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Some Remarks on Measure-theoretic Entropy for a Free Semigroup Action

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Abstract. In this paper, we study some properties about measure-theoretic entropy for a free semigroup action. We show some properties like conjugacy, power rule and affinity about the measure-theoretic entropy for a free semigroup action.

1. Introduction

The notion of entropy plays an important role in dynamic systems. In 1959, Kolmogorov and Sinai introduced the notion of measure-theoretic entropy. In 1965, the notion of topological entropy was introduced by Adler, Konheim and McAndrew [1]. Along with the deepening of the study, some researchers tried to find some suitable generalizations of topological entropy and measure-theoretic entropy for other systems and study these entropies. For example, the entropy of countable amenable group actions was studied by Ornstein and Weiss [15], Rudolph and Weiss [16], Dooley and Zhang [9] et al. The entropy of countable sofic group actions was studied by Bowen [4, 5], Kerr and Li [11], Chung and Zhang [7] et al. Kirillov [12] introduced the notion of entropy for the action of finitely generated groups of measure-preserving transformations. Bis [2] and Bufetov [6] introduced the notion of the topological entropy for a free semigroup action. Biś and Urbański [3], Ma and Wu [14], Wang, Ma and Lin [18, 19] and so on further studied the topological entropy for a free semigroup action. The notion of measure-theoretic entropy for a nonautonomous dynamical system was introduced by Zhu, Liu, Xu and Zhang [20]. Lin, Ma and Wang [13] introduced the notion of measure-theoretic entropy for a free semigroup action.

Since entropy appeared to be an important invariant in ergodic theory and dynamical systems, on the basis of [13], we further study the property of the measure-theoretic entropy for a free semigroup action. This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we give some properties of the measure-theoretic entropy for a free semigroup action.

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2. Preliminaries

Before studying the measure-theoretic entropy for a free semigroup action, we introduce some notations. Denote by F_m^+ the set of all finite words of symbols $0, 1, \dots, m-1$. For any $w \in F_m^+$, $|w|$ stands for the length of w, that is, the number of symbols in w. Obviously, F_m^+ with respect to this law of composition is a free semigroup with m generators. If $w, w' \in F_m^+$, then let ww' be the word obtained by writing w' to the right of w. We write $w \le w'$ if there exists a word $w'' \in F_m^+$ such that $w' = w''w$.

Denote by Σ_m the set of all two-side infinite sequences of symbols $0, 1, \dots, m-1$, i.e.,

$$\Sigma_m = \{\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots) \mid \omega_i = 0, 1, \dots, m-1 \text{ for all integers } i\}.$$

A metric on Σ_m is introduced by

$$d(\omega, \omega') = \frac{1}{2^k}, \quad \text{where } k = \inf\left\{|n| : \omega_n \neq \omega'_n\right\}.$$

Obviously, Σ_m is compact with respect to this metric. Recall that the Bernoulli shift $\sigma_m \colon \Sigma_m \to \Sigma_m$ is a homeomorphism of Σ_m given by the formula:

$$(\sigma_m \omega)_i = \omega_{i+1}.$$

Let $\omega \in \Sigma_m$, $w \in F_m^+$, a, b integers, and $a \le b$. We write $\omega|_{[a,b]} = w$ if $w = \omega_a \omega_{a+1} \cdots \omega_{b-1} \omega_b$.

Let $(X, \mathscr{B}, \mu)$ be a probability space. Suppose that a free semigroup with m generators acts on X; denote the maps corresponding to the generators by $f_0, f_1, \dots, f_{m-1}$; we assume that these maps are measure-preserving transformations. Let $w \in F_m^+$, $w = w_1 w_2 \cdots w_k$, where $w_i = 0, 1, \dots, m-1$ for all $i = 1, 2, \dots, k$. Let $f_w = f_{w_1} f_{w_2} \cdots f_{w_k}$, $f_w^{-1} = f_{w_k}^{-1} f_{w_{k-1}}^{-1} \cdots f_{w_1}^{-1}$. Obviously, $f_{ww'} = f_w f_{w'}$.

