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On the Grothendieck Groups of Toric Stacks

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Abstract. In this short note, we give an elementary proof for the fact that the Grothendieck group of complete toric Deligne-Mumford stack is torsion free.

1. Introduction

Complete toric Deligne-Mumford (DM) stacks are important objects in string theory, in particular in mirror symmetry. By far, the largest class of examples of mirror pairs of Calabi-Yau threefolds are constructed as pairs of complete intersections inside complete toric DM stacks using the Batyrev-Borisov construction [1, 2]. Many topological and geometric quantities of the complete intersection Calabi-Yaus can be computed using the combinatorial data of the ambient toric stacks. One example of such a calculation is the computation of hodge numbers in [1, 2].

In this note, we prove that the Grothendieck group of a smooth complete toric Deligne-Mumford stack is torsion free. This statement holds even when the generic point is stacky. We also construct an example of noncomplete toric stack with torsion in K-theory. The key ingredient is to prove the Grothendieck groups of toric stacks are Cohen-Macaulay. A similar result has been proved by Goldin, Harada and Holm in [7] using symplectic methods. Under slightly stronger assumption on the toric stacks, Kawamata proved in [8] that the derived category of a complete toric stack has a full exceptional collection of sheaves. The torsion freeness of the Grothendieck group follows from Kawamata's theorem. Kawamata's proof uses the machinery of minimal model program on toric stacks.

2. Grothendieck groups of reduced stacks

Let N be a free abelian group of rank d and $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Given a complete simplicial fan Σ in $N_{\mathbb{R}}$, one chooses an integral element v_i in each of the one-dimensional cones of Σ. This defines a stacky fan $\mathbf{\Sigma}$ in the sense of [3]. We denote the corresponding toric Deligne-Mumford stack by $\mathcal{X}_{\mathbf{\Sigma}}$. Recall that the Grothendieck group $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ is defined

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to be the free abelian group generated by all formal combinations of coherent sheaves on $\mathcal{X}_{\mathbf{\Sigma}}$ modding out by the short exact sequences. Each element v_i corresponds to a toric invariant divisor E_i. This divisor E_i determines an invertible sheaf $\mathcal{O}(E_i)$. We denote its equivalent class in $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ by R_i. The ring structure of $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ is given by tensor product of coherent sheaves. K-theory of smooth toric stacks has been studied in [4]. In particular they computed $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ explicitly by writing out its generators and relations.

Theorem 2.1. [4] *Let B be the quotient of the Laurent polynomial ring $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ by the ideal generated by the relations*

- $\prod_{i=1}^{n} x_i^{\langle m, v_i \rangle} = 1$ *for any dual vector $m \in M = \mathrm{Hom}(N, \mathbb{Z})$,*
- $\prod_{i \in S}(1 - x_i) = 0$ *for any set $S \subseteq [1, \ldots, n]$ such that $\{v_i \mid i \in S\}$ are not contained in any cone of Σ.*

Then the map from B to $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ which sends x_i to R_i is an isomorphism of rings.

Our main result is given as follows.

Theorem 2.2. *The Grothendieck group $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ of a complete smooth toric Deligne-Mumford stack $\mathcal{X}_{\mathbf{\Sigma}}$ is a free $\mathbb{Z}$-module.*

Proof. We denote the Laurent polynomial ring $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ by R. Let $A = R/I$, where I is generated by $\prod_{i \in S}(1 - x_i) = 0$ for any set $S \subseteq [1, \ldots, n]$ such that $\{v_i \mid i \in S\}$ are not contained in any cone of Σ. Let $B = A/J$, where J is generated by n Laurent polynomials $h_j = \prod_{i=1}^{n} x_i^{\langle m_j, v_i \rangle} - 1$ with m_j an integral basis of M.

We first want to replace h_j by $g_j = \prod_{\langle m_j, v_i \rangle > 0} x_i^{\langle m_j, v_i \rangle} - \prod_{\langle m_j, v_i \rangle < 0} x_i^{-\langle m_j, v_i \rangle}$. They generate the same ideal J but this collection avoids negative powers. To prove B is a free $\mathbb{Z}$-module we need to show that the multiplication map $B \to pB$ is an injection for any prime p. Let $K(g_1, \ldots, g_d)$ and $K(g_1, \ldots, g_d, p)$ be the Koszul complexes for sequences $g_1, \ldots, g_d$ and $g_1, \ldots, g_d, p$ of elements of the ring A. These two Koszul complexes are related by the following lemma.

