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Boundedness of Pseudodifferential Operators on Realized Homogeneous Besov Spaces

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Abstract. Using the notion of realizations, the realized homogeneous Besov spaces $\dot{\widetilde{B}}^s_{p,q}(\mathbb{R}^n)$ are subsets of tempered distributions. Then we will study the boundedness of some pseudodifferential operators on $\dot{\widetilde{B}}^s_{p,q}(\mathbb{R}^n)$, in the cases either $s < n/p$ or $s = n/p$ and $q = 1$.

1. Introduction and main results

For a function $a\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ we study the boundedness of the corresponding pseudodifferential operator (abbreviated ps.d.o) $f \to a(x,D)f$ on the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$, where $a(x,D)$ is defined by

$$a(x,D)f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \widehat{f}(\xi)\, \mathrm{d}\xi, \quad \forall\, x \in \mathbb{R}^n;$$

the function a is called the *symbol*. We will use the notation $a(x,\xi)$ instead of a.

The space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined modulo polynomials since $\|f\|_{\dot{B}^s_{p,q}} = 0$ for all *polynomial* functions f. Then a *ps.d.o cannot map e.g.,* $\dot{B}^{s+m}_{p,q}(\mathbb{R}^n)$ *into* $\dot{B}^s_{p,q}(\mathbb{R}^n)$ *in general*, since, for instance, *if we take the constant function* $f(x) := 1$ *we have* $a(x,D)f(x) = a(x,0)$ *and it suffices to choose the symbol* $a(x,\xi)$ *satisfying* $a(x,0) \notin \dot{B}^s_{p,q}(\mathbb{R}^n)$, (cf., see Example 2.6 below). For this reason, we will show the boundedness problem on the *realized* homogeneous Besov spaces $\dot{\widetilde{B}}^s_{p,q}(\mathbb{R}^n)$, and exactly from $\dot{\widetilde{B}}^{s+m}_{p,q}(\mathbb{R}^n)$ into $\dot{\widetilde{B}}^s_{p,q}(\mathbb{R}^n)$, where throughout the paper we will make use of the following convention: the parameters s, p, q and m will verify

$$s \in \mathbb{R}, \quad m \in \mathbb{R} \quad \text{and} \quad p, q \in [1,\infty]$$

unless otherwise stated. Recall that the notion of *the realization* has been initiated by G. Bourdaud in [2], where its essential purpose is to give for any element of an homogeneous

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space (e.g., $\dot{B}^s_{p,q}(\mathbb{R}^n)$) an unique representative which belongs to the tempered distributions space $\mathcal{S}'(\mathbb{R}^n)$. Actually, there are many references about this subject e.g., [3, Section 3], [4] and [11, Section 3]. The space $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ is a subset of $\mathcal{S}'(\mathbb{R}^n)$ (see Section 2.3 below), where under the following condition:

$$\text{either} \quad s < \frac{n}{p}, \quad \text{or} \quad s = \frac{n}{p} \quad \text{and} \quad q = 1, \tag{1.1}$$

all element of $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ can be characterize via the Littlewood-Paley decomposition; however, for other cases given to the parameters n, s, p and q, the characterization of $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ relies on a polynomial with degree depending on $s - n/p$ and q. On the one hand, (1.1) presents one of assumptions of the main results, and on the other hand, since in this paper we also work in $\dot{\tilde{B}}^{s+m}_{p,q}(\mathbb{R}^n)$, we mention the following condition:

$$\text{either} \quad s+m < \frac{n}{p}, \quad \text{or} \quad s+m = \frac{n}{p} \quad \text{and} \quad q = 1. \tag{1.2}$$

To formulate the results we introduce $\dot{S}^{m,N}_{1,0}(E)$ (so-called the homogeneous class of symbols) the set of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ functions $a(x,\xi)$, such that for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \le N$ there exists a constant $c := c(\alpha) > 0$, such that

$$\left\|\partial^\alpha_\xi a(\cdot\,,\xi)\right\|_E \le c\,|\xi|^{m-|\alpha|}, \quad \forall\,\xi \in \mathbb{R}^n \setminus \{0\}\,.$$

In the same way, we define the inhomogeneous class $S^{m,N}_{1,0}(E)$ of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ symbols $a(x,\xi)$ satisfying

$$\left\|\partial^\alpha_\xi a(\cdot\,,\xi)\right\|_E \le c(1+|\xi|)^{m-|\alpha|}, \quad \forall\,\xi \in \mathbb{R}^n,$$

with $|\alpha| \le N$ and $c := c(\alpha) > 0$. Noticing that the sets $\dot{S}^{m,N}_{1,0}(E)$ and $S^{m,N}_{1,0}(E)$ are Fréchet spaces with the seminorms

$$\dot{\Pi}_N(a) := \sup_{|\alpha|\le N}\ \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{-m+|\alpha|}\left\|\partial^\alpha_\xi a(\cdot\,,\xi)\right\|_E$$

and

$$\Pi_N(a) := \sup_{|\alpha|\le N}\ \sup_{\xi\in\mathbb{R}^n} (1+|\xi|)^{-m+|\alpha|}\left\|\partial^\alpha_\xi a(\cdot\,,\xi)\right\|_E,$$

respectively. Now in case of the homogeneous class, we deal with $E := L_\infty(\mathbb{R}^n) \cap \dot{B}^{n/p}_{p,q}(\mathbb{R}^n)$, endowed with the norm

$$\|f\|_E := \|f\|_\infty + \|[f]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,q}},$$

where $[f]_{\mathcal{P}}$ denotes the equivalence class of f modulo polynomials, and in case of the inhomogeneous class, we work with $E := L_\infty(\mathbb{R}^n) \cap \dot{B}^{n/p}_{p,q}(\mathbb{R}^n) \cap \dot{B}^{n/p-m}_{p,q}(\mathbb{R}^n)$ endowed with the corresponding norm. To explain the difference between these two classes, we give the following observation: Except the case $m = 0$, there are no embeddings between $\dot{B}^{n/p}_{p,q}(\mathbb{R}^n)$ and $\dot{B}^{n/p-m}_{p,q}(\mathbb{R}^n)$. We will prove essentially the following results:

Theorem 1.1. *Let $s > 0$ and $1 \leq p < \infty$. Assume that* (1.1) *and* (1.2) *hold. Let N be an even natural number satisfying $N > 3n/2+2$. If a symbol $a(x,\xi)$ belongs to $\dot{S}^{m,N}_{1,0}(L_\infty(\mathbb{R}^n) \cap \dot{B}^{n/p}_{p,q}(\mathbb{R}^n))$, then the ps.d.o $a(x,D)$ takes the space $\dot{\tilde{B}}^{s+m}_{p,q}(\mathbb{R}^n)$ into $\dot{\tilde{B}}^{s}_{p,q}(\mathbb{R}^n)$. Moreover, there exists a constant $c > 0$ such that the inequality*

$$\|[a(x,D)f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \leq c\dot{\Pi}_N(a)\, \|[f]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}} \tag{1.3}$$

holds, for all $f \in \dot{\tilde{B}}^{s+m}_{p,q}(\mathbb{R}^n)$.

Theorem 1.2. *Let $s > 0$, $1 \leq p < \infty$ and $m \geq 0$. Assume that* (1.1) *and* (1.2) *hold. Let N be an even natural number satisfying $N > 3n/2 + 2$. If a symbol $a(x,\xi)$ belongs to $S^{m,N}_{1,0}(L_\infty(\mathbb{R}^n) \cap \dot{B}^{n/p}_{p,q}(\mathbb{R}^n) \cap \dot{B}^{n/p-m}_{p,q}(\mathbb{R}^n))$, then the ps.d.o $a(x,D)$ takes the space $\dot{\tilde{B}}^{s+m}_{p,q}(\mathbb{R}^n)$ into $\dot{\tilde{B}}^{s}_{p,q}(\mathbb{R}^n)$. Moreover, there exists a constant $c > 0$ such that the inequality*

$$\|[a(x,D)f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \leq c\Pi_N(a)\, \|[f]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}}$$

holds, for all $f \in \dot{\tilde{B}}^{s+m}_{p,q}(\mathbb{R}^n)$.

In [5], R. R. Coifman and Y. Meyer proved that any ps.d.o of order 0 (i.e., $m = 0$) can be decomposed as the sum of a regularized operator with another one which is associated to an *elementary symbol* (i.e., an elementary symbol $a(x,\xi)$ is given by the following expression: $a(x,\xi) = \sum_{j\in\mathbb{N}} \chi_j(x)\theta(2^{-j}\xi)$, where $(\chi_j)_{j\in\mathbb{N}}$ is a bounded sequence in an appropriate functions space and $\theta \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$), see also e.g., [8,9]. Then the proofs of Theorems 1.1 and 1.2 are based on the following three assertions:

- a reduction to elementary symbols in *the homogeneous case*,
- an almost orthogonality estimate of type of G. Gibbons [6] which will be obtained by Nikol'skij representation method (see Lemma 3.2 below),
- a pointwise multipliers property of realized homogeneous Besov space (see Lemma 3.3 below).

We will also extend the principal result to, both, the symbols class $S^{m,N,M}_{\mu,\eta}(E)$ (see Theorem 4.1 below) and by taking (1.1) in other cases on the parameters n, s, p and q (see Theorem 4.3 below).

Notations. As usual, $\mathbb{N}$ denotes the set of natural numbers including 0, $\mathbb{Z}$ the set of integers and $\mathbb{R}$ the set of real numbers. All function spaces occurring in this work are defined on Euclidean space $\mathbb{R}^n$, then we omit $\mathbb{R}^n$ in notation. For $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t. For $b \in \mathbb{R}$ we put $b_+ := \max(0,b)$. We denote by $\|\cdot\|_p$ the L_p norm. The symbol $\hookrightarrow$ indicates that the embedding is continuous. For

$\alpha, \beta \in \mathbb{N}^n$ multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, we say that $\alpha \prec \beta$ if $\alpha_j \leq \beta_j$ for all $j = 1, \ldots, n$ and $\alpha_k < \beta_k$ for at least one index $1 \leq k \leq n$. The standard norms of f in $\mathcal{S}$ are given by

$$\zeta_M(f) := \sup_{|\alpha| \leq M} \sup_{x \in \mathbb{R}^n} (1 + |x|)^M \left| f^{(\alpha)}(x) \right|, \quad M \in \mathbb{N}.$$

If $f \in L_1$, the Fourier transform of f and its Fourier transform inverse on $\mathbb{R}^n$ are defined by

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, \mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1} f(x) := (2\pi)^{-n} \widehat{f}(-x).$$

They are extended to $\mathcal{S}'$ in the usual way. We denote by $\mathcal{P}_\infty$ the set of all polynomials on $\mathbb{R}^n$. We denote by $\mathcal{S}_\infty$ the orthogonal of $\mathcal{P}_\infty$ in $\mathcal{S}$, i.e., the set of all $u \in \mathcal{S}$ such that $\langle f, u \rangle = 0$ for all $f \in \mathcal{P}_\infty$. For all $f \in \mathcal{S}'$, then $[f]_\mathcal{P}$ is the equivalence class of f modulo $\mathcal{P}_\infty$, this notation has been defined before. The mapping which takes any $[f]_\mathcal{P}$ to the restriction of f to $\mathcal{S}_\infty$ turns out to be an isomorphism from $\mathcal{S}'/\mathcal{P}_\infty$ onto $\mathcal{S}'_\infty$; for this reason, $\mathcal{S}'_\infty$ is called the space of *distributions modulo polynomials*. The constants c are strictly positive and depend only on the fixed parameters n, s, p and q and probably on auxiliary functions, their values may vary from line to line. Finally, sometimes we will use the symbol $\lesssim$ instead of $\leq$, the notation $A \lesssim B$ means that $A \leq cB$.