Let $(X, \mathscr{B}, \mu)$ be a probability space. Let $\xi = \{A_1, \dots, A_k\}$ be a finite partition of $(X, \mathscr{B}, \mu)$. Let $\eta = \{C_1, \dots, C_l\}$ be another finite partition of $(X, \mathscr{B}, \mu)$. The join of ξ and η is the partition

$$\xi \vee \eta = \{A_i \cap C_j : 1 \le i \le k, 1 \le j \le l\}.$$

We write $\xi \le \eta$ to mean that each element of ξ is a union of elements of η. Under the convention that $0 \log 0 = 0$, the entropy of the partition ξ is

$$H_\mu(\xi) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

The conditional entropy of ξ relative to η is given by

$$H_\mu(\xi \mid \eta) = -\sum_{\mu(C_j) \neq 0} \sum_{i=1}^{k} \mu(A_i \cap C_j) \log \frac{\mu(A_i \cap C_j)}{\mu(C_j)}.$$

We denote the set of all finite partitions of X by $\mathcal{L}$, then $\rho(\xi,\eta) := H_\mu(\xi\,|\,\eta) + H_\mu(\eta\,|\,\xi)$ is a metric on $\mathcal{L}$.

Let $(X,\mathscr{B},\mu)$ be a probability space and $f_0, f_1, \ldots, f_{m-1}$ measure-preserving transformations on X. If all f_i, $i = 0,1,\ldots,m-1$, preserve the same probability measure μ, then we say that $f_0, f_1, \ldots, f_{m-1}$ preserve μ, or μ is an f_i-invariant measure. Denote by $M(f_0,\ldots,f_{m-1})$ the set of all probability measures which are invariant under all f_i.

The following example shows that $M(f_0,\ldots,f_{m-1})$ can be nonempty even if some f_i and f_j do not commute with each other.

Example 2.1. [13, Example 5.4] Let A and B be the endomorphisms on the two-dimensional torus $\mathbb{T}^2$ introduced by the matrices

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix}$$

respectively. Let H be the semigroup generated by A and B. Obviously, H is a non-Abelian semigroup. Let μ be the Haar measure defined on $\mathbb{T}^2$. Then we have $\mu \in M(A,B)$, i.e., $M(A,B) \neq \emptyset$.

If $\xi \in \mathcal{L}$, denote

$$h_\mu(f_0,\ldots,f_{m-1},\xi) = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{m^n}\sum_{|w|=n} H_\mu\left(\bigvee_{w'\le w} f_{w'}^{-1}\xi\right)\right].$$

In the paper [13], the measure-theoretic entropy for a free semigroup action is defined by

$$h_\mu(f_0,\ldots,f_{m-1}) = \sup_{\xi\in\mathcal{L}} h_\mu(f_0,\ldots,f_{m-1},\xi).$$

If we let $\mathscr{F} := \{f_0,\ldots,f_{m-1}\}$, then we also denote $h_\mu(f_0,\ldots,f_{m-1})$ by $h_\mu(\mathscr{F})$.

Remark 2.2. If $m = 1$, then $h_\mu(f_0)$ is the classical measure-theoretic entropy of a single transformation (see e.g., [17]).

Let X be a compact metric space with metric d. Assume that $f_0, f_1, \ldots, f_{m-1}$ are continuous maps on X. To each $w \in F_m^+$, a new metric d_w on X (named Bowen metric) is given by

$$d_w(x_1,x_2) = \max_{w'\le w} d(f_{w'}(x_1), f_{w'}(x_2)).$$

Let $\varepsilon > 0$, a subset E of X is said to be a $(w,\varepsilon,f_0,\ldots,f_{m-1})$-spanning subset if, for $\forall\, x \in X$, $\exists\, y \in E$ with $d_w(x,y) < \varepsilon$. The minimal cardinality of a $(w,\varepsilon,f_0,\ldots,f_{m-1})$-spanning subset of X is denoted by $B(w,\varepsilon,f_0,\ldots,f_{m-1})$. Let

$$B(n,\varepsilon,f_0,\ldots,f_{m-1}) = \frac{1}{m^n}\sum_{|w|=n} B(w,\varepsilon,f_0,\ldots,f_{m-1}).$$

