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Low Regularity Global Well-posedness for the Quantum Zakharov System in $1D$

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Abstract. In this paper, we consider the quantum Zakharov system in one spatial dimension. We prove the global well-posedness of the system with L^2-Schrödinger data and some wave data. The regularity of the wave data is in the largest set. We give counterexamples for the boundary of the set. As the quantum parameter tends to zero, we formally recover the result of Colliander-Holmer-Tzirakis for the classical Zakharov system.

1. Introduction

Zakharov system describes the propagation of Langmuir waves in an ionized plasma, that is the nonlinear interaction between the quantum Langmuir waves and the quantum ion-acoustic waves. Langmuir waves are rapid oscillations of the electron density in conducting media, such as plasmas. The system reads as follows:

$$\begin{cases} iE_t + \partial_x^2 E = nE, \quad x \in \mathbb{R}, \\ n_{tt} - \partial_x^2 n = \partial_x^2 |E|^2, \\ E(0) = E_0,\ n(0) = n_0,\ \partial_t n(0) = n_1, \end{cases} \tag{1.1}$$

where E is the slowly varying envelope of the rapidly oscillating electric field and n is the deviation of the ion density from its mean value. E is complex valued and n is real valued. The regular solutions of (1.1) satisfy the conservation of mass

$$\int |E(t)|^2 \, dx = \int |E(0)|^2 \, dx = \text{constant}$$

and the conservation of the Hamiltonian

$$\int |\partial_x E(t)|^2 + \frac{1}{2} n(t)^2 + n(t) |E(t)|^2 + \frac{1}{2} \nu(t)^2 \, dx = \text{constant},$$

where $\partial_t n = \partial_x \nu$ and $\partial_t \nu = \partial_x (n + |E|^2)$.

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The (1.1) has been investigated extensively and we list some of the works in the reference. In 1997, Ginibre-Tsutsumi-Velo proved a local well-posedness result for $(E_0, n_0, n_1) \in H^k \oplus H^\ell \oplus H^{\ell-1}$ provided that $-1 \le \ell - k < 1/2$, $0 \le \ell + 1/2 \le 2k$, see [4]. In 2008, Colliander-Holmer-Tzirakis showed the global well-posedness for (1.1) with $(E_0, n_0, n_1) \in L^2 \oplus H^{-1/2} \oplus H^{-3/2}$, see [1].

Taking quantum effects into account, we consider the quantum Zakharov system

$$\begin{cases} iE_t + \partial_x^2 E - \varepsilon^2 \partial_x^4 E = nE, \quad x \in \mathbb{R}, \\ n_{tt} - \partial_x^2 n + \varepsilon^2 \partial_x^4 n = \partial_x^2 |E|^2, \\ E(0) = E_0,\ n(0) = n_0,\ \partial_t n(0) = n_1. \end{cases} \tag{1.2}$$

The quantum parameter ε for some plasmas typically goes from value of order 10^{-5} to the values of order unity, see [3,6]. It is known that $\varepsilon = \frac{\hbar\omega_i}{\kappa_B T_e}$, where $\hbar$ is Planck's constant divided by 2π, ω_i is the ion plasma frequency, κ_B is the Boltzmann constant and T_e is the electron fluid temperature.

Analogous to (1.1), (1.2) also possesses the conservation of mass

$$\int |E(t)|^2\,dx = \int |E(0)|^2\,dx \tag{1.3}$$

and the conservation of Hamiltonian

$$\int |\partial_x E|^2 + \varepsilon^2 \left|\partial_x^2 E\right|^2 + \frac{1}{2}n^2 + n|E|^2 + \frac{1}{2}\nu^2 + \frac{\varepsilon^2}{2}|\partial_x n|^2\,dx = \text{constant},$$

where $\partial_t n = \partial_x \nu$ and $\partial_t \nu = \partial_x(n + |E|^2 - \varepsilon^2 \partial_x^2 n)$.

The works on the (1.2) are less than those on the (1.1), and we only mention two of them and some are listed in the reference. In 2016, Jiang-Lin-Shao proved the local well-posedness for (1.2) with initial data $(E_0, n_0, n_1) \in H^{2k} \oplus H^{2\ell} \oplus H^{2\ell-2}$ provided that $-3/4 < \ell - k < 3/4$, $-k - 3/4 < \ell < 2k + 3/4$ and $k > -3/8$, see [7]. In 2016, Fang-Shih-Wang improved the result of local well-posedness for (1.2) with initial data to a wider range, see [2]. In 2013, Guo-Zhang-Guo proved the global well-posedness of (1.2) in the energy and above energy spaces and investigate the semi-classical limit behavior of (1.2) as ε tends to zero, see [5]. Our main result is as follows.

Theorem 1.1. *Let $-3/4 \le \ell \le 3/4$ and $0 < \varepsilon \le 1$. The* (1.2) *is globally well-posed for initial data $(E_0, n_0, n_1) \in L^2 \times H^{2\ell} \times H^{2\ell-2}$ and the solution (E, n) satisfies* (1.3) *and*

$$\begin{aligned} &\|n(t)\|_{H^{2\ell}} + \|\partial_t n(t)\|_{H^{2\ell-2}} \\ &\le e^{c|t|\left(\|E_0\|_{L^2}^2+1\right)\varepsilon^{-2|\ell+1/2|}} \max\left(\|n_0\|_{H^{2\ell}} + \|n_1\|_{H^{2\ell-2}}, \|E_0\|_{L^2}^2\right). \end{aligned}$$

Thus, for $k \ge 0$, we establish global well-posedness in the largest space for which local well-posedness holds. We extend the local well-posedness in the work of Jiang-Lin-Shao to the lowest point on the boundary and also formally recover the work of Colliander-Holmer-Tzirakis as ε approaches zero.

The outline of the paper is as follows. In Sections 3 and 4, we state the homogeneous estimates, Duhamel estimates and multilinear estimates. We then invoke Strichartz estimates and (1.3) to derive global well-posedness of (1.2). In Section 5, we prove the multilinear estimates which is the key to this work. Also we give a counterexample for the key estimates. In the appendix, we show some technical lemmas.

2. Notations and solution formulae

Denote $\langle\xi\rangle = (1+\xi^2)^{1/2}$, $\xi_\varepsilon = \xi\sqrt{1+\varepsilon^2\xi^2}$ and $D_\varepsilon = \sqrt{1-\varepsilon^2\partial_x^2}$, which will be used through out the paper. For the fourth order Schrödinger equation

$$iE_t + \partial_x^2 E - \varepsilon^2\partial_x^4 E = F,$$

we obtain the solution formula

$$E(t,x) = U_\varepsilon(t)E_0(x) - i\int_0^t U_\varepsilon(t-s)F(s,x)\,ds, \tag{2.1}$$

where $U_\varepsilon(t) := e^{it\partial_x^2 D_\varepsilon^2}$ is the Schrödinger propagator. Denote the Duhamel operator

$$U_\varepsilon *_R F(t,x) = -i\int_0^t U_\varepsilon(t-s)F(s,x)\,ds.$$

For the fourth order wave equation

$$n_{tt} - \partial_x^2 n + \varepsilon^2\partial_x^4 n = \partial_x G,$$

we denote the modified fourth order wave propagators via Fourier transform by

$$F(W_{\varepsilon\pm}(t)(n_0,n_1))(\xi) := \frac{1}{2}e^{\mp it\xi_\varepsilon}\widehat{n}_0 \mp \frac{1}{2}\frac{e^{\mp it\xi_\varepsilon}}{i\xi_\varepsilon}\widehat{n}_{1H} \mp \frac{1}{2}\frac{e^{\mp it\xi_\varepsilon}-1}{i\xi_\varepsilon}\widehat{n}_{1L},$$

where n_{1H} high frequency part of n_1 and $n_{1L} = n_1 - n_{1H}$ and Duhamel operators

$$F(W_{\varepsilon\pm} *_R G)(t,\xi) := \frac{1}{2}\int_0^t e^{\mp i(t-s)\xi_\varepsilon}\widehat{G}(s,\xi)\,ds.$$

Hence the solution n is split into $n = n_+ + n_-$, where

$$n_\pm(t,x) := W_{\varepsilon\pm}(t)(n_0,n_1)(x) \mp W_{\varepsilon\pm} *_R G_\varepsilon(t,x) \tag{2.2}$$

and $G_\varepsilon = D_\varepsilon^{-1}G$. Also we denote

$$W_\varepsilon(t)(n_0,n_1)(x) := W_{\varepsilon+}(t)(n_0,n_1)(x) + W_{\varepsilon-}(t)(n_0,n_1)(x),$$
$$W_\varepsilon *_R G_\varepsilon := -W_{\varepsilon+} *_R G_\varepsilon + W_{\varepsilon-} *_R G_\varepsilon,$$

thus

$$n = W_\varepsilon(t)(n_0, n_1) + W_\varepsilon *_R G_\varepsilon. \tag{2.3}$$

Thus (1.2) can be rewritten as

$$\begin{cases} i\partial_t E + \partial_x^2 D_\varepsilon^2 E = (n_+ + n_-)E, & x \in \mathbb{R}, \\ \partial_t n_\pm \pm \partial_x D_\varepsilon n_\pm = \mp \frac{1}{2} D_\varepsilon^{-1} \partial_x |E|^2 + \frac{1}{2} n_{1L}, \end{cases} \tag{2.4}$$

and the solution formulae are as follows:

$$E(t,x) = U_\varepsilon(t) E_0(x) + U_\varepsilon *_R ((n_+ + n_-)E)(t,x) \tag{2.5}$$

and

$$n_\pm(t,x) = W_{\varepsilon\pm}(t)(n_0, n_1)(x) \mp W_{\varepsilon\pm} *_R \left(D_\varepsilon^{-1} \partial_x |E|^2 \right)(t,x). \tag{2.6}$$

Notice that the wave parts in (2.6) are mixtures of transport equation and Schrödinger equation.