The paper is organized as follows. In Section 2, we collect information about homogeneous, inhomogeneous and the realized version, of Besov spaces. In Section 3, we give the proofs of Theorems 1.1 and 1.2. In Section 4, we discuss some generalizations and remarks.

2. The Besov space

2.1. The Littlewood-Paley setting

We choose, once and for all, a standard cut-off function. More precisely, we assume that ρ is a radial C^∞ function satisfying $0 \leq \rho \leq 1$, $\rho(\xi) = 1$ if $|\xi| \leq 1$, $\rho(\xi) = 0$ if $|\xi| \geq 3/2$. Then $\widehat{\rho}$ is a radial and real-valued function. We define $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$ for all $\xi \in \mathbb{R}^n$, which is supported by the compact annulus $1/2 \leq |\xi| \leq 3/2$, and the following identities hold:

$$\sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1, \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\},$$

$$\rho(2^{-k}\xi) + \sum_{j > k} \gamma(2^{-j}\xi) = 1, \quad \forall\, k \in \mathbb{Z},\ \forall\, \xi \in \mathbb{R}^n.$$

For any $j \in \mathbb{Z}$, we introduce the ps.d.o $S_j := \rho(2^{-j}D)$ and $Q_j := \gamma(2^{-j}D)$. It is clear that S_j is defined on $\mathcal{S}'$ and that Q_j is defined on $\mathcal{S}'_\infty$ since $Q_j f = 0$ for all $f \in \mathcal{P}_\infty$. All these operators take values in the space of analytical functions of exponential type, see the Paley-Wiener Theorem. By abuse of notation, if $f \in \mathcal{S}'_\infty$, then for any $f_1, f_2 \in \mathcal{S}'$ with $f = [f_1]_\mathcal{P} = [f_2]_\mathcal{P}$, we have $Q_j f_1 = Q_j f_2$ (recall that $f_1 - f_2 \in \mathcal{P}_\infty$). For this reason we say:

$$\text{if } f \in \mathcal{S}'_\infty \text{ we set } Q_j f := Q_j f_1 \text{ for all } f_1 \in \mathcal{S}' \text{ such that } [f_1]_\mathcal{P} = f.$$

Using the Young inequality, the families of operators $(S_j)_{j\in\mathbb{Z}}$ and $(Q_j)_{j\in\mathbb{Z}}$ constitute bounded subsets of the normed space $\mathcal{L}(L_p)$ for any $p \in [1,\infty]$. We also have the following assertion which is proved in [11, Proposition 2.5].

Proposition 2.1. (i) *For any $N \in \mathbb{N}$, there exist a positive constant c and a natural number M such that $\|Q_j f\|_p \leq c 2^{-jN} \zeta_M(f)$ holds, for all $f \in \mathcal{S}$ and all $j \in \mathbb{N}$.*

(ii) *For any $N \in \mathbb{N}$, there exist a positive constant c and a natural number M such that $\|Q_j f\|_p + \|S_j f\|_p \leq c 2^{jN} \zeta_M(f)$ holds, for all $f \in \mathcal{S}_\infty$ and all negative integers j.*

As a consequence of this proposition is the convergence of the Littlewood-Paley decompositions of a distribution which is described in the following well known statement.

Proposition 2.2. (i) *For every $f \in \mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, respectively), it holds that $f = \sum_{j\in\mathbb{Z}} Q_j f$ in $\mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, respectively).*

(ii) *For every $f \in \mathcal{S}$ ($\mathcal{S}'$, respectively) and every $k \in \mathbb{Z}$, it holds that $f = S_k f + \sum_{j>k} Q_j f$ in $\mathcal{S}$ ($\mathcal{S}'$, respectively).*

2.2. Homogeneous Besov spaces

The Littlewood-Paley approach presents the basic definition of the Besov spaces.

Definition 2.3. The homogeneous Besov space $\dot{B}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j\in\mathbb{Z}} (2^{sj} \|Q_j f\|_p)^q \right)^{1/q} < \infty.$$

The space $\dot{B}^s_{p,q}$ is a Banach space, and the following chain of continuous embeddings holds:

$$\mathcal{S}_\infty \hookrightarrow \dot{B}^s_{p,q} \hookrightarrow \mathcal{S}'_\infty.$$

We also have the embeddings $\dot{B}^s_{p,q_1} \hookrightarrow \dot{B}^s_{p,q_2}$ if $q_1 < q_2$, and $\dot{B}^{s_1}_{p1,q} \hookrightarrow \dot{B}^{s_2}_{p2,q}$ if $p_1 < p_2$ and $s_1 - n/p_1 = s_2 - n/p_2$, see [7, Theorem 2.1].

On the other hand, one of the tools for the proofs of main results is the Nikol'skij type estimates. We recall here the following statement.

Proposition 2.4. *Let $s > 0$ and $1 \leq p < \infty$. Assume that* (1.1) *holds. Let $b > 0$. Let $(u_j)_{j\in\mathbb{Z}}$ be a sequence in $\mathcal{S}'$ satisfying the following two conditions:*

- *$\widehat{u}_j$ is supported by the ball $|\xi| \leq b2^j$,*
- *$A := \left(\sum_{j\in\mathbb{Z}}(2^{js}\left\|u_j\right\|_p)^q\right)^{1/q} < \infty$.*

Then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'$ to a limit u which satisfies $\|[u]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \leq cA$, and where c depends only on n, s, p, q and b.

Proof. The proof is given in [11, Proposition 2.15]. See also [10, Proposition 3.4] or [14, Proposition 2.3.2/1, p. 59] or [17] for the case of inhomogeneous Besov spaces. □

By replacing the discrete Littlewood-Paley decompositions by continuous ones, we obtain equivalent continuous norms of $\dot{B}^s_{p,q}$ which easily yield the following result.

Proposition 2.5. *There exist constants $c_1, c_2 > 0$, depending only on s, p, q, n, such that the inequality*

$$c_1 \left\|f\right\|_{\dot{B}^s_{p,q}} \leq \lambda^{n/p-s} \left\|f(\lambda(\cdot))\right\|_{\dot{B}^s_{p,q}} \leq c_2 \left\|f\right\|_{\dot{B}^s_{p,q}}$$

holds, for all $f \in \dot{B}^s_{p,q}$ and all $\lambda > 0$.

Now, in connection with the boundedness of the ps.d.o on homogeneous Besov spaces, we give the following example:

Example 2.6. Let g be a C^∞ positive function supported by the compact annulus $7/9 \leq |\xi| \leq 7/8$. We set

$$f(x) := \sum_{j\geq 0} 2^{j(2n/p-s)}(\mathcal{F}^{-1}g)(2^j x), \quad \forall\, x \in \mathbb{R}^n.$$

Clearly it holds $\widehat{Q_k f}(\xi) = 2^{k(2n/p-s-n)}g(2^{-k}\xi)\gamma(2^{-k}\xi)$ if $k \geq 0$ and $\widehat{Q_k f}(\xi) = 0$ if $k < 0$, which imply that $\left\|Q_k f\right\|_p = 2^{k(n/p-s)}\left\|\mathcal{F}^{-1}(g\gamma)\right\|_p$, $(\forall\, k \geq 0)$, then $[f]_{\mathcal{P}} \notin \dot{B}^s_{p,\infty}$ and consequently $[f]_{\mathcal{P}} \notin \dot{B}^s_{p,q}$. Now, if we put $a(x,0) := f(x)$, then the associated ps.d.o $a(x,D)$ *is not bounded* from $\dot{B}^{s+m}_{p,q}$ into $\dot{B}^s_{p,q}$. Recall that $(a(x,D)1)(x) = a(x,0)$ and the constant function 1 belongs to $\dot{B}^{s+m}_{p,q}$.

2.3. Realized homogeneous Besov spaces

The difficulty to handle distributions modulo polynomials requires the use of both the realizations and the convergence in $\mathcal{S}'$ in the weak sense. We will outline these two approaches and refer to G. Bourdaud [2–4] for definitions below.

Definition 2.7. Let E be a vector subspace of $\mathcal{S}'_\infty$, endowed with a norm which renders continuous the embedding $E \subset \mathcal{S}'_\infty$. A *realization* of E in $\mathcal{S}'$ is a continuous linear mapping $\sigma\colon E \to \mathcal{S}'$ such that $[\sigma(f)]_{\mathcal{P}} = f$ for all $f \in E$. The image set $\sigma(E)$ is called the realized space of E.

Definition 2.8. We say that a tempered distribution $f \in \mathcal{S}'$ vanishes at infinity in the weak sense if $\lim_{\lambda\to 0} f(\lambda^{-1}(\cdot)) = 0$ in $\mathcal{S}'$, i.e., $\lim_{\lambda\to 0} \langle f(\lambda^{-1}(\cdot)), \varphi\rangle = 0$ for all $\varphi \in \mathcal{S}$. The set of all such distributions is denoted by $\widetilde{C}_0$.

On the one hand, let us give examples of such distributions: (i) functions in L_p for $1 \le p < \infty$, (ii) derivatives of functions in L_∞, (iii) derivatives of the members of $\widetilde{C}_0$. On the other hand, in connection with the polynomial functions we have at our disposal the following easy lemma which is proved in [2, p. 46].

Lemma 2.9. *If f is a polynomial vanishing at infinity in the weak sense, then $f = 0$, i.e., $\widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$.*

We now turn to the realizations. A typical example of realization is given by the classical Littlewood-Paley decomposition. We recall the following assertion.