In the paper [6], the topological entropy for a free semigroup action is defined by

$$h(f_0,\dots,f_{m-1}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log B(n,\varepsilon,f_0,\dots,f_{m-1}).$$

By the Partial Variational Principle from the paper [13], we have

$$\sup_{\mu\in M(f_0,\dots,f_{m-1})} h_\mu(f_0,\dots,f_{m-1}) \le h(f_0,\dots,f_{m-1}).$$

Let $(X,\mathscr{B},\mu)$ be a probability space. Define an equivalence relation on $\mathscr{B}$ by saying A and B are equivalent if and only if $\mu(A\triangle B)=0$. Let $\widetilde{\mathscr{B}}$ denote the collection of equivalence classes. Then $\widetilde{\mathscr{B}}$ is a Boolean σ-algebra under the operation of complementation, union and intersection inherited from $\mathscr{B}$. The measure μ induces a measure $\widetilde{\mu}$ on $\widetilde{\mathscr{B}}$ by $\widetilde{\mu}(\widetilde{B})=\mu(B)$. (Here $\widetilde{B}$ is the equivalence class to which B belongs.) The pair $(\widetilde{\mathscr{B}},\widetilde{\mu})$ is called a measure algebra.

Let $(X_1,\mathscr{B}_1,\mu_1)$ and $(X_2,\mathscr{B}_2,\mu_2)$ be probability spaces with measure algebras $(\widetilde{\mathscr{B}_1},\widetilde{\mu_1})$, $(\widetilde{\mathscr{B}_2},\widetilde{\mu_2})$. The measure algebras are isomorphic if there is a bijection $\Phi\colon \widetilde{\mathscr{B}_2}\to\widetilde{\mathscr{B}_1}$ which preserves complements, countable unions and intersections and satisfies $\widetilde{\mu_1}(\Phi\widetilde{B})=\widetilde{\mu_2}(\widetilde{B})$, $\forall\,\widetilde{B}\in\mathscr{B}_2$.

Let T_i be a measure-preserving transformation of the probability space $(X_i,\mathscr{B}_i,\mu_i)$, $i=1,2$. We say that T_1 is conjugate to T_2 if there is a measure-algebra isomorphism $\Phi\colon(\widetilde{\mathscr{B}_2},\widetilde{\mu_2})\to(\widetilde{\mathscr{B}_1},\widetilde{\mu_1})$ such that $\Phi\widetilde{T_2}^{-1}=\widetilde{T_1}^{-1}\Phi$, where $\widetilde{T_i}^{-1}\colon(\widetilde{\mathscr{B}_i},\widetilde{\mu_i})\to(\widetilde{\mathscr{B}_i},\widetilde{\mu_i})$ defined by $\widetilde{T_i}^{-1}(\widetilde{B})=(T_i^{-1}(B))^{\sim}$, $i=1,2$ (see [17]).

3. Main results

In this section, we give some results about the measure-theoretic entropy for a free semigroup action. Let us consider the following situation: $(X_1,\mathscr{B}_1,\mu_1)$ and $(X_2,\mathscr{B}_2,\mu_2)$ are probability spaces. Assume that $f_0,\dots,f_{m-1}$ are measure-preserving transformations on $(X_1,\mathscr{B}_1,\mu_1)$ and $g_0,\dots,g_{m-1}$ are measure-preserving transformations on $(X_2,\mathscr{B}_2,\mu_2)$. We say that $f_0,\dots,f_{m-1}$ is conjugate to $g_0,\dots,g_{m-1}$ if there is a measure-algebra isomorphism $\Phi\colon(\widetilde{\mathscr{B}_2},\widetilde{\mu_2})\to(\widetilde{\mathscr{B}_1},\widetilde{\mu_1})$ such that for any $i=0,1,\dots,m-1$, $\Phi\widetilde{g_i}^{-1}=\widetilde{f_i}^{-1}\Phi$. Observe that if $m=1$, this definition coincides with the classical case [17].

