t) \to +\infty$ as $t \to +\infty$, and then integrate the obtained result over $\Omega \times [S,T]$. Then we have

$$\begin{aligned} 0 &= \int_S^T E(t)\phi'(t) \int_\Omega u(t)\left(u''(t) - M(t)Lu(t) + g(u'(t)) - |u(t)|^\rho u(t)\right) dxdt \\ &= \int_S^T E(t)\phi'(t) \int_\Omega u(t)u''(t)\, dxdt - \int_S^T E(t)\phi'(t)M(t) \int_\Omega u(t)Lu(t)\, dxdt \\ &\quad + \int_S^T E(t)\phi'(t) \int_\Omega u(t)g(u'(t))\, dxdt - \int_S^T E(t)\phi'(t) \int_\Omega |u(t)|^{\rho+2}\, dxdt. \end{aligned} \tag{3.1}$$

We note that

$$\begin{aligned} &\int_S^T E(t)\phi'(t) \int_\Omega u(t)u''(t)\, dxdt \\ &= \left[E(t)\phi'(t)(u(t),u'(t))\right]_S^T - \int_S^T \left(E'(t)\phi'(t) + E(t)\phi''(t)\right) \int_\Omega u(t)u'(t)\, dxdt \\ &\quad - \int_S^T E(t)\phi'(t) \left\|u'(t)\right\|_2^2 dt \end{aligned}$$

and

$$\begin{aligned} &- \int_S^T E(t)\phi'(t)M(t) \int_\Omega u(t)Lu(t)\, dxdt \\ &= \xi_1 \int_S^T E(t)\phi'(t)a(u(t),u(t))\, dt + \xi_2 \int_S^T E(t)\phi'(t)a^2(u(t),u(t))\, dt \\ &\quad + \frac{\xi_3}{4}\left[E(t)\phi'(t)a^2(u(t),u(t))\right]_S^T - \frac{\xi_3}{4}\int_S^T \left(E'(t)\phi'(t) + E(t)\phi''(t)\right) a^2(u(t),u(t))\, dt. \end{aligned}$$

By replacing above identities in (3.1) and having in mind the definition of the energy associated to problem (1.1), it follows that

$$\begin{aligned} &2\int_S^T E^2(t)\phi'(t)\, dt \\ &= 2\int_S^T E(t)\phi'(t)\left\|u'(t)\right\|_2^2 dt - \frac{\xi_2}{2}\int_S^T E(t)\phi'(t)a^2(u(t),u(t))\, dt \\ &\quad - \left[E(t)\phi'(t)\left((u(t),u'(t)) + \frac{\xi_3}{4}a^2(u(t),u(t))\right)\right]_S^T \\ &\quad + \int_S^T \left(E'(t)\phi'(t) + E(t)\phi''(t)\right)\left((u(t),u'(t)) + \frac{\xi_3}{4}a^2(u(t),u(t))\right) dt \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&- \int_S^T E(t)\phi'(t) \int_\Omega u(t)g(u'(t))\,dxdt + \frac{\rho}{\rho+2}\int_S^T E(t)\phi'(t)\,\|u(t)\|_{\rho+2}^{\rho+2}\,dt \\
:= {}& 2\int_S^T E(t)\phi'(t)\,\|u'(t)\|_2^2\,dt - \frac{\xi_2}{2}\int_S^T E(t)\phi'(t)a^2(u(t),u(t))\,dt \\
&+ I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Now we are going to estimate terms on the right-hand side of (3.2).

Estimate for $I_1 := -\left[E(t)\phi'(t)\left((u(t),u'(t)) + \frac{\xi_3}{4}a^2(u(t),u(t))\right)\right]_S^T$.

By using Young's and Poincaré's inequalities, (2.2) and (2.8), we obtain

$$
|u(t),u'(t)| \le CE(t) \tag{3.3}
$$

and

$$
a^2(u(t),u(t)) \le \left(\frac{2(\rho+2)}{\xi_1\rho}\right)^2 E(0)E(t). \tag{3.4}
$$

Since $E(t)$ is nonincreasing and $\phi(t)$ is nondecreasing, we have

$$
I_1 \le -C\left[E(t)\phi'(t)E(t)\right]_S^T \le CE^2(S).
$$

Estimate for $I_2 := \int_S^T (E'(t)\phi'(t) + E(t)\phi''(t))\left((u(t),u'(t)) + \frac{\xi_3}{4}a^2(u(t),u(t))\right)dt$.

