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Positive Toeplitz Operators Between Different Doubling Fock Spaces

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Abstract. Let $F^p(\phi)$ be the weighted Fock space on the complex plane $\mathbb{C}$, where ϕ is subharmonic with $\Delta\phi\, dA$ a doubling measure. In this paper, we characterize the positive Borel measure μ on $\mathbb{C}$ for which the induced Toeplitz operator T_μ is bounded (or compact) from one weighted Fock space $F^p(\phi)$ to another $F^q(\phi)$ for $0 < p, q < \infty$.

1. Introduction

Let $\mathbb{C}$ be the complex plane. Set $D(z,r) = \{w \in \mathbb{C} : |w-z| < r\}$ for $z \in \mathbb{C}$ and $r > 0$. A positive Borel measure ν on $\mathbb{C}$, written as $\nu \geq 0$, is called doubling if there exists some constant $C > 0$ such that

$$\nu(D(z,2r)) \leq C\nu(D(z,r))$$

for $z \in \mathbb{C}$ and $r > 0$. Let dA be the Lebesgue area measure on $\mathbb{C}$. As in [9,17], suppose ϕ is subharmonic, real-valued and not identically zero on $\mathbb{C}$ with $\nu = \Delta\phi\, dA$ doubling. For $z \in \mathbb{C}$, we denote by $\rho(z)$ the positive radius such that $\nu(D(z,\rho(z))) = 1$. The function ρ^{-2} can be viewed as a regularized version of $\Delta\phi$, see [9] or [17] for details.

Suppose $0 < p < \infty$, the space $L^p(\phi)$ consists of all Lebesgue measurable functions f on $\mathbb{C}$ for which

$$\|f\|_{p,\phi} = \left(\int_{\mathbb{C}} \left| f(z) e^{-\phi(z)} \right|^p dA(z) \right)^{1/p} < \infty.$$

Let $H(\mathbb{C})$ be the family of all entire functions on $\mathbb{C}$. The weighted Fock space $F^p(\phi)$ is defined as

$$F^p(\phi) = L^p(\phi) \cap H(\mathbb{C}).$$

It is clear that $F^p(\phi)$ is a Banach space under $\|\cdot\|_{p,\phi}$ if $p \geq 1$, and $F^p(\phi)$ is an F-space under $d(f,g) = \|f-g\|_{p,\phi}^p$ if $0 < p < 1$. Fock spaces in the present paper cover lots in the literature. When $\phi(z) = \frac{1}{2}|z|^2$, $F^2(\phi)$ is the classical Fock space, which has been studied

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by many authors, see [1–3, 10, 13, 15, 24] and more references therein. As mentioned in [14] and [6], when $\phi(z) = -m\ln(A+|z|^2)+|z|^2$ with some suitable $A>0$ and positive integer m $F^2(\phi)$ is just the Fock-Sobolev space introduced in [7]. The Fock-Sobolev space has also been investigated in [4–6, 8, 23]. For $\phi(z)=|z|^m$, $F^2(\phi)$ is the generalized Fock space in [20] and [21]. If $n=1$ and the weight φ is as in [14, 16, 22], then $0<c\le\Delta\varphi(z)\le C$ for all $z\in\mathbb{C}$ which implies $\Delta\varphi\, dA$ is doubling.

Let $K(\cdot\,,\cdot)$ be the Bergman kernel for $F^2(\phi)$, that is, for $f\in F^2(\phi)$

$$f(\cdot) = Pf(\cdot) = \int_{\mathbb{C}} K(\cdot\,,w)f(w)e^{-2\phi(w)}\,dA(w).$$

Suppose μ is a Borel measure on $\mathbb{C}$, Toeplitz operator T_μ with symbol μ is defined as

$$T_\mu f(\cdot) = \int_{\mathbb{C}} K(\cdot\,,w)f(w)e^{-2\phi(w)}\,d\mu(w)$$

if it is well (densely) defined.

When $d\mu = g\,dA$ for some restricted function g, for example g is bounded or $g\in$ BMO, the induced Toeplitz operator T_μ has been well studied, see [1–3, 10] and other references. Also, positive Toeplitz operators have been studied on Fock spaces by many people. For $\mu\ge 0$, in 2008 Isralowitz and Zhu characterized the boundedness, compactness and Schatten-p classes of Toeplitz operators T_μ on $F^2(\frac{1}{2}\,|z|^2)$, see [15]; Wang, Cao and Xia extended [15] to Fock-Sobolev spaces in [23]. In [13], we obtained some sufficient and necessary conditions on μ for which T_μ is bounded (or compact) from $F^p(\frac{1}{2}\,|z|^2)$ to $F^q(\frac{1}{2}\,|z|^2)$ for $1<p,q<\infty$. Denote $d=\partial+\overline{\partial}$ and $d^c=\frac{\sqrt{-1}}{4}(\overline{\partial}-\partial)$. With the restriction that $dd^c\varphi\simeq dd^c\,|z|^2$ on the weight φ in $\mathbb{C}^n$, in 2012, Schuster and Varolin [22] studied the boundedness and compactness of Toeplitz operators in terms of averaging functions and Berezin transforms. In 2014, the corresponding problems were discussed from $F^p(\varphi)$ to $F^q(\varphi)$ for $0<p,q<\infty$ in [14], between $F^p(\varphi)$ and $F^\infty(\varphi)$ for $0<p\le\infty$ in [16]. In 2015, Oliver and Pascuas [19] characterized the boundedness and compactness of positive Toeplitz operators on the weighted Fock space $F^p(\phi)$ for $1\le p<\infty$.

The purpose of this work is to extend those of [13–16, 19, 22, 23]. In Section 2, we will give some basic estimates about the Bergman kernel. Section 3 is devoted to characterize those $\mu\ge 0$ for which the induced operators T_μ are bounded (or compact) from $F^p(\phi)$ to $F^q(\phi)$ for $0<p,q<\infty$.

We would like to mention that the approach in [13–16, 19, 22, 23] does not work well in the present case. The research in [13, 15, 19, 22, 23] depends strongly on the restricted range of the exponent p, say $p=2$ or $1<p<\infty$, where the Banach space technique can be applied to. Also, the proof in [14, 16] relies on two points: one is the inclusion

$$F^p(\varphi)\subset F^q(\varphi)\quad\text{for } 0<p\le q;$$

and the other is that $Pf = f$ for any $f \in F^p(\varphi)$ while $0 < p \leq \infty$. However, these two points are not available in the present case. For example, take $\phi(z) = |z|^4$, $\Delta\phi\, dA$ is doubling, but

$$F^p(\phi) \setminus F^q(\phi) \neq \emptyset \quad \text{and} \quad F^q(\phi) \setminus F^p(\phi) \neq \emptyset$$

for $p \neq q$, see [11] for details.

In what follows, we use C to denote positive constants whose value may change from line to line but does not depend on the functions being considered. Two quantities A and B are called equivalent, denoted by "$A \simeq B$", if there exists some C such that $C^{-1}A \leq B \leq CA$.

2. Some basic estimates

In this section, we are going to give some basic estimates which will be used in the following sections.

For $r > 0$ and $z \in \mathbb{C}$, write $D^r(z) = D(z, r\rho(z))$, and $D(z) = D^1(z)$ for short. By [17], there exist some absolute constants γ and $C > 0$ such that, for $z \in \mathbb{C}$ and $w \in D^r(z)$,

$$\rho(w) \simeq \rho(z) \text{ if } r \leq 1, \quad \text{and} \quad \frac{1}{Cr^\gamma} \leq \frac{\rho(w)}{\rho(z)} \leq Cr^\gamma \text{ if } r > 1. \tag{2.1}$$

Then, for fixed $r > 0$ there exists some constant $\alpha > 0$ such that

$$\frac{1}{\alpha}\rho(z) \leq \rho(w) \leq \alpha\rho(z) \tag{2.2}$$

for $z \in \mathbb{C}$ and $w \in D^r(z)$. From (2.2) and the triangle inequality, for $r > 0$ we have $m_1 = m_1(r)$, $m_2 = m_2(r)$ that

$$D^r(z) \subseteq D^{m_1 r}(w) \quad \text{and} \quad D^r(w) \subseteq D^{m_2 r}(z) \quad \text{whenever } w \in D^r(z). \tag{2.3}$$

Clearly, $m_j > 1$ for $j = 1, 2$. And furthermore,

$$\tau = \sup_{0<r\leq 1} [m_1(r) + m_2(r)] < \infty. \tag{2.4}$$

In 2009, Marzo and Ortega-Cerdà [18] obtained pointwise estimates on the Bergman kernel $K(\cdot\,,\cdot)$ as follows.