Theorem 3.1. *The measure-theoretic entropy for a free semigroup action is a conjugacy invariant.*

Proof. Let $(X_1,\mathscr{B}_1,\mu_1)$ and $(X_2,\mathscr{B}_2,\mu_2)$ be two probability spaces. Let $f_0,\dots,f_{m-1}$ be measure-preserving transformations on $(X_1,\mathscr{B}_1,\mu_1)$ and $g_0,\dots,g_{m-1}$ measure-preserving transformations on $(X_2,\mathscr{B}_2,\mu_2)$.

Since $(X_1, \mathscr{B}_1, \mu_1)$ is conjugate to $(X_2, \mathscr{B}_2, \mu_2)$, then there is an isomorphism of measure algebras $\Phi\colon (\widetilde{\mathscr{B}_2}, \widetilde{\mu_2}) \to (\widetilde{\mathscr{B}_1}, \widetilde{\mu_1})$ such that $\Phi\widetilde{g_i}^{-1} = \widetilde{f_i}^{-1}\Phi$ $(\forall\, i = 0, 1, \ldots, m-1)$. Let $\xi = \{A_1, \ldots, A_r\}$ be any finite partition of X_2. Choose $B_i \in \mathscr{B}_1$, such that $\widetilde{B_i} = \Phi(\widetilde{A_i})$ and so that $\eta = \{B_1, \ldots, B_r\}$ forms a partition of $(X_1, \mathscr{B}_1, \mu_1)$.

For any $w \in F_m^+$, $|w| = n$, $\bigcap_{w' \le w} f_{w'}^{-1} B_{w'}$ has the same measure as $\bigcap_{w' \le w} g_{w'}^{-1} A_{w'}$, where $B_{w'} \in \eta$, $A_{w'} \in \xi$, since

$$\begin{aligned}\Phi\left(\bigcap_{w' \le w} (g_{w'}^{-1} A_{w'})^{\sim}\right) &= \Phi\left(\bigcap_{w' \le w} \widetilde{g_{w'}}^{-1} \widetilde{A_{w'}}\right) = \bigcap_{w' \le w} \Phi\widetilde{g_{w'}}^{-1}(\widetilde{A_{w'}}) \\ &= \bigcap_{w' \le w} \widetilde{f_{w'}}^{-1} \Phi(\widetilde{A_{w'}}) = \bigcap_{w' \le w} \widetilde{f_{w'}}^{-1} \widetilde{B_{w'}} = \bigcap_{w' \le w} (f_{w'}^{-1} B_{w'})^{\sim}.\end{aligned}$$

Thus, $H_{\mu_1}\left(\bigvee_{w' \le w} f_{w'}^{-1}\eta\right) = H_{\mu_2}\left(\bigvee_{w' \le w} g_{w'}^{-1}\xi\right)$ which implies that

$$h_{\mu_1}(f_0, \ldots, f_{m-1}, \eta) = h_{\mu_2}(g_0, \ldots, g_{m-1}, \xi).$$

And then

$$\sup_{\xi} h_{\mu_2}(g_0, \ldots, g_{m-1}, \xi) \le \sup_{\eta} h_{\mu_1}(f_0, \ldots, f_{m-1}, \eta).$$

That is

$$h_{\mu_1}(f_0, \ldots, f_{m-1}) \ge h_{\mu_2}(g_0, \ldots, g_{m-1}).$$

By symmetry we then get that

$$h_{\mu_1}(f_0, \ldots, f_{m-1}) = h_{\mu_2}(g_0, \ldots, g_{m-1}).$$ □

Remark 3.2. If $m = 1$, the above result coincides with the result that the classical measure-theoretic entropy (see, [17, Theorem 4.11]).

It is well known that there is a power rule for the measure-theoretic entropy of the classical measure-preserving system, that is, for any transformation f which preserves μ we have $h_\mu(f^k) = kh_\mu(f)$, where $k \in \mathbb{N}$ [17]. For the measure-theoretic entropy for a free semigroup action, we can get the following result.

