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Existence and Stability of Coexistence States for a Reaction-diffusion-advection Model

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Abstract. In this paper, we consider a two-species Lotka-Volterra competition model in one-dimensional spatially inhomogeneous environments. It is assumed that two competitors have the same movement strategy but slightly differing in their inter- and intra-specific competition rates. By using the Lyapunov-Schmidt reduction technique as well as some analytic skills, we find that the existence and stability of coexistence states can be determined by some scalar functions, and hence the unique coexistence state of the system is established in certain cases.

1. Introduction

Reaction-diffusion-advection equations have received increasing attention from ecologists and mathematicians. The evolution of dispersal has been an important topic in population dynamics and has been extensively studied by many scholars, see [1–7, 9–24] and the references therein. In this paper, a two species Lotka-Volterra competition model in an advective homogeneous environment is considered. In particular, we assume that two species are identical in dispersal strategy and are competing for the same resources which are evenly distributed across space, but they have different competition abilities, which are incorporated by a perturbation approach, as measured by a perturbation parameter τ in model (1.4) below.

We first recall the classical Lotka-Volterra competition model

$$(1.1)\qquad \begin{cases} u_t = \mu u_{xx} - \alpha u_x + u(r - u - v), & 0 < x < L,\ t > 0, \\ v_t = \mu v_{xx} - \alpha v_x + v(r - u - v), & 0 < x < L,\ t > 0, \\ \mu u_x(0,t) - \alpha u(0,t) = \mu u_x(L,t) - \alpha u(L,t) = 0, & t > 0, \\ \mu v_x(0,t) - \alpha v(0,t) = \mu v_x(L,t) - \alpha v(L,t) = 0, & t > 0, \end{cases}$$

where $u(x,t)$ and $v(x,t)$ represent the population densities of two competing species at location x and time t, L is the size of the habitat, α is the advection rate. The diffusion

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rate μ is a positive constant and the constant $r > 0$ accounts for the intrinsic growth rate of species.

It is well-known that for any $\alpha \in R$, the problem

$$(1.2)\qquad \begin{cases} \mu\theta_{xx} - \alpha\theta_x + \theta(r-\theta) = 0, & 0 < x < L, \\ \mu\theta_x(0) - \alpha\theta(0) = \mu\theta_x(L) - \alpha\theta(L) = 0 \end{cases}$$

has a unique positive solution which we denote by $\theta_{\mu,\alpha}$. In particular, $\theta_{\mu,\alpha}$ is non-degenerate and linearly stable.

In system (1.1), two species are supposed to be completely identical (the same dispersal strategy and population dynamics), i.e., no one has more competitive advantages, then biologically it is expected that two competitors will coexist eventually. Indeed, mathematically this is easy to confirm as we can see that system (1.1) is degenerate and there is a continuum of positive steady states $(s\theta_{\mu,\alpha}, (1-s)\theta_{\mu,\alpha})$ for any $s \in (0,1)$, which are globally attractive. It seems natural and interesting to inquire what will happen if two competing species are slightly different. In [23], Lou and Zhou considered the two species are almost identical except for their diffusion rates

$$\begin{cases} u_t = \mu_1 u_{xx} - \alpha u_x + u(r-u-v), & 0 < x < L,\ t > 0, \\ v_t = \mu_2 v_{xx} - \alpha v_x + v(r-u-v), & 0 < x < L,\ t > 0, \\ \mu_1 u_x(0,t) - \alpha u(0,t) = \mu_1 u_x(L,t) - \alpha u(L,t) = 0, & t > 0, \\ \mu_2 v_x(0,t) - \alpha v(0,t) = \mu_2 v_x(L,t) - \alpha v(L,t) = 0, & t > 0. \end{cases}$$

They showed that if $0 < \mu_1 < \mu_2$, $\alpha, r, L > 0$, then $(0, \theta_{\mu_2,\alpha})$ must be globally asymptotically stable, that is, the faster diffuser always wipes out its slower competitor in a spatial homogeneous environment, which is a contrast to that the species with small random diffusion rate is always the winner in non-advective but spatially heterogeneous environments.

Moreover, Lou, Xiao and Zhou [22] considered the two species are almost identical except for their advection rates

$$(1.3)\qquad \begin{cases} u_t = \mu u_{xx} - \alpha u_x + u(r-u-v), & 0 < x < L,\ t > 0, \\ v_t = \mu v_{xx} - \beta v_x + v(r-u-v), & 0 < x < L,\ t > 0, \\ \mu u_x(0,t) - \alpha u(0,t) = \mu u_x(L,t) - \alpha u(L,t) = 0, & t > 0, \\ \mu v_x(0,t) - \beta v(0,t) = \mu v_x(L,t) - \beta v(L,t) = 0, & t > 0. \end{cases}$$

They proved that if $0 = \alpha < |\beta|$, then $(\theta_{\mu,0}, 0) = (r, 0)$ is globally asymptotically stable. That is, the species without directed movement will win the the competition. Moreover, they also proved that $(\theta_{\mu,0}, 0) = (r, 0)$ is globally asymptotically stable for any diffusion rate. Furthermore, they also showed that if $0 < \alpha < \beta$, then $(\theta_{\mu,\alpha}, 0)$ is globally asymptotically stable; if $\alpha < 0 < \beta$, then system (1.3) has a stable coexistence state. And they conjectured that this coexistence state should be unique, and hence it must be globally asymptotically stable.

Furthermore, Zhou [26] also considered a more general case: two competitors differing in both their diffusion and advection rates. In particular, they showed that if two species drift along the same direction and one competitor takes both larger diffusion and advection rates, then it will be completely wiped out provided its ratio of advection and diffusion rate is also larger; while this ratio relation is reversed, either one can become the winner, even two species could coexist under some suitable conditions; if two species drift along the same direction and one competitor takes smaller advection rate but larger diffusion rate, then it will always win; if two species drift along opposite directions, then two species will coexist, regardless of the size of diffusion and advection rates. In [25], Zhao and Zhou studied one species assumes pure random diffusion while another species undergoes mixed movement in a one-dimensional habitat with spatial variation, and they showed that either pure random diffusion or mixed movement wins, or both species can coexist eventually, and system could admit two coexistence states in some special cases.