From (3.3) and (3.4), we have

$$
\begin{aligned}
|I_2| &\le C\int_S^T \left|E'(t)\phi'(t) + E(t)\phi''(t)\right| E(t)\,dt \\
&\le C\int_S^T -E'(t)E(t)\,dt + CE^2(S)\int_S^T -\phi''(t)\,dt \\
&\le CE^2(S).
\end{aligned}
$$

Estimate for $I_3 := -\int_S^T E(t)\phi'(t)\int_\Omega u(t)g(u'(t))\,dxdt$.

By Young's and Poincaré's inequalities, (2.2) and (2.8), we have

$$
\begin{aligned}
|I_3| &\le \frac{a_0\xi_1\rho\epsilon}{2(\rho+2)C_P}\int_S^T E(t)\phi'(t)\,\|u(t)\|_2^2\,dt + C(\epsilon)\int_S^T E(t)\phi'(t)\int_\Omega \left|g(u'(t)\right|^2 dxdt \\
&\le \frac{a_0\xi_1\rho\epsilon}{2(\rho+2)}\int_S^T E(t)\phi'(t)\,\|\nabla u(t)\|_2^2\,dt + C(\epsilon)\int_S^T E(t)\phi'(t)\int_\Omega \left|g(u'(t)\right|^2 dxdt \\
&\le \epsilon\int_S^T E^2(t)\phi'(t)dt + C(\epsilon)\int_S^T E(t)\phi'(t)\int_\Omega \left|g(u'(t)\right|^2 dxdt,
\end{aligned}
$$

where C_P is a Poincaré constant.

Estimate for $I_4 := \frac{\rho}{\rho+2}\int_S^T E(t)\phi'(t)\,\|u(t)\|_{\rho+2}^{\rho+2}\,dt$.

By (2.9), there exists a positive constant $\alpha > 1$ such that $\xi_1 a(u(t), u(t)) = \alpha \|u(t)\|_{\rho+2}^{\rho+2}$. Hence, from (2.8) we have

$$I_4 = \frac{\rho \xi_1}{\alpha(\rho+2)} \int_S^T E(t)\phi'(t) a(u(t), u(t))\, dt \le \frac{2}{\alpha} \int_S^T E(t)^2 \phi'(t)\, dt.$$

By replacing all estimates $I_1, \ldots, I_4$ in (3.4), and taking ϵ sufficiently small, we get that

$$\begin{aligned} \int_S^T E^2(t)\phi'(t)\, dt \le CE^2(S) + C \underbrace{\int_S^T E(t)\phi'(t) \left\|u'(t)\right\|_2^2 dt}_{:=I_5} \\ + C \underbrace{\int_S^T E(t)\phi'(t) \int_\Omega \left|g(u'(t))\right|^2 dx dt}_{:=I_6}. \end{aligned} \tag{3.5}$$

The last two terms of the right-hand side of (3.5) can be estimated by the same arguments of [10, 12] as follows:

$$I_5, I_6 \le CE^2(S) + CE(S) \int_S^T \phi'(t) \left(G^{-1}(\phi'(t))\right)^2 dt. \tag{3.6}$$

Now let us set $\phi(t)$ be the concave function such that its inverse is defined by

$$\phi^{-1}(t) = 1 + \int_1^t \frac{1}{G(1/s)}\, ds$$

for all $t \ge 1$. Then $\phi(t)$ satisfies all the required properties and can be extended on $[0, 1)$ such that it remains concave nondecreasing. Moreover,

$$\begin{aligned} \int_S^\infty \phi'(t) \left(G^{-1}(\phi'(t))\right)^2 dt &= \int_{\phi(S)}^\infty \left(G^{-1}(\phi'(\phi^{-1}(s)))\right)^2 ds \\ &= \int_{\phi(S)}^\infty \left(G^{-1}\left(\frac{1}{(\phi^{-1})'(s)}\right)\right)^2 ds = \int_{\phi(S)}^\infty \frac{1}{s^2}\, ds \\ &= \frac{1}{\phi(S)} \le \beta^{-1}\left(\frac{1}{S}\right). \end{aligned} \tag{3.7}$$

By replacing (3.6) in (3.5) and applying Lemma 2.5 with $\sigma = \sigma' = 1$ and (3.7), we obtain

$$E(t) \le CE(0) \left(\beta^{-1}\left(\frac{1}{t}\right)\right)^2.$$

References

[1] A. V. Balakrishnan and L. W. Taylor, *Distributed parameter nonlinear damping models for flight structures*, in Proceedings "Damping 89", Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAFB, (1989).