Denote the Sobolev spaces H^ℓ, H_ε^ℓ and A_ε^ℓ, used in the paper with the norms

$$\|f\|_{H^\ell}^2 := \int \langle \xi \rangle^{2\ell} \left| \widehat{f}(\xi) \right|^2 d\xi, \quad \|f\|_{H_\varepsilon^\ell}^2 := \int \langle \xi_\varepsilon \rangle^{2\ell} \left| \widehat{f}(\xi) \right|^2 d\xi,$$

and

$$\|f\|_{A_\varepsilon^\ell}^2 := \int_{|\xi_\varepsilon| \le 1} \left| \widehat{f}(\xi) \right|^2 d\xi + \int_{1 \le |\xi_\varepsilon|} |\xi_\varepsilon|^{2\ell} \left| \widehat{f}(\xi) \right|^2 d\xi.$$

To characterize the quantum parameter ε, we also use the following equivalent norm

$$\int_{|\xi| \le 6} \left| \widehat{f}(\xi) \right|^2 d\xi + \int_{6 < |\xi| \le 12\varepsilon^{-1}} |\xi|^{2\ell} \left| \widehat{f}(\xi) \right|^2 d\xi + \int_{12\varepsilon^{-1} < |\xi|} \varepsilon^{2\ell} |\xi|^{4\ell} \left| \widehat{f}(\xi) \right|^2 d\xi.$$

For $0 < \varepsilon \le 1$, we obtain the relations between the above norms as

$$\varepsilon^{\max\{0,\ell\}} \|f\|_{H^{2\ell}} \lesssim \|f\|_{H_\varepsilon^\ell} \sim \|f\|_{A_\varepsilon^\ell} \lesssim \varepsilon^{\min\{\ell,0\}} \|f\|_{H^{2\ell}}.$$

We define the norm

$$\|n(t)\|_{W_\varepsilon} := \|(n(t), \partial_t n(t))\|_{W_\varepsilon} := \left(\|n(t)\|_{A_\varepsilon^\ell}^2 + \|\partial_t n(t)\|_{A_\varepsilon^{\ell-1}}^2 \right)^{1/2}.$$

Also we define the fourth order Schrödinger-Bourgain space $X_{0,\alpha}^{S_\varepsilon}$, $\alpha \in \mathbb{R}$, with the norm

$$\|E\|_{X_{0,\alpha}^{S_\varepsilon}} := \left(\int \left\langle \tau + \xi_\varepsilon^2 \right\rangle^{2\alpha} \left| \widehat{E}(\tau,\xi) \right|^2 d\tau d\xi \right)^{1/2},$$

and the reduced fourth order wave Bourgain spaces $X_{\ell,\alpha}^{W_{\varepsilon\pm}}$, $\alpha \in \mathbb{R}$, with the norm

$$\|n\|_{X_{\ell,\alpha}^{W_{\varepsilon\pm}}} := \left(\int \langle \xi_\varepsilon \rangle^{2\ell} \langle \tau \pm \xi_\varepsilon \rangle^{2\alpha} |\widehat{n}(\tau,\xi)|^2 d\tau d\xi \right)^{1/2}.$$

Now we state the estimates needed for proof of Theorem 1.1.

3. Homogeneous estimates and Duhamel estimates

Proofs of some lemmas stated below are analogous to those in [1, 4, 8], with necessary changes and adaptations. Let ψ be a cut-off function such that $\psi(t)$ is 1 for $|t| \le 1$, 0 for $|t| > 2$ and $\psi_T(t) = \psi(t/T)$. Also let $\chi_S(\tau)$ be the indicator function on S, that is 1 if $\tau \in S$, 0 if $\tau \notin S$.

Lemma 3.1 (Homogeneous estimates)**.** *Suppose $T \le 1$. For (2.1), we have*

(a1) $\|U_\varepsilon(t)E_0\|_{C([0,T];L^2)} = \|E_0\|_{L^2}$.

(a2) *If $0 \le b_1 \le 1/2$, then* $\|\psi_T(t)U_\varepsilon(t)E_0\|_{X^{S_\varepsilon}_{0,b_1}} \lesssim T^{1/2-b_1}\|E_0\|_{L^2}$.

(a3) *(Strichartz estimates). If $(q,r) = (4/\theta, 2/(1-\theta))$ and $\theta \in [0,1]$, then*

$$\left\|D_\varepsilon^{\theta/2}U_\varepsilon(t)E_0\right\|_{L^q_tL^r_x} \lesssim \|E_0\|_{L^2}.$$

For (2.3) *and* (2.2)*, we have*

(b1) $\|W_\varepsilon(t)(n_0,n_1)\|_{C([0,T];W_\varepsilon)} \le (1+T)\|(n_0,n_1)\|_{W_\varepsilon}$.

(b2) *If $0 \le b \le 1/2$, then* $\|\psi_T(t)W_{\varepsilon\pm}(t)(n_0,n_1)\|_{X^{W_{\varepsilon\pm}}_{\ell,b}} \lesssim T^{1/2-b}\|(n_0,n_1)\|_{W_\varepsilon}$.

Lemma 3.2 (Duhamel estimates)**.** *Suppose $T \le 1$. For (2.1), we have*

(a1) *If $0 \le c_1 < 1/2$, then* $\|U_\varepsilon *_R F\|_{C([0,T];L^2_x)} \lesssim T^{1/2-c_1}\|F\|_{X^{S_\varepsilon}_{0,-c_1}}$.

(a2) *If $0 \le c_1 < 1/2$, $0 \le b_1$ and $b_1 + c_1 \le 1$, then*

$$\|\psi_T(t)U_\varepsilon *_R F\|_{X^{S_\varepsilon}_{0,b_1}} \lesssim T^{1-b_1-c_1}\|F\|_{X^{S_\varepsilon}_{0,-c_1}}.$$

(a3) *(Strichartz estimates). If $(q,r) = (4/\theta, 2/(1-\theta))$, $\theta \in [0,1]$ and $d > 1/2$, then*

$$\left\|D_\varepsilon^\theta U_\varepsilon *_R F\right\|_{L^q[0,T]L^r_x} + \left\|D_\varepsilon^\theta U_\varepsilon *_R F\right\|_{C([0,T];L^2_x)} \lesssim \|F\|_{L^{q'}[0,T]L^{r'}_x},$$

$$\left\|D_\varepsilon^{\theta/2}E\right\|_{L^q_tL^r_x} \lesssim \|E\|_{X^{S_\varepsilon}_{0,d}}. \tag{3.1}$$

For (2.2) *and* (2.3)*, we have*

(b1) *If $0 \le c < 1/2$, then* $\|W_\varepsilon *_R G\|_{C([0,T];W_\varepsilon)} \lesssim T^{1/2-c}\left(\|G\|_{X^{W_{\varepsilon+}}_{\ell,-c}} + \|G\|_{X^{W_{\varepsilon-}}_{\ell,-c}}\right)$.

(b2) *If $0 \le c < 1/2$, $0 \le b$, $b + c \le 1$ then*

$$\|\psi_T(t)W_{\varepsilon\pm} *_R G\|_{X^{W_{\varepsilon\pm}}_{\ell,b}} \lesssim T^{1-b-c}\|G\|_{X^{W_{\varepsilon\pm}}_{\ell,-c}}.$$

We skip the proofs of Lemmas 3.1 and 3.2 and the readers are referred to [1, 4, 8]. For more complete discussion of Strichartz estimates for the fourth order Schrödinger equations, see [9].

4. Local and global well-posedness for QZ system

We prove Theorem 1.1 whose proof essentially follows from those in [1, 4], however some adaptations and changes are required. We use the conservation law (1.3) to control the growth of $E(t)$ from one local time step to the next. We track the growth of $n(t)$ in the norm W_ε using the estimates from the local theory. We now state the estimates.

Lemma 4.1 (Multilinear Estimates)**.** *Let* $-3/4 \le \ell \le 3/4$ *and* $0 < \varepsilon \le 1$*. We have the following estimates.*

(a) *If* $\max\{1/4, -\ell/2\} < b_1, c_1 < 1/2$, $1/4 \le b < 1/2$ *and* $b + b_1 + c_1 \ge \max\{1, (-2\ell + 3)/4\}$*, then*

$$\|n_\pm E\|_{X^{S_\varepsilon}_{0,-c_1}} \lesssim \varepsilon^{\min\{\ell+1/2,0\}} \|n_\pm\|_{X^{W_{\varepsilon\pm}}_{\ell,b}} \|E\|_{X^{S_\varepsilon}_{0,b_1}} . \tag{4.1}$$

(b) *If* $\max\{1/4, \ell/2\} < b_1 < 1/2$, $1/4 \le c < 1/2$ *and* $2b_1 + c \ge \max\{1, (2\ell + 3)/4\}$*, then*

$$\left\|D_\varepsilon^{-1}\partial_x(E_1\overline{E}_2)\right\|_{X^{W_{\varepsilon\pm}}_{\ell,-c}} \lesssim \varepsilon^{\min\{-\ell-1/2,0\}} \|E_1\|_{X^{S_\varepsilon}_{0,b_1}} \|E_2\|_{X^{S_\varepsilon}_{0,b_1}} . \tag{4.2}$$

(c) *If* $3/8 < b_1, c_1 < 1/2$ *and* $0 \le b$, $c < 1/2$*, then*

$$\left\|D_\varepsilon^{-1}\partial_x(E_1\overline{E}_2)\right\|_{X^{W_{\varepsilon\pm}}_{-3/4,-c}} \lesssim \varepsilon^{-1/4} \|E_1\|_{X^{S_\varepsilon}_{0,b_1}} \|E_2\|_{X^{S_\varepsilon}_{0,b_1}}$$

and

$$\|n_\pm E\|_{X^{S_\varepsilon}_{0,-c_1}} \lesssim \|n_\pm\|_{X^{W_{\varepsilon\pm}}_{3/4,b}} \|E\|_{X^{S_\varepsilon}_{0,b_1}} .$$

For the initial data $(E(0), n(0), \partial_t n(0)) \in L^2 \times H_\varepsilon^{-1/2} \times H_\varepsilon^{-3/2}$, we can prove multilinear estimates which are analogous to those in [1].

Corollary 4.2. [1, Lemma 3.1]

(a) *If* $1/4 < b, b_1, c_1 < 1/2$ *and* $b + b_1 + c_1 \ge 1$*, then*

$$\|n_\pm E\|_{X^{S_\varepsilon}_{0,-c_1}} \lesssim \|n_\pm\|_{X^{W_{\varepsilon\pm}}_{-1/2,b}} \|E\|_{X^{S_\varepsilon}_{0,b_1}} .$$

(b) *If* $1/4 < b_1, c < 1/2$ *and* $2b_1 + c \ge 1$*, then*

$$\left\|D_\varepsilon^{-1}\partial_x(E_1\overline{E}_2)\right\|_{X^{W_{\varepsilon\pm}}_{-1/2,-c}} \lesssim \|E_1\|_{X^{S_\varepsilon}_{0,b_1}} \|E_2\|_{X^{S_\varepsilon}_{0,b_1}} .$$

Remark 4.3. Let $\ell = -1/2$. Observe that the constants in the estimates of Lemmas 4.1(a) and 4.1(b) are independent of ε. Taking the limit of the estimates in Lemmas 4.1(a) and 4.1(b) as ε tends to zero, we get the multilinear estimates given in [1], and thus we formally recover the result of (1.1) in [1].