(A) There exist $C, \epsilon > 0$ such that

$$|K(w,z)| \leq C\frac{e^{\phi(w)+\phi(z)}}{\rho(w)\rho(z)} e^{-\left(\frac{|z-w|}{\rho(z)}\right)^\epsilon}, \quad w, z \in \mathbb{C}. \tag{2.5}$$

(B) There exists some $r_0 > 0$ such that for $z \in \mathbb{C}$ and $w \in D^{r_0}(z)$, we have

$$|K(w,z)| \simeq \frac{e^{\phi(w)+\phi(z)}}{\rho(z)^2}. \tag{2.6}$$

With these two basic estimates we are going to give some lemmas. When $k \geq 0$ and $p = 1$, Lemma 2.1 is similar to Lemma 2.7 in [18].

Lemma 2.1. *Given $p, t > 0$ and real number k, there is $C > 0$ such that*

$$\int_{\mathbb{C}} \rho(w)^k e^{-p\left(\frac{|z-w|}{\rho(z)}\right)^t} dA(w) \leq C\rho(z)^{k+2}, \quad z \in \mathbb{C}.$$

Proof. By a straightforward calculation, we have

$$\begin{aligned}
\int_{\mathbb{C}} \rho(w)^k e^{-p\left(\frac{|z-w|}{\rho(z)}\right)^t} dA(w) &= \left(\int_{D(z)} + \int_{\mathbb{C}\setminus D(z)}\right) \rho(w)^k e^{-p\left(\frac{|z-w|}{\rho(z)}\right)^t} dA(w) \\
&\leq \int_{D(z)} \rho(w)^k \, dA(w) + \int_{\mathbb{C}\setminus D(z)} \rho(w)^k \, dA(w) \int_{p\left(\frac{|z-w|}{\rho(z)}\right)^t}^{\infty} e^{-s} \, ds \\
&\leq C\rho(z)^{k+2} + \int_p^{\infty} e^{-s} \, ds \int_{D^{(s/p)^{\frac{1}{t}}}(z)} \rho(w)^k \, dA(w) \\
&\leq C\rho(z)^{k+2} + \int_p^{\infty} \sup_{w \in D^{(s/p)^{\frac{1}{t}}}(z)} \rho(w)^k A\left(D^{(s/p)^{\frac{1}{t}}}(z)\right) e^{-s} \, ds.
\end{aligned}$$

From (2.1) we know

$$\begin{aligned}
\int_p^{\infty} \sup_{w \in D^{(s/p)^{\frac{1}{t}}}(z)} \rho(w)^k A\left(D^{(s/p)^{1/t}}(z)\right) e^{-s} \, ds &\leq C\rho(z)^{k+2} \int_p^{\infty} \left(\frac{s}{p}\right)^{k\gamma/t+2/t} e^{-s} \, ds \\
&= C\rho(z)^{k+2}.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{C}} \rho(w)^k e^{-p\left(\frac{|z-w|}{\rho(z)}\right)^t} dA(w) \leq C\rho(z)^{k+2}.$$

The proof is ended. □

The next lemma is about the $L^p(\phi)$-norm of the Bergman kernel $K(\cdot\,,\cdot)$. While $p \geq 1$, Lemma 2.2 is just Proposition 2.9 in [19].

Lemma 2.2. *For $0 < p < \infty$, we have*

$$\|K(\cdot\,, z)\|_{p,\phi} \simeq e^{\phi(z)} \rho(z)^{2/p-2}, \quad z \in \mathbb{C}.$$

Proof. Notice that, $|K(\cdot\,,\cdot)|$ is symmetric in the two variables, by (2.5) and Lemma 2.1, we obtain

$$\begin{aligned}
\int_{\mathbb{C}} |K(w,z)|^p \, e^{-p\phi(w)} \, dA(w) &\leq C \frac{e^{p\phi(z)}}{\rho(z)^p} \int_{\mathbb{C}} \rho(w)^{-p} e^{-p\left(\frac{|z-w|}{\rho(z)}\right)^{\epsilon}} dA(w) \\
&\leq C e^{p\phi(z)} \rho(z)^{2-2p}.
\end{aligned}$$

The other direction follows easily from (2.6). The proof is completed. □

For $p > 0$ and $z \in \mathbb{C}$, set $k_{p,z}(\cdot) = K(\cdot\,, z)/\|K(\cdot\,, z)\|_{p,\phi}$ to be the normalized Bergman kernel for $F^p(\phi)$.

Lemma 2.3. *The set $\{k_{p,z} : z \in \mathbb{C}\}$ is bounded in $F^p(\phi)$, and $k_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $z \to \infty$.*

Proof. By definition, we know $\|k_{p,z}\|_{p,\phi} = 1$. As that on page 869 in [17] we have $\eta, C > 0$ and $\beta \in (0, 1)$ such that

$$\frac{1}{C}\,|z|^{-\eta} \le \rho(z) \le C\,|z|^{\beta} \quad \text{for } |z| > 1. \tag{2.7}$$

Write

$$c = \begin{cases} -\eta(1 - \frac{2}{p}) & \text{if } p < 2, \\ \beta(1 - \frac{2}{p}) & \text{if } p \ge 2. \end{cases}$$

Thus, Lemma 2.2 and (2.5) yield

$$|k_{p,z}(w)| \le C e^{\phi(w)} \rho(w)^{-1}\,|z|^c\, e^{-\left(\frac{|z|-|w|}{\rho(w)}\right)^{\epsilon}}.$$

Hence, $k_{p,z} \to 0$ uniformly on any compact subset of $\mathbb{C}$ as $z \to \infty$. The proof is completed. □

For our later use, we need the concepts of averaging functions and Berezin transforms. If $E \subset \mathbb{C}$ measurable, write $A(E) = \int_E dA$. For $\mu \ge 0$ and $r > 0$, the average of μ is defined as

$$\widehat{\mu}_r(z) = \mu(D^r(z))/A(D^r(z)), \quad z \in \mathbb{C}.$$

Lemma 2.4. *Suppose $0 < p < \infty$, $\mu \ge 0$, $r > 0$. There exists some constant C such that*

$$\int_{\mathbb{C}} \left|f(z)e^{-\phi(z)}\right|^p d\mu(z) \le C \int_{\mathbb{C}} \left|f(z)e^{-\phi(z)}\right|^p \widehat{\mu}_r(z)\, dA(z)$$

for $f \in H(\mathbb{C})$.