Theorem 3.3. *Let $(X, \mathscr{B}, \mu)$ be a probability space and $f_0, \ldots, f_{m-1}$ preserve μ. Let $\mathscr{F} := \{f_0, \ldots, f_{m-1}\}$ and $\mathscr{F}^k := \{g_0, \ldots, g_{m^k-1}\}$ ($k \in \mathbb{N}$), where $g_i \in \{f_w \mid f_w = f_{w_0} \circ f_{w_1} \circ \cdots \circ f_{w_{k-1}}, w \in F_m^+, |w| = k, w_j = 0, \ldots, m-1, \forall\, j = 0, \ldots, k-1\}$, then $h_\mu(\mathscr{F}^k) \le kh_\mu(\mathscr{F})$.*

Proof. Let ξ be any finite partition of X. For any $w \in F_m^+$, $|w| = nk$, $w = w_0 w_1 \cdots w_{k-1} w_k \cdots w_{nk-1}$, denote $w^0 = w_0^0 w_1^0 \cdots w_{n-1}^0$, where $w_i^0 = w_{ik} w_{ik+1} \cdots w_{ik+k-1}$, then $g_{w^0} =$

$g_{w_0^0} \circ g_{w_1^0} \circ \cdots \circ g_{w_{n-1}^0} = f_w$. We have

$$\begin{aligned} h_\mu(\mathscr{F}^k, \xi) &= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{(m^k)^n} \sum_{|w^0|=n} H_\mu\left(\bigvee_{w'\le w^0} g_{w'}^{-1}\xi\right)\right] \\ &= k \lim_{n\to\infty} \frac{1}{nk}\left[\frac{1}{m^{nk}} \sum_{|w^0|=n} H_\mu\left(\bigvee_{w'\le w^0} g_{w'}^{-1}\xi\right)\right] \\ &\le k \lim_{n\to\infty} \frac{1}{nk}\left[\frac{1}{m^{nk}} \sum_{|w|=nk} H_\mu\left(\bigvee_{w'\le w} f_{w'}^{-1}\xi\right)\right] \\ &= k h_\mu(\mathscr{F}, \xi) \\ &\le k h_\mu(\mathscr{F}). \end{aligned}$$

□

It is natural to ask if we can get the opposite inequality, i.e., $h_\mu(\mathscr{F}^k) \ge k h_\mu(\mathscr{F})$? And then $h_\mu(\mathscr{F}^k) = k h_\mu(\mathscr{F})$ holds. But up to now we haven't solved it.

Lemma 3.4. *Let $(X, \mathscr{B}, \mu)$ be a probability space and $f_0, \ldots, f_{m-1}$ transformations preserve μ. If ξ is a finite partition of X, for any $w \in F_m^+$, $|w| = n-1$, $n \in \mathbb{N}$, we have*

$$H_\mu\left(\bigvee_{w'\le w} f_{w'}^{-1}\xi\right) = H_\mu(\xi) + \sum_{|w^*|=1}^{n-1} H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w'\le w^* \\ |w'|\ge 1}} f_{w'}^{-1}\xi\right),$$

where w^* satisfies that there is a w^{**} such that $w = w^* w^{**}$.

Proof. We show by induction that the formula holds for all n.

For $n = 1$ it is clear, and if we assume it true for $n = p$ then it also holds for $n = p+1$ because for any $w, w^0 \in F_m^+$, $w = i_{p-1}\cdots i_1 i_0$, $w^0 = i_{p-1}\cdots i_1$, we have

$$\begin{aligned} H_\mu\left(\bigvee_{w'\le w} f_{w'}^{-1}\xi\right) &= H_\mu\left(\left(f_{i_0}^{-1} \bigvee_{w'\le w^0} f_{w'}^{-1}\xi\right) \bigvee \xi\right) \\ &= H_\mu\left(f_{i_0}^{-1} \bigvee_{w'\le w^0} f_{w'}^{-1}\xi\right) + H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w'\le w \\ |w'|\ge 1}} f_{w'}^{-1}\xi\right) \\ &= H_\mu\left(\bigvee_{w'\le w^0} f_{w'}^{-1}\xi\right) + H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w'\le w \\ |w'|\ge 1}} f_{w'}^{-1}\xi\right) \\ &= H_\mu(\xi) + \sum_{|w^*|=1}^{p-1} H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w'\le w^* \\ |w'|\ge 1}} f_{w'}^{-1}\xi\right) + H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w'\le w \\ |w'|\ge 1}} f_{w'}^{-1}\xi\right) \end{aligned}$$