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For the system (2.4), we have the solution formulae (2.5) and (2.6). Thus we consider the maps Λ_{S_ε} and $\Lambda_{W_{\varepsilon\pm}}$ such that

$$\Lambda_{S_\varepsilon}(E,n_\pm) = \psi_T U_\varepsilon E_0 + \psi_T U_\varepsilon *_R [(n_+ + n_-)E], \tag{4.3}$$

$$\Lambda_{W_{\varepsilon\pm}}(E) = \psi_T W_{\varepsilon\pm}(n_0,n_1) \mp \psi_T W_{\varepsilon\pm} *_R (D_\varepsilon^{-1}\partial_x |E|^2). \tag{4.4}$$

For some $0 < T < 1$, we seek a fixed point $(E(t),n_\pm(t)) = (\Lambda_{S_\varepsilon}(E,n_\pm),\Lambda_{W_{\varepsilon\pm}}(E))$. Estimating (4.3) in $X^{S_\varepsilon}_{0,b_1}$, applying the estimate in Lemmas 3.1(a2), 3.2(a2) and following through with Lemma 4.1(a) and (c); and estimating (4.4) in $X^{W_{\varepsilon\pm}}_{\ell,b}$, applying the estimates in Lemmas 3.1(b2), 3.2(b2) and following through with Lemma 4.1(b) and (c), we obtain

$$\begin{aligned}\|\Lambda_{S_\varepsilon}(E,n_\pm)\|_{X^{S_\varepsilon}_{0,b_1}} &\lesssim T^{1/2-b_1}\|E_0\|_{L^2} + T^{1-b_1-c_1}\|n_\pm\|_{X^{W_{\varepsilon\pm}}_{\ell,b}}\|E\|_{X^{S_\varepsilon}_{0,b_1}} \\ &\lesssim T^{1/2-b_1}\|E_0\|_{L^2}\left(1 + T^{3/2-b-b_1-c_1}\|(n_0,n_1)\|_{W_\varepsilon}\right)\end{aligned}$$

and

$$\left\|\Lambda_{W_{\varepsilon\pm}}(E)\right\|_{X^{W_{\varepsilon\pm}}_{\ell,b}} \lesssim T^{1/2-b}\left(\|(n_0,n_1)\|_{W_\varepsilon} + T^{3/2-2b_1-c}\|E_0\|^2_{L^2}\right).$$

Also we have

$$\begin{aligned}&\left\|\Lambda_{S_\varepsilon}(E,n_\pm) - \Lambda_{S_\varepsilon}(E',n'_\pm)\right\|_{X^{S_\varepsilon}_{0,b_1}} \\ &\lesssim T^{3/2-2b_1-c_1}\|E_0\|_{L^2}\left\|n_\pm - n'_\pm\right\|_{X^{W_{\varepsilon\pm}}_{\ell,b}} + T^{3/2-b-b_1-c_1}\|(n_0,n_1)\|_{W_\varepsilon}\left\|E-E'\right\|_{X^{S_\varepsilon}_{0,b_1}}\end{aligned}$$

and

$$\left\|\Lambda_{W_{\varepsilon\pm}}(E) - \Lambda_{W_{\varepsilon\pm}}(E')\right\|_{X^{W_{\varepsilon\pm}}_{\ell,b}} \lesssim T^{3/2-b_1-b-c}\|E_0\|_{L^2}\left\|E-E'\right\|_{X^{S_\varepsilon}_{0,b_1}}.$$

By taking T such that

$$T^{3/2-2b_1-c_1}\|E_0\|_{L^2} \lesssim 1, \quad T^{3/2-b_1-b-c}\|E_0\|_{L^2} \lesssim 1,$$

$$T^{3/2-b-b_1-c_1}\|(n_0,n_1)\|_{W_\varepsilon} \lesssim 1, \tag{4.5}$$

and

$$T^{3/2-2b_1-c}\|E_0\|^2_{L^2} \lesssim \|(n_0,n_1)\|_{W_\varepsilon}, \tag{4.6}$$

one can obtain sufficient conditions for a contraction argument yielding the existence of a fixed point $E \in X^{S_\varepsilon}_{0,b_1}$, $n_\pm \in X^{W_{\varepsilon\pm}}_{\ell,b}$ of (4.3)–(4.4) such that

$$\|E\|_{X^{S_\varepsilon}_{0,b_1}} \lesssim T^{1/2-b_1}\|E_0\|_{L^2} \quad \text{and} \quad \|n_\pm\|_{X^{W_{\varepsilon\pm}}_{\ell,b}} \lesssim T^{1/2-b}\|(n_0,n_1)\|_{W_\varepsilon}. \tag{4.7}$$

To prove the global existence for $-1/2 \le \ell \le 3/4$, we interpolate between Lemma 4.1(a) with $\ell' = -1/2$ and $b' = b_1' = c_1' = 1/3$, and Lemma 4.1(c) with $\ell'' = 3/4$, $b'' = 14/128$ and $b_1'' = c_1'' = 49/128$. Then we obtain

$$\|n_\pm E\|_{X^{S_\varepsilon}_{0,-c_1}} \lesssim \|n_\pm\|_{X^{W_\varepsilon\pm}_{\ell,b}} \|E\|_{X^{S_\varepsilon}_{0,b_1}} ,$$

where $b = \frac{14}{128}\theta + \frac{1}{3}(1-\theta)$, $b_1 = c_1 = \frac{49}{128}\theta + \frac{1}{3}(1-\theta)$ and $\ell = \frac{5}{4}\theta - \frac{1}{2}$. On the other hand, we need the interpolation between Lemma 4.1(b) with $\ell' = -1/2$ and $c' = b_1' = 1/3$, and Lemma 4.1(b) with $\ell'' = 3/4$, $b_1'' = 49/128$ and $c'' = 46/128$. Hence we get

$$\left\|D_\varepsilon^{-1}\partial_x(E_1\overline{E}_2)\right\|_{X^{W_\varepsilon\pm}_{\ell,-c}} \lesssim \varepsilon^{-(\ell+1/2)} \|E_1\|_{X^{S_\varepsilon}_{0,b_1}} \|E_2\|_{X^{S_\varepsilon}_{0,b_1}} ,$$

where $b_1 = \frac{49}{128}\theta + \frac{1}{3}(1-\theta)$, $c = \frac{46}{128}\theta + \frac{1}{3}(1-\theta)$ and $\ell = \frac{5}{4}\theta - \frac{1}{2}$.

To prove the global well-posedness for $-3/4 \le \ell \le -1/2$, we analogously interpolate between Lemma 4.1(a) with $\ell' = -1/2$ and $c_1' = b_1' = b' = 1/3$, and Lemma 4.1(a) with $\ell'' = -3/4$, $c_1'' = b_1'' = 49/128$ and $b'' = 46/128$. Then we obtain

$$\|n_\pm E\|_{X^{S_\varepsilon}_{0,-c_1}} \lesssim \varepsilon^{\ell+1/2} \|n_\pm\|_{X^{W_\varepsilon\pm}_{\ell,b}} \|E\|_{X^{S_\varepsilon}_{0,b_1}} ,$$

where $c_1 = b_1 = \frac{49}{128}\theta + \frac{1}{3}(1-\theta)$, $b = \frac{46}{128}\theta + \frac{1}{3}(1-\theta)$ and $\ell = -\frac{1}{2} - \frac{1}{4}\theta$. On the other hand, we need the interpolation between Lemma 4.1(b) with $c' = b_1' = 1/3$ and $\ell' = -1/2$, and Lemma 4.1(c) with $\ell'' = -3/4$, $b_1'' = 49/128$ and $c'' = 14/128$. Hence we get

$$\left\|D_\varepsilon^{-1}\partial_x(E_1\overline{E}_2)\right\|_{X^{W_\varepsilon\pm}_{\ell,-c}} \lesssim \varepsilon^{\ell+1/2} \|E_1\|_{X^{S_\varepsilon}_{0,b_1}} \|E_2\|_{X^{S_\varepsilon}_{0,b_1}} ,$$

where $b_1 = \frac{49}{128}\theta + \frac{1}{3}(1-\theta)$, $c = \frac{14}{128}\theta + \frac{1}{3}(1-\theta)$ and $\ell = -\frac{1}{2} - \frac{1}{4}\theta$.

By applying Lemmas 3.1(a1), 3.2(a1) and (4.7), we can show that $E \in C([0,T]; L^2_x)$. Then we invoke the conservation law (1.3) to conclude $\|E(T)\|_{L^2_x} = \|E_0\|_{L^2}$. Thus we are concerned with the possibility of growth of $\|n(t)\|_{W_\varepsilon}$ in time. Assume that after some number of iterations we have $\|n(t)\|_{W_\varepsilon} \gg \|E(t)\|^2_{L^2_x} = \|E_0\|^2_{L^2_x}$. We reset this time position as the initial time $t = 0$ so that $\|E_0\|^2_{L^2_x} \ll \|(n_0, n_1)\|_{W_\varepsilon}$. Let us take the quantum parameter into account. Thus (4.6) and (4.5) become

$$\varepsilon^{-|\ell+1/2|} T^{3/2-b-b_1-c_1} \|(n_0,n_1)\|_{W_\varepsilon} \lesssim 1 \tag{4.8}$$

and

$$\varepsilon^{-|\ell+1/2|} T^{3/2-2b_1-c} \|E_0\|^2_{L^2} \lesssim \|(n_0,n_1)\|_{W_\varepsilon} . \tag{4.9}$$

Then (4.9) is satisfied and by (4.8), we choose a time increment of size

$$T \sim \|(n_0,n_1)\|_{W_\varepsilon}^{-1/(3/2-b-b_1-c_1)} \varepsilon^{|\ell+1/2|/(3/2-b-b_1-c_1)}. \tag{4.10}$$

Using (4.5), the quantity $b+b_1+c_1$ we choose satisfies the conditions in Lemma 4.1(a). Since

$$n = W_\varepsilon(t)(n_0, n_1) + W_\varepsilon *_R \left(D_\varepsilon^{-1}\partial_x |E|^2\right)$$

we can apply Lemmas 3.1(b1), 3.2(b1), (4.7) and (4.10) to obtain

$$\begin{aligned}\|n(T)\|_{W_\varepsilon} &\le (1+T)\,\|(n_0,n_1)\|_{W_\varepsilon} + \varepsilon^{-|\ell+1/2|}T^{3/2-2b_1-c}\,\|E_0\|_{L^2}^2 \\ &\le \|(n_0,n_1)\|_{W_\varepsilon} + C\varepsilon^{-|\ell+1/2|}T^{3/2-2b_1-c}\left(\|E_0\|_{L^2}^2 + 1\right),\end{aligned}$$

where C is some fixed constant and $1-(3/2-b-b_1-c_1) = 3/2-2b_1-c$. Now we can carry out m iterations on time intervals, each of length (4.10) to get

$$\|n(mT)\|_{W_\varepsilon} \le \|(n_0,n_1)\|_{W_\varepsilon} + mCT^{3/2-2b_1-c}\left(\|E_0\|_{L^2}^2+1\right)\varepsilon^{-|\ell+1/2|}.$$

Thus, for some m, the upper bound in the above reaches $2\,\|(n_0,n_1)\|_{W_\varepsilon}$. Hence we get

$$m \sim \frac{\|(n_0,n_1)\|_{W_\varepsilon}\,\varepsilon^{|\ell+1/2|}}{T^{3/2-2b_1-c}(\|E_0\|_{L^2}^2+1)}. \tag{4.11}$$

The total time the solution n advances after these m iterations, by (4.10) and (4.11), is

$$mT \sim \frac{\varepsilon^{2|\ell+1/2|}}{\|E_0\|_{L^2}^2+1}$$

which is independent of $\|n(t)\|_{W_\varepsilon}$.