In this paper, we consider the following model

$$
\begin{cases}
u_t = \mu u_{xx} - \alpha u_x + u(r - (1+\tau m)u - (1+\tau g)v), & 0 < x < L,\ t > 0,\\
v_t = \mu v_{xx} - \alpha v_x + v(r - (1+\tau h)u - v), & 0 < x < L,\ t > 0,\\
\mu u_x(0,t) - \alpha u(0,t) = \mu u_x(L,t) - \alpha u(L,t) = 0, & t > 0,\\
\mu v_x(0,t) - \alpha v(0,t) = \mu v_x(L,t) - \alpha v(L,t) = 0, & t > 0,
\end{cases}
\tag{1.4}
$$

where τ is a positive constant and m, g and h are smooth functions. We assume that two competing species have the same diffusion rate, advection rate and the distribution of resources, however, they have the different intraspecific competition rates and interspecific competition rates.

For $0 < \tau \ll 1$, we show that the existence and stability of coexistence states of system (1.4) can be determined by the scalar functions of μ and α defined as follows:

$$
H(\mu,\alpha) = \int_0^L h(x) e^{-\frac{\alpha}{\mu}x}\theta^3_{\mu,\alpha}, \quad M(\mu,\alpha) = \int_0^L m(x) e^{-\frac{\alpha}{\mu}x}\theta^3_{\mu,\alpha}, \quad G(\mu,\alpha) = \int_0^L g(x) e^{-\frac{\alpha}{\mu}x}\theta^3_{\mu,\alpha}.
$$

However, for the sake of simplicity, we fix the parameter α in this paper, and we regard the functions $\theta_{\mu,\alpha}$, $H(\mu,\alpha)$, $M(\mu,\alpha)$ and $G(\mu,\alpha)$ as only a function of the diffusion rate μ. We simply denote them by θ_μ, $H(\mu)$, $M(\mu)$ and $G(\mu)$, respectively.

For the monotone dynamical system (see [8]), we know

(a) if there is no coexistence state, then one of the semi-trivial equilibria is unstable and the other one is the global attractor;

(b) if there is a unique coexistence state and it is stable, then it is the global attractor;

(c) if all coexistence states are asymptotically stable, then there is at most a coexistence state, which is the global attractor (if it exists).

For the linearized stability of a steady state (u, v) of system (1.4), we only need to consider the following eigenvalue problem

(1.5)
$$\begin{cases} \mu\varphi_{xx} - \alpha\varphi_x + [r - 2(1+\tau m)u - (1+\tau g)v]\varphi - u(1+\tau g)\psi + \lambda\varphi = 0, & 0 < x < L, \\ \mu\psi_{xx} - \alpha\psi_x + [r - (1+\tau h)u - 2v]\psi - v(1+\tau h)\varphi + \lambda\psi = 0, & 0 < x < L, \\ \mu\varphi_x(0) - \alpha\varphi(0) = \mu\varphi_x(L) - \alpha\varphi(L) = 0, \\ \mu\psi_x(0) - \alpha\psi(0) = \mu\psi_x(L) - \alpha\psi(L) = 0. \end{cases}$$

It is easy to see that problem (1.5) has a principal eigenvalue $\lambda_1 \in R$, which is simple and has the least real part among all eigenvalues. Moreover, we may choose the corresponding eigenfunction (φ, ψ) to satisfy $\varphi > 0 > \psi$ on $[0, L]$. The linearized stability of (u, v) can be expressed in terms of the principal eigenvalue: (u, v) is linearly stable if $\lambda_1 > 0$; it is unstable if $\lambda_1 < 0$.

Finally, we state main results of this paper. In particular, we show how $H(\mu)$, $M(\mu)$ and $G(\mu)$ determine the structure of coexistence states and their stability of system (1.4).

Theorem 1.1. *Suppose that functions $H(\mu) - M(\mu)$ and $G(\mu)$ have no common roots. Let μ_1 and μ_2 be two consecutive simple roots of the function $(H(\mu) - M(\mu))G(\mu)$. Then*

(i) *if $(H(\mu) - M(\mu))G(\mu) < 0$ in (μ_1, μ_2), then system (1.4) has no coexistence state for $0 < \tau \ll 1$;*

(ii) *if $(H(\mu) - M(\mu))G(\mu) > 0$ in (μ_1, μ_2), then for $0 < \tau \ll 1$, there exist $\underline{\mu} = \underline{\mu}(\tau)$ nearby μ_1 and $\overline{\mu} = \overline{\mu}(\tau)$ nearby μ_2 such that for each $\mu \in (\underline{\mu}, \overline{\mu})$, system (1.4) has a coexistence state $(u(\mu, \tau), v(\mu, \tau))$. In particular, $(u(\mu, \tau), v(\mu, \tau))$ will reduce to the semi-trivial equilibria when $\mu = \underline{\mu}$ or $\mu = \overline{\mu}$.*

Theorem 1.2. *Assume that conditions of Theorem 1.1 hold. Then we have*

(i) *if $H(\mu) - M(\mu) < 0$ and $G(\mu) < 0$ in (μ_1, μ_2), then system (1.4) has a unique coexistence state $(u(\mu, \tau), v(\mu, \tau))$, and it is the global attractor of system (1.4);*

(ii) *if $H(\mu) - M(\mu) > 0$ and $G(\mu) > 0$ in (μ_1, μ_2), then system (1.4) has a unstable coexistence state $(u(\mu, \tau), v(\mu, \tau))$.*

The rest of this paper is organized as follows. In Section 2, we study the existence and non-existence of coexistence states of system (1.4). In Section 3, the stability of coexistence states of system (1.4) is established.

2. Existence and non-existence of coexistence states

In this section, we study the existence and non-existence of coexistence states of system (1.4) by the Lyapunov-Schmidt reduction technique. In particular, Theorem 1.1 can be established by the following Theorem 2.1.

To a large extent, the dynamics of system (1.4) are determined by the stability or instability of steady states since system (1.4) is a monotonic dynamical system. To deal with the existence of coexistence states of system (1.4), we only need to show that the following system (2.1) has a positive solution under some suitable conditions.