Proposition 2.10. *Assume that* (1.1) *holds. If $u \in \dot{B}^s_{p,q}$, then the series $\sum_{j\in\mathbb{Z}} Q_j u$ converges in $\mathcal{S}'$. We put $f := \sum_{j\in\mathbb{Z}} Q_j u$. Then f belongs to $\widetilde{C}_0$ and is the unique tempered distribution satisfying $[u]_{\mathcal{P}} = f$ in $\mathcal{S}'_\infty$.*

Proof. See [4, Proposition 4.6] and [11, Theorems 1.2, 4.1, Section 4.2]. □

Now, we are able to define the realized homogeneous Besov space.

Definition 2.11. Assume that (1.1) holds. The realized homogeneous Besov space $\dot{\widetilde{B}}^s_{p,q}$ is the set of all $g \in \mathcal{S}'$ such that $[g]_{\mathcal{P}} \in \dot{B}^s_{p,q}$ and $g \in \widetilde{C}_0$.

Clearly in Proposition 2.10, if we define $\sigma(u) := \sum_{j\in\mathbb{Z}} Q_j u$ for all $u \in \dot{B}^s_{p,q}$, then σ is a realization on $\dot{B}^s_{p,q}$ which commutes with translations and dilations, i.e., $\sigma(u)(x-a) = \sigma(u(\cdot - a))(x)$ and $\sigma(u)(x/\lambda) = \sigma(u(\cdot/\lambda))(x)$ for all $x, a \in \mathbb{R}^n$ and all $\lambda > 0$, and we have

$$\sigma(\dot{B}^s_{p,q}) = \dot{\widetilde{B}}^s_{p,q}, \tag{2.1}$$

indeed, we have "$\subset$" by the definition. Let now $g \in \dot{\widetilde{B}}^s_{p,q}$, then $[g]_{\mathcal{P}} \in \dot{B}^s_{p,q}$ and $g - \sigma([g]_{\mathcal{P}}) \in \mathcal{P}_\infty$, by Lemma 2.9 we conclude that $g = \sigma([g]_{\mathcal{P}})$, and this proves "$\supset$".

The space $\dot{\widetilde{B}}^s_{p,q}$ is endowed with the same norm of $\dot{B}^s_{p,q}$, i.e., $\|f\|_{\dot{\widetilde{B}}^s_{p,q}} := \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$, and is a Banach space in $\mathcal{S}'$ defined by "true" distributions not distributions modulo polynomials, as was mentioned at the introduction. We also note, that some properties of $\dot{\widetilde{B}}^s_{p,q}$ as, embeddings, interpolations, etc, can be found in, e.g., [2, 4, 11, 18].

Remark 2.12. Based on Proposition 2.10 and conditions (1.1)–(1.2), we note that in Theorems 1.1 and 1.2, the two spaces $\dot{\widetilde{B}}^s_{p,q}$ and $\dot{\widetilde{B}}^{s+m}_{p,q}$ are well-defined.

2.4. Inhomogeneous Besov spaces

By using the inhomogeneous Littlewood-Paley decomposition instead of the homogeneous one (see Section 2.1), we obtain the inhomogeneous, or ordinary, Besov spaces.

Definition 2.13. The inhomogeneous Besov space $B^s_{p,q}$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{B^s_{p,q}} := \|S_0 f\|_p + \left(\sum_{j\geq 1} (2^{sj}\, \|Q_j f\|_p)^q\right)^{1/q} < \infty.$$

The links exist between the homogeneous space and its inhomogeneous counterpart at least for $s > 0$. Indeed we have the following assertion, see e.g., Triebel [16, Theorem 2.3.3].

Proposition 2.14. *If $s > 0$, then $B^s_{p,q}$ is the set of $f \in L_p$ such that $[f]_{\mathcal{P}} \in \dot{B}^s_{p,q}$, and $\|f\|_p + \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$ is an equivalent norm in $B^s_{p,q}$.*

For Besov spaces we do not go into details, instead we refer to, e.g., [1, 10, 13–16].

3. Proofs of Theorems 1.1 and 1.2

Theorems 1.1 and 1.2 can be easily derived from the following statement.

Proposition 3.1. *Let $s > 0$ and $1 \leq p < \infty$. Assume that* (1.1) *and* (1.2) *hold. Let $a(x,\xi)$ be a $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function.*

(i) *$a(x,\xi)$ can be decomposed into a sum $\tau(x,\xi) + \theta(x,\xi)$ where $\tau(x,\xi) = 0$ if $|\xi| \geq 3/2$ and $\theta(x,\xi) := \sum_{K\in\mathbb{Z}^n} \vartheta_K(x,\xi)$ with $\theta(x,\xi) = 0$ if $|\xi| \leq 1$, and the functions $\vartheta_K(x,\xi)$ are elementary symbols-type.*

(ii) *Let N be an even natural number satisfying $N > 3n/2 + 2$. If $a(x,\xi)$ belongs to $S^{m,N}_{1,0}(L_\infty \cap \dot{B}^{n/p}_{p,q})$, then the ps.d.o $\theta(x,D)$ takes the space $\dot{\widetilde{B}}^{s+m}_{p,q}$ into $\dot{\widetilde{B}}^s_{p,q}$. Moreover, there exists a constant $c > 0$ such that the inequality*

$$\|[\theta(x,D)f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \leq c\Pi_N(a)\, \|[f]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}}$$

holds, for all $f \in \dot{\widetilde{B}}^{s+m}_{p,q}$.

(iii) *Assume in addition $m \geq 0$. Let N be an even natural number satisfying $N > n$. If $a(x,\xi)$ belongs to $S^{m,N}_{1,0}(L_\infty \cap \dot{B}^{n/p-m}_{p,q})$, then the ps.d.o $\tau(x,D)$ takes the space $\dot{\widetilde{B}}^{s+m}_{p,q}$ into $\dot{\widetilde{B}}^s_{p,q}$. Moreover, there exists a constant $c > 0$ such that the inequality*

$$\|[\tau(x,D)f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \leq c\Pi_N(a)\, \|[f]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}}$$

holds, for all $f \in \dot{\widetilde{B}}^{s+m}_{p,q}$.

As mentioned in Section 1 we will formulate two properties, an almost orthogonality estimate-type and a pointwise multipliers in realized homogeneous Besov spaces, respectively, and which are needed for the proof of the above proposition.

Lemma 3.2. *Let $s > 0$ and $1 \leq p < \infty$. Assume that* (1.1) *holds. Let $b > 0$. Let $(u_j)_{j\in\mathbb{Z}}$ and $(\chi_j)_{j\in\mathbb{Z}}$ be sequences in $\mathcal{S}'$ satisfying the following three conditions:*

(i) *$\widehat{u}_j$ is supported by the ball $|\xi| \leq b2^j$,*

(ii) $A := \left(\sum_{j\in\mathbb{Z}}(2^{js}\left\|u_j\right\|_p)^q\right)^{1/q} < \infty$,

(iii) $B := \sup_{j\in\mathbb{Z}}\left(\|\chi_j\|_\infty + \|[\chi_j]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,q}}\right) < \infty$.

Then the series $\sum_{j\in\mathbb{Z}}\chi_j u_j$ converges in $\mathcal{S}'$ to a limit u which satisfies

$$\|[u]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \leq cAB, \tag{3.1}$$

where c depends only on n, s, p, q and b.

Proof. Step 1*: Convergence in $\mathcal{S}'$.* The proof is the same as in [11, Proposition 2.15/Substep 1.2]. We omit the details.

Step 2*: Proof of* (3.1). In $\sum_{j\in\mathbb{Z}}\chi_j u_j$ we split the area of summation with respect to j. That is, by Proposition 2.2 we write $\chi_j = S_j\chi_j + \sum_{k>j}Q_k\chi_j$ (recall that $\chi_j \in \mathcal{S}'$), which implies that $\sum_{j\in\mathbb{Z}}\chi_j u_j = V + W$, where

$$V := \sum_{j\in\mathbb{Z}} u_j S_j \chi_j \quad \text{and} \quad W := \sum_{k\in\mathbb{Z}}\sum_{j<k} u_j Q_k \chi_j, \tag{3.2}$$

with $\mathcal{F}(u_j S_j\chi_j)$ and $\mathcal{F}\left(\sum_{j<k} u_j Q_k\chi_j\right)$ are supported by the balls $|\xi| \leq (3/2+b)2^j$ and $|\xi| \leq (3/2+b/2)2^k$, respectively. This allows to apply Proposition 2.4 to the series V and W.

Substep 2.1*: Estimate of $\|[V]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$.* It is immediate by using $\|S_j\chi_j\|_\infty \lesssim B$, $\forall\, j \in \mathbb{Z}$.

Substep 2.2*: Estimate of $\|[W]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$ in the case $0 < s < n/p$.* We will use the following inequality

$$\left\|\sum_{j<k} u_j Q_k \chi_j\right\|_p \leq \sum_{j<k}\left\|u_j\right\|_\infty \left\|Q_k\chi_j\right\|_p. \tag{3.3}$$

By Bernstein inequality (cf., [15, Remark 1.3.2/1]) we obtain

$$\begin{aligned}\|[W]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} &\lesssim \left(\sum_{k\in\mathbb{Z}}\left(2^{sk}\sum_{j<k}\left\|u_j\right\|_\infty\left\|Q_k\chi_j\right\|_p\right)^q\right)^{1/q} \\ &\lesssim \sup_{\ell\in\mathbb{Z}}\|\chi_\ell\|_{\dot{B}^{n/p}_{p,\infty}}\left(\sum_{k\in\mathbb{Z}}\left(\sum_{j<k}2^{(n/p-s)(j-k)}(2^{sj}\left\|u_j\right\|_p)\right)^q\right)^{1/q}.\end{aligned}$$

Using the assumption on s, and applying Young inequality in $\ell_q(\mathbb{Z})$, i.e.,

$$\left\| \left(\sum_{j\in\mathbb{Z}} b_{k-j}\varepsilon_j \right)_{k\in\mathbb{Z}} \right\|_{\ell_q(\mathbb{Z})} \leq \|(b_j)_{j\in\mathbb{Z}}\|_{\ell_1(\mathbb{Z})} \|(\varepsilon_j)_{j\in\mathbb{Z}}\|_{\ell_q(\mathbb{Z})}, \tag{3.4}$$

with $b_j := 2^{(s-n/p)j}$ if $j > 0$, $b_j := 0$ if $j \leq 0$ and $\varepsilon_j := 2^{sj} \|u_j\|_p$, we conclude that

$$\|[W]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \lesssim AB.$$

Substep 2.3*: Estimate of* $\|[W]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$ *in the case* $s = n/p$ *and* $q = 1$. Applying (3.3) and Bernstein inequality, successively, we get

$$\begin{aligned} \|[W]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,1}} &\lesssim \sum_{k\in\mathbb{Z}} 2^{kn/p} \sum_{j<k} 2^{jn/p} \|u_j\|_p \|Q_k\chi_j\|_p \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{jn/p} \|u_j\|_p \sum_{k>j} 2^{kn/p} \|Q_k\chi_j\|_p \\ &\lesssim AB. \end{aligned}$$

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Let* $s > 0$ *and* $1 \leq p < \infty$*. Assume that* (1.1) *and* (1.2) *hold. Assume in addition* $m \geq 0$*. Then there exists a constant* $c > 0$ *such that the inequality*

$$\|[hg]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \leq c \left(\|[h]_{\mathcal{P}}\|_{\dot{B}^{n/p-m}_{p,q}} + \|h\|_\infty \right) \|[g]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}}$$

holds, for all $g \in C^\infty \cap \dot{\tilde{B}}^{s+m}_{p,q}$ *all* $h \in L_\infty \cap \dot{\tilde{B}}^{n/p-m}_{p,q}$.