We can keep repeating this procedure. Each time the solution n advancing a time of length about the size of $(\|E_0\|_{L^2}^2+1)^{-1}\varepsilon^{2|\ell+1/2|}$, which is independent of the size of $\|n(t)\|_{W_\varepsilon}$, the size of $\|n(t)\|_{W_\varepsilon}$ will at most double. This implies that the solution grows at most exponentially in time as stated in Theorem 1.1. □

Remark 4.4. Let $k=0$. For $\ell = 3/4$, we choose $b = 14/128$, $b_1 = c_1 = 49/128$ and $c = 46/128$ to satisfy all the conditions stated in Lemmas 3.1, 3.2, 4.1(b) and 4.1(c). Notice that the choice also meets the optimal condition $2b_1 + c = 9/8$.

For $\ell = -1/2$, we choose $b = b_1 = c = c_1 = 1/3$ to satisfy all the required conditions in Lemmas 4.1(a), 4.1(b), and this choice is the same as that in [1] which meets the optimal conditions $b+b_1+c_1 = 1$ and $2b_1+c = 1$.

For $\ell = -3/4$, we choose $b = 46/128$, $b_1 = c_1 = 49/128$ and $c = 14/128$ which satisfy all the conditions stated in Lemmas 3.1, 3.2, 4.1(a) and 4.1(c). Notice that the choice also meets the optimal condition $b+b_1+c_1 = 9/8$.

The range $-3/4 \le \ell \le 3/4$ for local well-posedness is optimal. We can modify the counterexamples in [4] to show that the multilinear estimates fail when $|\ell| > 3/4$.

5. Proof for the multilinear estimates

We need the following calculus lemmas whose proofs are elementary. Denote $A := \sqrt{1+(x-y)^2}$, $A_1 := \sqrt{1+x^2}$, $A_2 := \sqrt{1+y^2}$ and $B_\pm := x \pm y$. Let

$$f(x,y) = \frac{2A(A_1+A_2)}{(A_1+A_2)^2+B_+^2}$$

and

$$F(x,y) = \frac{(1+B_-^2)y - B_-A_2(A_1+A_2)}{\left((A_1+A_2)^2+B_+^2\right)AA_2} - \frac{2A(A_1+A_2)\left((A_1+A_2)y+B_+A_2\right)}{\left((A_1+A_2)^2+B_+^2\right)^2 A_2}.$$

Lemma 5.1. *For any $x, y \in \mathbb{R}$, we have*

$$0 < f(x,y) < 2, \quad f(x,y) \sim \frac{1+|x-y|}{1+|x|+|y|}, \quad \frac{\partial}{\partial y} f(x,y) = 2F(x,y) \quad \textit{and} \quad |F(x,y)| < 2.$$

The proof is given in the next section.

Remark 5.2. The bound for $|F|$ is not optimal, however we will not pursuit this.

Define $[\lambda]_+ = \lambda$ if $\lambda > 0$, δ if $\lambda = 0$, 0 if $\lambda < 0$.

Lemma 5.3. *Let $\xi = \xi_1 - \xi_2$ and $|\xi| > 6$. For any $\sigma_1, \xi_1 \in \mathbb{R}$ and $\alpha \geq 0$,*

$$\sup_{\sigma_1} \int_{\left|\xi_{1\varepsilon} - \frac{1}{2}f\right| \geq 2\left|\xi_{2\varepsilon} - \frac{1}{2}f\right|} \left\langle \sigma_1 - \xi_{1\varepsilon}^2 + \xi_\varepsilon + \xi_{2\varepsilon}^2 \right\rangle^{-\alpha} d\xi_2 \lesssim \langle \xi_{1\varepsilon} \rangle^{[1-2\alpha]_+}.$$

The proof of the lemma will be given in the end.

Lemma 5.4. [4, Lemma 4.2] *Let $0 \leq a_- \leq a_+$, $a_+ + a_- > 1/2$ and $\alpha = 2a_- - [1-2a_+]_+$. Then the following estimate holds for all $s \in \mathbb{R}$*

$$\int_y \langle y - s \rangle^{-2a_+} \langle y + s \rangle^{-2a_-} dy \leq c \langle s \rangle^{-\alpha}.$$

We now prove the multilinear estimates.

Proof of Lemma 4.1(a). We prove the case of n_+ only. By duality argument, the estimate is equivalent to

$$|\langle n_+ E, g \rangle| \lesssim \|n_+\|_{X_{\ell,b}^{W_{\varepsilon+}}} \|E\|_{X_{0,b_1}^{S_\varepsilon}} \|g\|_{X_{0,c_1}^{S_\varepsilon}} \tag{5.1}$$

for all function $g \in X_{0,c_1}^{S_\varepsilon}$. We set $\widehat{v} = \langle \xi_\varepsilon \rangle^\ell \langle \tau + \xi_\varepsilon \rangle^b \widehat{n}_+$, $\widehat{v}_2 = \left\langle \tau + \xi_\varepsilon^2 \right\rangle^{b_1} \widehat{E}$, $\widehat{v}_1 = \left\langle \tau + \xi_\varepsilon^2 \right\rangle^{c_1} \widehat{g}$. Thus we can rewrite the left-hand side of (5.1) and denote the integral by S which gives the following bound

$$|S| \leq \int \frac{1}{\langle \xi_\varepsilon \rangle^\ell} \frac{|\widehat{v}(\tau,\xi)|}{\langle \sigma \rangle^b} \frac{|\widehat{v}_2(\tau_2,\xi_2)|}{\langle \sigma_2 \rangle^{b_1}} \frac{|\widehat{v}_1(\tau_1,\xi_1)|}{\langle \sigma_1 \rangle^{c_1}} d\tau_2 d\xi_2 d\tau_1 d\xi_1,$$

where

$$\tau = \tau_1 - \tau_2, \quad \xi = \xi_1 - \xi_2, \quad \sigma = \tau + \xi_\varepsilon, \quad \sigma_2 = \tau_2 + \xi_{2\varepsilon}^2 \quad \text{and} \quad \sigma_1 = \tau_1 + \xi_{1\varepsilon}^2. \tag{5.2}$$

We split S into two parts, one is on $|\xi| \le 6$ and denoted by S_1, while the other part is on $|\xi| > 6$ and denoted by S_2. The proof of the estimate for S_1 is the same as that of Lemma 4.1(a) in [1]. Thus we obtain $|S_1| \lesssim \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2}$ provided that $b_1 > 1/4$ and $c \ge 1/4$.

For $|\xi| > 6$, we split the region on which the integral S_2 is taken into three parts.

Region σ dominant, $|\sigma| \ge \max(|\sigma_1|, |\sigma_2|)$. We can rewrite the bound for $|S_2|$ as follows:

$$\int \frac{|\widehat{v}|}{\langle\sigma\rangle^b} \langle\xi_\varepsilon\rangle^{-\ell} \int \frac{|\widehat{v_1}|}{\langle\sigma_1\rangle^{c_1}} \frac{|\widehat{v_2}|}{\langle\sigma_2\rangle^{b_1}} \, d\sigma_2 d\xi_2 d\sigma d\xi.$$

Using the Cauchy-Schwarz inequality, we get the following bound

$$\left(\sup_{\sigma,\xi} \langle\sigma\rangle^{-2b} \langle\xi_\varepsilon\rangle^{-2\ell} \int \langle\sigma_1\rangle^{-2c_1} \langle\sigma_2\rangle^{-2b_1} \, d\xi_2 d\sigma_2\right)^{1/2} \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2},$$

which gives the desired result if the above supremum

$$\sup_{\sigma,\xi} \langle\sigma\rangle^{-2b} \langle\xi_\varepsilon\rangle^{-2\ell} \int \langle\sigma_1\rangle^{-2c_1} \langle\sigma_2\rangle^{-2b_1} \, d\xi_2 d\sigma_2 \tag{5.3}$$

is finite. The inner integral is taken over fixed σ, ξ and σ_2. Since $\xi_1 = \xi + \xi_2$, we have $d\xi_1 = d\xi_2$, and since

$$\sigma_1 - \sigma - \sigma_2 = (\xi + \xi_2)^2 + \varepsilon^2(\xi + \xi_2)^4 - \xi_2^2 - \varepsilon^2\xi_2^4 - \xi\sqrt{1 + \varepsilon^2\xi^2}, \tag{5.4}$$

we have

$$\left|\frac{d\sigma_1}{d\xi_2}\right| = |\xi| \left(2 + \varepsilon^2(4\xi^2 + 12\xi\xi_2 + 12\xi_2^2)\right) \sim |\xi| \left(1 + \varepsilon^2(\xi^2 + \xi_2^2)\right). \tag{5.5}$$

Thus

$$\langle\xi_\varepsilon\rangle^{-2\ell} \left|\frac{d\xi_2}{d\sigma_1}\right| \lesssim \frac{|\xi_\varepsilon|^{-2\ell}}{|\xi| \left(1 + \varepsilon^2(\xi^2 + \xi_2^2)\right)} \lesssim |\xi|^{-2\ell-1} \left(1 + \varepsilon^2\xi^2\right)^{-\ell-1}.$$

The integral in the supremum (5.3) becomes

$$\langle\sigma\rangle^{-2b} \int_0^{|\sigma|} \int_0^{|\sigma|} \langle\sigma_1\rangle^{-2c_1} \langle\sigma_2\rangle^{-2b_1} \, d\sigma_1 d\sigma_2 \lesssim \langle\sigma\rangle^{-2b + [1-2c_1]_+ + [1-2b_1]_+}.$$

For $6 < |\xi| < 12\varepsilon^{-1}$, we have

$$(5.3) \lesssim \sup_{\sigma,\xi} |\xi|^{-2\ell-1} \langle\sigma\rangle^{-2b+[1-2c_1]_+ + [1-2b_1]_+} \lesssim \varepsilon^{\min\{2\ell+1,0\}}$$

provided that $-2b+[1-2c_1]_+ + [1-2b_1]_+ \leq 0$. If $b_1, c_1 < 1/2$, then the exponent becomes $2-2b-2b_1-2c_1$, and it is sufficient to have $b+b_1+c_1 \geq 1$.