Let $\omega = e^{-\frac{\alpha}{\mu}x}u$ and $\chi = e^{-\frac{\alpha}{\mu}x}v$. Then the steady state of system (1.4) can be rewritten as

$$(2.1)\qquad \begin{cases} \mu\left\{e^{\frac{\alpha}{\mu}x}\omega_x\right\}_x + e^{\frac{\alpha}{\mu}x}\omega\left(r-(1+\tau m)e^{\frac{\alpha}{\mu}x}\omega-(1+\tau g)e^{\frac{\alpha}{\mu}x}\chi\right)=0, & 0<x<L,\\ \mu\left\{e^{\frac{\alpha}{\mu}x}\chi_x\right\}_x + e^{\frac{\alpha}{\mu}x}\chi\left(r-(1+\tau h)e^{\frac{\alpha}{\mu}x}\omega-e^{\frac{\alpha}{\mu}x}\chi\right)=0, & 0<x<L,\\ \omega_x(0)=\omega_x(L)=\chi_x(0)=\chi_x(L)=0. \end{cases}$$

We let $\widetilde{\theta}_\mu$ be the unique positive solution of the following problem

$$\begin{cases} \mu\left\{e^{\frac{\alpha}{\mu}x}\widetilde{\theta}_x\right\}_x + e^{\frac{\alpha}{\mu}x}\widetilde{\theta}\left(r-e^{\frac{\alpha}{\mu}x}\widetilde{\theta}\right)=0, & 0<x<L,\\ \widetilde{\theta}_x(0)=\widetilde{\theta}_x(L)=0 \end{cases}$$

and let $\widetilde{\omega}$ be the unique positive solution of the following problem

$$\begin{cases} \mu\left\{e^{\frac{\alpha}{\mu}x}\omega_x\right\}_x + e^{\frac{\alpha}{\mu}x}\omega\left(r-(1+\tau m)e^{\frac{\alpha}{\mu}x}\omega\right)=0, & 0<x<L,\\ \omega_x(0)=\omega_x(L)=0. \end{cases}$$

In particular, we have $\widetilde{\theta}_\mu = e^{-\frac{\alpha}{\mu}x}\theta_\mu$, where θ_μ is the unique positive solution of (1.2).

Similarly, we set the following scalar functions:

$$\widetilde{H}(\mu)=\int_0^L h(x)e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu^3,\quad \widetilde{M}(\mu)=\int_0^L m(x)e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu^3,\quad \widetilde{G}(\mu)=\int_0^L g(x)e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu^3.$$

We also regard the above three functions as a function of the diffusion rate μ. In fact, we claim that $H(\mu)=\widetilde{H}(\mu)$, $M(\mu)=\widetilde{M}(\mu)$ and $G(\mu)=\widetilde{G}(\mu)$ under the change of variable $\widetilde{\theta}_\mu = e^{-\frac{\alpha}{\mu}x}\theta_\mu$, respectively.

We know that system (2.1) has a nontrivial nonnegative solution

$$\gamma_\mu = \left\{\left(s\widetilde{\theta}_\mu,(1-s)\widetilde{\theta}_\mu\right): s\in[0,1]\right\}$$

for $\tau=0$, and we construct a positive solution (ω,χ) of system (2.1) near the curve γ_μ for $0<\tau\ll 1$. Moreover, we also claim that all positive solutions of system (2.1) are close to the curve γ_μ when $0<\tau\ll 1$ by the following Lemma 2.2.

For any $p > 1$, the Sobolev space $W^{2,p}(0,L) \hookrightarrow C^1[0,L]$. Set

$$X = \left\{(y,z) \in W^{2,p}(0,L) \times W^{2,p}(0,L) : \omega_x(0) = \omega_x(L) = \chi_x(0) = \chi_x(L) = 0\right\},$$
$$X_1 = \operatorname{span}\left\{\left(\widetilde{\theta}_\mu, -\widetilde{\theta}_\mu\right)\right\}, \quad Y = L^p(0,L) \times L^p(0,L),$$
$$X_2 = \left\{(y,z) \in X : \int_0^L (y-z)\widetilde{\theta}_\mu = 0\right\}.$$

Theorem 2.1. *Suppose that functions $\widetilde{H}(\mu) - \widetilde{M}(\mu)$ and $\widetilde{G}(\mu)$ have no common roots for $0 < \tau \ll 1$, and let μ_1 and μ_2 be two consecutive simple roots of function $\left(\widetilde{H}(\mu) - \widetilde{M}(\mu)\right)$ $\widetilde{G}(\mu)$. Then there exists a neighborhood U of the curve γ_μ, and*

(i) *if $\left(\widetilde{H}(\mu) - \widetilde{M}(\mu)\right)\widetilde{G}(\mu) < 0$ in (μ_1, μ_2), then system (2.1) has no positive solution near γ_μ;*

(ii) *if $\left(\widetilde{H}(\mu) - \widetilde{M}(\mu)\right)\widetilde{G}(\mu) > 0$ in (μ_1, μ_2), then for $0 < \tau \ll 1$, the set of solutions of system (2.1) in U consists of the semi-trivial solutions $(\mu, \widetilde{w}, 0)$, $\left(\mu, 0, \widetilde{\theta}_\mu\right)$, and the set $\Xi \cap U$, where Ξ is given by*

$$\Xi = \left\{(\mu, \omega(\mu,\tau), \chi(\mu,\tau)) : \mu_1 - \delta \le \mu \le \mu_2 + \delta\right\}.$$

Here,

$$\omega(\mu,\tau) = s^*(\mu,\tau)\left[\widetilde{\theta}_\mu + \overline{y}(\mu,\tau)\right], \quad \chi(\mu,\tau) = [1 - s^*(\mu,\tau)]\left[\widetilde{\theta}_\mu + \overline{z}(\mu,\tau)\right]. \tag{2.2}$$

In particular, $s^*(\mu,0) = s_0(\mu) := \frac{\widetilde{G}(\mu)}{\widetilde{G}(\mu) + \widetilde{H}(\mu) - \widetilde{M}(\mu)}$, $\overline{y}(\mu,0) = \overline{z}(\mu,0) = 0$. Moreover, there exist smooth functions $\underline{\mu}(\tau)$ and $\overline{\mu}(\tau)$ for $0 \le \tau \ll 1$ such that $\underline{\mu}(0) = \mu_1$, $\overline{\mu}(0) = \mu_2$.

Proof. It is easy to see that any nonnegative solution (ω, χ) of system (2.1) for $0 < \tau \ll 1$ can be written as $(\omega, \chi) = \left(s\widetilde{\theta}_\mu, (1-s)\widetilde{\theta}_\mu\right) + (y,z)$, where $s \in [0,1]$ and $(y,z) \in X_2$ near $(0,0)$. In particular, we find a positive solution of system (2.1) near γ_μ.