Proof. Given $r > 0$, from (2.2), there is some $\delta > 0$ such that

$$\chi_{D^\delta(z)}(w) \le \chi_{D^r(w)}(z)$$

for $z, w \in \mathbb{C}$. Checking the proof of Lemma 19 in [17] carefully, we have some $C > 0$ such that

$$\left|f(z)e^{-\phi(z)}\right|^p \le \frac{C}{A\left(D^\delta(z)\right)} \int_{D^\delta(z)} \left|f(w)e^{-\phi(w)}\right|^p dA(w) \tag{2.8}$$

for $f \in H(\mathbb{C})$ and $z \in \mathbb{C}$. Hence, Fubini's theorem and (2.2) give

$$\begin{aligned}\int_{\mathbb{C}} \left|f(z)e^{-\phi(z)}\right|^p d\mu(z) &\leq C \int_{\mathbb{C}} \frac{1}{A(D^{\delta}(z))} \int_{\mathbb{C}} \chi_{D^{\delta}(z)}(w) \left|f(w)e^{-\phi(w)}\right|^p dA(w) d\mu(z) \\ &\leq C \int_{\mathbb{C}} \left|f(w)e^{-\phi(w)}\right|^p dA(w) \int_{\mathbb{C}} \frac{\chi_{D^r(w)}(z)}{\rho(w)^2} d\mu(z) \\ &= C \int_{\mathbb{C}} \left|f(w)e^{-\phi(w)}\right|^p \widehat{\mu}_r(w)\, dA(w).\end{aligned}$$

The proof is completed. □

Given $t > 0$, we set the t-Berezin transform of μ to be

$$\widetilde{\mu}_t(z) = \int_{\mathbb{C}} |k_{t,z}(w)|^t e^{-t\phi(w)}\, d\mu(w), \quad z \in \mathbb{C}. \tag{2.9}$$

Given a measurable function f and $d\mu = f\, dA$, we write $\widehat{f}_r(z) = \widehat{\mu}_r$ and $\widetilde{f}_t = \widetilde{\mu}_t$ for short. When $\phi(z) = \frac{1}{2}|z|^2$, the t-Berezin transform is closely connected with the heat flow on $\mathbb{C}$ which is very important for PDE and relative topics, see [1]. And $\widetilde{\mu}_2$ is just the classical Berezin transform on Fock spaces.

Given $r > 0$, we call a sequence $\{a_k\}_{k=1}^{\infty}$ in $\mathbb{C}$ is an r-lattice if $\{D^r(a_k)\}_k$ covers $\mathbb{C}$ and the disks $\left\{D^{r/5}(a_k)\right\}_k$ are pairwise disjoint. For $r > 0$, the existence of some r-lattice comes from a standard covering lemma, see [17] for details. Given an r-lattice $\{a_k\}_k$ and $m > 0$, there exists some integer N such that each $z \in \mathbb{C}$ can be in at most N disks of $\{D^{mr}(a_k)\}_k$. Equivalently,

$$\sum_{k=1}^{\infty} \chi_{D^{mr}(a_k)}(z) \leq N \quad \text{for } z \in \mathbb{C}, \tag{2.10}$$

see [12].

As usual we set the Lebesgue space $L^p = L^p(\mathbb{C}, dA)$. Similar to Lemma 2.1 of [13], we know that both operators $f \mapsto \widehat{f}_r$ and $f \mapsto \widetilde{f}_t$ are bounded on L^p for $1 \leq p \leq \infty$. That is, if $1 \leq p \leq \infty$,

$$\left\|\widehat{f}_r\right\|_{L^p} \leq C \|f\|_{L^p} \quad \text{and} \quad \left\|\widetilde{f}_t\right\|_{L^p} \leq C \|f\|_{L^p}. \tag{2.11}$$

The following lemma, Lemma 2.5, shows the equivalence between the L^p-norm of averaging functions and t-Berezin transforms. When the weight ϕ satisfies $\rho(\cdot) \simeq 1$, this can be seen in Lemma 2.3 of [14].

Lemma 2.5. *Suppose $0 < p < \infty$, $\mu \geq 0$. Then the following statements are equivalent:*

(A) $\widetilde{\mu}_t \in L^p$ *for some (or any)* $t > 0$;

(B) $\widehat{\mu}_\delta \in L^p$ *for some (or any)* $\delta > 0$;

(C) *The sequence* $\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2/p}\right\}_k \in l^p$ *for some (or any) r-lattice* $\{a_k\}_k$.

Furthermore,

$$\|\widetilde{\mu}_t\|_{L^p} \simeq \|\widehat{\mu}_\delta\|_{L^p} \simeq \left\|\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2/p}\right\}_k\right\|_{l^p}. \tag{2.12}$$

Proof. First, given $p \in (0,\infty)$, $s \in \mathbb{R}$, and r-lattice $\{a_k\}_k$, δ-lattice $\{b_k\}_k$ we claim

$$\left\|\left\{\widehat{\mu}_r(a_j)\rho(a_j)^{s+2/p}\right\}_j\right\|_{l^p} \simeq \left\|\left\{\widehat{\mu}_\delta(b_j)\rho(b_j)^{s+2/p}\right\}_j\right\|_{l^p}. \tag{2.13}$$

To see this, for $z \in \mathbb{C}$ set

$$J_z = \left\{j : D^\delta(z) \cap D^r(a_j) \neq \emptyset\right\},$$

and set $|J_z|$ to be the cardinality of J_z. Notice that $\left\{D^{r/5}(a_j)\right\}_k$ are pairwise disjoint, and there is some $\alpha > 0$ such that

$$\frac{1}{\alpha}\rho(z) \le \rho(a_j) \le \alpha\rho(z) \quad \text{and} \quad D^{r/5}(a_j) \subset D^{2\alpha r+\delta}(z) \tag{2.14}$$

for $j \in J_z$. By $A\left(\bigcup_{j\in J_z} D^{r/5}(a_j)\right) \le A(D^{2\alpha r+\delta}(z))$ to know $|J_z| \le M$ with some integer $M > 0$ independent of z. Define

$$S_{j,k} = \begin{cases} 1 & \text{if } D^\delta(b_k) \cap D^r(a_j) \neq \emptyset, \\ 0 & \text{if } D^\delta(b_k) \cap D^r(a_j) = \emptyset. \end{cases}$$

Then, $\sum_{j=1}^\infty S_{j,k} = |J_{b_k}| \le M$. Symmetrically, set

$$L_z = \left\{k : D^\delta(b_k) \cap D^r(z) \neq \emptyset\right\}.$$

We have $D^r(a_j) \subset \bigcup_{k\in L_{a_j}} D^\delta(b_k)$. Similarly

$$\rho(b_k) \simeq \rho(a_j) \quad \text{for } k \in L_{a_j}.$$

Then,

$$\widehat{\mu}_r(a_j)\rho(a_j)^{s+2/p} \le C \sum_{k\in L_{a_j}} \widehat{\mu}_\delta(b_k)\rho(b_k)^{s+2/p}. \tag{2.15}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^\infty \widehat{\mu}_r(a_j)^p\rho(a_j)^{sp+2} &\le C\sum_{j=1}^\infty\left(\sum_{k\in L_{a_j}} \widehat{\mu}_\delta(b_k)\rho(b_k)^{s+2/p}\right)^p \\ &\le C\sum_{k=1}^\infty\sum_{j=1}^\infty S_{j,k}\widehat{\mu}_\delta(b_k)^p\rho(b_k)^{sp+2} \\ &\le CM\sum_{k=1}^\infty \widehat{\mu}_\delta(b_k)^p\rho(b_k)^{sp+2}.\end{aligned}$$

By symmetry, we obtain (2.13).