For $12\varepsilon^{-1} < |\xi|$, we get

$$(5.3) \lesssim \sup_{\sigma,\xi} \varepsilon^{-2\ell-2} |\xi|^{-4\ell-3} \langle\sigma\rangle^{-2b+[1-2c_1]_+ + [1-2b_1]_+} \lesssim \varepsilon^{2\ell+1}$$

provided that $\ell \geq -3/4$ and $-2b+[1-2c_1]_+ + [1-2b_1]_+ \leq 0$. If $b_1, c_1 < 1/2$, then the exponent becomes $2-2b-2b_1-2c_1$, and it is sufficient to have $b+b_1+c_1 \geq 1$.

Region σ_1 dominant, $|\sigma_1| \geq \max(|\sigma|, |\sigma_2|)$. We can rewrite the bound for $|S_2|$ as follows:

$$\int \frac{|\widehat{v}_1|}{\langle\sigma_1\rangle^{c_1}} \int \langle\xi_\varepsilon\rangle^{-\ell} \frac{|\widehat{v}|}{\langle\sigma\rangle^{b}} \frac{|\widehat{v}_2|}{\langle\sigma_2\rangle^{b_1}}\, d\sigma_2 d\xi_2 d\sigma_1 d\xi_1.$$

Using the Cauchy-Schwarz inequality, we get the following bound

$$\left(\sup_{\sigma_1,\xi_1} \langle\sigma_1\rangle^{-2c_1} \int \langle\xi_\varepsilon\rangle^{-2\ell} \langle\sigma\rangle^{-2b} \langle\sigma_2\rangle^{-2b_1}\, d\xi_2 d\sigma_2\right)^{1/2} \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2},$$

which gives the desired result if the above supremum

$$\sup_{\sigma_1,\xi_1} \langle\sigma_1\rangle^{-2c_1} \int \langle\xi_\varepsilon\rangle^{-2\ell} \langle\sigma\rangle^{-2b} \langle\sigma_2\rangle^{-2b_1}\, d\xi_2 d\sigma_2 \tag{5.6}$$

is finite. We observe that

$$\sigma - \sigma_1 + \sigma_2 = -\xi_{1\varepsilon}^2 + \xi_{2\varepsilon}^2 + (\xi_1-\xi_2)_\varepsilon = -\left(\xi_{1\varepsilon} - \frac{1}{2}f\right)^2 + \left(\xi_{2\varepsilon} - \frac{1}{2}f\right)^2, \tag{5.7}$$

where $f(\varepsilon\xi_1, \varepsilon\xi_2) = (\xi_1-\xi_2)_\varepsilon/(\xi_{1\varepsilon}-\xi_{2\varepsilon})$. Again we need to split the integral into two parts.

Subregion: $\left|\xi_{1\varepsilon} - \frac{1}{2}f\right| \leq 2\left|\xi_{2\varepsilon} - \frac{1}{2}f\right|$. Then $|\xi_\varepsilon| \leq 3f\left|\xi_{2\varepsilon} - \frac{1}{2}f\right|$. The inner integral in (5.6) over ξ_2 is taken with σ_1, ξ_1, σ_2 fixed. Denote $x := \varepsilon\xi_1$ and $y := \varepsilon\xi_2$. Thus

$$f(\varepsilon\xi_1, \varepsilon\xi_2) = f(x,y) = 2\frac{A(A_1+A_2)}{(A_1+A_2)^2 + B_+^2} \quad \text{and} \quad \frac{d}{d\xi_2} f(\varepsilon\xi_1, \varepsilon\xi_2) = 2\varepsilon F(\varepsilon\xi_1, \varepsilon\xi_2),$$

see Lemma 5.1. From (5.7), we have

$$\frac{d\sigma}{d\xi_{2\varepsilon}} = \left(\xi_{1\varepsilon} - \frac{1}{2}f\right) 2\varepsilon F \frac{d\xi_2}{d\xi_{2\varepsilon}} + \left(\xi_{2\varepsilon} - \frac{1}{2}f\right)\left(2 - 2\varepsilon F \frac{d\xi_2}{d\xi_{2\varepsilon}}\right)$$

and

$$\frac{d\xi_{2\varepsilon}}{d\xi_2} = \frac{1+2\varepsilon^2\xi_2^2}{\sqrt{1+\varepsilon^2\xi_2^2}} \sim 1 + \varepsilon|\xi_2|. \tag{5.8}$$

Notice that

$$\left|\frac{d\sigma}{d\xi_{2\varepsilon}}\right| \geq \left|\xi_{2\varepsilon} - \frac{1}{2}f\right| G, \tag{5.9}$$

where $G := 2\left(1 - 3\varepsilon\left|F\right|\frac{d\xi_2}{d\xi_{2\varepsilon}}\right)$. From Lemma 5.1, $|F| < 2$ together with $|\xi_2| > 6$, we have $3\varepsilon\left|F\right|\frac{d\xi_2}{d\xi_{2\varepsilon}} \le \frac{6\varepsilon}{\sqrt{1+72\varepsilon^2}} < \frac{1}{2}$. Thus $1 \le G \le 2$ for $\varepsilon \le 1$. Using (5.8) and (5.9) we have

$$\langle\xi_\varepsilon\rangle^{-2\ell}\frac{d\xi_2}{d\sigma} \sim |\xi_\varepsilon|^{-2\ell}\frac{d\xi_2}{d\xi_{2\varepsilon}}\frac{d\xi_{2\varepsilon}}{d\sigma} \lesssim |\xi|^{-2\ell-1}\left(1+\varepsilon^2\xi^2\right)^{-\ell-1}.$$

For $6 < |\xi| < 12\varepsilon^{-1}$, we obtain

$$\begin{aligned}(5.6) &\lesssim \sup_{\sigma_1,\xi_1}\langle\sigma_1\rangle^{-2c_1}\int_0^{|\sigma_1|}\int_0^{|\sigma_1|}|\xi|^{-2\ell-1}\langle\sigma\rangle^{-2b}\langle\sigma_2\rangle^{-2b_1}\,d\sigma d\sigma_2\\ &\lesssim \sup_{\sigma_1}\varepsilon^{\min\{2\ell+1,0\}}\langle\sigma_1\rangle^{-2c_1+[1-2b]_++[1-2b_1]_+}\\ &\lesssim \varepsilon^{\min\{2\ell+1,0\}}\end{aligned}$$

provided that $-2c_1+[1-2b]_++[1-2b_1]_+ \le 0$. If $b_1, b < 1/2$, then the exponent becomes $2-2b-2b_1-2c_1$, and it is sufficient to have $b+b_1+c_1 \ge 1$.

For $12\varepsilon^{-1} < |\xi|$, the supremum is bounded by

$$\begin{aligned}(5.6) &\lesssim \sup_{\sigma_1,\xi_1}\langle\sigma_1\rangle^{-2c_1}\int_0^{|\sigma_1|}\int_0^{|\sigma_1|}\varepsilon^{-2\ell-2}|\xi|^{-4\ell-3}\langle\sigma\rangle^{-2b}\langle\sigma_2\rangle^{-2b_1}\,d\sigma d\sigma_2\\ &\lesssim \sup_{\sigma_1}\varepsilon^{2\ell+1}\langle\sigma_1\rangle^{-2c_1+[1-2b]_++[1-2b_1]_+}\\ &\lesssim \varepsilon^{2\ell+1}\end{aligned}$$

provided that $\ell \ge -3/4$ and $-2c_1+[1-2b]_++[1-2b_1]_+ \le 0$. If $b_1, b < 1/2$, then the exponent becomes $2-2b-b_1-2c_1$, and it is sufficient to have $b+b_1+c_1 \ge 1$.

Subregion: $\left|\xi_{1\varepsilon}-\frac{1}{2}f\right| \ge 2\left|\xi_{2\varepsilon}-\frac{1}{2}f\right|$. Since $\xi = \xi_1 - \xi_2$, we have

$$\frac{1}{2}f\left|\xi_{1\varepsilon}-\frac{1}{2}f\right| \le |\xi_\varepsilon| \le \frac{3}{2}f\left|\xi_{1\varepsilon}-\frac{1}{2}f\right| \quad\text{and}\quad \langle\xi_\varepsilon\rangle \sim \left\langle f\left(\xi_{1\varepsilon}-\frac{1}{2}f\right)\right\rangle. \tag{5.10}$$

Also we have

$$\frac{3}{4}\left(\xi_{1\varepsilon}-\frac{1}{2}f\right)^2 \le \left(\xi_{1\varepsilon}-\frac{1}{2}f\right)^2 - \left(\xi_{2\varepsilon}-\frac{1}{2}f\right)^2 = \sigma_1-\sigma_2-\sigma \le 3|\sigma_1| \tag{5.11}$$

and thus

$$\left(\xi_{1\varepsilon}-\frac{1}{2}f\right)^2 \le 4|\sigma_1|, \quad f\left|\xi_{2\varepsilon}-\frac{1}{2}f\right| \le \frac{1}{2}f\left|\xi_{1\varepsilon}-\frac{1}{2}f\right| \le |\xi_\varepsilon|. \tag{5.12}$$

From Lemma 5.1, we have

$$f \sim \frac{1+\varepsilon|\xi|}{1+\varepsilon(|\xi_1|+|\xi_2|)}, \tag{5.13}$$

and also we have $C_1 < f < C_2$ on the subregion for some positive constants C_1 and C_2. Thus

$$|\xi|\sqrt{1+\varepsilon^2\xi^2} = |\xi_\varepsilon| \sim f\left|\xi_{1\varepsilon}-\frac{1}{2}f\right| \sim \frac{1+\varepsilon|\xi|}{1+\varepsilon|\xi_1|}|\xi_1|\sqrt{1+\varepsilon^2\xi_1^2} \sim (1+\varepsilon|\xi|)|\xi_1|. \tag{5.14}$$

Hence the supremum (5.6) is bounded by

$$\sup_{\sigma_1,\xi_1} \langle \xi_{1\varepsilon} \rangle^{-2\ell-4c_1} \int \langle \sigma \rangle^{-2b} \langle \sigma_2 \rangle^{-2b_1} \, d\sigma_2 d\xi_2.$$