First, we define the map F:

$$F(y,z,\mu,\tau,s) = \begin{pmatrix} \mu\left\{e^{\frac{\alpha}{\mu}x} y_x\right\}_x + e^{\frac{\alpha}{\mu}x} y(r - e^{\frac{\alpha}{\mu}x}\widetilde{\theta}_\mu) - s e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu(y+z) + f_1(y,z,\mu,\tau,s) \\ \mu\left\{e^{\frac{\alpha}{\mu}x} z_x\right\}_x + e^{\frac{\alpha}{\mu}x} z(r - e^{\frac{\alpha}{\mu}x}\widetilde{\theta}_\mu) - (1-s) e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu(y+z) + f_2(y,z,\mu,\tau,s) \end{pmatrix},$$

where

$$\begin{aligned} f_1(y,z,\mu,\tau,s) = {} & -e^{\frac{2\alpha}{\mu}x} y(y+z) - s\tau g e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu\left[(1-s)\widetilde{\theta}_\mu + z\right] - \tau g e^{\frac{2\alpha}{\mu}x} y\left[(1-s)\widetilde{\theta}_\mu + z\right] \\ & - s\tau m e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu(s\widetilde{\theta}_\mu + y) - \tau m e^{\frac{2\alpha}{\mu}x} y(s\widetilde{\theta}_\mu + y), \\ f_2(y,z,\mu,\tau,s) = {} & -e^{\frac{2\alpha}{\mu}x} z(y+z) - (1-s)\tau h e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu(s\widetilde{\theta}_\mu + y) - \tau h e^{\frac{2\alpha}{\mu}x} z(s\widetilde{\theta}_\mu + y). \end{aligned}$$

Obviously, for small $\delta_1 > 0$, the map $F\colon X\times(\mu_1-\delta_1,\mu_2+\delta_1)\times(-\delta_1,\delta_1)\times(-\delta_1,1+\delta_1)\to Y$. Hence, we only need to consider the equation $F(y,z,\mu,\tau,s)=(0,0)^T$. In particular, the map F is a smooth function and have the following properties:

$$F(0,0,\mu,0,s)=0,\quad F(0,0,\mu,\tau,0)=0,\quad F\left(\widetilde{\omega}-\widetilde{\theta}_\mu,0,\mu,\tau,1\right)=0. \tag{2.3}$$

In order to use the Lyapunov-Schmidt procedure, we let $L(\mu,s)=D_{(y,z)}F(0,0,\mu,0,s)\in L(X,Y)$. Then

$$\begin{aligned} &L(\mu,s)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\\ &=\begin{pmatrix}\mu\left\{e^{\frac{\alpha}{\mu}x}\phi_x\right\}_x+e^{\frac{\alpha}{\mu}x}\phi\left(r-e^{\frac{\alpha}{\mu}x}\widetilde{\theta}_\mu\right)-se^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu\phi & -se^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu\psi\\ -(1-s)e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu\phi & \mu\left\{e^{\frac{\alpha}{\mu}x}\psi_x\right\}_x+e^{\frac{\alpha}{\mu}x}\psi\left(r-e^{\frac{\alpha}{\mu}x}\widetilde{\theta}_\mu\right)-(1-s)e^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu\psi\end{pmatrix},\end{aligned}$$

where we denote $L(\mu,s)$ by L. Moreover, L is a Fredholm operator of index zero, and 0 is a principal eigenvalue of L since $\widetilde{\theta}_\mu>0$. Hence,

$$\ker(L)=\operatorname{span}\left\{\left(\widetilde{\theta}_\mu,-\widetilde{\theta}_\mu\right)\right\}=X_1.$$

Next, we define the projection operator $P=P(\mu,s)$ on Y by

$$P\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\frac{1}{\int_0^L\widetilde{\theta}_\mu^2}\left[(1-s)\int_0^L\widetilde{\theta}_\mu\xi-s\int_0^L\widetilde{\theta}_\mu\eta\right]\begin{pmatrix}\widetilde{\theta}_\mu\\ -\widetilde{\theta}_\mu\end{pmatrix}.$$

It is clear that $R(P)=X_1$, $P^2=P$, $PL=0$, which result in

$$R(L)=\left\{(\xi,\eta)\in Y:(1-s)\int_0^L\widetilde{\theta}_\mu\xi-s\int_0^L\widetilde{\theta}_\mu\eta=0\right\},$$

where $R(L)$ stands for the range of L.

Now, we express the system for

$$P(\mu,s)F(y,z,\mu,\tau,s)=0, \tag{2.4a}$$

$$[I-P(\mu,s)]F(y,z,\mu,\tau,s)=0, \tag{2.4b}$$

where $(y,z)\in X_2$. Obviously, $L(\mu,s)$ is an isomorphism from X_2 to $R(L(\mu,s))$. Hence, we can find continuously differentiable functions $(y,z)=(y_1(\mu,\tau,s),z_1(\mu,\tau,s))$ near $(0,0)$ for (2.4b) by the implicit function theorem. Moreover, there exists small $\delta_2>0$ with a smooth function

$$(y_1(\mu,\tau,s),z_1(\mu,\tau,s))\colon(\mu_1-\delta_2,\mu_2+\delta_2)\times(-\delta_2,\delta_2)\times(-\delta_2,1+\delta_2)\to X_2,$$

by the finite covering argument. Thus, we need to consider (2.4a) for (μ,τ,s). That is,

$$P(\mu,s)F(y_1(\mu,\tau,s),z_1(\mu,\tau,s),\mu,\tau,s)=0.$$

By (2.3), we get

$$y_1(\mu,0,s)=0, \qquad z_1(\mu,0,s)=0, \tag{2.5a}$$

$$y_1(\mu,\tau,0)=0, \qquad z_1(\mu,\tau,0)=0, \tag{2.5b}$$

$$y_1(\mu,\tau,1)=\widetilde{\omega}-\widetilde{\theta}_\mu, \qquad z_1(\mu,\tau,1)=0. \tag{2.5c}$$

Recall the definition of the projection operator P, there exists a smooth function $\kappa(\mu,\tau,s)$ satisfying

$$\kappa(\mu,\tau,s)\begin{pmatrix}\widetilde{\theta}_\mu\\ -\widetilde{\theta}_\mu\end{pmatrix}=P(\mu,s)F(y_1(\mu,\tau,s),z_1(\mu,\tau,s),\mu,\tau,s). \tag{2.6}$$

Thus, we only need to solve the equation $\kappa(\mu,\tau,s)=0$. Moreover, $\kappa(\mu,0,s)=\kappa(\mu,\tau,0)=\kappa(\mu,\tau,1)=0$ by (2.5). Hence, we can find a smooth function $\kappa_1(\mu,\tau,s)$ such that

$$\kappa(\mu,\tau,s)=\tau s(1-s)\kappa_1(\mu,\tau,s).$$

Therefore, it suffices to solve the equation $\kappa_1(\mu,\tau,s)=0$.