Similar to that in Theorem 4.4 of [15], with (2.2) and (2.13), we can check the equivalence between (B) and (C). Moreover, for fixed $\delta, r > 0$, we obtain

$$\|\widehat{\mu}_\delta\|_{L^p} \simeq \|\widehat{\mu}_r\|_{L^p}. \tag{2.16}$$

To prove (A) $\Rightarrow$ (B), we take r_0 as in (2.6). Then

$$\widehat{\mu}_{r_0}(z) \le C\widetilde{\mu}_t(z). \tag{2.17}$$

From (2.16), the conclusion (B) follows. Now we prove the implication (B) $\Rightarrow$ (A). If $1 \le p < \infty$, taking $f(w) = k_{t,z}(w)$ and $t = p$ in Lemma 2.4, we know

$$\widetilde{\mu}_t(z) \le C\widetilde{[\widehat{\mu}_\delta]}_t(z), \quad z \in \mathbb{C}.$$

By (2.11), $f \mapsto \widetilde{f}_t$ is bounded on L^p,

$$\|\widetilde{\mu}_t\|_{L^p} \le C \left\|\widetilde{[\widehat{\mu}_\delta]}_t\right\|_{L^p} \le C \|\widehat{\mu}_\delta\|_{L^p}.$$

Next, suppose $0 < p < 1$ and $\widehat{\mu}_\delta \in L^p$ for some $\delta > 0$. For any r-lattice $\{a_k\}_k$, from the proof above we have

$$\left\|\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2/p}\right\}_k\right\|_{l^p} \le C \|\widehat{\mu}_\delta\|_{L^p}. \tag{2.18}$$

Also, we have some constant $m > 1$ such that

$$\bigcup_{z\in D^r(a_k)} D^r(z) \subseteq D^{mr}(a_k) \quad \text{for } z \in \mathbb{C}.$$

We can divide $\{a_k\}_k$ into J subsequence $\{a_{j,k}\}_k$, $j = 1, 2, \ldots, J$, each $\{a_{j,k}\}_k$ is an $mr-$lattice. From (2.18), we have

$$\sum_{k=1}^{\infty} \widehat{\mu}_{mr}(a_k)^p \rho(a_k)^2 = \sum_{j=1}^{J}\sum_{k=1}^{\infty} \widehat{\mu}_{mr}(a_{j,k})^p \rho(a_{j,k})^2 \le C \|\widehat{\mu}_\delta\|_{L^p}^p.$$

Hence, Lemma 2.2, (2.8), (2.5) imply

$$\begin{aligned}
|\widetilde{\mu}_t(z)|^p &\le Ce^{-tp\phi(z)}\rho(z)^{2tp-2p}\left(\int_{\mathbb{C}} |K(w,z)|^t e^{-t\phi(w)}\widehat{\mu}_r(w)\,dA(w)\right)^p \\
&\le Ce^{-tp\phi(z)}\rho(z)^{2tp-2p}\left(\sum_{j=1}^{\infty}\int_{D^r(a_j)} |K(w,z)|^t e^{-t\phi(w)}\widehat{\mu}_r(w)\,dA(w)\right)^p \\
&\le Ce^{-tp\phi(z)}\rho(z)^{2tp-2p}\left(\sum_{j=1}^{\infty}\widehat{\mu}_{mr}(a_j)\int_{D^r(a_j)} |K(w,z)|^t e^{-t\phi(w)}\,dA(w)\right)^p
\end{aligned}$$

$$\le Ce^{-tp\phi(z)}\rho(z)^{2tp-2p}\sum_{j=1}^{\infty}\widehat{\mu}_{mr}(a_j)^p\left(\int_{D^r(a_j)}|K(w,z)|^t\,e^{-t\phi(w)}\,dA(w)\right)^p$$
$$\le C\rho(z)^{tp-2p}\sum_{j=1}^{\infty}\widehat{\mu}_{mr}(a_j)^p\rho(a_j)^{2p}\sup_{w\in D^r(a_j)}\rho(w)^{-tp}e^{-tp\left(\frac{|z-w|}{\rho(w)}\right)^{\epsilon}}.$$

For $z\in D^{2r}(a_j)$, (2.2) implies

$$\rho(z)^{tp-2p}\rho(a_j)^{2p}\sup_{w\in D^r(a_j)}\rho(w)^{-tp}e^{-tp\left(\frac{|z-w|}{\rho(w)}\right)^{\epsilon}}\le C.$$

If $z\in\mathbb{C}\setminus D^{2r}(a_j)$ and $w\in D^r(a_j)$, (2.2) shows

$$e^{-tp\left(\frac{|z-w|}{\rho(w)}\right)^{\epsilon}}\le e^{-tp\left(\frac{\left|z-a_j\right|-\left|w-a_j\right|}{\rho(w)}\right)^{\epsilon}}\le e^{-\alpha^{\epsilon}tp\left(\frac{\left|z-a_j\right|}{\rho(a_j)}-r\right)^{\epsilon}}\le e^{-(\alpha/2)^{\epsilon}tp\left(\frac{\left|z-a_j\right|}{\rho(a_j)}\right)^{\epsilon}}.$$

These, (2.2) and Lemma 2.1 yield

$$\int_{\mathbb{C}}\rho(z)^{tp-2p}\sum_{j=1}^{\infty}\widehat{\mu}_{mr}(a_j)^p\rho(a_j)^{2p}\sup_{w\in D^r(a_j)}\rho(w)^{-tp}e^{-tp\left(\frac{|z-w|}{\rho(w)}\right)^{\epsilon}}\,dA(z)$$
$$=\sum_{j=1}^{\infty}\widehat{\mu}_{mr}(a_j)^p\rho(a_j)^{2p}\left(\int_{D^{2r}(a_j)}+\int_{\mathbb{C}\setminus D^{2r}(a_j)}\right)\rho(z)^{tp-2p}$$
$$\times\sup_{w\in D^r(a_j)}\rho(w)^{-tp}e^{-tp\left(\frac{|z-w|}{\rho(w)}\right)^{\epsilon}}\,dA(z)$$
$$\le C\sum_{j=1}^{\infty}\widehat{\mu}_{mr}(a_j)^p\left(\rho(a_j)^2+\rho(a_j)^{2p-tp}\int_{\mathbb{C}\setminus D^{2r}(a_j)}\rho(z)^{tp-2p}e^{-(\alpha/2)^{\epsilon}tp\left(\frac{\left|z-a_j\right|}{\rho(a_j)}\right)^{\epsilon}}\,dA(z)\right)$$
$$\le C\sum_{j=1}^{\infty}\widehat{\mu}_{mr}(a_j)^p\rho(a_j)^2.$$

Therefore,

$$\|\widetilde{\mu}_t\|_{L^p}^p\le C\sum_{j=1}^{\infty}\widehat{\mu}_{mr}(a_j)^p\rho(a_j)^2\le C\,\|\widehat{\mu}_\delta\|_{L^p}^p.$$

The quantity equivalence (2.12) comes from a carefully checking of the implication above. The proof is completed. □

The next lemma, Lemma 2.6, is some partial result about atomic decomposition on $F^p(\phi)$.

Lemma 2.6. *Let $\{a_k\}_k$ be an r-lattice. For $0<p\le\infty$ and $\{\lambda_k\}_k\in l^p$, set*

$$f(z)=\sum_{k=1}^{\infty}\lambda_k k_{2,a_k}(z)\rho(a_k)^{1-2/p}. \tag{2.19}$$

Then $f\in F^p(\phi)$ and $\|f\|_{p,\phi}\le C\,\|\{\lambda_k\}_k\|_{l^p}$.

Proof. The proof is similar to that of Lemma 2.4 from [14]. If $0 < p \le 1$, Lemma 2.2 gives

$$\|f\|_{p,\phi}^p \le \sum_{k=1}^{\infty} |\lambda_k|^p \|k_{2,a_k}\|_{p,\phi}^p \rho(a_k)^{p-2} \le C \|\{\lambda_k\}_k\|_{l^p}^p .$$

For $1 < p \le \infty$, define $F(z) = \sum_{k=1}^{\infty} |\lambda_k| \rho(a_k)^{-2/p} \chi_{D^r(a_k)}(z)$. With the 1-Berezin transform, from (2.9) and (2.8) we get

$$|f(z)| e^{-\phi(z)} \le C e^{-\phi(z)} \sum_{k=1}^{\infty} |\lambda_k| \rho(a_k)^{2-2/p} |K(z,a_k)| e^{-\phi(a_k)} \le C\widetilde{F}_1(z).$$

By (2.10) and the boundedness of $F \to \widetilde{F}_1$ on L^p, we see

$$\|f\|_{p,\phi} \le C \left\|\widetilde{F}_1\right\|_{L^p} \le C \|F\|_{L^p} \le C \|\{\lambda_k\}_k\|_{l^p} .$$

This completes the proof. □

3. Toeplitz operators

In this section, we are going to characterize those $\mu \ge 0$ for which the induced Toeplitz operator T_μ is bounded (or compact) from one weighted Fock space to another. To this purpose, we need the relatively compact subsets in $F^p(\phi)$. With the same proof as that of Lemma 3.2 in [14], we know a bounded subset $E \subset F^p(\phi)$ is relatively compact if and only if for each $\varepsilon > 0$ there is some $S > 0$ such that

$$\sup_{f\in E} \int_{|z|\ge S} \left| f(z)e^{-\phi(z)}\right|^p dA(z) < \varepsilon. \tag{3.1}$$

This observation on the compact subsets in Fock spaces is crucial to our study on the compactness of T_μ from $F^p(\phi)$ to $F^q(\phi)$. While $1 < p = q < \infty$, our result coincides with that in [19]. But the proof in [19] strongly depends on some basic facts about compactness of operators in the setting of Banach spaces, see [22, Proposition 4.3] as well.