Proof. Step 1. Let g and h be functions given as in this lemma. By assumption, the functions g and h belong to $\mathcal{S}'$. Clearly also that S_jg, Q_kg and Q_kh are C^∞ functions for all $j, k \in \mathbb{Z}$. Then in $\mathcal{S}'$, we can write the following expressions $g = S_jg + \sum_{k>j} Q_kg$, $S_jg = \sum_{k\leq j} Q_kg$, $h = \sum_{j\in\mathbb{Z}} Q_jh$ and $S_jh = \sum_{k\leq j} Q_kh$. Thus, for all $\varphi \in \mathcal{S}$, we get

$$\begin{aligned} \left\langle \sum_{j\in\mathbb{Z}} (S_jg)(Q_jh), \varphi \right\rangle &= \sum_{j\in\mathbb{Z}} \langle Q_jh, (S_jg)\varphi \rangle = \sum_{j\in\mathbb{Z}} \left\langle Q_jh, \left(g - \sum_{k>j} Q_kg \right) \varphi \right\rangle \\ &= \langle h, g\varphi \rangle - \sum_{k\in\mathbb{Z}} \sum_{j\leq k-1} \langle Q_jh, (Q_kg)\varphi \rangle \\ &= \langle hg, \varphi \rangle - \left\langle \sum_{k\in\mathbb{Z}} (S_{k-1}h)(Q_kg), \varphi \right\rangle. \end{aligned}$$

Hence, we arrive at the decomposition of hg in $\mathcal{S}'$ into a sum $hg = A_1 + A_2$ where $A_1 := \sum_{j\in\mathbb{Z}} (S_jg)(Q_jh)$ and $A_2 := \sum_{k\in\mathbb{Z}} (S_{k-1}h)(Q_kg)$.

Step 2*: Estimate of* $\|[A_i]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$, $i = 1, 2$. We begin by noting that the functions $\mathcal{F}\{(S_j g)(Q_j h)\}$ and $\mathcal{F}\{(S_{k-1}h)(Q_k g)\}$ are supported by the balls $|\xi| \le 3 \cdot 2^j$ and $|\xi| \le (9/4)2^k$, respectively, then we can apply Proposition 2.4. First, we obtain

$$\begin{aligned}
&\|[A_1]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \\
&\lesssim \left(\sum_{j\in\mathbb{Z}} \left(2^{sj} \|S_j g\|_\infty \|Q_j h\|_p \right)^q \right)^{1/q} \\
&\lesssim \left\{ \sum_{j\in\mathbb{Z}} \left(2^{(n/p-m)j} \|Q_j h\|_p \right)^q \left(2^{(s+m-n/p)j} \sum_{k\le j} 2^{(n/p-s-m)k} 2^{(s+m)k} \|Q_k g\|_p \right)^q \right\}^{1/q} \\
&\lesssim \|[h]_{\mathcal{P}}\|_{\dot{B}^{n/p-m}_{p,q}} \|[g]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}},
\end{aligned}$$

where owing to assumptions (1.1) and (1.2) we have used *the convolution inequality*, see e.g., (3.4) or [17, Lemma 3.8]. Secondly, assume that $m > 0$, then we have

$$\begin{aligned}
\|[A_2]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} &\lesssim \left(\sum_{k\in\mathbb{Z}} \left(2^{sk} \|S_{k-1} h\|_\infty \|Q_k g\|_p \right)^q \right)^{1/q} \\
&\lesssim \left\{ \sum_{k\in\mathbb{Z}} \left(2^{-mk} \sum_{j\le k-1} 2^{mj} 2^{(n/p-m)j} \|Q_j h\|_p \right)^q \left(2^{(s+m)k} \|Q_k g\|_p \right)^q \right\}^{1/q} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}} \left(2^{-mk} \sum_{j\le k} 2^{mj} 2^{(n/p-m)j} \|Q_j h\|_p \right)^q \right)^{1/q} \|[g]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,\infty}} \\
&\lesssim \|[h]_{\mathcal{P}}\|_{\dot{B}^{n/p-m}_{p,q}} \|[g]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}}.
\end{aligned}$$

Assume now $m = 0$. Then

$$\|[A_2]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \lesssim \|h\|_\infty \|[g]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$$

holds. The proof of Lemma 3.3 is complete. □

Proof of Proposition 3.1*.* For the simplicity we will divide the proof in several steps.

Step 1*: Proof of* (i). As mentioned in the introduction on the method of Coifman and Meyer for the reduction to elementary symbols (cf. [5]), we will adapt this approach to *the homogeneous case*. We give the following construction:

We introduce a radial and positive function $\widetilde{\gamma} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ supported by the compact annulus $1/4 \le |\xi| \le 7/4$, and satisfies $\widetilde{\gamma}\gamma = \gamma$ (see the beginning of Section 2.1 for the definition of ρ and γ). We write $a(x,\xi) = \tau(x,\xi) + \theta(x,\xi)$ where $\tau(x,\xi) := a(x,\xi)\rho(\xi)$ and $\theta(x,\xi) := a(x,\xi)(1 - \rho(\xi))$, with

- $\tau(x,\xi)=0$ if $|\xi|\geq 3/2$,
- $\theta(x,\xi)=0$ if $|\xi|\leq 1$.

We now go to decompose $\theta(x,\xi)$. We first set

$$\theta_j(x,\xi):=2^{-mj}\gamma(2^{-|j|}\xi)\theta(x,2^{j-|j|}\xi),\quad j\in\mathbb{Z},\ \forall\, x,\xi\in\mathbb{R}^n, \tag{3.5}$$

which is well-defined. Then by definition of θ we have

$$\theta(x,\xi)=\theta(x,\xi)\sum_{j\in\mathbb{Z}}\gamma(2^{-j}\xi)=\sum_{j\in\mathbb{Z}}2^{mj}\theta_j(x,2^{-j+|j|}\xi). \tag{3.6}$$

Let us define the function

$$w_j(x,\xi):=\sum_{K\in\mathbb{Z}^n}\theta_j(x,2^{|j|}(\xi-2\pi K)), \tag{3.7}$$

which is 2π-periodic with respect to ξ, satisfying $w_j(x,\xi)\widetilde{\gamma}(\xi)=\theta_j(x,2^{|j|}\xi)$. Consequently, a Fourier expansion of w_j gives

$$\theta_j(x,2^{|j|}\xi)=\widetilde{\gamma}(\xi)\sum_{K\in\mathbb{Z}^n}C_{K,j}(x)e^{iK\cdot\xi}, \tag{3.8}$$

where

$$C_{K,j}(x):=(2\pi)^{-n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}e^{-iK\cdot\xi}w_j(x,\xi)\,\mathrm{d}\xi.$$

Now, by inserting (3.8) in (3.6), it follows

$$\theta(x,\xi):=\sum_{K\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}}2^{mj}C_{K,j}(x)\widetilde{\gamma}(2^{-j}\xi)e^{i2^{-j}K\cdot\xi}. \tag{3.9}$$

On the one hand, for $\xi\in\mathbb{R}^n$ and $K\in\mathbb{Z}^n$ satisfying $1/2\leq|\xi-2\pi K|\leq 3/2$ and $-\pi\leq\xi_\ell\leq\pi$, $\ell=1,\dots,n$, we have $|K|=0$. Hence, $w_j(x,\xi)$ coincides with $\theta_j(x,2^{|j|}\xi)$ since the function $\xi\mapsto\theta_j(x,2^{|j|}(\xi-2\pi K))$ has a support in $1/2\leq|\xi-2\pi K|\leq 3/2$. Thus

$$C_{K,j}(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{-iK\cdot\xi}\theta_j(x,2^{|j|}\xi)\,\mathrm{d}\xi. \tag{3.10}$$

On the other hand, for $N\in\mathbb{N}$, which will be fixed later on, we set

$$\chi_{K,j}(x):=(2\pi)^n(1+|K|^2)^{N/2}C_{K,j}(x), \tag{3.11}$$

$$\Theta_K(\xi):=(2\pi)^{-n}(1+|K|^2)^{-N/2}\widetilde{\gamma}(\xi)e^{iK\cdot\xi} \tag{3.12}$$

and

$$\vartheta_K(x,\xi):=\sum_{j\in\mathbb{Z}}2^{mj}\chi_{K,j}(x)\Theta_K(2^{-j}\xi),\quad K\in\mathbb{Z}^n,\ \forall\, x,\xi\in\mathbb{R}^n. \tag{3.13}$$

By substituting these formulas, (3.11)–(3.13), into (3.9) we get the desired decomposition:

$$\theta(x,\xi) = \sum_{K\in\mathbb{Z}^n} \vartheta_K(x,\xi), \quad \forall\, x,\xi \in \mathbb{R}^n. \tag{3.14}$$

Step 2: Proof of (ii).