Differentiating with respect to τ at $\tau=0$ for (2.6), we have

$$\begin{aligned}\kappa_\tau(\mu,0,s)\begin{pmatrix}\widetilde{\theta}_\mu\\ -\widetilde{\theta}_\mu\end{pmatrix}&=P(\mu,s)L(\mu,s)\begin{pmatrix}y_{1,\tau}(\mu,0,s)\\ z_{1,\tau}(\mu,0,s)\end{pmatrix}+P(\mu,s)F_\tau(0,0,\mu,0,s)\\&=P(\mu,s)F_\tau(0,0,\mu,0,s),\end{aligned}$$

where

$$F_\tau(0,0,\mu,0,s)=-s\begin{pmatrix}(1-s)ge^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu^2+sme^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu^2\\ (1-s)he^{\frac{2\alpha}{\mu}x}\widetilde{\theta}_\mu^2\end{pmatrix}.$$

Hence,

$$P(\mu,s)F_\tau(0,0,\mu,0,s)=s(1-s)\frac{s\left(\widetilde{H}(\mu)-\widetilde{M}(\mu)\right)-(1-s)\widetilde{G}(\mu)}{\int_0^L\widetilde{\theta}_\mu^2}\begin{pmatrix}\widetilde{\theta}_\mu\\ -\widetilde{\theta}_\mu\end{pmatrix}.$$

That is,

$$\kappa_1(\mu,0,s)=\frac{s\left(\widetilde{H}(\mu)-\widetilde{M}(\mu)\right)-(1-s)\widetilde{G}(\mu)}{\int_0^L\widetilde{\theta}_\mu^2}. \tag{2.7}$$

If $\left(\widetilde{H}(\mu)-\widetilde{M}(\mu)\right)\widetilde{G}(\mu)<0$ in (μ_1,μ_2), for example, $\left(\widetilde{H}(\widetilde{\mu})-\widetilde{M}(\widetilde{\mu})\right)\widetilde{G}(\widetilde{\mu})<0$ for any $\widetilde{\mu}\in(\mu_1,\mu_2)$. Then there exists small $\delta_3>0$ and we can show that the equation $\kappa_1(\widetilde{\mu},\tau,s)=0$ has no positive solution in the domain $(\widetilde{\mu}-\delta_3,\widetilde{\mu}+\delta_3)\times(-\delta_3,\delta_3)\times(-\delta_3,1+\delta_3)$ when $\widetilde{\mu}\in(\mu_1,\mu_2)$. Moreover, we can also show that the equation $\kappa_1(\mu,\tau,s)=0$ has no positive solution in the domain $(\mu_1-\delta_3,\mu_2+\delta_3)\times(-\delta_3,\delta_3)\times(-\delta_3,1+\delta_3)$ by virtue of the finite covering argument. Hence, system (2.1) has no positive solution near γ_μ for $0<\tau\ll 1$.

If $\left(\widetilde{H}(\mu)-\widetilde{M}(\mu)\right)\widetilde{G}(\mu)>0$ in (μ_1,μ_2), for example, $\left(\widetilde{H}(\widetilde{\mu})-\widetilde{M}(\widetilde{\mu})\right)\widetilde{G}(\widetilde{\mu})>0$ for any $\widetilde{\mu}\in(\mu_1,\mu_2)$. Then $s_0(\widetilde{\mu})=\frac{\widetilde{G}(\widetilde{\mu})}{\widetilde{G}(\widetilde{\mu})+\widetilde{H}(\widetilde{\mu})-\widetilde{M}(\widetilde{\mu})}$ is the unique zero of $\kappa_1(\widetilde{\mu},0,\cdot)$ by (2.7), and $\kappa_{1,s}(\widetilde{\mu},0,\cdot)=\frac{\widetilde{G}(\widetilde{\mu})+\widetilde{H}(\widetilde{\mu})-\widetilde{M}(\widetilde{\mu})}{\int_0^L\widetilde{\theta}_\mu^2}\neq 0$ for any $\widetilde{\mu}\in[\mu_1,\mu_2]$. By virtue of the implicit function theorem and the finite covering argument, there exists small $\delta_4>0$ such that any solution of the equation $\kappa_1(\mu,\tau,s)=0$ in the neighborhood $(\mu,\tau,s)\in(\mu_1-\delta_4,\mu_2+\delta_4)\times(-\delta_4,\delta_4)\times(-\delta_4,1+\delta_4)$ are given by a smooth function $s=s^*(\mu,\tau)$. In particular, $s^*(\mu,0)=s_0(\mu)$. Hence, the set of solutions of the equation $\kappa(\mu,\tau,s)=0$ consists exactly of the surfaces $\tau=0$, $s=0$, $s=1$ and $s=s^*(\mu,\tau)$.

It is clear that any solution of system (2.1) can be written as in (2.2) with some functions $\overline{y}(\mu,\tau)$, $\overline{z}(\mu,\tau)$ since there exist some smooth functions $\widetilde{y}_1$, $\widetilde{z}_1$ such that $(y_1(\mu,\tau,s), z_1(\mu,\tau,s))=(s\widetilde{y}_1(\mu,\tau,s),(1-s)\widetilde{z}_1(\mu,\tau,s))$, where $\overline{y}(\mu,\tau)=\widetilde{y}_1(\mu,\tau,s^*(\mu,\tau))$, $\overline{z}(\mu,\tau)=\widetilde{z}_1(\mu,\tau,s^*(\mu,\tau))$.