Lemma 2.3. [6] *Let $\phi\colon K(g_1, \ldots, g_d) \to K(g_1, \ldots, g_d)$ be the map of multiplication by p. Then $K(g_1, \ldots, g_d, p)$ equals* $\mathrm{Cone}(\phi)[-1]$. *Here* Cone *means a mapping cone of complexes.*

Proof. See Corollary 17.11 of [6]. ☐

According to this lemma, we get a long exact sequence of cohomology groups:

$$
\begin{aligned}
(2.1)\qquad & \cdots \to H^i(K(g_1, \ldots, g_d, p)) \to H^i(K(g_1, \ldots, g_d)) \\
& \xrightarrow{\phi} H^i(K(g_1, \ldots, g_d)) \to H^{i+1}(K(g_1, \ldots, g_d, p)) \to \cdots .
\end{aligned}
$$

We will show that all the cohomology groups of $K(g_1, \dots, g_d)$ and $K(g_1, \dots, g_d, p)$ vanish except at one position. More precisely, the only non-vanishing piece of (2.1) is

$$0 \to H^n(K(g_1, \dots, g_d)) \cong B \xrightarrow{p} H^n(K(g_1, \dots, g_d)) \cong B \\ \to H^{n+1}(K(g_1, \dots, g_d, p)) \cong B/pB \to 0.$$

To prove this we need a result about Cohen-Macaulay properties of Stanley-Reisner rings.

Theorem 2.4. *Let $A' = \mathbb{Z}[x_1, \dots, x_n]/I$. Then A' is Cohen-Macaulay.*

Proof. If we make a change of variables x_i to $1-x_i$, then we see that A' is nothing but the Stanley-Reisner ring associated to supporting polytope of Σ. It is a general fact that the Stanley-Reisner ring of polytopes are CM over any field (see, [5, Chapter 5]). Furthermore one can show it is actually CM over $\mathbb{Z}$ (see, [5, Exercise 5.1.25]). We will sketch the solution of this exercise in the following remark. □

Remark 2.5. Consider the flat morphism $\mathbb{Z} \to A'$. For any maximal ideal $\mathfrak{q} \subset A'$, we have $\mathfrak{q} \cap \mathbb{Z} = (p)$. In order to show A' is CM it suffices to check it for each fiber, i.e., $A'_{\mathfrak{q}}/pA'_{\mathfrak{q}}$ is CM for all the maximal ideal $\mathfrak{q}$. If (p) is not (0) then $A'_{\mathfrak{q}}/pA'_{\mathfrak{q}} = (A' \otimes \mathbb{Z}/p\mathbb{Z})_{\mathfrak{q}}$. It is CM because Stanley-Reisner ring over the field is CM. So we just need to show that for any maximal ideal $\mathfrak{q}$, the restriction $\mathfrak{q} \cap \mathbb{Z}$ is not (0). Suppose this is the case, we will have an inclusion $\mathbb{Z} \to A'/\mathfrak{q}$. However, since we assume $\mathfrak{q} \cap \mathbb{Z} = (0)$, the field $A'/\mathfrak{q}$ must have characteristic zero. But this contradicts the fact that A' is finitely generated over $\mathbb{Z}$ because $\mathbb{Q}$ is not finitely generated over $\mathbb{Z}$.

Corollary 2.6. *The ring A is Cohen-Macaulay.*

Proof. Because A is a localization of A' and being CM ring is a local property, A is CM by Theorem 2.4. □

Remark 2.7. It follows from the general theory of Stanley-Reisner ring [5, Theorem 5.1.16] that A' has Krull dimension $d+1$.