Substep 2.1: Estimate of $\|\chi_{K,j}\|_\infty + \|[\chi_{K,j}]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,q}}$. We first choose N as an even integer (i.e., we put $N := 2M$ with $M \in \mathbb{N}$) and note that by inserting (3.10) in (3.11) we get

$$\chi_{K,j}(x) = \int_{\mathbb{R}^n} e^{-iK\cdot\xi}(I-\Delta_\xi)^M \left(\theta_j(x, 2^{|j|}\xi)\right) \mathrm{d}\xi.$$

The expression $(I-\Delta_\xi)^M \left(\theta_j(x,2^{|j|}\xi)\right)$ is the following sum (cf., see (3.5))

$$\sum_{|\mu|\le M}\sum_{\alpha+\eta=2\mu} \frac{M!}{(M-|\mu|)!\mu!}\frac{(2\mu)!}{\alpha!\eta!}(-1)^{|\mu|}2^{(|\alpha|-m)j}\gamma^{(\eta)}(\xi)\partial_\xi^\alpha\theta(x,2^j\xi),$$

which yields

$$\begin{aligned}\chi_{K,j}(x) = \sum_{|\mu|\le M}\sum_{\alpha+\eta=2\mu} &\frac{M!}{(M-|\mu|)!\mu!}\frac{(2\mu)!}{\alpha!\eta!}(-1)^{|\mu|}2^{(|\alpha|-m)j}\\ &\times\int_{\mathbb{R}^n} e^{-iK\cdot\xi}\gamma^{(\eta)}(\xi)\partial_\xi^\alpha\theta(x,2^j\xi)\,\mathrm{d}\xi.\end{aligned} \tag{3.15}$$

As

$$\partial_\xi^\alpha\theta(x,\xi) = (1-\rho(\xi))\partial_\xi^\alpha a(x,\xi) - \sum_{\beta\prec\alpha}\frac{\alpha!}{(\alpha-\beta)!\beta!}\rho^{(\alpha-\beta)}(\xi)\partial_\xi^\beta a(x,\xi)$$

(see Section 1 for the definition of $\prec$), then using the assumption on the symbol $a(x,\xi)$, we arrive at

$$2^{(|\alpha|-m)j}\left|\partial_\xi^\alpha\theta(x,2^j\xi)\right| \lesssim \Pi_N(a)\left(g_{\alpha,j}(\xi) + \sum_{\beta\prec\alpha} h_{\beta,j}(\xi)\right) \tag{3.16}$$

where

$$g_{\alpha,j}(\xi) := 2^{(|\alpha|-m)j}(1+2^j|\xi|)^{m-|\alpha|}\left|1-\rho(2^j\xi)\right| \tag{3.17}$$

and

$$h_{\beta,j}(\xi) := 2^{(|\alpha|-m)j}(1+2^j|\xi|)^{m-|\beta|}\left|\rho^{(\alpha-\beta)}(2^j\xi)\right|, \quad \beta\prec\alpha. \tag{3.18}$$

On $\operatorname{supp} g_{\alpha,j}$ we have $1\le 2^j|\xi|$, which implies that:

- if $m-|\alpha|>0$, then

$$|g_{\alpha,j}(\xi)| \le (1+\|\rho\|_\infty)2^{(|\alpha|-m)j}(2\cdot 2^j|\xi|)^{m-|\alpha|} \lesssim |\xi|^{m-|\alpha|};$$

- if $m-|\alpha|\le 0$, then

$$|g_{\alpha,j}(\xi)| \le (1+\|\rho\|_{\infty})2^{(|\alpha|-m)j}(2^j\,|\xi|)^{m-|\alpha|} \lesssim |\xi|^{m-|\alpha|}\,.$$

Thus in all cases we obtain

$$|g_{\alpha,j}(\xi)| \lesssim |\xi|^{m-|\alpha|}\,, \quad \forall\, j\in\mathbb{Z}. \tag{3.19}$$

Similar to the function $h_{\beta,j}$, on $\operatorname{supp} h_{\beta,j}$ we have $1\le 2^j\,|\xi|\le 3/2$ (i.e., $|\xi|\sim 2^{-j}$) which implies, both,

$$2^{(|\alpha|-m)j} \le \max(1,(3/2)^{|\alpha|-m})\,|\xi|^{m-|\alpha|}\,, \quad (1+2^j\,|\xi|)^{m-|\beta|} \le \max(2^{m-|\beta|},(5/2)^{m-|\beta|})$$

and

$$|h_{\beta,j}(\xi)| \lesssim |\xi|^{m-|\alpha|}\,, \quad \forall\, j\in\mathbb{Z}. \tag{3.20}$$

Now, we are able to estimate $\|\chi_{K,j}\|_{\infty}$. Indeed, by (3.19) and (3.20) we obtain

$$\begin{aligned}\|\chi_{K,j}\|_{\infty} &\lesssim \Pi_N(a)\sum_{|\mu|\le M}\sum_{\alpha+\eta=2\mu}\int_{1/2\le|\xi|\le 3/2}\left|\gamma^{(\eta)}(\xi)\right|\,|\xi|^{m-|\alpha|}\,\mathrm{d}\xi \\ &\lesssim \Pi_N(a), \quad \forall\, K\in\mathbb{Z}^n,\ \forall\, j\in\mathbb{Z}.\end{aligned} \tag{3.21}$$

We now turn to the estimate of $\|[\chi_{K,j}]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,q}}$. From (3.15), we write (for all $K\in\mathbb{Z}^n$ and all $k,j\in\mathbb{Z}$)

$$\begin{aligned}Q_k\chi_{K,j}(x) = \sum_{|\mu|\le M}\sum_{\alpha+\eta=2\mu}&\frac{M!}{(M-|\mu|)!\mu!}\,\frac{(2\mu)!}{\alpha!\eta!}(-1)^{|\mu|}2^{(|\alpha|-m)j} \\ &\times\int_{\mathbb{R}^n} e^{-iK\cdot\xi}\gamma^{(\eta)}(\xi)Q_k(\partial_\xi^\alpha\theta(x,2^j\xi))\,\mathrm{d}\xi.\end{aligned}$$

Again, on $\operatorname{supp}\gamma^{(\eta)}$ and because $p,q\ge 1$, it follows

$$\|[\chi_{K,j}]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,q}} \lesssim \sum_{|\mu|\le M}\sum_{\alpha=2\mu}2^{(|\alpha|-m)j}\int_{1/2\le|\xi|\le 3/2}\left\|[\partial_\xi^\alpha\theta(\cdot\,,2^j\xi)]_{\mathcal{P}}\right\|_{\dot{B}^{n/p}_{p,q}}\,\mathrm{d}\xi. \tag{3.22}$$

Consequently, as in (3.16), the assumption $a(x,\xi)\in S^{m,N}_{1,0}(L_\infty\cap\dot{B}^{n/p}_{p,q})$ yields

$$2^{(|\alpha|-m)j}\left\|[\partial_\xi^\alpha\theta(\cdot\,,2^j\xi)]_{\mathcal{P}}\right\|_{\dot{B}^{n/p}_{p,q}} \lesssim \Pi_N(a)\left(g_{\alpha,j}(\xi)+\sum_{\beta\prec\alpha}h_{\beta,j}(\xi)\right),$$

see (3.17) and (3.18) for definitions of $g_{\alpha,j}$ and $h_{\alpha,j}$ respectively. Then by inserting (3.19) and (3.20) into (3.22), and by using the fact that $1/2\le|\xi|\le 3/2$, we get

$$\|[\chi_{K,j}]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,q}} \lesssim \Pi_N(a), \quad \forall\, K\in\mathbb{Z}^n,\ \forall\, j\in\mathbb{Z}. \tag{3.23}$$

Now, the last inequality together with (3.21) yield that the sequence $(\chi_{K,j})_{j\in\mathbb{Z}}$ satisfies the assumption (iii) of Lemma 3.2 uniformly with respect to K.

Substep 2.2: Estimate of $\|[\theta(x,D)f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$ *for* $f \in \tilde{\dot{B}}^{s+m}_{p,q}$. Since Θ_K (see (3.12)) is defined on $\mathbb{R}^n \setminus \{0\}$, then by considering the intersection of the supports, i.e.,

$$\widetilde{\gamma}(2^{-j}\xi)\gamma(2^{-j-\ell}\xi) = 0 \quad \text{if } \ell \le -3 \text{ or } \ell \ge 2,$$

it holds that

$$\Theta_K(2^{-j}\xi) = \Theta_K(2^{-j}\xi) \sum_{-2\le\ell\le1} \gamma(2^{-j-\ell}\xi), \quad \forall\, \xi \in \mathbb{R}^n.$$

The assumption (1.2), Proposition 2.10 and formula (2.1) (i.e., $f = \sum_{j\in\mathbb{Z}} Q_j f$) yield

$$\Theta_K(2^{-j}D)f = \sum_{-2\le\ell\le1} (2^{jn}\mathcal{F}^{-1}\Theta_K(2^j\,\cdot)) * Q_{j+\ell}f.$$

We continue, since the function $\xi \mapsto \Theta_K(2^{-j}\xi)\gamma(2^{-j-\ell}\xi)$ has a support in the compact annulus $(1/8)2^j \le |\xi| \le 3\cdot 2^j$ (here we note that the assumption (i) of Lemma 3.2 is satisfied), then Lemma 3.2 can be applied to the elementary symbol defined by the formula (3.13), and it follows

$$\begin{aligned}
&\|[\vartheta_K(x,D)f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \\
&\lesssim \Pi_N(a) \sum_{-2\le\ell\le1} \left\{ \sum_{j\in\mathbb{Z}} \left(2^{j(s+m)} \left\| (2^{(j-\ell)n}\mathcal{F}^{-1}\Theta_K(2^{j-\ell}\,\cdot)) * Q_j f \right\|_p \right)^q \right\}^{1/q}.
\end{aligned} \tag{3.24}$$

Then the first observation is that by the Young inequality we have

$$\left\| (2^{(j-\ell)n}\mathcal{F}^{-1}\Theta_K(2^{j-\ell}\,\cdot)) * Q_j f \right\|_p \lesssim \left\|\mathcal{F}^{-1}\Theta_K\right\|_1 \|Q_j f\|_p, \quad \forall\, j \in \mathbb{Z},\ \forall\, K \in \mathbb{Z}^n. \tag{3.25}$$

The second observation that for an even natural number N_0, we can still write

$$\mathcal{F}^{-1}\Theta_K(x) = (2\pi)^{-n}(1+|x|^2)^{-N_0/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(I-\Delta_\xi)^{N_0/2}\Theta_K(\xi)\,\mathrm{d}\xi,$$

which, by the Cauchy-Schwartz inequality and Parseval equality, yields that

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\Theta_K\right\|_1 &\lesssim \left(\int_{\mathbb{R}^n} (1+|x|^2)^{-N_0}\,\mathrm{d}x \right)^{1/2} \left(\int_{1/4\le|\xi|\le 7/4} \left| (I-\Delta_\xi)^{N_0/2}\Theta_K(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \\
&\lesssim \sup_{|\alpha|\le N_0} \left\| \Theta_K^{(\alpha)} \right\|_\infty, \quad \forall\, K \in \mathbb{Z}^n
\end{aligned} \tag{3.26}$$

with the condition $N_0 > n/2$. Then we put

$$N_0 := 2N_1 \quad \text{with} \quad N_1 = \left[\frac{n}{4}\right] + 1. \tag{3.27}$$

We still get, from (3.12),

$$\left|\Theta_K^{(\alpha)}(\xi)\right| \lesssim (1+|K|^2)^{-N/2} \sum_{\beta+\eta=\alpha} |K|^{|\beta|} \left|\widetilde{\gamma}^{(\eta)}(\xi)\right|, \quad \forall\, K \in \mathbb{Z}^n,\ \forall\, \xi \in \mathbb{R}^n.$$