[2] R. W. Bass and D. Zes, *Spillover, nonlinearity and flexible structures*, in *The Fourth NASA Workshop on Computational Control of Flexible Aerospace Systems*, NASA Conference Publication 10065 (ed. L.W. Taylor), (1991), 1–14. https://doi.org/10.1109/cdc.1991.261683

[3] Y. Boukhatem and B. Benabderrahmane, *Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions*, Nonlinear Anal. **97** (2014), 191–209. https://doi.org/10.1016/j.na.2013.11.019

[4] M. M. Cavalcanti, A. Khemmoudj and M. Medjden, *Uniform stabilization of the damped Cauchy-Ventcel problem with variable coefficients and dynamic boundary conditions*, J. Math. Anal. Appl. **328** (2007), no. 2, 900–930. https://doi.org/10.1016/j.jmaa.2006.05.070

[5] T. G. Ha, *General decay rate estimates for viscoelastic wave equation with Balakrishnan-Taylor damping*, Z. Angew. Math. Phys. **67** (2016), no. 2, Art. 32, 17 pp. https://doi.org/10.1007/s00033-016-0625-3

[6] ______, *Global existence and general decay estimates for the viscoelastic equation with acoustic boundary conditions*, Discrete Contin. Dyn. Syst. **36** (2016), no. 12, 6899–6919. https://doi.org/10.3934/dcds.2016100

[7] I. Lasiecka, R. Triggiani and P.-F. Yao, *Inverse/observability estimates for second-order hyperbolic equations with variable coefficients*, J. Math. Anal. Appl. **235** (1999), no. 1, 13–57. https://doi.org/10.1006/jmaa.1999.6348

[8] J. Li and S. Chai, *Energy decay for a nonlinear wave equation of variable coefficients with acoustic boundary conditions and a time-varying delay in the boundary feedback*, Nonlinear Anal. **112** (2015), 105–117. https://doi.org/10.1016/j.na.2014.08.021

[9] L. Lu and S. Li, *Higher order energy decay for damped wave equations with variable coefficients*, J. Math. Anal. Appl. **418** (2014), no. 1, 64–78. https://doi.org/10.1016/j.jmaa.2014.03.081

[10] P. Martinez, *A new method to obtain decay rate estimates for dissipative systems with localized damping*, Rev. Mat. Complut. **12** (1999), no. 1, 251–283. https://doi.org/10.5209/rev_rema.1999.v12.n1.17227

[11] C. Mu and J. Ma, *On a system of nonlinear wave equations with Balakrishnan-Taylor damping*, Z. Angew. Math. Phys. **65** (2014), no. 1, 91–113. https://doi.org/10.1007/s00033-013-0324-2

[12] J. Y. Park, T. G. Ha and Y. H. Kang, *Energy decay rates for solutions of the wave equation with boundary damping and source term*, Z. Angew. Math. Phys. **61** (2010), no. 2, 235–265. https://doi.org/10.1007/s00033-009-0009-z

[13] N.-e. Tatar and A. Zaraï, *Exponential stability and blow up for a problem with Balakrishnan-Taylor damping*, Demonstratio Math. **44** (2011), no. 1, 67–90.

[14] J. Wu, *Well-posedness for a variable-coefficient wave equation with nonlinear damped acoustic boundary conditions*, Nonlinear Anal. **75** (2012), no. 18, 6562–6569. https://doi.org/10.1016/j.na.2012.07.032

[15] P.-F. Yao, *On the observability inequalities for exact controllablility of wave equations with variable coefficients*, SIAM J. Control Optim. **37** (1999), no. 5, 1568–1599. https://doi.org/10.1137/s0363012997331482

[16] Y. You, *Inertial manifolds and stabilization of nonlinear beam equations with Balakrishnan-Taylor damping*, Abstr. Appl. Anal. **1** (1996), no. 1, 83–102. https://doi.org/10.1155/s1085337596000048

[17] A. Zaraï and N.-E. Tatar, *Global existence and polynomial decay for a problem with Balakrishnan-Taylor damping*, Arch. Math. (Brno) **46** (2010), no. 3, 157–176.

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