The inner integral over σ_2 is taken with fixed ξ_2, ξ_1, σ_1. By Lemma 5.4,

$$\int \left\langle \sigma_2 - [\sigma_1 - \xi_{1\varepsilon}^2 + \xi_\varepsilon + \xi_{2\varepsilon}^2] \right\rangle^{-2b} \langle \sigma_2 \rangle^{-2b_1} \, d\sigma_2 \lesssim \left\langle \sigma_1 - \xi_{1\varepsilon}^2 + \xi_\varepsilon + \xi_{2\varepsilon}^2 \right\rangle^{-\alpha}$$

if $b + b_1 > 1/2$, where $\alpha = 2b - [1 - 2b_1]_+$ if $b_1 \geq b$, $2b_1 - [1 - 2b]_+$ if $b \geq b_1$. Applying Lemma 5.3, we get

$$\sup_{\sigma_1} \int_{|\xi_{1\varepsilon} - \frac{1}{2}f| \geq 2|\xi_{2\varepsilon} - \frac{1}{2}f|} \left\langle \sigma_1 - \xi_{1\varepsilon}^2 + \xi_\varepsilon + \xi_{2\varepsilon}^2 \right\rangle^{-\alpha} \lesssim \langle \xi_{1\varepsilon} \rangle^{[1-2\alpha]_+}.$$

For $6 < |\xi| < 12\varepsilon^{-1}$, we derive

$$(5.6) \lesssim \sup_{\xi_1} \langle \xi_{1\varepsilon} \rangle^{-2\ell-1} \langle \xi_{1\varepsilon} \rangle^{1-4c_1+[1-2\alpha]_+} \lesssim \varepsilon^{\min\{2\ell+1,0\}}$$

provided that $1 - 4c_1 + [1 - 2\alpha]_+ \leq 0$.

For $12\varepsilon^{-1} < |\xi|$, the supremum is bounded by

$$(5.6) \lesssim \sup_{\xi_1} \langle \xi_{1\varepsilon} \rangle^{-2\ell-1} \langle \xi_{1\varepsilon} \rangle^{1-4c_1+[1-2\alpha]_+} \lesssim \varepsilon^{2\ell+1}$$

provided that $-2\ell - 4c_1 + [1 - 2\alpha]_+ \leq 0$. We now discuss the above two exponents and combine them into $\max\{1, -2\ell\} - 4c_1 + [1 - 2\alpha]_+ \leq 0$. Suppose that $b, b_1 < 1/2$ but $b + b_1 > 1/2$. Then $\alpha = -1 + 2b + 2b_1$.

Case 1. $\alpha > 1/2$ if and only if $b + b_1 > 3/4$. Then we need $c_1 \geq \max\{1/4, -\ell/2\}$.

Case 2. $\alpha = 1/2$ if and only if $b + b_1 = 3/4$. Then we need $c_1 > \max\{1/4, -\ell/2\}$.

Case 3. $\alpha < 1/2$ if and only if $b + b_1 < 3/4$. Then the exponent is $\max\{1, -2\ell\} + 3 - 4b - 4b_1 - 4c_1$. We need $b + b_1 + c_1 \geq \max\{1, (-2\ell + 3)/4\}$.

Combine the above we need conditions as follows:

$$b, b_1 < \frac{1}{2}, \quad b + b_1 > \frac{1}{2}, \quad c_1 > \max\left\{\frac{1}{4}, -\frac{\ell}{2}\right\} \quad \text{and} \quad b + b_1 + c_1 \geq \max\left\{1, \frac{-2\ell+3}{4}\right\}.$$

Region σ_2 dominant, $|\sigma_2| \geq \max(|\sigma|, |\sigma_1|)$. This case is analogous to the case of region σ_1 dominant so that we skip the proof. The conditions resulted from this case are as follows:

$$b, c_1 < \frac{1}{2}, \quad b + c_1 > \frac{1}{2}, \quad b_1 > \max\left\{\frac{1}{4}, -\frac{\ell}{2}\right\} \quad \text{and} \quad b + b_1 + c_1 \geq \max\left\{1, \frac{-2\ell+3}{4}\right\}. \quad \square$$

Proof of Lemma 4.1(b). The proof for the part (b1) is proceeded in the same vein with that of the part (a). We prove the case of + only. By duality argument, the estimate is equivalent to

$$(5.15) \qquad \left| \left\langle D_\varepsilon^{-1} \partial_x (E_1 \overline{E_2}), g \right\rangle \right| \lesssim \|E_1\|_{X_{0,b_1}^{S_\varepsilon}} \|E_2\|_{X_{0,b_1}^{S_\varepsilon}} \|g\|_{X_{-\ell,c}^{W_{\varepsilon+}}}$$

for all function $g \in X^{W_{\varepsilon+}}_{-\ell,c}$. We set

$$\widehat{v} = \langle \xi_\varepsilon \rangle^{-\ell} \langle \tau + \xi_\varepsilon \rangle^c \widehat{g}, \quad \widehat{v}_1 = \left\langle \tau + \xi_\varepsilon^2 \right\rangle^{b_1} \widehat{E}_1, \quad \widehat{v}_2 = \left\langle \tau + \xi_\varepsilon^2 \right\rangle^{b_1} \widehat{E}_2. \tag{5.16}$$

Thus we can rewrite the left-hand side of (5.15) and denote it by W. Then we get the following bound

$$|W| \lesssim \int \frac{|\xi|}{\sqrt{1+\varepsilon^2\xi^2}} \langle \xi_\varepsilon \rangle^\ell \frac{|\widehat{v}|}{\langle \sigma \rangle^c} \frac{|\widehat{v}_2|}{\langle \sigma_1 \rangle^{b_1}} \frac{|\widehat{v}_1|}{\langle \sigma_1 \rangle^{b_1}} \, d\tau_2 d\xi_2 d\tau_1 d\xi_1,$$

where τ, ξ, σ, σ_2 and σ_1 are given as in (5.2). We then split W into two parts, one is on $|\xi| \leq 6$ and denoted by W_1, while the other part is on $|\xi| > 6$ and denoted by W_2. The proof of the estimate for W_1 is the same as that of Lemma 4.1(b) in [1]. Thus we obtain $|W_1| \lesssim \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2}$ provided that $b_1 > 1/4$ and $c \geq 1/4$.

For $|\xi| > 6$, we split the region on which the integral W_2 is split into three parts.

Region σ dominant, $|\sigma| \geq \max(|\sigma_1|, |\sigma_2|)$. We can rewrite the bound for $|W_2|$ as follows:

$$\int \frac{|\widehat{v}|}{\langle \sigma \rangle^c} \int \frac{|\xi|}{\widehat{D}_\varepsilon} \langle \xi_\varepsilon \rangle^\ell \frac{|\widehat{v}_1|}{\langle \sigma_1 \rangle^{b_1}} \frac{|\widehat{v}_2|}{\langle \sigma_2 \rangle^{b_1}} \, d\sigma_2 d\xi_2 d\sigma d\xi.$$

Analogously we want to show that the following supremum

$$\sup_{\sigma,\xi} \langle \sigma \rangle^{-2c} \int \frac{|\xi|^2}{\widehat{D}_\varepsilon^2} \langle \xi_\varepsilon \rangle^{2\ell} \langle \sigma_1 \rangle^{-2b_1} \langle \sigma_2 \rangle^{-2b_1} \, d\xi_2 d\sigma_2 \tag{5.17}$$

is finite. Using (5.4) and (5.5), we get

$$\frac{|\xi|^2}{\widehat{D}_\varepsilon^2} \langle \xi_\varepsilon \rangle^{2\ell} \left| \frac{d\xi_2}{d\sigma_1} \right| \lesssim \frac{|\xi|^2}{\widehat{D}_\varepsilon^2} \frac{|\xi_\varepsilon|^{2\ell}}{|\xi| \left(1 + \varepsilon^2(\xi^2 + \xi_2^2)\right)} \lesssim |\xi|^{2\ell+1} \left(1 + \varepsilon^2\xi^2\right)^{\ell-2}.$$

For $6 < |\xi| < 12\varepsilon^{-1}$, we have

$$(5.17) \lesssim \sup_{\sigma,\xi} |\xi|^{2\ell+1} \langle \sigma \rangle^{-2c+[1-2b_1]_+ + [1-2b_1]_+} \lesssim \varepsilon^{\min\{-2\ell-1,0\}}$$

provided that $-2c + [1-2b_1]_+ + [1-2b_1]_+ \leq 0$. If $b_1, c < 1/2$, then the exponent becomes $2 - 4b_1 - 2c$, and it is sufficient to have $2b_1 + c \geq 1$. For $12\varepsilon^{-1} < |\xi|$, we obtain

$$(5.17) \lesssim \sup_{\sigma,\xi} \varepsilon^{2\ell-4} |\xi|^{4\ell-3} \langle \sigma \rangle^{-2c+[1-2b_1]_+ + [1-2b_1]_+} \lesssim \varepsilon^{-2\ell-1}$$

provided that $\ell \leq 3/4$ and $-2c + [1-2b_1]_+ + [1-2b_1]_+ \leq 0$. If $b_1, c < 1/2$, then the exponent becomes $2 - 4b_1 - 2c$, and it is sufficient to have $2b_1 + c \geq 1$.