But, if $|\beta| \le N_0$ we have $|K|^{|\beta|}(1+|K|^2)^{-N/2} \le (1+|K|^2)^{(N_0-N)/2}$ for all $K \in \mathbb{Z}^n$, which implies that

$$\sup_{|\alpha|\le N_0} \left\|\Theta_K^{(\alpha)}\right\|_\infty \lesssim (1+|K|^2)^{(N_0-N)/2}, \quad \forall\, K \in \mathbb{Z}^n. \tag{3.28}$$

By inserting (3.28) into (3.26) and (3.25) into (3.24), we obtain that the right-hand side of (3.24) is bounded by

$$c(1+|K|^2)^{(N_0-N)/2}\Pi_N(a)\left\|[f]_{\mathcal{P}}\right\|_{\dot{B}^{s+m}_{p,q}},$$

where the constant c is independent of K and $a(x,\xi)$. We now turn to $\theta(x,D)f$, see (3.14). We have

$$\begin{aligned} \left\|[\theta(x,D)f]_{\mathcal{P}}\right\|_{\dot{B}^s_{p,q}} &\lesssim \sum_{K\in\mathbb{Z}^n} \left\|[\vartheta_K(x,D)f]_{\mathcal{P}}\right\|_{\dot{B}^s_{p,q}} \\ &\lesssim \Pi_N(a)\left\|[f]_{\mathcal{P}}\right\|_{\dot{B}^{s+m}_{p,q}} \sum_{K\in\mathbb{Z}^n} (1+|K|^2)^{(N_0-N)/2}. \end{aligned} \tag{3.29}$$

Clearly the last series converges if $N - N_0 > n$, i.e., $N > n + 2[n/4] + 2$ since the condition (3.27). Then it suffices to choose N as an even integer satisfying

$$N > \frac{3n}{2} + 2,$$

and we deduce that $\left\|[\theta(x,D)f]_{\mathcal{P}}\right\|_{\dot{B}^s_{p,q}}$ is bounded by $c\Pi_N(a)\left\|[f]_{\mathcal{P}}\right\|_{\dot{B}^{s+m}_{p,q}}$.

Substep 2.3*: Proof of* $\theta(x,D)f \in \mathcal{S}'$ *for* $f \in \dot{\tilde{B}}^{s+m}_{p,q}$. Let $\varphi \in \mathcal{S}$. We put $\widetilde{Q}_j := \widetilde{\gamma}(2^{-j}D)$, where the function $\widetilde{\gamma}$ is defined in the beginning of Step 1, and write $\varphi = S_0\varphi + \sum_{j\ge1} Q_j\widetilde{Q}_j\varphi$. It holds

$$\begin{aligned} |\langle\theta(x,D)f,\varphi\rangle| &= \left|\langle S_0(\theta(x,D)f),\varphi\rangle + \sum_{j\ge1}\left\langle Q_j(\theta(x,D)f),\widetilde{Q}_j\varphi\right\rangle\right| \\ &\le \|S_0(\theta(x,D)f)\|_\infty \|\varphi\|_1 + \sum_{j\ge1} \|Q_j(\theta(x,D)f)\|_\infty \left\|\widetilde{Q}_j\varphi\right\|_1 \\ &\lesssim \|S_0(\theta(x,D)f)\|_\infty \|\varphi\|_1 + \sum_{j\ge1} 2^{jn/p}\|Q_j(\theta(x,D)f)\|_p \left\|\widetilde{Q}_j\varphi\right\|_1. \end{aligned} \tag{3.30}$$

First, we have

$$\|S_0(\theta(x,D)f)\|_\infty \lesssim \sum_{j\le0} 2^{j(n/p-s)}\left(2^{js}\|Q_j(\theta(x,D)f)\|_p\right). \tag{3.31}$$

Secondly, by Proposition 2.1, $\left\|\widetilde{Q}_j\varphi\right\|_1 \lesssim 2^{-jL}\zeta_M(\varphi)$ for any $L \in \mathbb{N}$ and some $M \in \mathbb{N}$, then if $0 < s < n/p$, we choose an integer L satisfying $L > n/p - s$, and obtain

$$\begin{aligned}|\langle\theta(x,D)f,\varphi\rangle| &\lesssim \left(\sup_{k\in\mathbb{Z}} 2^{ks}\left\|Q_k(\theta(x,D)f)\right\|_p\right)\left(\sum_{j\le 0}2^{j(n/p-s)}+\sum_{j\ge 1}2^{j(n/p-s-L)}\right)\\ &\lesssim \left\|[\theta(x,D)f]_{\mathcal{P}}\right\|_{\dot{B}^s_{p,\infty}}\\ &\lesssim \left\|[\theta(x,D)f]_{\mathcal{P}}\right\|_{\dot{B}^s_{p,q}}.\end{aligned}$$

If $s = n/p$ and $q = 1$, then in (3.30) we use $\left\|\widetilde{Q}_j\varphi\right\|_1 \lesssim \|\varphi\|_1$ for all $j \ge 1$, and from (3.31) it also holds

$$|\langle\theta(x,D)f,\varphi\rangle| \lesssim \|\varphi\|_1 \left\|[\theta(x,D)f]_{\mathcal{P}}\right\|_{\dot{B}^{n/p}_{p,1}} \lesssim \left\|[\theta(x,D)f]_{\mathcal{P}}\right\|_{\dot{B}^{n/p}_{p,1}}.$$

Substep 2.4: Proof of $\theta(x,D)f \in \widetilde{C}_0$ *for* $f \in \dot{\tilde{B}}^{s+m}_{p,q}$. Let $\varphi \in \mathcal{S}$ and $\lambda > 0$.

- *The case* $s < n/p$. As in the previous substep and using the same notations, it holds that

$$\begin{aligned}&|\langle\theta(x,D)f(\cdot/\lambda),\varphi\rangle|\\ &\le \sum_{j\in\mathbb{Z}}\left\|Q_j(\theta(x,D)f(\cdot/\lambda))\right\|_\infty \left\|\widetilde{Q}_j\varphi\right\|_1\\ &\lesssim \sum_{j\in\mathbb{Z}}2^{sj}\left\|Q_j(\theta(x,D)f(\cdot/\lambda))\right\|_p 2^{j(n/p-s)}\left\|\widetilde{Q}_j\varphi\right\|_1\\ &\lesssim \left\|[\theta(x,D)f(\cdot/\lambda)]_{\mathcal{P}}\right\|_{\dot{B}^s_{p,\infty}}\sum_{j\in\mathbb{Z}}2^{j(n/p-s)}\left\|\widetilde{Q}_j\varphi\right\|_1,\\ &\lesssim \left\|[\theta(x,D)f(\cdot/\lambda)]_{\mathcal{P}}\right\|_{\dot{B}^s_{p,q}}\left(\zeta_M(\varphi)\sum_{j\ge 1}2^{j(n/p-s-L)}+\|\varphi\|_1\sum_{j\le 0}2^{j(n/p-s)}\right),\end{aligned}$$

for some integer $L > n/p - s$ and some $M \in \mathbb{N}$. Consequently, Proposition 2.5 gives

$$|\langle\theta(x,D)f(\cdot/\lambda),\varphi\rangle| \lesssim \lambda^{n/p-s}\left\|[f]_{\mathcal{P}}\right\|_{\dot{B}^{s+m}_{p,q}}(\zeta_M(\varphi)+\|\varphi\|_1).$$

Then the result follows by taking $\lambda \to 0$.

- *The case* $s = n/p$ *and* $q = 1$. The embedding $\dot{B}^{n/p}_{p,1} \hookrightarrow \dot{B}^0_{\infty,1}$ implies that $[\theta(x,D)f]_{\mathcal{P}}$ belongs to $\dot{B}^0_{\infty,1}$. Then we have $\sum_{j\in\mathbb{Z}}\|Q_j(\theta(x,D)f)\|_\infty < \infty$. Now, let $\varepsilon > 0$ be fixed arbitrarily. We write, for a $J \in \mathbb{N}$,

$$|\langle\theta(x,D)f(\cdot/\lambda),\varphi\rangle| \lesssim \sum_{|j|>J}\left\|Q_j(\theta(x,D)f)\right\|_\infty + \sum_{|j|\le J}\left|\langle Q_j(\theta(x,D)f)(\cdot/\lambda),\varphi\rangle\right|.$$

Choosing J such that the first term (i.e., with $\sum_{|j|>J}\cdots$) is less than ε. For the second term (i.e., with $\sum_{|j|\le J}\cdots$), we apply the following lemma which is proved in [2, p. 46] or [4, Proposition 4.4]:

Lemma 3.4. *If a function $h \in L_\infty$ satisfies that* $\operatorname{supp} \widehat{h}$ *is a compact set in* $\mathbb{R}^n \setminus \{0\}$, *then* $h \in \widetilde{C}_0$.

Thus $\lim_{\lambda \to 0} \sum_{|j| \le J} |\langle Q_j(\theta(x,D)f)(\cdot/\lambda), \varphi \rangle| = 0$.

Summarizing, from Substep 2.1 through 2.4, we have proved that the operator $\theta(x,D)$ is bounded from $\dot{\widetilde{B}}^{s+m}_{p,q}$ into $\dot{\widetilde{B}}^{s}_{p,q}$ and the proof of (ii) is complete.