When $p = q > 1$ the following lemma, Lemma 3.1, can be found in [19].

Lemma 3.1. *Suppose $\mu \ge 0$ satisfying $\widehat{\mu}_\delta \rho^\sigma \in L^\infty$ for some $\delta > 0$ and $\sigma \in \mathbb{R}$. Then T_μ is well-defined on $F^p(\phi)$ for $0 < p < \infty$. And, for $R > 0$, Toeplitz operator T_{μ_R} is compact from $F^p(\phi)$ to $F^q(\phi)$ for $0 < p, q < \infty$, where μ_R is defined by*

$$\mu_R(V) = \mu\left(V \cap \overline{D(0,R)}\right) \quad \textit{for } V \subseteq \mathbb{C} \textit{ measurable.} \tag{3.2}$$

Proof. Suppose $\widehat{\mu}_\delta \rho^\sigma \in L^\infty$. For $f \in F^p(\phi)$ and $z \in \mathbb{C}$, from (2.8) to know

$$|f(z)| e^{-\phi(z)} \le C\rho(z)^{-2/p} \|f\|_{p,\phi} . \tag{3.3}$$

Applying (2.8) to the weight 2ϕ and the holomorphic function $K(\cdot\,, z)f(\cdot)$ to get

$$|T_\mu f(z)| \le C \int_{\mathbb{C}} |K(w,z)|\,|f(w)|\, e^{-2\phi(w)} \widehat{\mu}_\delta(w)\, dA(w). \tag{3.4}$$

Then by Lemma 2.1,

$$\begin{aligned}
|T_\mu f(z)| &\le C\, \|f\|_{p,\phi}\, \|\widehat{\mu}_\delta \rho^\sigma\|_{L^\infty} \int_{\mathbb{C}} \rho(w)^{-\sigma-2/p}\, |K(w,z)|\, e^{-\phi(w)}\, dA(w) \\
&\le C e^{\phi(z)} \rho(z)^{-1}\, \|\widehat{\mu}_\delta \rho^\sigma\|_{L^\infty}\, \|f\|_{p,\phi} \int_{\mathbb{C}} \rho(w)^{-1-\sigma-2/p} e^{-\left(\frac{|z-w|}{\rho(z)}\right)^\epsilon}\, dA(w) \\
&\le C e^{\phi(z)} \rho(z)^{-\sigma-2/p}\, \|\widehat{\mu}_\delta \rho^\sigma\|_{L^\infty}\, \|f\|_{p,\phi} < \infty.
\end{aligned}$$

This means that T_μ is well-defined on $F^p(\phi)$.

Next, we show the compactness of T_{μ_R}. To see this, we claim there are some $\eta, \theta, \epsilon > 0$, such that for $f \in F^p(\phi)$

$$\int_{|z|\ge S} \left| T_{\mu_R} f(z) e^{-\phi(z)} \right|^q dA(z) \le C\, \|f\|_{p,\phi}^q \int_{|z|\ge S} |z|^{\eta q}\, e^{-\theta |z|^\epsilon}\, dA(z)$$

when S is large enough. In fact, there is some positive constant M, whenever $|w| \le R$ we have

$$M^{-1} \le \rho(w) \le M$$

and

$$|z-w| \ge |z| - |w| \ge |z| - R \ge \frac{|z|}{2} \quad \text{if } |z| \ge \frac{R}{2}.$$

The estimates (2.7) and (3.3) imply, when S is large enough,

$$\begin{aligned}
&\int_{|z|\ge S} \left| T_{\mu_R} f(z) e^{-\phi(z)} \right|^q dA(z) \\
&\le \int_{|z|\ge S} \left(\int_{\mathbb{C}} \chi_{|w|\le R}(w) \left| f(w) K(z,w) e^{-2\phi(w)} e^{-\phi(z)} \right| d\mu(w) \right)^q dA(z) \\
&\le C \int_{|z|\ge S} \left(\int_{|w|\le R+\delta M} |f(w)|\, |K(z,w)|\, e^{-2\phi(w)} e^{-\phi(z)} \widehat{\mu}_\delta(w)\, dA(w) \right)^q dA(z) \\
&\le C\, \|f\|_{p,\phi}^q\, \|\widehat{\mu}_\delta \rho^\sigma\|_{L^\infty}^q \\
&\quad \times \int_{|z|\ge S} \left(\int_{|w|\le R+\delta M} \rho(z)^{-1} \rho(w)^{-2/p-\sigma-1} e^{-\left(\frac{|z-w|}{\rho(w)}\right)^\epsilon}\, dA(w) \right)^q dA(z) \\
&\le C_1\, \|f\|_{p,\phi}^q \int_{|z|\ge S} |z|^{\eta q}\, e^{-\theta |z|^\epsilon}\, dA(z)
\end{aligned}$$

where the constant C_1 is independent of f and S. Hence,

$$\|T_{\mu_R} f\|_{q,\phi}^q = \left(\int_{|z|\le S} + \int_{|z|>S} \right) \left| T_{\mu_R} f(z) e^{-\phi(z)} \right|^q dA(z) \le C\, \|f\|_{p,\phi}^q\,.$$

Thus, T_{μ_R} is bounded from $F^p(\phi)$ to $F^q(\phi)$. Suppose E is the unit ball of $F^p(\phi)$, then $\{T_{\mu_R} f : f \in E\}$ is a bounded subset in $F^q(\phi)$. To prove the compactness, for $\varepsilon > 0$, since $\int_0^\infty r^{\eta q+2n-1} e^{-\theta r^\epsilon}\, dr < \infty$, there exists some S large enough such that

$$\int_S^\infty r^{\eta q+2n-1} e^{-\theta r^\epsilon}\, dr < \frac{\varepsilon}{C_1 + 1}.$$

This implies

$$\sup_{f\in E} \int_{|z|\geq S} \left| T_{\mu_R} f(z) e^{-\phi(z)} \right|^q dA(z) \leq C_1 \int_S^\infty r^{\eta q+2n-1} e^{-\theta r^\epsilon}\, dr < \varepsilon.$$

The proof is completed. □

We are now in the position to characterize the boundedness (and the compactness) of positive Toeplitz operators T_μ from one weighted Fock space $F^p(\phi)$ to another $F^q(\phi)$. Because the inclusion between any two spaces $F^p(\phi)$ and $F^q(\phi)$ is no longer valid while $p \neq q$, and also $F^p(\phi)$ is not a Banach space with $0 < p < 1$, the approach in [13–16, 19, 22, 23] does not work here.