Now, we consider $s_0(\mu_i)=0$ or $1-s_0(\mu_i)=0$, $i=1,2$. Without loss of generality, we consider the case $s_0(\mu_1)=0$, i.e., $\widetilde{G}(\mu_1)=0$. Since $s^*(\mu,0)=\frac{\widetilde{G}(\mu)}{\widetilde{G}(\mu)+\widetilde{H}(\mu)-\widetilde{M}(\mu)}$, we have $s^*(\mu_1,0)=0$ and $s^*_\mu(\mu_1,0)=\frac{\widetilde{G}'(\mu_1)}{\widetilde{H}(\mu_1)-\widetilde{M}(\mu_1)}\neq 0$. Hence, by the implicit function theorem, there exists $\delta_5>0$ and a smooth function $\mu=\underline{\mu}(\tau)$ is the unique solution of $s^*(\mu,\tau)=0$ near μ_1 such that $\underline{\mu}(0)=\mu_1$. Similarly, there exists a smooth function $\mu=\overline{\mu}$ is the unique solution of $s^*(\mu,\tau)=0$ near μ_2 such that $\overline{\mu}(0)=\mu_2$. The case $1-s_0(\mu_i)=0$, $i=1,2$, can be treated similarly, we omit it here. The proof is complete. $\square$

The following lemma shows that all positive solutions of system (2.1) are close to the curve γ_μ when $0<\tau\ll 1$. In particular, we have

Lemma 2.2. *Let* (ω,χ) *be any positive solution of system* (2.1). *Then we have*

$$(\omega,\chi)\to\left(s\widetilde{\theta}_\mu,(1-s)\widetilde{\theta}_\mu\right)\quad in\ C^2(\overline{\Omega})$$

for some $s\in[0,1]$ *as* $\tau\to 0$.

Proof. It is clear that solutions (ω_i,χ_i) of system (2.1) with $\mu=\mu_i$, $\tau=\tau_i$ are uniformly bounded in L^∞ by the maximum principle, and we only need to show that solutions (ω_i,χ_i) of system (2.1) with $\mu=\mu_i$, $\tau=\tau_i$ converges to the curve γ_μ if $\tau_i\to 0_+$, $\mu_i\to\mu$. By the standard elliptic regularity theory, we may assume that $(\omega_i,\chi_i)\to(\omega,\chi)$ in $C^1(\overline{\Omega})$, and

(ω, χ) satisfies

$$\begin{cases} \mu\left\{e^{\frac{\alpha}{\mu}x}\omega_x\right\}_x + e^{\frac{\alpha}{\mu}x}\omega\left(r - e^{\frac{\alpha}{\mu}x}\omega - e^{\frac{\alpha}{\mu}x}\chi\right) = 0, \quad 0 < x < L, \\ \mu\left\{e^{\frac{\alpha}{\mu}x}\chi_x\right\}_x + e^{\frac{\alpha}{\mu}x}\chi\left(r - e^{\frac{\alpha}{\mu}x}\omega - e^{\frac{\alpha}{\mu}x}\chi\right) = 0, \quad 0 < x < L, \\ \omega_x(0) = \omega_x(L) = \chi_x(0) = \chi_x(L) = 0. \end{cases}$$

Therefore, we have $(\omega, \chi) = (0, 0)$ or $(\omega, \chi) = \left(s\widetilde{\theta}_\mu, (1-s)\widetilde{\theta}_\mu\right)$, where $s \in [0, 1]$. Moreover, we claim that one can rule out the case $(\omega, \chi) = (0, 0)$, and hence the case $(\omega, \chi) = \left(s\widetilde{\theta}_\mu, (1-s)\widetilde{\theta}_\mu\right)$ holds. The proof is complete. □

3. Stability of coexistence states

In this section, we study the stability of coexistence states of system (1.4) for $0 < \tau \ll 1$. In particular, the stability of any coexistence state $(u(\mu, \tau), v(\mu, \tau))$ of system (1.4) is determined by the principal eigenvalue $\lambda_1(\mu, \tau)$ near 0 of problem (1.5) for $0 < \tau \ll 1$. For the sake of convenience, we can write problem (1.5) as

(3.1)
$$\begin{cases} \mu\left\{e^{\frac{\alpha}{\mu}x}\left(e^{-\frac{\alpha}{\mu}x}\varphi\right)_x\right\}_x + [r - 2(1+\tau m)u - (1+\tau g)v]\varphi - u(1+\tau g)\psi + \lambda\varphi = 0, & 0 < x < L, \\ \mu\left\{e^{\frac{\alpha}{\mu}x}\left(e^{-\frac{\alpha}{\mu}x}\psi\right)_x\right\}_x + [r - (1+\tau h)u - 2v]\psi - v(1+\tau h)\varphi + \lambda\psi = 0, & 0 < x < L, \\ \mu\varphi_x(0) - \alpha\varphi(0) = \mu\varphi_x(L) - \alpha\varphi(L) = 0, \\ \mu\psi_x(0) - \alpha\psi(0) = \mu\psi_x(L) - \alpha\psi(L) = 0. \end{cases}$$

For $0 < \tau \ll 1$, we set the principal eigenfunction (φ, ψ) as

$$\varphi(\mu, \tau) = \theta_\mu + \tau\varphi_1(\mu, \tau), \quad \psi(\mu, \tau) = -\theta_\mu + \tau\psi_1(\mu, \tau), \tag{3.2}$$

in (3.2) with some smooth functions $\varphi_1(\mu, \tau)$ and $\psi_1(\mu, \tau)$, and we only need to consider the following three possibilities here: μ close to μ_1, μ close to μ_2 and μ bounded away from both μ_1 and μ_2. First, we prove the following lemma.

Lemma 3.1. *Suppose that $0 < \tau \ll 1$. Then we have*

$$\begin{aligned} \frac{\lambda_1(\mu, \tau)}{\tau}\int_0^L (v\varphi - u\psi)e^{-\frac{\alpha}{\mu}x} = 2\int_0^L muv\varphi e^{-\frac{\alpha}{\mu}x} - 2\int_0^L huv\varphi e^{-\frac{\alpha}{\mu}x} \\ + 2\int_0^L guv\psi e^{-\frac{\alpha}{\mu}x} - \int_0^L hu^2\psi e^{-\frac{\alpha}{\mu}x} \\ + \int_0^L mu^2\psi e^{-\frac{\alpha}{\mu}x} + \int_0^L gv^2\varphi e^{-\frac{\alpha}{\mu}x}. \end{aligned} \tag{3.3}$$