Step 3*: Proof of* (iii). Arguing so as in Step 1 (i.e., (3.7)–(3.14)) to decompose $\tau(x,\xi)$ as an elementary symbol. We introduce a radial and positive function $\widetilde{\rho} \in \mathcal{D}$, supported by the ball $|\xi| \le 7/4$ and satisfying $\widetilde{\rho}\rho = \rho$. We also introduce a 2π-periodic function denoted by

$$v(x,\xi) := \sum_{K \in \mathbb{Z}^n} \tau(x, \xi - 2\pi K).$$

Since $\tau(x,\xi - 2\pi K)\widetilde{\rho}(\xi) = 0$ if $K \neq 0$, then from the Fourier expansion of v, we have

$$v(x,\xi)\widetilde{\rho}(\xi) = \tau(x,\xi) = \widetilde{\rho}(\xi) \sum_{K \in \mathbb{Z}^n} C_K(x) e^{iK\cdot\xi}. \tag{3.32}$$

If $\xi \in \mathbb{R}^n$ and $K \in \mathbb{Z}^n$ satisfy both $|\xi - 2\pi K| \le 3/2$ and $-\pi \le \xi_\ell \le \pi$, $\ell = 1, \ldots, n$, then $K = 0$ and $v(x,\xi) = \tau(x,\xi)$. This yields

$$C_K(x) := (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-iK\cdot\xi} v(x,\xi)\, \mathrm{d}\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iK\cdot\xi} \tau(x,\xi)\, \mathrm{d}\xi.$$

We continue, for an even natural number $N \in \mathbb{N}$ $(N := 2M)$, which will be chosen later on, we set

$$\chi_K(x) := (2\pi)^{-n} (1 + |K|^2)^{N/2} C_K(x),$$

i.e.,

$$\chi_K(x) = \int_{\mathbb{R}^n} e^{-iK\cdot\xi} (I - \Delta_\xi)^{N/2} \tau(x,\xi)\, \mathrm{d}\xi.$$

Consequently, as in (3.14) and from (3.32), we arrive at the following expression

$$\tau(x,D)f(x) = \sum_{K \in \mathbb{Z}^n} (1 + |K|^2)^{-N/2} \chi_K(x) \mathcal{F}^{-1}(\widetilde{\rho}\widehat{f})(x + K), \tag{3.33}$$

wherein Lemma 3.3 can be applied with $h := \chi_K$ and $g := \mathcal{F}^{-1}(\widetilde{\rho}\widehat{f})(\cdot + K)$:

• $h \in L_\infty \cap \dot{\widetilde{B}}^{n/p-m}_{p,q}$, *i.e., estimate of* $\|\chi_K\|_\infty + \|[\chi_K]_{\mathcal{P}}\|_{\dot{B}^{n/p-m}_{p,q}}$. This can be done as the estimate of $(\chi_{K,j})_{j \in \mathbb{Z}}$ in Substep 2.1. Indeed, instead of (3.21) and (3.22), we still get

$$\|\chi_K\|_\infty \lesssim \Pi_N(a) \sum_{|\mu| \le M} \sum_{\alpha + \eta = 2\mu} \int_{|\xi| \le 3/2} \left| \rho^{(\eta)}(\xi) \right| (1 + |\xi|)^{m - |\alpha|}\, \mathrm{d}\xi \lesssim \Pi_N(a)$$

for all $K \in \mathbb{Z}^n$, and

$$\begin{aligned}\|[\chi_K]_{\mathcal{P}}\|_{\dot{B}^{n/p-m}_{p,q}} &\lesssim \sum_{|\mu|\le M}\sum_{\alpha=2\mu}\int_{|\xi|\le 3/2}\left\|[\partial_\xi^\alpha \tau(\cdot\,,\xi)]_{\mathcal{P}}\right\|_{\dot{B}^{n/p-m}_{p,q}}\,\mathrm{d}\xi \\ &\lesssim \Pi_N(a)\sum_{|\mu|\le M}\sum_{\alpha=2\mu}\int_{|\xi|\le 3/2}(1+|\xi|)^{m-|\alpha|}\,\mathrm{d}\xi \\ &\lesssim \Pi_N(a)\end{aligned}$$

for all $K \in \mathbb{Z}^n$, respectively.

• $g \in C^\infty \cap \dot{\widetilde{B}}^{s+m}_{p,q}$. We first see the estimate of $\left\|[(\mathcal{F}^{-1}\widetilde{\rho}) * f(\cdot+K)]_{\mathcal{P}}\right\|_{\dot{B}^{s+m}_{p,q}}$. By the Young inequality it holds $\left\|Q_j\left((\mathcal{F}^{-1}\widetilde{\rho}) * f\right)\right\|_p \le \left\|\mathcal{F}^{-1}\widetilde{\rho}\right\|_1 \|Q_j f\|_p$. Consequently by using the fact that $\|\cdot\|_{\dot{B}^{s+m}_{p,q}}$ is translation invariant, we obtain

$$\left\|[(\mathcal{F}^{-1}\widetilde{\rho}) * f(\cdot+K)]_{\mathcal{P}}\right\|_{\dot{B}^{s+m}_{p,q}} = \left\|[(\mathcal{F}^{-1}\widetilde{\rho}) * f]_{\mathcal{P}}\right\|_{\dot{B}^{s+m}_{p,q}} \lesssim \|[f]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}}\,.$$

Secondly, we prove that $g \in \widetilde{C}_0$. Indeed, we apply the following assertion: If $f \in \widetilde{C}_0$, $\varphi \in \mathcal{S}$ and $a \in \mathbb{R}^n$ then

$$\left\langle f\left(\lambda^{-1}(\cdot)-a\right),\varphi\right\rangle = \left\langle f(\lambda^{-1}(\cdot)),\varphi(\cdot+\lambda a)\right\rangle \to 0 \quad \text{as } \lambda \to 0$$

since $\varphi(\cdot+\lambda a)$ still belongs to $\mathcal{S}$. Thus by a simple calculation, for all $\varphi \in \mathcal{S}$, we have

$$\left\langle g(\lambda^{-1}(\cdot)),\varphi\right\rangle = \left\langle f(\lambda^{-1}(\cdot)),\mathcal{F}^{-1}\widetilde{\rho} * \varphi_\lambda\left(K-\lambda^{-1}(\cdot)\right)\right\rangle \quad \text{with } \varphi_\lambda := \varphi(\lambda(\cdot)),$$

and $\left\langle g(\lambda^{-1}(\cdot)),\varphi\right\rangle \to 0$ as $\lambda \to 0$ since $\mathcal{F}^{-1}\widetilde{\rho} * \varphi_\lambda\left(K-\lambda^{-1}(\cdot)\right) \in \mathcal{S}$. Finally it is clear that g is a C^∞ function.

Applying now Lemma 3.3 to the expression (3.33), and choosing N an even integer greater than n, we obtain

$$\|[\tau(x,D)f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \lesssim \Pi_N(a)\,\|[f]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}} \sum_{K\in\mathbb{Z}^n}(1+|K|^2)^{-N/2} \lesssim \Pi_N(a)\,\|[f]_{\mathcal{P}}\|_{\dot{B}^{s+m}_{p,q}}\,.$$

We note that the proof of, both, $\tau(x,D)f \in \mathcal{S}'$ and $\tau(x,D)f \in \widetilde{C}_0$ for all $f \in \dot{\widetilde{B}}^{s+m}_{p,q}$ is similar to the case of the operator $\theta(x,D)$, see Substeps 2.3 and 2.4. Hence the proof of Proposition 3.1 is complete. □

Proof of Theorem 1.1. This is similar to the proofs of Proposition 3.1(i) and (ii) where we do not need of the decomposition $a(x,\xi) = \tau(x,\xi) + \theta(x,\xi)$, but we take $\theta(x,\xi) := a(x,\xi)$, e.g., instead of (3.5) we directly write

$$\theta_j(x,\xi) := 2^{-mj}\gamma(2^{-|j|}\xi)a(x,2^{j-|j|}\xi), \quad j \in \mathbb{Z},\ \forall\, x,\xi \in \mathbb{R}^n,$$

and obtain the functions $\vartheta_K(x,\xi)$ and the decomposition $a(x,\xi)=\sum_{K\in\mathbb{Z}^n}\vartheta_K(x,\xi)$ as in (3.13) and (3.14), respectively. In the same way of (3.21) and (3.23) we have

$$\|\chi_{K,j}\|_\infty+\|[\chi_{K,j}]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,q}}\lesssim\dot{\Pi}_N(a),\quad\forall\,K\in\mathbb{Z}^n,\ \forall\,j\in\mathbb{Z}.$$

The rest of the proof is unchanged. □

Proof of Theorem 1.2. This follows from Proposition 3.1. □

4. Some generalizations and remarks

We begin by an extension on the symbol classes: For $(\mu,\eta)\in\mathbb{R}^2$, $(M,N)\in\mathbb{N}^2$ and a Banach space E, we introduce the classes $\dot{S}^{m,N,M}_{\mu,\eta}(E)$ and $S^{m,N,M}_{\mu,\eta}(E)$ of functions $a(x,\xi)$ in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$ and in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, respectively. These classes are defined (as Fréchet spaces) by the seminorms

$$\dot{\Pi}_{N,M}(a):=\sup_{\substack{|\alpha|\le N\\|\beta|\le M}}\ \sup_{\xi\in\mathbb{R}^n\setminus\{0\}}|\xi|^{-m+\mu|\alpha|-\eta|\beta|}\left\|\partial_\xi^\alpha\partial_x^\beta a(\cdot\,,\xi)\right\|_E,$$

$$\Pi_{N,M}(a):=\sup_{\substack{|\alpha|\le N\\|\beta|\le M}}\ \sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^{-m+\mu|\alpha|-\eta|\beta|}\left\|\partial_\xi^\alpha\partial_x^\beta a(\cdot\,,\xi)\right\|_E,$$

respectively. Then Theorems 1.1 and 1.2 can be generalized in the following sense:

Theorem 4.1. *Let s, p, q, m, N be real numbers given as in Theorem* 1.1 *(*1.2, *respectively). Let $M\in\mathbb{N}$, $\eta\in\mathbb{R}$ and $\mu\ge 1$. If a symbol $a(x,\xi)$ belongs to $\dot{S}^{m,N,M}_{\mu,\eta}(L_\infty\cap\dot{B}^{n/p}_{p,q})$ ($S^{m,N,M}_{\mu,\eta}(L_\infty\cap\dot{B}^{n/p}_{p,q}\cap\dot{B}^{n/p-m}_{p,q})$, respectively), then the ps.d.o $a(x,D)$ takes the space $\tilde{\dot{B}}^{s+m}_{p,q}$ into $\tilde{\dot{B}}^{s}_{p,q}$, with an estimate similar to* (1.3) *wherein $\dot{\Pi}_N(a)$ is replaced by $\dot{\Pi}_{N,M}(a)$ ($\Pi_{N,M}(a)$, respectively).*

Proof. We only check the similar estimates of functions $g_{\alpha,j}(\xi)$ and $h_{\beta,j}(\xi)$ given in (3.17) and (3.18), respectively, which must be replaced by

$$g_{\alpha,j}(\xi):=2^{(|\alpha|-m)j}(1+2^j|\xi|)^{m-\mu|\alpha|}\left|1-\rho(2^j\xi)\right|,\quad|\alpha|\le N$$

and

$$h_{\nu,j}(\xi):=2^{(|\alpha|-m)j}(1+2^j|\xi|)^{m-\mu|\alpha|}\left|\rho^{(\alpha-\nu)}(2^j\xi)\right|,\quad|\alpha|\le N,\ \nu\prec\alpha.$$

Owing to the support of $1-\rho(2^j\xi)$ we have $1\le 2^j|\xi|$, then as in (3.19) we obtain:

- if $m-\mu|\alpha|>0$, then

$$\begin{aligned}|g_{\alpha,j}(\xi)|&\le(1+\|\rho\|_\infty)2^{(|\alpha|-m)j}(2\cdot 2^j|\xi|)^{m-\mu|\alpha|}\\&\lesssim 2^{(1-\mu)|\alpha|j}|\xi|^{m-\mu|\alpha|}\\&\lesssim|\xi|^{m-|\alpha|},\end{aligned}$$