Theorem 3.2. *Let $0 < p \leq q < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent:*

(A) *$T_\mu \colon F^p(\phi) \to F^q(\phi)$ is bounded;*

(B) *$\widetilde{\mu}_t \rho^{2(p-q)/(pq)} \in L^\infty$ for some (or any) $t > 0$;*

(C) *$\widehat{\mu}_\delta \rho^{2(p-q)/(pq)} \in L^\infty$ for some (or any) $\delta > 0$;*

(D) *The sequence $\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)}\right\}_k \in l^\infty$ for some (or any) r-lattice $\{a_k\}_k$.*

Furthermore,

$$\begin{aligned} \|T_\mu\|_{F^p(\phi)\to F^q(\phi)} &\simeq \left\| \widetilde{\mu}_t \rho^{2(p-q)/(pq)} \right\|_{L^\infty} \simeq \left\| \widehat{\mu}_\delta \rho^{2(p-q)/(pq)} \right\|_{L^\infty} \\ &\simeq \left\| \left\{ \widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)} \right\}_k \right\|_{l^\infty}. \end{aligned} \tag{3.5}$$

Proof. It is trivial that (D) follows from (C) because of (2.13), moreover

$$\left\| \left\{ \widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)} \right\}_k \right\|_{l^\infty} \leq \left\| \widehat{\mu}_\delta \rho^{2(p-q)/(pq)} \right\|_{L^\infty}. \tag{3.6}$$

Estimate (2.17) tells us that (B) implies (C) for r_0 with r_0 in (2.6). Notice that, (2.16) is still true for $p = \infty$. These imply

$$\left\| \widehat{\mu}_\delta \rho^{2(p-q)/(pq)} \right\|_{L^\infty} \simeq \left\| \widehat{\mu}_{r_0} \rho^{2(p-q)/(pq)} \right\|_{L^\infty} \leq C \left\| \widetilde{\mu}_t \rho^{2(p-q)/(pq)} \right\|_{L^\infty} \tag{3.7}$$

for all $\delta > 0$.

Now we prove that (D) implies (B). By (2.2), we have some $m > 0$ such that $D^r(z) \subset D^{mr}(a)$ for $z \in D^r(a)$ and $a \in \mathbb{C}$. For any $t > 0$, set $s = tpq/(pq-p+q)$. The inequality (2.8) tells us, for $f \in F^s(\phi)$,

$$\sup_{z\in D^r(a)} \left|f(z)e^{-\phi(z)}\right|^s \le \frac{C}{\rho(a)^2}\int_{D^{mr}(a)} \left|f(w)e^{-\phi(w)}\right|^s dA(w). \tag{3.8}$$

By Lemma 2.2,

$$|k_{t,z}(w)|^t \rho(z)^{2(p-q)/(pq)} \simeq |k_{s,z}(w)|^t .$$

Then from (3.8) and (2.10) we obtain

$$\begin{aligned}
&\widetilde{\mu}_t(z)\rho(z)^{2(p-q)/(pq)} \\
&\simeq \int_{\mathbb{C}} |k_{s,z}(w)|^t e^{-t\phi(w)}\, d\mu(w) \\
&\le \sum_{k=1}^{\infty}\int_{D^r(a_k)} |k_{s,z}(w)|^t e^{-t\phi(w)}\, d\mu(w) \\
&\le \sum_{k=1}^{\infty}\mu(D^r(a_k))\left(\sup_{w\in D^r(a_k)}\left|k_{s,z}(w)e^{-\phi(w)}\right|^s\right)^{(pq-p+q)/(pq)} \\
&\le C\sum_{k=1}^{\infty}\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)}\left(\int_{D^{mr}(a_k)}\left|k_{s,z}(w)e^{-\phi(w)}\right|^s dA(w)\right)^{(pq-p+q)/(pq)} \\
&\le C\sup_k\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)}\left(\sum_{k=1}^{\infty}\int_{D^{mr}(a_k)}\left|k_{s,z}(w)e^{-\phi(w)}\right|^s dA(w)\right)^{(pq-p+q)/(pq)} \\
&\le CN^{(pq-p+q)/(pq)}\sup_k\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)}\left\|k_{s,z}\right\|_{s,\phi}^t .
\end{aligned}$$

This gives

$$\left\|\widetilde{\mu}_t\rho^{2(p-q)/(pq)}\right\|_{L^\infty} \le C\left\|\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)}\right\}_k\right\|_{l^\infty}. \tag{3.9}$$

That is, (D) implies (B).

To prove the implication from (A) to (B) we suppose the statement (A) is valid. By Lemma 2.2, (2.8) and the fact that

$$|K_{2,z}(w)|^2 \rho(z)^{2(p-q)/(pq)} \simeq e^{-\phi(z)}k_{p,z}(w)K(z,w)$$

we have

$$\begin{aligned}
\widetilde{\mu}_2(z)\rho(z)^{2(p-q)/(pq)} &\le C\rho(z)^{2/q}\left|T_\mu k_{p,z}(z)\right| e^{-\phi(z)} \\
&\le C\left(\int_{D(z)}\left|T_\mu k_{p,z}(w)e^{-\phi(w)}\right|^q dA(w)\right)^{1/q}.
\end{aligned} \tag{3.10}$$

Then

$$\widetilde{\mu}_2(z)\rho(z)^{2(p-q)/(pq)} \leq C \left\| T_\mu k_{p,z} \right\|_{q,\phi} \leq C \left\| T_\mu \right\|_{F^p(\phi)\to F^q(\phi)} . \tag{3.11}$$

This and the equivalence between (B) and (C) shows the estimate (3.11) remains true when $\widetilde{\mu}_2$ is replaced by $\widetilde{\mu}_t$ for any $t > 0$.

Now we are going to prove the implication (C) $\Rightarrow$ (A). Lemma 3.1 tells us that T_μ is well-defined on $F^p(\phi)$. Given $\delta > 0$, we first claim there is some positive constant C such that

$$\| T_\mu f \|_{q,\phi}^q \leq C \int_{\mathbb{C}} |f(w)|^q \, e^{-q\phi(w)} \widehat{\mu}_\delta(w)^q \, dA(w) \tag{3.12}$$

for $f \in F^p(\phi)$. In fact, if $q > 1$, (3.4) and Hölder's inequality tell us

$$\begin{aligned}
|T_\mu f(z)|^q \, e^{-q\phi(z)} &\leq C \left(\int_{\mathbb{C}} \widehat{\mu}_\delta(w) \, |f(w)| \, |K(w,z)| \, e^{-2\phi(w)} e^{-\phi(z)} \, dA(w) \right)^q \\
&\leq C \int_{\mathbb{C}} |f(w)|^q \, e^{-q\phi(w)} \widehat{\mu}_\delta(w)^q \left| K(w,z) e^{-\phi(w)} e^{-\phi(z)} \right| dA(w) \\
&\quad \times \left(\int_{\mathbb{C}} \left| K(w,z) e^{-\phi(w)} e^{-\phi(z)} \right| dA(w) \right)^{q/q'} \\
&\leq C \int_{\mathbb{C}} |f(w)|^q \, e^{-q\phi(w)} \widehat{\mu}_\delta(w)^q \left| K(w,z) e^{-\phi(w)} e^{-\phi(z)} \right| dA(w).
\end{aligned}$$

Integrating both sides above, applying Fubini's Theorem and Lemma 2.2 to get (3.12). To deal with the case $q \leq 1$, for given $\delta > 0$ we pick some $r > 0$ so that $\tau^2 r \leq \min\{\delta, 1\}$ with τ as in (2.4), and let $\{a_k\}_k$ be some r-lattice. By (2.8) we know, for $f \in F^p(\phi)$,

$$\begin{aligned}
|T_\mu f(z)|^q &\leq \left(\sum_{k=1}^\infty \int_{D^r(a_k)} |f(w) K(w,z)| \, e^{-2\phi(w)} \, d\mu(w) \right)^q \\
&\leq \sum_{k=1}^\infty \left(\int_{D^r(a_k)} |f(w) K(w,z)| \, e^{-2\phi(w)} \, d\mu(w) \right)^q \\
&\leq \sum_{k=1}^\infty \widehat{\mu}_r(a_k)^q \rho(a_k)^{2q} \left(\sup_{w \in D^r(a_k)} |f(w) K(w,z)| \, e^{-2\phi(w)} \right)^q .
\end{aligned}$$

From (2.8), there are some constant $C > 0$ such that $|T_\mu f(z)|^q$ is no more than C times

$$\sum_{k=1}^\infty \widehat{\mu}_r(a_k)^q \rho(a_k)^{2q-2} \int_{D^{\tau r}(a_k)} |f(w)|^q \, |K(w,z)|^q \, e^{-2q\phi(w)} \, dA(w).$$