$$= H_\mu(\xi) + \sum_{|w^*|=1}^{p} H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w' \leq w^* \\ |w'| \geq 1}} f_{w'}^{-1}\xi\right),$$

where w^* satisfies that there is a w^{**} such that $w = w^*w^{**}$. That is, it also holds for $n = p+1$, thus the formula holds for any $n \in \mathbb{N}$. □

Theorem 3.5. *Let $(X, \mathscr{B}, \mu)$ be a probability space and $f_0, \ldots, f_{m-1}$ transformations preserve μ. If ξ is a finite partition of X, we have*

$$h_\mu(f_0, \ldots, f_{m-1}, \xi) = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{m^n}\sum_{|w|=n}\sum_{|w^*|=1}^{n} H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w' \leq w^* \\ |w'| \geq 1}} f_{w'}^{-1}\xi\right)\right],$$

where $w \in F_m^+$, $w = i_{n-1}\cdots i_0$ and w^* satisfies that there is a w^{**} such that $w = w^*w^{**}$.

Proof. By Lemma 3.4, for any $w \in F_m^+$, $|w| = n$, w^* satisfies that there is a $w^{**} \in F_m^+$ such that $w = w^*w^{**}$, we have

$$H_\mu\left(\bigvee_{w' \leq w} f_{w'}^{-1}\xi\right) = H_\mu(\xi) + \sum_{|w^*|=1}^{n} H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w' \leq w^* \\ |w'| \geq 1}} f_{w'}^{-1}\xi\right).$$

Thus

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{m^n}\sum_{|w|=n} H_\mu\left(\bigvee_{w' \leq w} f_{w'}^{-1}\xi\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{m^n}\sum_{|w|=n}\sum_{|w^*|=1}^{n} H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w' \leq w^* \\ |w'| \geq 1}} f_{w'}^{-1}\xi\right)\right].$$

That is,

$$h_\mu(f_0, \ldots, f_{m-1}, \xi) = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{m^n}\sum_{|w|=n}\sum_{|w^*|=1}^{n} H_\mu\left(\xi \,\middle|\, \bigvee_{\substack{w' \leq w^* \\ |w'| \geq 1}} f_{w'}^{-1}\xi\right)\right],$$

where $w \in F_m^+$, $w = i_{n-1}\cdots i_0$ and w^* satisfies that there is a w^{**} such that $w = w^*w^{**}$. □

Similar to the classical measure-preserving systems, we can show that the measure-theoretic entropy map for a free semigroup action is affine.

Theorem 3.6. *Let $(X, \mathscr{B})$ be a measurable space and $f_0, \ldots, f_{m-1}$ measurable transformations of X. Then for any f_i-invariant probability measure μ_1, μ_2 and $p \in [0,1]$, where $i = 0, \ldots, m-1$, we have*

$$h_{p\mu_1+(1-p)\mu_2}(f_0, \ldots, f_{m-1}) = p h_{\mu_1}(f_0, \ldots, f_{m-1}) + (1-p) h_{\mu_2}(f_0, \ldots, f_{m-1}).$$

Proof. Without loss of generality, assume $0 < p < 1$. As in the proof of Theorem 8.1 of [17], for any finite partition ξ of X we have

$$0 \leq H_{p\mu_1+(1-p)\mu_2}(\xi) - pH_{\mu_1}(\xi) - (1-p)H_{\mu_2}(\xi) \leq \log 2.$$