Lemma 2.8. [6] *Suppose M is a finitely generated module over ring A and $I = (x_1, \dots, x_n) \subset A$ is a proper ideal. If* $\operatorname{depth}(I) = r$ *then $H^i(M \otimes K(x_1, \dots, x_n)) = 0$ for $i < r$, while $H^r(M \otimes K(x_1, \dots, x_n)) = M/IM$.*

Lemma 2.9. *The quotient A/J is a finitely generated abelian group.*

Proof. Let $\mathbf{k}$ be any field and f be an arbitrary map from A/J to $\mathbf{k}$. Maximal ideals of A/J are in one to one correspondence with such map f. We want to solve for $(x_1, \dots, x_n)$ that satisfy equations in ideal I and J in the field $\mathbf{k}$. Recall that elements of ideal I are in form of $\prod_{i \in S}(1-x_i)$ for any subset $S \subseteq [1, \dots, n]$ such that one-dimensional rays $v_i, i \in S$

are not contained in any cone of Σ. So x_i equals 1 outside some cone σ. Then equations in J reduce to $\prod_{v_i\in\sigma} x_i^{\langle m,v_i\rangle} = 1$. We can choose the dual vector m such that $\langle m, v_i\rangle = 0$ for all but one i. Say $\langle m, v_i\rangle = d_i$. The number d_i only depends on the fan but not on the field $\mathbf{k}$. This implies that $1 - x_i^{d_i}$ maps to 0 for any map f from A/J to $\mathbf{k}$, i.e., $1 - x_i^{d_i}$ is contained in any maximal ideal of A/J. Because A/J is a finitely generated $\mathbb{Z}$-algebra the Jacobson radical coincides with nilradical. So $(1 - x_i^{d_i})^N = 0$ for any i. We can pick a large enough integer N uniformly for any x_i such that there exists a $\mathbb{Z}$-basis consisting of monomials with powers of each x_i between 0 and Nd_i. This proves the statement of the lemma. □

By Theorem 2.4, Remark 2.7 and Lemma 2.9 we can prove

Corollary 2.10. *The ideal $J = (g_1, \ldots, g_d)$ has depth d.*

Proof. Because A is CM, by the definition of CM rings $\mathrm{depth}(J) = \mathrm{codim}(J)$. The quotient A/J is finitely generated over $\mathbb{Z}$, therefore, of Krull dimension one. By Remark 2.7 $\mathrm{codim}(J) = d$ and $\mathrm{depth}(J) = d$. □

Corollary 2.10 together with Lemma 2.8 imply the Koszul complex $K(g_1, \ldots, g_d)$ has only one nonzero cohomology $H^d(K(g_1, \ldots, g_d)) = B = A/J$. On the other hand, Lemma 2.9 implies B/pB is a finite-dimensional vector space over $\mathbb{Z}/p$. By the similar argument with the corollary above we get $\mathrm{depth}(J, p) = d + 1$. Then $H^i(K(g_1, \ldots, g_d, p)) = B/pB$ when $i = d + 1$ and zero otherwise. Now by applying the long exact sequence (2.1) we prove the multiplication map by p is an injection. This finishes the proof of Theorem 2.2. □

Remark 2.11. The proof of Theorem 2.2 can be generalized to the noncomplete toric stacks satisfying a condition called “shellability”. This is a combinatorial condition on the underlying simplicial complex of the toric stack (see [5] for details of this definition). It is proved in [5] that Stanley-Reisner rings of shellable simplicial complexes are Cohen-Macaulay. However, we will see in Section 4 that Grothendieck groups of open toric stacks are not necessarily free.

3. Grothendieck groups of non-reduced stacks

Now we remove the assumption that N is a free abelian group. Then the corresponding toric stack will have nontrivial stabilizer at the generic point. We will generalize Theorem 2.2 to this setting. Recall that the derived Gale dual of the homomorphism $\beta\colon \mathbb{Z}^n \to N$ is the homomorphism $\beta^\vee : (\mathbb{Z}^n)^\vee \to \mathrm{DG}(\beta)$. When N is torsion free, $\mathrm{DG}(\beta)$ is the Picard group. The general definition of $\mathrm{DG}(\beta)$ involves a projective resolution of N.