From (2.3) and (2.4), we have $D^r(a_k) \subseteq D^{\tau^2 r}(w)$ if $w \in D^{\tau r}(a_k)$. This, together with

(2.2) and (2.10), implies

$$\begin{aligned}
|T_\mu f(z)|^q &\leq C \sum_{k=1}^{\infty} \int_{D^{\tau r}(a_k)} \widehat{\mu}_{\tau^2 r}(w)^q \rho(w)^{2q-2} \,|f(w)|^q \,|K(w,z)|^q \,e^{-2q\phi(w)} \,dA(w) \\
&\leq CN \int_{\mathbb{C}} \widehat{\mu}_{\tau^2 r}(w)^q \rho(w)^{2q-2} \,|f(w)|^q \,|K(w,z)|^q \,e^{-2q\phi(w)} \,dA(w) \\
&\leq C \int_{\mathbb{C}} \widehat{\mu}_{\delta}(w)^q \rho(w)^{2q-2} \,|f(w)|^q \,|K(w,z)|^q \,e^{-2q\phi(w)} \,dA(w).
\end{aligned}$$

Similarly, integrating both sides of the above with respect to $e^{-q\phi(z)}\,dA(z)$ and applying Fubini's Theorem to get (3.12).

Now we prove (C) $\Leftrightarrow$ (A). Suppose (C) is true, by $p \leq q$, (3.12) and (3.3) we obtain

$$\begin{aligned}
\|T_\mu f\|_{q,\phi}^q &\leq C \int_{\mathbb{C}} |f(w)|^p \,e^{-p\phi(w)} \widehat{\mu}_\delta(w)^q \left(\rho(w)^{-2/p} \,\|f\|_{p,\phi}\right)^{q-p} dA(w) \\
&\leq C \left\|\widehat{\mu}_\delta \rho^{2(p-q)/(pq)}\right\|_{L^\infty}^q \|f\|_{p,\phi}^q
\end{aligned}$$

for $f \in F^p(\phi)$. Therefore, T_μ is bounded from $F^p(\phi)$ to $F^q(\phi)$ and

$$\|T_\mu\|_{F^p(\phi)\to F^q(\phi)} \leq C \left\|\widehat{\mu}_\delta \rho^{2(p-q)/(pq)}\right\|_{L^\infty}. \tag{3.13}$$

The estimates of (3.5) come from (3.6), (3.7), (3.9), (3.11) and (3.13). The proof is ended. $\square$

For the compactness of T_μ while $p \leq q$ we have the following Theorem 3.3.

Theorem 3.3. *Let $0 < p \leq q < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent:*

(A) $T_\mu \colon F^p(\phi) \to F^q(\phi)$ *is compact;*

(B) $\widetilde{\mu}_t(z)\rho(z)^{2(p-q)/(pq)} \to 0$ *as $z \to \infty$ for some (or any) $t > 0$;*

(C) $\widehat{\mu}_\delta(z)\rho(z)^{2(p-q)/(pq)} \to 0$ *as $z \to \infty$ for some (or any) $\delta > 0$;*

(D) $\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)} \to 0$ *as $k \to \infty$ for some (or any) r-lattice $\{a_k\}_k$.*

Proof. The proof of the implication that "(B) $\Rightarrow$ (C)" and "(C) $\Rightarrow$ (D)" can be carried out as the same part of Theorem 3.2.

Now we assume μ satisfies condition (D) for some r-lattice $\{a_k\}_k$. Then, for $\varepsilon > 0$ there exists some integer $K > 0$ such that $\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)} < \varepsilon$ whenever $k > K$. Notice that, $\bigcup_{k=1}^K \overline{B^{mr}(a_k)}$ is a compact subset of $\mathbb{C}$, and $\{k_{s,z}\}_{z\in\mathbb{C}} \subseteq F^s(\phi)$ uniformly converges to 0 on $\bigcup_{k=1}^K \overline{B^{mr}(a_k)}$ as $z \to \infty$, where $s = tpq/(pq - p + q)$. From Lemma 2.2, (3.8) and (2.10), when $|z|$ is sufficiently large we have

$$\begin{aligned}
&\widetilde{\mu}_t(z)\rho(z)^{2(p-q)/(pq)} \\
&\simeq \int_{\mathbb{C}} |k_{s,z}(w)|^t e^{-t\phi(w)}\, d\mu(w) \\
&\leq \int_{\bigcup_{k=1}^K \overline{B^{mr}(a_k)}} |k_{s,z}(w)|^t e^{-t\phi(w)}\, d\mu(w) \\
&\quad + \sum_{k=K+1}^{\infty} \mu(B^r(a_k)) \left(\sup_{w \in B^r(a_k)} \left| k_{s,z}(w) e^{-\phi(w)} \right|^s \right)^{(pq-p+q)/pq} \\
&< \varepsilon + C \sum_{k=K+1}^{\infty} \widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)} \left(\int_{B^{mr}(a_k)} \left| k_{s,z}(w) e^{-\phi(w)} \right|^s dA(w) \right)^{(pq-p+q)/pq} \\
&< \varepsilon + C \sup_{k \geq K+1} \widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)} \\
&\qquad \times \left(\sum_{k=K+1}^{\infty} \int_{B^{mr}(a_k)} \left| k_{s,z}(w) e^{-\phi(w)} \right|^s dA(w) \right)^{(pq-p+q)/(pq)} \\
&< \varepsilon + C N^{(pq-p+q)/(pq)} \|k_{s,z}\|_{s,\phi}^t \varepsilon = C\varepsilon
\end{aligned}$$

where C is independent of ε. This yields that $\widetilde{\mu}_t(z)\rho(z)^{2(p-q)/(pq)} \to 0$ as $z \to \infty$. So, μ satisfies (B) for any $t > 0$.

To prove "(A) $\Rightarrow$ (B)", we suppose T_μ is compact from $F^p(\phi)$ to $F^q(\phi)$. Since $\{k_{p,z} : z \in \mathbb{C}\}$ is bounded in $F^p(\phi)$, $\{T_\mu k_{p,z} : z \in \mathbb{C}\}$ is relatively compact in $F^q(\phi)$. By (3.1), for any $\varepsilon > 0$ there exists some $S > 0$ such that

$$\sup_{z \in \mathbb{C}} \int_{|w| > S} \left| T_\mu k_{p,z}(w) e^{-\phi(w)} \right|^q dA(w) < \varepsilon^q.$$

When $|z|$ is sufficiently large and $w \in D(z)$,

$$|w| \geq |z| - |w - z| \geq |z| - \rho(z) \geq |z| - C|z|^\beta \geq |z|^\beta > S,$$

where $\beta \in (0,1)$ as in (2.7). Hence, $D(z) \subseteq \{w : |w| > S\}$. By (3.10), we obtain

$$\widetilde{\mu}_2(z)\rho(z)^{2(p-q)/(pq)} \leq C \left(\int_{D(z)} \left| T_\mu k_{p,z}(w) e^{-\phi(w)} \right|^q dA(w) \right)^{1/q} < C\varepsilon$$

when $|z|$ is sufficiently large. Hence,

$$\lim_{z \to \infty} \widetilde{\mu}_2(z)\rho(z)^{2(p-q)/(pq)} = 0.$$

The equivalence between (B) and (C) shows the above limit is still valid if μ_2 is replaced by μ_t for any $t > 0$.

Finally, we suppose the statement (C) is true. Set μ_R as (3.2). Lemma 3.1 shows that T_{μ_R} is compact from $F^p(\phi)$ to $F^q(\phi)$. And also, $\mu - \mu_R \geq 0$. By (C) and (3.5), for $\delta > 0$ fixed we have

$$\|T_\mu - T_{\mu_R}\|_{F^p(\phi)\to F^q(\phi)} \simeq \left\|(\widehat{\mu - \mu_R})_\delta \rho^{2(p-q)/(pq)}\right\|_{L^\infty} \to 0$$

as $R \to \infty$. Therefore, T_μ is compact from $F^p(\phi)$ to $F^q(\phi)$. The proof is completed. $\square$

Now we are in the position to characterize the boundedness (and equivalently the compactness) of T_μ for $q < p$.