Region σ_1 dominant, $|\sigma_1| \geq \max(|\sigma|, |\sigma_2|)$. We can rewrite the bound for $|W_2|$ as follows:

$$\int \frac{|\widehat{v}_1|}{\langle \sigma_1 \rangle^{b_1}} \int \frac{|\xi|}{\widehat{D}_\varepsilon} \langle \xi_\varepsilon \rangle^\ell \frac{|\widehat{v}|}{\langle \sigma \rangle^c} \frac{|\widehat{v}_2|}{\langle \sigma_2 \rangle^{b_1}} \, d\sigma_2 d\xi_2 d\sigma_1 d\xi_1.$$

Analogously we want to show that the following supremum is finite:

$$\sup_{\sigma_1,\xi_1} \langle \sigma_1 \rangle^{-2b_1} \int \frac{|\xi|^2}{\widehat{D}_\varepsilon^2} \langle \xi_\varepsilon \rangle^{2\ell} \langle \sigma \rangle^{-2c} \langle \sigma_2 \rangle^{-2b_1} \, d\xi_2 d\sigma_2. \tag{5.18}$$

Subregion: $\left|\xi_{1\varepsilon} - \frac{1}{2}f\right| \leq 2\left|\xi_{2\varepsilon} - \frac{1}{2}f\right|$. Using $|\xi_\varepsilon| \leq 3f\left|\xi_{2\varepsilon} - \frac{1}{2}f\right|$ and (5.9), we have

$$\frac{|\xi|^2}{\widehat{D}_\varepsilon^2} \langle \xi_\varepsilon \rangle^{2\ell} \frac{d\xi_2}{d\sigma} \lesssim \frac{|\xi|^2}{\widehat{D}_\varepsilon^2} |\xi_\varepsilon|^{2\ell} \frac{1}{\left|\xi_{2\varepsilon} - \frac{1}{2}f\right| G} \frac{1}{1 + \varepsilon|\xi_2|} \lesssim |\xi|^{2\ell+1} \left(1 + \varepsilon^2\xi^2\right)^{\ell-2}.$$

For $6 < |\xi| < 12\varepsilon^{-1}$, we get

$$\begin{aligned} (5.18) &\lesssim \sup_{\sigma_1,\xi_1} \langle \sigma_1 \rangle^{-2b_1} \int_0^{|\sigma_1|} \int_0^{|\sigma_1|} |\xi|^{2\ell+1} \langle \sigma \rangle^{-2c} \langle \sigma_2 \rangle^{-2b_1} \, d\sigma d\sigma_2 \\ &\lesssim \varepsilon^{\max\{-2\ell-1,0\}} \sup_{\sigma_1} \langle \sigma_1 \rangle^{-2b_1+[1-2c]_+ + [1-2b_1]_+} \\ &\lesssim \varepsilon^{\max\{-2\ell-1,0\}} \end{aligned}$$

provided that $-2b_1 + [1-2c]_+ + [1-2b_1]_+ \leq 0$. If $b_1, c < 1/2$, then the exponent becomes $2 - 4b_1 - 2c$, and it is sufficient to have $2b_1 + c \geq 1$.

For $12\varepsilon^{-1} < |\xi|$, the supremum is bounded by

$$\begin{aligned} (5.18) &\lesssim \sup_{\sigma_1,\xi_1} \langle \sigma_1 \rangle^{-2b_1} \int_0^{|\sigma_1|} \int_0^{|\sigma_1|} \varepsilon^{2\ell-4} |\xi|^{4\ell-3} \langle \sigma \rangle^{-2c} \langle \sigma_2 \rangle^{-2b_1} \, d\sigma d\sigma_2 \\ &\lesssim \sup_{\sigma_1} \varepsilon^{-2\ell-1} \langle \sigma_1 \rangle^{-2b_1+[1-2c]_+ + [1-2b_1]_+} \\ &\lesssim \varepsilon^{-2\ell-1} \end{aligned}$$

provided that $\ell \leq 3/4$ and $-2b_1 + [1-2c]_+ + [1-2b_1]_+ \leq 0$. If $b_1, c < 1/2$, then the exponent becomes $2 - 4b_1 - 2c$, and it is sufficient to have $2b_1 + c \geq 1$.

Subregion: $\left|\xi_{1\varepsilon} - \frac{1}{2}f\right| \geq 2\left|\xi_{2\varepsilon} - \frac{1}{2}f\right|$. Using Lemma 5.1 and (5.10)–(5.14), the supremum (5.18) is bounded by

$$\sup_{\sigma_1,\xi_1} |\xi_1|^2 \widehat{D}_{1,\varepsilon}^{-2} \langle \xi_{1\varepsilon} \rangle^{2\ell+2-4b_1} \int \langle \sigma \rangle^{-2c} \langle \sigma_2 \rangle^{-2b_1} \, d\sigma_2 d\xi_2.$$

Using Lemmas 5.3 and 5.4, we have the following bound

$$\sup_{\xi_1} |\xi_1|^2 \widehat{D}_{1,\varepsilon}^{-2} \langle \xi_{1\varepsilon} \rangle^{2\ell-1} \langle \xi_{1\varepsilon} \rangle^{1-4b_1+[1-2\beta]_+},$$

where $\beta = 2c - [1-2b_1]_+$ if $b_1 \geq c$, $2b_1 - [1-2c]_+$ if $c \geq b_1$.

For $6 < |\xi| < 12\varepsilon^{-1}$, we have

$$(5.18) \lesssim \sup_{\xi_1} \langle \xi_{1\varepsilon} \rangle^{2\ell+1} \langle \xi_{1\varepsilon} \rangle^{1-4b_1+[1-2\beta]_+} \lesssim \varepsilon^{\min\{-2\ell-1,0\}}$$

provided that $1 - 4b_1 + [1-2\beta]_+ \le 0$. For $12\varepsilon^{-1} < |\xi|$, we get

$$(5.18) \lesssim \sup_{\xi_1} \varepsilon^{-2} \langle \xi_{1\varepsilon} \rangle^{2\ell-1} \langle \xi_{1\varepsilon} \rangle^{1-4b_1+[1-2\beta]_+} \lesssim \varepsilon^{-2\ell-1}$$

provided that $2\ell - 4b_1 + [1-2\beta]_+ \le 0$. We now combine the above two parts and consider the exponent $\max\{1, 2\ell\} - 4b_1 + [1-2\beta]_+ \le 0$. Suppose that $c, b_1 < 1/2$ but $c + b_1 > 1/2$. Then $\beta = -1 + 2c + 2b_1$.
Case 1. $\beta > 1/2$ if and only if $c + b_1 > 3/4$. Then we need $b_1 \ge \max\{1/4, \ell/2\}$.
Case 2. $\beta = 1/2$ if and only if $c + b_1 = 3/4$. Then we need $b_1 > \max\{1/4, \ell/2\}$.
Case 3. $\beta < 1/2$ if and only if $c + b_1 < 3/4$. Then the exponent is $\max\{4, 2\ell+3\} - 8b_1 - 4c$. We need $2b_1 + c \ge \max\{1, (2\ell+3)/4\}$.
Combine the above we need conditions as follows:

$$c, b_1 < 1/2, \quad c + b_1 > \frac{1}{2}, \quad b_1 > \max\left\{\frac{1}{4}, \frac{\ell}{2}\right\} \quad \text{and} \quad 2b_1 + c \ge \max\left\{1, \frac{2\ell+3}{4}\right\}.$$

Region σ_2 dominant, $|\sigma_2| \ge \max(|\sigma|, |\sigma_1|)$. This case is analogous to the case of region σ_1 dominant so that we omit the proof. The conditions resulted from this case are the same. □

Proof of Lemma 4.1(c). For the second part of (c), we prove the case of + only. By duality argument, the estimate is equivalent to

$$\left|\left\langle D_\varepsilon^{-1}\partial_x(E_1\overline{E_2}), g\right\rangle\right| \lesssim \|E_1\|_{X^{S_\varepsilon}_{0,b_1}} \|E_2\|_{X^{S_\varepsilon}_{0,b_1}} \|g\|_{X^{W_{\varepsilon+}}_{3/4,c}}$$

for all function $g \in X^{W_{\varepsilon+}}_{3/4,c}$. We set $\widehat{v}$, $\widehat{v}_1$ and $\widehat{v}_2$ as in (5.16). Thus we can rewrite the left-hand side of the formula and then use the fact that $\widehat{D}_\varepsilon \lesssim \widehat{D}_{1\varepsilon}\widehat{D}_{2\varepsilon}$ and assume that $b_1 > 3/8$ and $c \ge 0$ to obtain

$$\begin{aligned} |S| &\lesssim \int \frac{|\xi|^{1/4}}{\widehat{D}_\varepsilon^{7/4}} \frac{|\widehat{v}(\tau,\xi)|}{\langle\sigma\rangle^c} \frac{|\widehat{v}_2(\tau_2,\xi_2)|}{\langle\sigma_2\rangle^{b_1}} \frac{|\widehat{v}_1(\tau_1,\xi_1)|}{\langle\sigma_1\rangle^{b_1}} \, d\tau_2 d\xi_2 d\tau_1 d\xi_1 \\ &\lesssim \varepsilon^{-1/4} \left\| \left(\frac{\widehat{D}_{1\varepsilon}^{1/4} |\widehat{v}_1|}{\langle\sigma_1\rangle^{3/8+}} \right)^\vee \right\|_{L_t^4 L_x^4} \left\| \left(\frac{|\widehat{v}|}{\widehat{D}_\varepsilon^{7/4}} \right)^\vee \right\|_{L_t^2 L_x^2} \left\| \left(\frac{\widehat{D}_{2\varepsilon}^{1/4} |\widehat{v}_2|}{\langle\sigma_2\rangle^{3/8+}} \right)^\vee \right\|_{L_t^4 L_x^4}, \end{aligned}$$

where the variables τ, ξ, σ, σ_2 and σ_1 are given as in (5.2). To bound the first and the third quantities above, we need

$$\left\| |\widehat{v}_j|^\vee \right\|_{L_t^2 L_x^2} = \|v_j\|_{L_t^2 L_x^2}. \tag{5.19}$$

Then we interpolate between (5.19) and (3.1) with $\theta = 2/3$ to get

$$\left\| \left[\langle\sigma_j\rangle^{-(3/8+)} |\widehat{v}_j| \right]^\vee \right\|_{L_t^4 L_x^4} \lesssim \left\| D_{j\varepsilon}^{-1/4} v_j \right\|_{L_t^2 L_x^2} \quad \text{for } j = 1, 2.$$

To estimate the second quantity, we get

$$\left\| \left[\widehat{D}_{\varepsilon}^{-7/4} |\widehat{v}| \right]^{\vee} \right\|_{L_t^2 L_x^2} \lesssim \|v\|_{L_t^2 L_x^2} .$$

Thus we obtain $|S| \lesssim \varepsilon^{-1/4} \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2}$.

The proof for the first part is analogous. □

Proposition 5.5. *Assume that the estimates in* (4.1) *and* (4.2) *hold for some* b, b_1, c, c_1 *and for all* $v, v_1, v_2 \in L^2$. *Then one must have* $-3/4 \le \ell \le 3/4$.