We refer to [3] for details. Theorem 2.1 can be generalized to the case when N has torsion. Notice the ring $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]/J$ is the representation ring of the algebraic group $\operatorname{Hom}(\operatorname{DG}(\beta), \mathbb{C}^*)$ when N is torsion free. If N has torsion then $\operatorname{Hom}(\operatorname{DG}(\beta), \mathbb{C}^*)$ maps to $(\mathbb{C}^*)^n$ with finite kernel. After replacing $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]/J$ by the representation ring of $\operatorname{Hom}(\operatorname{DG}(\beta), \mathbb{C}^*)$ we can generalize Theorem 2.1 to non-reduced case (see [4, Section 6] for more details).

Theorem 3.1. *Let N be a finitely generated abelian group and $\mathbf{\Sigma}$ is a stacky fan in N. The Grothendieck Group $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ is a free $\mathbb{Z}$-module.*

Proof. Let's denote the N_{free} for the quotient $N/\operatorname{torsion}(N)$ and $\mathcal{X}_{\text{red}}$ for the reduced stack associated to N_{free}. Recall the Grothendieck group $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ is the quotient of representation ring of $\operatorname{Hom}(\operatorname{DG}(\beta), \mathbb{C}^*)$ by the ideal I generated by Stanley-Reisner relations. Let's denote the Gale dual group of the reduced stack $\mathcal{X}_{\text{red}}$ by $\operatorname{DG}(\beta_{\text{red}})$. The quotient map $N \to N_{\text{free}}$ induces an inclusion on Gale dual groups $\operatorname{DG}(\beta_{\text{red}}) \to \operatorname{DG}(\beta)$, whose cokernel is isomorphic to $\operatorname{torsion}(N)$. Now we see the Grothendieck groups $K_0(\mathcal{X}_{\mathbf{\Sigma}})$ and $K_0(\mathcal{X}_{\text{red}})$ are isomorphic to the group rings $\mathbb{Z}[\operatorname{DG}(\beta)]$ and $\mathbb{Z}[\operatorname{DG}(\beta_{\text{red}})]$. If we fix a lifting from $\operatorname{torsion}(N)$ to $\operatorname{DG}(\beta)$, then we get a coset decomposition $\operatorname{DG}(\beta) = \bigsqcup_{y \in \operatorname{torsion}(N)} (y \operatorname{DG}(\beta_{\text{red}}))$. This induces a coset decomposition of the group ring $\mathbb{Z}[\operatorname{DG}(\beta)]$ such that each coset is isomorphic to $\mathbb{Z}[\operatorname{DG}(\beta_{\text{red}})]$. Since $\mathbb{Z}[\operatorname{DG}(\beta_{red})]$ is torsion free by Theorem 2.2, we prove the theorem. □

4. Grothendieck groups of noncomplete stacks

Theorem 2.1 holds for noncomplete toric stacks too. But our proof for freeness of K-theory relies on the shellability of the underlying simplicial complex of the toric stack. There are many noncomplete toric stacks whose underlying simplicial complexes are *not* shellable. For example, we can take $\mathbb{P}^1 \times \mathbb{P}^1$. Denote its four toric invariant divisors by E_1, E_2, E_3 and E_4. Let point P (resp. Q) be the intersection of E_1 and E_2 (resp. E_3 and E_4). Simplicial complex of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{P, Q\}$ is not shellable.

Actually, there are examples of noncomplete toric stacks such that their Grothendieck groups have torsions. The following example is due to Lev Borisov.

Example 4.1. Let's take a dimension five weighted projective stack $\mathbb{P}(1,1,1,1,2,2)$. Denote its toric invariant divisors by $E_1, E_2, \ldots, E_6$, where $E_1, \ldots, E_4$ have weights one and E_5, E_6 have weights two. Let $\mathcal{X}$ be the substack $\mathbb{P}(1,1,1,1,2,2) \setminus \{(E_1 \cap E_2 \cap E_3 \cap E_4) \cup (E_5 \cap E_6)\}$. By Theorem 2.1,

$$K_0(\mathcal{X}) = \frac{\mathbb{Z}[t, t^{-1}]}{\langle (1-t)^4, (1-t^2)^2 \rangle}.$$

It is easy to check that $t(1-t)^2$ is a torsion element.

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