- if $m - \mu |\alpha| \leq 0$, then

$$\begin{aligned} |g_{\alpha,j}(\xi)| &\leq (1 + \|\rho\|_\infty) 2^{(|\alpha|-m)j} (2^j |\xi|)^{m-\mu|\alpha|} \\ &\lesssim 2^{(1-\mu)|\alpha|j} |\xi|^{m-\mu|\alpha|} \\ &\lesssim |\xi|^{m-|\alpha|}, \end{aligned}$$

and in all cases we obtain $|g_{\alpha,j}(\xi)| \lesssim |\xi|^{m-|\alpha|}$. Similarly for $h_{\nu,j}(\xi)$ (here as in (3.20)) since on the support of $\rho^{(\alpha-\nu)}(2^j\xi)$ we have $|\xi| \sim 2^{-j}$. Consequently, the argument proof of Theorem 4.1 remains completely similar to the proofs of Theorems 1.1 and 1.2. □

Now, we see an extension on the realized spaces: The condition (1.1) can be extended in the following sense. For

$$s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}, \quad \text{or} \quad s - \frac{n}{p} \in \mathbb{N} \setminus \{0\} \quad \text{and} \quad q = 1, \tag{4.1}$$

we denote by ν the integer ≥ 1 defined by $s - n/p \leq \nu < s - n/p + 1$, i.e.,

$$\nu := \begin{cases} [s - n/p] + 1 & \text{if } s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}, \\ s - n/p & \text{if } s - n/p \in \mathbb{N} \setminus \{0\} \text{ and } q = 1. \end{cases} \tag{4.2}$$

Then *the realized homogeneous Besov space* $\dot{\widetilde{B}}^s_{p,q}$ in this case is defined as the following, cf., [2, p. 47] or [18, p. 113] or [11, p. 150]:

Definition 4.2. Assume that (4.1) holds. The space $\dot{\widetilde{B}}^s_{p,q}$ is the set of all $f \in \mathcal{S}'$ satisfying the following four conditions: (i) $[f]_{\mathcal{P}} \in \dot{B}^s_{p,q}$, (ii) f is of class $C^{\nu-1}$, (iii) $f^{(\alpha)}(0) = 0$ for $|\alpha| < \nu$, (iv) $f^{(\alpha)} \in \widetilde{C}_0$ for $|\alpha| = \nu$.

We have the following result:

Theorem 4.3. *Let $s > 0$ and $1 \leq p < \infty$. Assume that (4.1) holds. Let N be an even natural number satisfying $N > 3n/2 + 2$. Let m be a real number satisfying*

$$s + m - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}, \quad \textit{or} \quad s + m - \frac{n}{p} \in \mathbb{N} \setminus \{0\} \quad \textit{and} \quad q = 1. \tag{4.3}$$

If a symbol $a(x,\xi)$ belongs to $\dot{S}^{m,N}_{1,0}(L_\infty \cap \dot{B}^{n/p}_{p,q})$, then the ps.d.o $a(x,D)$ takes the space $\dot{\widetilde{B}}^{s+m}_{p,q}$ into $\dot{\widetilde{B}}^s_{p,q}$. Moreover, an estimate similar to (1.3) is obtained for all $f \in \dot{\widetilde{B}}^{s+m}_{p,q}$.

The proof is similar to that of Theorem 1.1. We give a sketchy proof. According to Propositions 2.4, 2.10 and Lemma 3.2 we have the following assertions, in which the two propositions below are proved in [11, Proposition 2.17] and [11, Theorems 1.2, 4.5, Section 4.2], respectively.

Proposition 4.4. *Let $s > 0$ and $1 \le p < \infty$. Assume that (4.1) holds. Let $b > 0$. Let $(u_j)_{j\in\mathbb{Z}}$ be a sequence in $\mathcal{S}'$ satisfying the following two conditions:*

- $\widehat{u}_j$ *is supported by the ball* $|\xi| \le b2^j$,
- $A := \left(\sum_{j\in\mathbb{Z}}(2^{js}\left\|u_j\right\|_p)^q\right)^{1/q} < \infty.$

We put $w_j(x) := u_j(x) - \sum_{|\alpha|\le\nu-1} u_j^{(\alpha)}(0)x^\alpha/\alpha!$ for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$, (ν is defined in (4.2)). Then the series $\sum_{j\in\mathbb{Z}} w_j$ converges in $\mathcal{S}'$ to a limit w which satisfies $\|[w]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \le cA$, and where c depends only on n, s, p, q and b.

Proposition 4.5. *Assume that (4.1) holds. If $u \in \dot{B}^s_{p,q}$, we put $v_j(x) := Q_j u(x) - \sum_{|\alpha|\le\nu-1}(Q_j u)^{(\alpha)}(0)x^\alpha/\alpha!$ for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$. Then the series $\sum_{j\in\mathbb{Z}} v_j$ converges in $\mathcal{S}'$. We put $f := \sum_{j\in\mathbb{Z}} v_j$. Then $f^{(\alpha)}$ ($|\alpha| = \nu$) belongs to $\widetilde{C}_0$, and f is the unique tempered distribution satisfying $[u]_{\mathcal{P}} = f$ in $\mathcal{S}'_\infty$.*

Remark 4.6. As in Remark 2.12, owing to conditions (4.1) and (4.3), Proposition 4.5 guarantees that the two spaces $\dot{\widetilde{B}}{}^s_{p,q}$ and $\dot{\widetilde{B}}{}^{s+m}_{p,q}$ are well-defined as subspaces of $\mathcal{S}'$.

Lemma 4.7. *Let $s > 0$ and $1 \le p < \infty$. Assume that (4.1) holds. Let $b > 0$. Let $(u_j)_{j\in\mathbb{Z}}$ and $(\chi_j)_{j\in\mathbb{Z}}$ be sequences in $\mathcal{S}'$ satisfying the following three conditions:*

(i) $\widehat{u}_j$ *is supported by the ball* $|\xi| \le b2^j$,

(ii) $A := \left(\sum_{j\in\mathbb{Z}}(2^{js}\left\|u_j\right\|_p)^q\right)^{1/q} < \infty,$

(iii) $B := \sup_{j\in\mathbb{Z}}\left(\left\|\chi_j\right\|_\infty + \|[\chi_j]_{\mathcal{P}}\|_{\dot{B}^{n/p}_{p,q}}\right) < \infty.$

We put $z_j(x) := \chi_j u_j(x) - \sum_{|\alpha|\le\nu-1}(\chi_j u_j)^{(\alpha)}(0)x^\alpha/\alpha!$ for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$. Then the series $\sum_{j\in\mathbb{Z}} z_j$ converges in $\mathcal{S}'$ to a limit z which satisfies $\|[z]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} \le cAB$, and where c depends only on n, s, p, q and b.

Proof. This can be done as the proof of Lemma 3.2, we only give a little change in definitions of V and W of (3.2). We put $\sum_{j\in\mathbb{Z}} z_j = \sum_{j\in\mathbb{Z}} V_j + \sum_{k\in\mathbb{Z}} W_k$, where

$$V_j(x) := u_j(x)S_j\chi_j(x) - \sum_{|\alpha|\le\nu-1}(u_jS_j\chi_j)^{(\alpha)}(0)\frac{x^\alpha}{\alpha!},$$
$$W_k(x) := \sum_{j<k} u_j(x)Q_k\chi_j(x) - \sum_{|\alpha|\le\nu-1}\sum_{j<k}(u_jQ_k\chi_j)^{(\alpha)}(0)\frac{x^\alpha}{\alpha!},$$

then instead of Proposition 2.4 we apply Proposition 4.4 to the sequences $(V_j)_{j\in\mathbb{Z}}$ and $(W_k)_{k\in\mathbb{Z}}$. We omit the details. □

Proof of Theorem 4.3. Steps 1 and 2 of the proof of Proposition 3.1 are globally unchanged, we briefly outline. Since $\|g\|_{\dot{B}^s_{p,q}} = 0$ for all $g \in \mathcal{P}_\infty$, then

$$\|[a(x,D)f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}} = \left\| \left[a(x,D)f - \sum_{|\alpha|\leq \nu-1} (a(x,D)f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \right]_{\mathcal{P}} \right\|_{\dot{B}^s_{p,q}}.$$

Using notations of the proof of Proposition 3.1, then it reduces to estimate $\|[U(f)]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$ where

$$U(f) := \sum_{K\in\mathbb{Z}^n} \left(\vartheta_K(x,D)f(x) - \sum_{|\alpha|\leq\nu-1} (\vartheta_K(x,D)f)^{(\alpha)}(0)\frac{x^\alpha}{\alpha!} \right).$$

We argue so as in (3.29), then it suffices to apply Lemma 4.7 to the elementary symbol $\vartheta_K(x,\xi)$, see (3.13)–(3.14), we obtain the correct bound. To prove that $U(f) \in \mathcal{S}'$ and $U(f)^{(\alpha)} \in \widetilde{C}_0$ for $|\alpha| = \nu$ and all $f \in \dot{\widetilde{B}}{}^{s+m}_{p,q}$, we apply the same method used in proofs of Substeps 2.3 and 2.4 of Proposition 3.1. □

Remark 4.8. Theorem 4.3 holds also for symbols of the class $\dot{S}^{m,N,M}_{\mu,\eta}(L_\infty \cap \dot{B}^{n/p}_{p,q})$ with $N > 3n/2 + 2$ (N is an even integer), $M \in \mathbb{N}$, $\eta \in \mathbb{R}$ and $\mu \geq 1$, cf., Theorem 4.1.

Remark 4.9. It seems clearly that we can replace Besov spaces by Triebel-Lizorkin spaces in all the above results, see [4, 11].

Remark 4.10. It would be interesting to extend the boundedness of the ps.d.o on localized of realized homogeneous Besov spaces $(\dot{\widetilde{B}}{}^s_{p,q})_{\ell_r(\mathbb{Z}^n)}$, $1 \leq r \leq \infty$, which is defined as the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{(\dot{\widetilde{B}}{}^s_{p,q})_{\ell_r(\mathbb{Z}^n)}} := \left(\sum_{K\in\mathbb{Z}^n} \|[\psi(\cdot - K)f]_{\mathcal{P}}\|^r_{\dot{B}^s_{p,q}} \right)^{1/r} < \infty,$$

where ψ is a C^∞ function supported by the ball $|\xi| \leq R$, with $R > \sqrt{n}$, and satisfies that $\sum_{K\in\mathbb{Z}^n} \psi(\xi - K) = 1$ for all $\xi \in \mathbb{R}^n$, cf., [12].

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