If η is any finite partition of X, then for any $w \in F_m^+$, $|w| = n$, by putting $\xi = \bigvee_{w' \leq w} f_{w'}^{-1}\eta$ in the above formula, we have

$$\begin{aligned} 0 &\leq \frac{1}{m^n} \sum_{|w|=n} H_{p\mu_1+(1-p)\mu_2}\left(\bigvee_{w'\leq w} f_{w'}^{-1}\eta\right) - p\left[\frac{1}{m^n}\sum_{|w|=n} H_{\mu_1}\left(\bigvee_{w'\leq w} f_{w'}^{-1}\eta\right)\right] \\ &\quad - (1-p)\left[\frac{1}{m^n}\sum_{|w|=n} H_{\mu_2}\left(\bigvee_{w'\leq w} f_{w'}^{-1}\eta\right)\right] \\ &\leq \frac{1}{m^n}\sum_{|w|=n}\log 2 \\ &= \log 2. \end{aligned}$$

Thus

$$\begin{aligned} 0 &\leq h_{p\mu_1+(1-p)\mu_2}(f_0,\ldots,f_{m-1},\eta) - ph_{\mu_1}(f_0,\ldots,f_{m-1},\eta) - (1-p)h_{\mu_2}(f_0,\ldots,f_{m-1},\eta) \\ &\leq \lim_{n\to\infty}\frac{1}{n}\log 2 = 0. \end{aligned}$$

That is,

$$h_{p\mu_1+(1-p)\mu_2}(f_0,\ldots,f_{m-1},\eta) = ph_{\mu_1}(f_0,\ldots,f_{m-1},\eta) + (1-p)h_{\mu_2}(f_0,\ldots,f_{m-1},\eta). \tag{3.1}$$

Clearly,

$$h_{p\mu_1+(1-p)\mu_2}(f_0,\ldots,f_{m-1}) \leq ph_{\mu_1}(f_0,\ldots,f_{m-1}) + (1-p)h_{\mu_2}(f_0,\ldots,f_{m-1}). \tag{3.2}$$

We now show the opposite inequality. Let $\varepsilon > 0$, choose $\eta_1, \eta_2 > 0$ such that

$$h_{\mu_1}(f_0,\ldots,f_{m-1},\eta_1) > \begin{cases} h_{\mu_1}(f_0,\ldots,f_{m-1}) - \varepsilon & h_{\mu_1}(f_0,\ldots,f_{m-1}) < \infty, \\ \dfrac{1}{\varepsilon} & h_{\mu_1}(f_0,\ldots,f_{m-1}) = \infty, \end{cases}$$

$$h_{\mu_2}(f_0,\ldots,f_{m-1},\eta_2) > \begin{cases} h_{\mu_2}(f_0,\ldots,f_{m-1}) - \varepsilon & h_{\mu_2}(f_0,\ldots,f_{m-1}) < \infty, \\ \dfrac{1}{\varepsilon} & h_{\mu_2}(f_0,\ldots,f_{m-1}) = \infty. \end{cases}$$

Putting $\eta = \eta_1 \vee \eta_2$ in (3.1) gives

$$\begin{aligned} h_{p\mu_1+(1-p)\mu_2}(f_0,\ldots,f_{m-1},\eta) &= ph_{\mu_1}(f_0,\ldots,f_{m-1},\eta) + (1-p)h_{\mu_2}(f_0,\ldots,f_{m-1},\eta) \\ &\geq ph_{\mu_1}(f_0,\ldots,f_{m-1},\eta_1) + (1-p)h_{\mu_2}(f_0,\ldots,f_{m-1},\eta_2). \end{aligned}$$

If $h_{\mu_1}(f_0,\dots,f_{m-1}), h_{\mu_2}(f_0,\dots,f_{m-1}) < \infty$, then

$$h_{p\mu_1+(1-p)\mu_2}(f_0,\dots,f_{m-1},\eta) > ph_{\mu_1}(f_0,\dots,f_{m-1}) + (1-p)h_{\mu_2}(f_0,\dots,f_{m-1}) - \varepsilon.$$