Proof. Let $L \ge \varepsilon^{-1}$. For S and W, We define

$$\widehat{v} = \chi \left(2L + \frac{\sqrt{1+4\varepsilon^2 L^2}}{2\varepsilon^2 L^2} \le \xi \le 2L + \frac{\sqrt{1+4\varepsilon^2 L^2}}{2\varepsilon^2 L^2} + 2L^{-3} \right) \chi(|\sigma| \lesssim \varepsilon^{-1}),$$

$$\widehat{v}_1 = \chi \left(L + \frac{\sqrt{1+4\varepsilon^2 L^2}}{2\varepsilon^2 L^2} \le \xi_1 \le L + \frac{\sqrt{1+4\varepsilon^2 L^2}}{2\varepsilon^2 L^2} + L^{-3} \right) \chi(|\sigma_1| \le 1),$$

$$\widehat{v}_2 = \chi \left(L \le -\xi_2 \le L + L^{-3} \right) \chi(|\sigma_2| \le 1).$$

Thus we can compute that

$$\|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2} \sim \varepsilon^{-1/2} L^{-9/2}, \quad S \sim (\varepsilon L^2)^{-\ell} L^{-6} \quad \text{and} \quad W \sim \varepsilon^{-1} (\varepsilon L^2)^{\ell} L^{-6}.$$

If (4.1) and (4.2) are to hold, then we have $-3/4 \le \ell \le 3/4$. □

6. Appendix

Proof of Lemma 5.1. It is clear that $f > 0$ and for f on $x + y \neq 0$, we have

$$f(x,y) = 2 \frac{A(A_1 + A_2)}{(A_1 + A_2)^2 + (x+y)^2} < 2 \frac{A}{(A_1 + A_2)} \le 2.$$

For f on $x + y = 0$, we get $f(x,-x) = \sqrt{1+4x^2}/\sqrt{1+x^2} < 2$. Since F has two parts, we call the first part F_1 and the second part F_2. For F_1, we have

$$|F_1| \le \frac{(1+B_-^2)\,|y|}{(A_1+A_2)^2 (1+B_-^2)^{1/2} A_2} + \frac{|B_-|\, A_2 (A_1 + A_2)}{(A_1+A_2)^2 (1+B_-^2)^{1/2} A_2} < \frac{1}{2} + \frac{1}{2} = 1.$$

For F_2, we again decompose it into two parts such that $F_2 = F_{21} \cdot F_{22}$, where

$$F_{21} = -\frac{\sqrt{1+B_-^2}(A_1 + A_2)}{(A_1+A_2)^2 + B_+^2} \quad \text{and} \quad F_{22} = \frac{2\left((A_1+A_2)y + B_+ A_2\right)}{\left((A_1+A_2)^2 + B_+^2\right) A_2}.$$

Through some calculations, we derive that $|F_{21}| < 1$ and $|F_{22}| < 1$, thus we have $|F_2| \le 1$. □

Proof of Lemma 5.3. Recall that $\xi_\varepsilon = \xi\sqrt{1+\varepsilon^2\xi^2}$, $\xi = \xi_1 - \xi_2$, $|\xi| > 6$ and $|\xi_1| > 4$. Since the sets

$$\left\{\xi_2 : \left|\xi_{1\varepsilon} - \frac{1}{2}f\right| \geq 2\left|\xi_{2\varepsilon} - \frac{1}{2}f\right|\right\} \subset \{\xi_2 : |\xi_{1\varepsilon}| \geq |\xi_{2\varepsilon}|\} = \{\xi_2 : |\xi_1| \geq |\xi_2|\},$$

it is sufficient to prove the estimate

$$\sup_{\sigma_1}\int_{|\xi_2|\leq|\xi_1|}\left\langle \sigma_1 - \xi_{1\varepsilon}^2 + \xi_\varepsilon + \xi_{2\varepsilon}^2\right\rangle^{-\alpha} d\xi_2 \lesssim \langle\xi_{1\varepsilon}\rangle^{[1-2\alpha]_+}.$$

Denote $Y(\xi_2) = \sigma_1 - \xi_{1\varepsilon}^2 + \xi_\varepsilon + \xi_{2\varepsilon}^2$. Thus we have

$$Y'(\xi_2) = -\frac{1+2\varepsilon^2\xi^2}{\sqrt{1+\varepsilon^2\xi^2}} + 2\xi_2 + 4\varepsilon^2\xi_2^3 \quad \text{and} \quad Y''(\xi_2) = \varepsilon\frac{3\varepsilon\xi + 2\varepsilon^3\xi^3}{\sqrt{(1+\varepsilon^2\xi^2)^3}} + 2 + 12\varepsilon^2\xi_2^2.$$

Then we can show that $Y''(\xi_2) > 0$ for all ξ_2, which implies that $Y'(\xi_2)$ is increasing and $Y(\xi_2)$ in convex. Together with the facts $Y'(1/2) < 0$ and $Y'(|\xi_1|/2) > 0$, Intermediate Value Theorem asserts that there is a unique number K in $[1/2, |\xi_1|/2]$ such that $Y'(K) = 0$, and thus $Y(K)$ is the minimum of Y. Denote $A = |\xi_1|$, $I = [-A, A]$ and the two roots of $Y(\xi_2)$ by R_1 and R_2 if there is any. Let $R_1 \leq R_2$ and $B = \min\{R_2, A\}$. For all possible values of σ_1 and ξ_1, we distinguish the following cases.

1. For $Y(K) \geq 0$, we split I into $I_1 = [-A, -1/2]$, $I_2 = [-1/2, 1/2]$, $I_3 = [1/2, K]$ and $I_4 = [K, A]$. Then we define the function $z(\xi_2) = z_j(\xi_2)$ on I_j for $j = 1, 2, 3, 4$, where

$$z_1(\xi_2) = \frac{1}{4}\left(\xi_2 + \frac{1}{2}\right)_\varepsilon^2,\ z_2(\xi_2) = 0,\ z_3(\xi_2) = \frac{1}{16}(\xi_{2\varepsilon} - K_\varepsilon)^2,\ z_4(\xi_2) = \frac{1}{4}(\xi_2 - K)_\varepsilon^2.$$

2. For $Y(K) < 0$ and $1/2 \leq R_1$, we split I into $I_1 = [-A, -1/2]$, $I_2 = [-1/2, 1/2]$, $I_3 = [1/2, R_1]$, $I_4 = [R_1, K]$, $I_5 = [K, B]$ and $I_6 = [B, A]$. Then we define the function $z(\xi_2) = z_j(\xi_2)$ on I_j for $j = 1, 2, 3, 4, 5, 6$, where

$$z_1(\xi_2) = \frac{1}{4}\left(\xi_2 + \frac{1}{2}\right)_\varepsilon^2, \quad z_2(\xi_2) = 0,$$
$$z_3(\xi_2) = \frac{1}{16}(\xi_{2\varepsilon} - R_{1\varepsilon})^2, \quad z_4(\xi_2) = -\frac{1}{16}\frac{(K_\varepsilon - R_{1\varepsilon})^2}{K - R_1}(\xi_2 - R_1),$$
$$z_5(\xi_2) = \frac{1}{4}\left[(B-K) + \varepsilon^2(B-K)^3\right](\xi_2 - B), \quad z_6(\xi_2) = \frac{1}{4}(\xi_2 - B)_\varepsilon^2.$$

3. For $Y(K) < 0$ and $-1/2 \leq R_1 \leq 1/2$, we split I into $I_1 = [-A, -1/2]$, $I_2 = [-1/2, 1/2]$, $I_3 = [1/2, K]$, $I_4 = [K, B]$ and $I_5 = [B, A]$. Then we define the function $z(\xi_2) = z_j(\xi_2)$ on I_j for $j = 1, 2, 3, 4, 5$, where

$$z_1(\xi_2) = \frac{1}{4}\left(\xi_2 + \frac{1}{2}\right)_\varepsilon^2, \quad z_2(\xi_2) = 0, \quad z_3(\xi_2) = -\frac{1}{16}\frac{(K_\varepsilon - (1/2)_\varepsilon)^2}{K - 1/2}\left(\xi_2 - \frac{1}{2}\right),$$
$$z_4(\xi_2) = \frac{1}{4}\left[(B-K) + \varepsilon^2(B-K)^3\right](\xi_2 - B), \quad z_5(\xi_2) = \frac{1}{4}(\xi_2 - B)_\varepsilon^2.$$

4. For $Y(K) < 0$ and $-A \leq R_1 \leq -1/2$, we split I into $I_1 = [-A, R_1]$, $I_2 = [R_1, -1/2]$, $I_3 = [-1/2, 1/2]$, $I_4 = [1/2, K]$, $I_5 = [K, B]$ and $I_6 = [B, A]$. Then we define the function $z(\xi_2) = z_j(\xi_2)$ on I_j for $j = 1, 2, 3, 4, 5, 6$, where
$$z_1(\xi_2) = \frac{1}{4}(\xi_2 + R_1)_\varepsilon^2, \quad z_2(\xi_2) = \frac{1}{4}\left[\left(R_1 + \frac{1}{2}\right) + \varepsilon^2\left(R_1 + \frac{1}{2}\right)^3\right](\xi_2 - R_1),$$
$$z_3(\xi_2) = 0, \quad z_4(\xi_2) = -\frac{1}{16}\frac{(K_\varepsilon - (1/2)_\varepsilon)^2}{K - 1/2}\left(\xi_2 - \frac{1}{2}\right),$$
$$z_5(\xi_2) = \frac{1}{4}\left[(B - K) + \varepsilon^2(B - K)^3\right](\xi_2 - B), \quad z_6(\xi_2) = \frac{1}{4}(\xi_2 - B)_\varepsilon^2.$$

5. For $Y(K) < 0$ and $R_1 \leq -A$, we split I into $I_1 = [-A, -1/2]$, $I_2 = [-1/2, 1/2]$, $I_3 = [1/2, K]$, $I_4 = [K, B]$ and $I_5 = [B, A]$. Then we define the function $z(\xi_2) = z_j(\xi_2)$ on I_j for $j = 1, 2, 3, 4, 5$, where
$$z_1(\xi_2) = \frac{1}{4}\left[\left(-A + \frac{1}{2}\right) + \varepsilon^2\left(-A + \frac{1}{2}\right)^3\right](\xi_2 + A), \quad z_2(\xi_2) = 0,$$
$$z_3(\xi_2) = -\frac{1}{16}\frac{(K_\varepsilon - (1/2)_\varepsilon)^2}{K - 1/2}\left(\xi_2 - \frac{1}{2}\right),$$
$$z_4(\xi_2) = \frac{1}{4}\left[(B - K) + \varepsilon^2(B - K)^3\right](\xi_2 - B), \quad z_5(\xi_2) = \frac{1}{4}(\xi_2 - B)_\varepsilon^2.$$

Through some straight forward calculations we obtain $0 \leq |z(\xi_2)| \leq |Y(\xi_2)|$. Hence we can replace the function Y by z such that the integral become manageable:
$$\int_I \langle Y(\xi_2)\rangle^{-\alpha}\, d\xi_2 \leq \sum_{j=1} \int_{I_j} \langle z_j(\xi_2)\rangle^{-\alpha}\, d\xi_2 \lesssim \langle \xi_{1\varepsilon}\rangle^{[1-2\alpha]_+}. \qquad \square$$

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