Proof. Multiplying the first equation of problem (3.1) by $e^{-\frac{\alpha}{\mu}x}v$ and integrating by parts, then we obtain

$$\begin{aligned}(3.4)\qquad -\lambda_1(\mu,\tau)\int_0^L v\varphi e^{-\frac{\alpha}{\mu}x} &= \tau\int_0^L huv\varphi e^{-\frac{\alpha}{\mu}x} - 2\tau\int_0^L muv\varphi e^{-\frac{\alpha}{\mu}x} - \int_0^L uv\varphi e^{-\frac{\alpha}{\mu}x}\\ &\quad - \tau\int_0^L gv^2\varphi e^{-\frac{\alpha}{\mu}x} - \int_0^L uv\psi e^{-\frac{\alpha}{\mu}x} - \tau\int_0^L guv\psi e^{-\frac{\alpha}{\mu}x}.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}(3.5)\qquad -\lambda_1(\mu,\tau)\int_0^L u\psi e^{-\frac{\alpha}{\mu}x} &= \tau\int_0^L mu^2\psi e^{-\frac{\alpha}{\mu}x} + \tau\int_0^L guv\psi e^{-\frac{\alpha}{\mu}x} - \tau\int_0^L hu^2\psi e^{-\frac{\alpha}{\mu}x}\\ &\quad - \int_0^L uv\psi e^{-\frac{\alpha}{\mu}x} - \int_0^L uv\varphi e^{-\frac{\alpha}{\mu}x} - \tau\int_0^L huv\varphi e^{-\frac{\alpha}{\mu}x}.\end{aligned}$$

Hence, we get (3.3) from (3.4) and (3.5). The proof is complete. □

Next, we study the stability of coexistence states of system (1.4) when μ bounded away from both μ_1 and μ_2. In particular, we have the following result.

Lemma 3.2. *For any $\eta > 0$, we have*

$$(3.6)\qquad \lim_{\tau\to 0_+}\frac{\lambda_1(\mu,\tau)}{\tau} = -\frac{(H(\mu)-M(\mu))G(\mu)}{G(\mu)+H(\mu)-M(\mu)}\frac{1}{\int_0^L e^{-\frac{\alpha}{\mu}x}\theta_\mu^2}$$

for $\mu \in [\mu_1+\eta, \mu_2-\eta]$.

Proof. It is clear that $(u(\mu,\tau), v(\mu,\tau)) \to (s_0(\mu)\theta_\mu, (1-s_0(\mu))\theta_\mu)$ and $(\varphi(\mu,\tau), \psi(\mu,\tau)) \to (\theta_\mu, -\theta_\mu)$ as $\tau \to 0_+$. Then we have

$$\int_0^L (v\varphi - u\psi)e^{-\frac{\alpha}{\mu}x} \to \int_0^L e^{-\frac{\alpha}{\mu}x}\theta_\mu^2,$$

$$\begin{aligned}&2\int_0^L muv\varphi e^{-\frac{\alpha}{\mu}x} - 2\int_0^L huv\varphi e^{-\frac{\alpha}{\mu}x} + 2\int_0^L guv\psi e^{-\frac{\alpha}{\mu}x}\\ &- \int_0^L hu^2\psi e^{-\frac{\alpha}{\mu}x} + \int_0^L mu^2\psi e^{-\frac{\alpha}{\mu}x} + \int_0^L gv^2\varphi e^{-\frac{\alpha}{\mu}x}\\ &\to 2s_0(1-s_0)[M(\mu)-H(\mu)-G(\mu)] + s_0^2[H(\mu)-M(\mu)] + (1-s_0)^2G(\mu)\\ &= -\frac{(H(\mu)-M(\mu))G(\mu)}{G(\mu)+H(\mu)-M(\mu)}.\end{aligned}$$

Hence, we get (3.6). The proof is complete. □

Now, we consider the case $G(\mu_1) = 0$, i.e., $s^*(\underline{\mu},\tau) = 0$, where $\underline{\mu} = \underline{\mu}(\tau)$. For this case, we have $\big(u(\underline{\mu},\tau), v(\underline{\mu},\tau)\big) = (0, \theta_{\underline{\mu}})$. The case $H(\mu_1) - M(\mu_1) = 0$ can be treated similarly.

Lemma 3.3. *Suppose that $G(\mu_{1+}) = 0$ and it is a simple root. Then we have*

$$(3.7)\qquad \lim_{(\mu,\tau)\to(\mu_{1+},0_+)}\frac{\lambda_1(\mu,\tau)}{\tau(\mu-\underline{\mu})} = -\frac{G'(\mu_{1+})}{\int_0^L e^{-\frac{\alpha}{\mu_{1+}}x}\theta_{\mu_{1+}}^2}.$$

Proof. It is easy to see that $\underline{\mu}$ is a simple bifurcation point. Then $\lambda_1(\underline{\mu},\tau)=0$ and

$$\begin{cases}\mu\left\{e^{\frac{\alpha}{\mu}x}\left(e^{-\frac{\alpha}{\mu}x}\varphi\right)_x\right\}_x+\left[r-(1+\tau g)\theta_{\underline{\mu}}\right]\varphi=0, & 0<x<L,\\ \mu\varphi_x(0)-\alpha\varphi(0)=\mu\varphi_x(L)-\alpha\varphi(L)=0.\end{cases}\tag{3.8}$$

Multiplying (3.8) by $e^{-\frac{\alpha}{\mu}x}v(\underline{\mu},\tau)$ and integrating by parts, we have $\int_0^L ge^{-\frac{\alpha}{\underline{\mu}}x}v^2(\underline{\mu},\tau)\varphi(\underline{\mu},\tau)=0$. Let $I(\mu,\tau)$ be the right-hand side of (3.3). Then $I(\underline{\mu},\tau)=0$. Hence, $I(\mu,\tau)=(\mu-\underline{\mu})I'_\mu(\mu^*,\tau)$ for some $\mu^*=\mu^*(\mu,\tau)\in(\underline{\mu},\mu)$.

Next we differentiate I with respect to μ. Then we have

$$\begin{aligned}I'_\mu(\mu,\tau)&=2\int_0^L m(\varphi_\mu uv+\varphi u_\mu v+\varphi uv_\mu)e^{-\frac{\alpha}{\mu}x}-2\int_0^L h(\varphi_\mu uv+\varphi u_\mu v+\varphi uv_\mu)e^{-\frac{\alpha}{\mu}x}\\&+2\int_0^L g(\psi_\mu uv+\psi u_\mu v+\psi uv_\mu)e^{-\frac{\alpha}{\mu}x}-\int_0^L h(2uu_\mu\psi+u^2\psi_\mu)e^{-\frac{\alpha}{\mu}x}\\&+\int_0^L m(2uu_\mu\psi+u^2\psi_\mu)e^{-\frac{\alpha}{\mu}x}+\int_0^L g(2vv_\mu\varphi+v^2\varphi_\mu)e^{-\frac{\alpha}{\mu}x}\\&+2\frac{\alpha}{\mu^2}\int_0^L muv\varphi e^{-\frac{\alpha}{\mu}x}x-2\frac{\alpha}{\mu^2}\int_0^L huv\varphi e^{-\frac{\alpha}{\mu}x}x+2\frac{\alpha}{\mu^2}\int_0^L guv\psi e^{-\frac{\alpha}{\mu}x}x\\&-\frac{\alpha}{\mu^2}\int_0^L hu^2\psi x+\frac{\alpha}{\mu^2}\int_0^L mu^2\psi e^{-\frac{\alpha}{\mu}x}x+\frac{\alpha}{\mu^2}\int_0^L gv^2\varphi e^{-\frac{\alpha}{\mu}x}x.\end{aligned}$$