If $h_{\mu_1}(f_0,\dots,f_{m-1}) = \infty$ or $h_{\mu_2}(f_0,\dots,f_{m-1}) = \infty$, then

$$h_{p\mu_1+(1-p)\mu_2}(f_0,\dots,f_{m-1},\eta) > \frac{1}{\varepsilon}\cdot\min\{p,1-p\}.$$

Therefore

$$h_{p\mu_1+(1-p)\mu_2}(f_0,\dots,f_{m-1}) \geq ph_{\mu_1}(f_0,\dots,f_{m-1}) + (1-p)h_{\mu_2}(f_0,\dots,f_{m-1}). \tag{3.3}$$

From (3.2) and (3.3), the desired equality holds. □

Example 3.7. Let K be a unit circle, $f_1 \colon x \mapsto x + a_1 \pmod 1$, $f_2 \colon x \mapsto x + a_2 \pmod 1$, where $a_1, a_2 \in K$. Let G be a free semigroup generated by f_1 and f_2. Then G is equicontinuous and preserves Haar measure μ. We can get $h(f_1,f_2) = 0$, and then by the Partial Variational Principle we have $h_\mu(f_1,f_2) = 0$.

Proof. By the definition of equicontinuity, $\forall\,\varepsilon > 0$, $\exists\,\delta > 0$, $\forall\,x,y \in K$, if $d(x,y) < \delta$, then $d(f(x),f(y)) < \varepsilon$. For any $f \in G$, since K is a compact space, $\exists\,M > 0$ such that for any $w \in F_2^+$, $B(w,\varepsilon,f_1,f_2) \leq M$, we have $B(n,\varepsilon,f_1,f_2) \leq M$. And then $\limsup_{n\to\infty} \frac{1}{n} \log B(n,\varepsilon,f_1,f_2) = 0$, therefore $h(f_1,f_2) = 0$.

By the Partial Variational Principle of [13], we have $\sup_{\mu \in M(f_1,f_2)} h_\mu(f_1,f_2) \leq h(f_1,f_2)$. Thus $h_\mu(f_1,f_2) = 0$. □

For convenience, we give the following lemma.

Lemma 3.8. [19, Theorem 5.9] *Let $A_0,\dots,A_{m-1}$ be surjective endomorphisms of $\mathbb{T}^p$. If for each $0 \leq i \leq m-1$ all eigenvalues of the matrix $[A_i]$ which represents A_i are of modulus greater than or equal to 1, then*

$$\log\frac{1}{m}\left(\sum_{i=0}^{m-1}\prod_{j=1}^{p}\left|\lambda_j^{(i)}\right|\right) \leq h(A_0,\dots,A_{m-1}) \leq \log\frac{1}{m}\left(\sum_{i=0}^{m-1}\Lambda_i^p\right)$$

where $\lambda_1^{(i)},\lambda_2^{(i)},\dots,\lambda_p^{(i)}$ are the eigenvalues of $[A_i]$, $0 \leq i \leq m-1$, counted with their multiplicities, and Λ_i is the biggest eigenvalues of $\sqrt{[A_i][A_i]^T}$, $0 \leq i \leq m-1$. In particular for the case $p = 1$, we have

$$h(A_0,\dots,A_{m-1}) = \log\frac{1}{m}\left(\sum_{i=0}^{m-1}\left|\lambda_1^{(i)}\right|\right),$$

where $\lambda_1^{(i)}$ is the degree of the endomorphism A_i of S^1, for every $0 \leq i \leq m-1$, where S^1 denotes the unite circle.

Example 3.9. Let S^1 be the unit circle, $f_i\colon x \mapsto \lambda_i x \pmod 1$, $\lambda_i \in \mathbb{N}$, $i = 0, 1, \ldots, m-1$. By Lemma 3.8, we have $h(f_0, \ldots, f_{m-1}) = \log \frac{1}{m} \sum_{i=0}^{m-1} \lambda_i$. And then by the Partial Variational Principle of [13], we have

$$h_\mu(f_0, \ldots, f_{m-1}) \le h(f_0, \ldots, f_{m-1}) = \log \frac{1}{m} \sum_{i=0}^{m-1} \lambda_i,$$

where μ is the Haar measure.

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