Theorem 3.4. *Let $0 < q < p < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent:*

(A) $T_\mu \colon F^p(\phi) \to F^q(\phi)$ *is bounded;*

(B) $T_\mu \colon F^p(\phi) \to F^q(\phi)$ *is compact;*

(C) $\widetilde{\mu}_t \in L^{pq/(p-q)}$ *for some (or any) $t > 0$;*

(D) $\widehat{\mu}_\delta \in L^{pq/(p-q)}$ *for some (or any) $\delta > 0$;*

(E) $\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)}\right\}_k \in l^{pq/(p-q)}$ *for some (or any) r-lattice $\{a_k\}_k$.*

Furthermore,

$$\begin{aligned}\|T_\mu\|_{F^p(\phi)\to F^q(\phi)} &\simeq \|\widehat{\mu}_t\|_{L^{pq/(p-q)}} \simeq \|\widehat{\mu}_\delta\|_{L^{pq/(p-q)}} \\ &\simeq \left\|\left\{\widehat{\mu}_r(a_k)\rho(a_k)^{2(p-q)/(pq)}\right\}_k\right\|_{l^{pq/(p-q)}}.\end{aligned} \tag{3.14}$$

Proof. The equivalence among the statements (C), (D) and (E) follows from Lemma 2.5. It is trivial that (B) $\Rightarrow$ (A). To finish our proof, we are going to prove the implications (A) $\Rightarrow$ (E), (D) $\Rightarrow$ (A) and (D) $\Rightarrow$ (B).

To get (A) $\Rightarrow$ (E), we borrow some idea from [14]. First, we claim that (E) is true for $r = r_0$ with r_0 in (2.6). For any r_0-lattice $\{a_k\}_k$ and sequence $\{\lambda_k\}_k \in l^p$, set f as (2.19). Lemma 2.6 shows $f \in F^p(\phi)$ with $\|f\|_{p,\phi} \leq C \|\{\lambda_k\}_k\|_{l^p}$. By Khinchine's inequality and the boundedness of T_μ, we have

$$\begin{aligned}&\int_{\mathbb{C}} \left(\sum_{k=1}^{\infty} \left|\lambda_k \rho(a_k)^{1-2/p} T_\mu(k_{2,a_k})(z)\right|^2\right)^{q/2} e^{-q\phi(z)}\, dA(z) \\ &\leq C \|T_\mu\|^q_{F^p(\phi)\to F^q(\phi)} \|\{\lambda_k\}_k\|^q_{l^p}.\end{aligned}$$

Meanwhile, there is

$$
\begin{aligned}
&\int_{\mathbb{C}}\left(\sum_{k=1}^{\infty}\left|\lambda_k\rho(a_k)^{1-2/p}T_\mu(k_{2,a_k})(z)\right|^2\right)^{q/2} e^{-q\phi(z)}\,dA(z)\\
&\geq C\sum_{j=1}^{\infty}|\lambda_j|^q\,\rho(a_j)^{2+2q-2q/p}\left|\int_{D^{r_0}(a_j)}|K(w,a_j)|^2\,e^{-2\phi(w)}\,d\mu(w)\right|^q e^{-2q\phi(a_j)}\\
&\geq C\sum_{j=1}^{\infty}|\lambda_j|^q\,\widehat{\mu}_{r_0}(a_j)^q\rho(a_j)^{2-2q/p},
\end{aligned}
$$

the last inequality follows from (2.2) and (2.6). Setting $\beta_j = |\lambda_j|^q$, then $\{\beta_j\}_{j=1}^{\infty} \in l^{p/q}$. Therefore,

$$
\begin{aligned}
\sum_{j=1}^{\infty}\beta_j\widehat{\mu}_{r_0}(a_j)^q\rho(a_j)^{2-2q/p} &\leq C\,\|T_\mu\|^q_{F^p(\phi)\to F^q(\phi)}\left\|\{\lambda_j\}_j\right\|^q_{l^p}\\
&= C\,\|T_\mu\|^q_{F^p(\phi)\to F^q(\phi)}\left\|\{\beta_j\}_j\right\|_{l^{p/q}}.
\end{aligned}
$$

The duality argument shows

$$
\left\{\widehat{\mu}_{r_0}(a_j)^q\rho(a_j)^{2-2q/p}\right\}_{j=1}^{\infty} \in l^{p/(p-q)}
$$

and

$$
\left\|\left\{\widehat{\mu}_{r_0}(a_j)^q\rho(a_j)^{2-2q/p}\right\}_j\right\|_{l^{p/(p-q)}} \leq C\,\|T_\mu\|^q_{F^p(\phi)\to F^q(\phi)}.
$$

This and Lemma 2.5 imply

$$
\left\|\left\{\widehat{\mu}_r(b_j)\rho(b_j)^{2(p-q)/(pq)}\right\}_j\right\|_{l^{pq/(p-q)}} \leq C\,\|T_\mu\|_{F^p(\phi)\to F^q(\phi)} \tag{3.15}
$$

for any r-lattice $\{b_j\}$. From this, the conclusion (E) follows.

Now we prove (D) $\Rightarrow$ (A). Suppose $\widehat{\mu}_\delta \in L^{pq/(p-q)}$ for some $\delta > 0$. By Lemma 2.5, we know $\left\{\widehat{\mu}_\delta(a_k)\rho(a_k)^{2(p-q)/(pq)}\right\}_k \in l^\infty$ for some δ-lattice $\{a_k\}_k$. Theorem 3.2 gives $\widehat{\mu}_\delta\rho^{2(p-q)/(pq)} \in L^\infty$, which shows that T_μ is well-defined on $F^p(\phi)$, see Lemma 3.1. Notice that $p/q > 1$. By (3.12), Hölder's inequality and Lemma 2.2, we obtain

$$
\begin{aligned}
\|T_\mu f\|^q_{q,\phi} &\leq C\left\{\int_{\mathbb{C}}\left(|f(w)|^q\,e^{-q\phi(w)}\right)^{p/q}dA(w)\right\}^{q/p}\left\{\int_{\mathbb{C}}\widehat{\mu}_\delta(w)^{pq/(p-q)}\,dA(w)\right\}^{(p-q)/p}\\
&\leq C\,\|\widehat{\mu}_\delta\|^q_{L^{pq/(p-q)}}\,\|f\|^q_{p,\phi}
\end{aligned}
$$

for $f \in F^p(\phi)$. Hence, T_μ is bounded from $F^p(\phi)$ to $F^q(\phi)$ and

$$
\|T_\mu\|_{F^p(\phi)\to F^q(\phi)} \leq C\,\|\widehat{\mu}_\delta\|_{L^{pq/(p-q)}}. \tag{3.16}
$$

To prove (D) $\Rightarrow$ (B), we take μ_R as (3.2). Then $\mu - \mu_R \geq 0$, and for $\delta > 0$ we have $\left\| \widehat{(\mu - \mu_R)_\delta} \right\|_{L^{pq/(p-q)}} \to 0$ as $R \to \infty$. By (3.16),

$$\|T_\mu - T_{\mu_R}\|_{F^p(\phi)\to F^q(\phi)} = \left\|T_{(\mu-\mu_R)}\right\|_{F^p(\phi)\to F^q(\phi)} \simeq \left\| \widehat{(\mu - \mu_R)_\delta} \right\|_{L^{pq/(p-q)}} \to 0$$

whenever $R \to \infty$. Since T_{μ_R} is compact from $F^p(\phi)$ to $F^q(\phi)$, the operator $T_\mu \colon F^p(\phi) \to F^q(\phi)$ is compact as well.

The norm equivalence (3.14) comes from Lemma 2.5, (3.15) and (3.16). The proof is completed. $\square$

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