Clearly, $u\to 0$, $v\to\theta_{\mu_{1+}}$, $\varphi\to\theta_{\mu_{1+}}$, $\psi\to-\theta_{\mu_{1+}}$, $u_\mu\to s_0'(\mu_{1+})\theta_{\mu_{1+}}$, $v_\mu\to -s_0'(\mu_{1+})\theta_{\mu_{1+}}+\theta'_{\mu_{1+}}$, $\varphi_\mu\to\theta'_{\mu_{1+}}$, and ψ_μ is uniformly bounded if $\tau\to 0_+$ and $\mu\to\mu_{1+}$. Hence,

$$I'_\mu(\mu_{1+},0)=-2s_0'(\mu_{1+})(H(\mu_{1+})-M(\mu_{1+}))+G'(\mu_{1+})$$

since $G(\mu_{1+})=0$, and $s_0'(\mu_{1+})=\frac{G'(\mu_{1+})}{H(\mu_{1+})-M(\mu_{1+})}$. Therefore,

$$I'_\mu(\mu_{1+},0)=-G'(\mu_{1+}).$$

Hence, we obtain (3.7). The proof is complete. □

Suppose that $G(\mu_{2-})=0$. Then $s^*(\overline{\mu},\tau)=0$, where $\overline{\mu}=\overline{\mu}(\tau)$. The case $H(\mu_{2-})-M(\mu_{2-})=0$ can be treated similarly.

Lemma 3.4. *Suppose that $G(\mu_{2-})=0$ and it is a simple root. Then we have*

$$\lim_{(\mu,\tau)\to(\mu_{2-},0_+)}\frac{\lambda_1(\mu,\tau)}{\tau(\mu-\overline{\mu})}=-\frac{G'(\mu_{2-})}{\int_0^L e^{-\frac{\alpha}{\mu_{2-}}x}\theta^2_{\mu_{2-}}}.$$

Finally, the stability of coexistence states $(u(\mu,\tau),v(\mu,\tau))$ of system (1.4) for $0<\tau\ll 1$ can be established in the following theorem. In particular, we have

Theorem 3.5. *Suppose that μ_1, μ_2 are consecutive simple zeros of function $(H(\mu) - M(\mu))G(\mu)$, and $(H(\mu) - M(\mu))G(\mu) > 0$ in (μ_1, μ_2). Then there exists $\tau^* > 0$ such that for any $\tau \in (0, \tau^*)$ and for $\mu \in (\underline{\mu}, \overline{\mu})$, we have*

(i) *if $H(\mu) - M(\mu) < 0$ and $G(\mu) < 0$ in (μ_1, μ_2), then $\lambda_1(\mu, \tau) > 0$;*

(ii) *if $H(\mu) - M(\mu) > 0$ and $G(\mu) > 0$ in (μ_1, μ_2), then $\lambda_1(\mu, \tau) < 0$.*

Proof. We only need to consider the case (i), the case (ii) can be treated similarly. We argue by contradiction. Suppose that there exist $\tau_i \to 0_+$ and $\mu_i \in (\underline{\mu}(\tau_i), \overline{\mu}(\tau_i))$ such that $\lambda_1(\mu_i, \tau_i) \le 0$ for $i = 1, 2, \ldots$. Since $\underline{\mu}(\tau_i) \to \mu_1$ and $\overline{\mu}_i \to \mu_2$, we may assume that $\mu_i \to \mu^* \in [\mu_1, \mu_2]$.

If $\mu^* \in (\mu_1, \mu_2)$, then we have

$$\lim_{(\mu_i,\tau_i)\to(\mu^*,0_+)} \frac{\lambda_1(\mu_i, \tau_i)}{\tau_i} = -\frac{(H(\mu^*) - M(\mu^*))G(\mu^*)}{G(\mu^*) + H(\mu^*) - M(\mu^*)} \frac{1}{\int_0^L e^{-\frac{\alpha}{\mu^*}x}\theta_{\mu^*}^2} > 0$$

by Lemma 3.2. Hence, $\lambda_1(\mu_i, \tau_i) > 0$ for large i, a contradiction.

If $\mu^* = \mu_1$. Suppose that $G(\mu_{1+}) = 0$. We see that $G'(\mu_{1+}) < 0$ since $G(\mu) < 0$ in (μ_1, μ_2) and μ_1 is a simple root. By Lemma 3.3, we obtain

$$\lim_{(\mu_i,\tau_i)\to(\mu_{1+},0_+)} \frac{\lambda_1(\mu_i, \tau_i)}{\tau_i(\mu_i - \underline{\mu}(\mu_i))} = -\frac{G'(\mu_{1+})}{\int_0^L e^{-\frac{\alpha}{\mu_{1+}}x}\theta_{\mu_{1+}}^2} > 0.$$

Since $\mu_i > \underline{\mu}(\tau_i)$, we have $\lambda_1(\mu_i, \tau_i) > 0$ for all large i, a contradiction. The case $\mu^* = \mu_2$ can be treated similarly by Lemma 3.4, we omit it here. The proof is complete. □

Now, we can establish the results of Theorem 1.2.

Proof of Theorem 1.2. By virtue of Theorem 2.1, Theorem 3.5 and Lemma 2.2, we know that as $\tau \to 0$, all coexistence states of system (1.4) lie in the neighbourhood of the curve γ_μ, and all coexistence states of system (1.4) in the neighbourhood of the curve γ_μ are asymptotically stable if $H(\mu) - M(\mu) < 0$ and $G(\mu) < 0$ in (μ_1, μ_2). Hence, system (1.4) has a unique coexistence state in this case and it is the global attractor of system (1.4) by the theory of monotone dynamical systems. And if $H(\mu) - M(\mu) > 0$ and $G(\mu) > 0$ in (μ_1, μ_2), then system (1.4) has a unstable coexistence state. □

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