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The Geometric Mean of Min Matrices

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Abstract. In this paper, we consider the geometric mean of min matrices. We establish a closed form for the geometric mean of two positive definite min matrices. We further show that the set of min matrices is geodesically convex in the Cartan–Hadamard manifold of positive definite matrices. We also present monotonic one-parameter families of min matrices.

1. Introduction

The *min matrix* is the $m \times m$ symmetric matrix whose (i, j) entry is $\min\{i, j\}$:

$$M_{\min} = [\min\{i,j\}]_{m\times m} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & m \end{bmatrix}.$$

It is positive definite with $\det M = 1$. In [3], Bhatia considers the generalized min matrices of the form

$$M(\lambda_1, \ldots, \lambda_m) = [\min\{\lambda_i, \lambda_j\}]_{m\times m}, \quad 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$$

and shows that $M(\lambda_1, \ldots, \lambda_m)$ is positive semidefinite and infinitely divisible, that is, its positive Hadamard powers are positive semidefinite. He offers beautiful and readily understandable proofs in many different and interesting directions of Fourier analysis, differential equation, probability, and number theories. From

$$\det M(\lambda_1, \ldots, \lambda_m) = \prod_{k=1}^{m} (\lambda_k - \lambda_{k-1}), \quad \lambda_0 = 0, \tag{1.1}$$

the generalized min matrix is positive definite if and only if $\lambda_{k-1} < \lambda_k$, $k = 1, \ldots, m$. Let us also call a matrix M of this sort a (positive) min matrix.

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In this paper, we are concerned with linear and geometric structures on the set of min matrices and particularly an explicit formula for the Karcher mean (also known as Riemannian or Cartan mean) of min matrices, which is a standing open problem for general positive definite matrices [5, 7, 8, 15]. The Karcher mean is a natural extension of two variable geometric mean $A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ of positive definite matrices A and B and is the central object within the theory of multivariate geometric means and operator means [1, 2, 4, 6, 13, 14, 16, 17].

We recall shortly the Karcher mean. Let $\mathbb{S}_m$ be the Euclidean space of $m \times m$ real symmetric matrices equipped with the trace inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$, and let $\mathbb{P}_m \subset \mathbb{S}_m$ be the open convex cone of positive definite matrices, which is a smooth Riemannian manifold with the Riemannian trace metric $\langle X, Y\rangle_A = \mathrm{tr}\, A^{-1}XA^{-1}Y$, where $X, Y \in \mathbb{S}_m$ and $A \in \mathbb{P}_m$. Then $\mathbb{P}_m$ is a Cartan–Hadamard Riemannian manifold and the Riemannian distance between $A, B \in \mathbb{P}_m$ with respect to the above metric is given by $\delta(A, B) = \|\log A^{-1/2}BA^{-1/2}\|_2$, where $\|X\|_2 = (\mathrm{tr}\, X^2)^{1/2}$ for $X \in \mathbb{S}_m$. The unique geodesic (up to parametrization) joining A and B is given as the curve of *weighted geometric means* [10,12], i.e., $t \in [0, 1] \mapsto A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$. The Karcher mean on $\mathbb{P}_m$ is uniquely defined by

$$\Lambda(A_1, \ldots, A_N) := \underset{X\in\mathbb{P}_m}{\arg\min} \sum_{k=1}^{N} \delta^2(X, A_k)$$

and is characterized by the *Karcher equation*

$$\sum_{j=1}^{N} \log(X^{-1/2}A_jX^{-1/2}) = 0.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Let* $\mathbf{x}_k = (x_1^{(k)}, \ldots, x_m^{(k)}) \in \mathbb{R}^m$ *such that* $0 < x_1^{(k)} \le \cdots \le x_m^{(k)}$, $k = 1, \ldots, N$. *The Karcher mean of min matrices* $A_k = M(x_1^{(k)}, \ldots, x_m^{(k)})$ *is*

$$\Lambda(A_1, \ldots, A_N) = M(\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_m),$$

where

$$\lambda_j = \sqrt[N]{\prod_{k=1}^{N} \left(x_j^{(k)} - x_{j-1}^{(k)}\right)}, \quad x_0^{(k)} = 0$$

for $j = 1, 2, \ldots, m$.

The main theorem together with the determinant formula (1.1) ensures that the set of min matrices is a closed subset of $\mathbb{P}_m$ and closed under the geometric mean operation, that is, a geodesically convex subset of the Riemannian manifold $\mathbb{P}_m$. As a special family of min matrices, we introduce a one-parameter family of min matrices

$P_x := M(x, x^2, \ldots, x^m)$ over $x > 1$. The curve $x \mapsto P_x$ on $[1, \infty)$, called the *power min curve*, and $x \mapsto x^{-1}P_x = M(1, x, \ldots, x^{m-1})$ which appears in the study of stiffness matrices arisen from finite element methods [9], seem to be new in the context of matrix analysis. We show that they are monotonically increasing for the Löwner ordering. An alternative closed form for the Karcher mean of 2×2 power min matrices is derived in Section 4.

In what follows, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_{++} = (0, \infty)$ and $\mathbb{R}^m$ is the real Hilbert algebra equipped with the usual inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^m x_k y_k$ and the multiplicative operation $\mathbf{xy} = (x_1y_1, \ldots, x_my_m)$, so called the Hadamard product.

2. Min matrices

Consider the linear automorphism $L\colon \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$L(x_1, \ldots, x_m) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \cdots + x_m)$$

whose inverse map is $L^{-1}(x_1, \ldots, x_m) = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_m - x_{m-1})$. The matrix representations of L and L^{-1} are

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}, \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots\cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots\cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \quad 1 \end{bmatrix}. \tag{2.1}$$

The lower triangular matrix L appears in the Cholesky decomposition of the min matrix $M_{\min} = LL^T$:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The linear transformation L sends the positive cone $\mathbb{R}^m_{++}$ onto the open convex cone $\mathbb{R}^{m\uparrow}_{++}$:

$$L(\mathbb{R}^m_{++}) = \mathbb{R}^{m\uparrow}_{++} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : 0 < x_1 < x_2 < \cdots < x_m\}. \tag{2.2}$$

Let $\le$ be the standard partial ordering on $\mathbb{R}^m$ that arises from the closed cone $\mathbb{R}^m_+$:

$$\mathbf{x} \le \mathbf{y} \quad \Longleftrightarrow \quad \mathbf{y} - \mathbf{x} \in \mathbb{R}^m_+.$$

The closure of the open convex cone $\mathbb{R}^{m\uparrow}_{++}$ gives rise to an alternative partial ordering on $\mathbb{R}^m$:

$$\mathbf{x} \preceq \mathbf{y} \quad \Longleftrightarrow \quad \mathbf{y} - \mathbf{x} \in \overline{\mathbb{R}^{m\uparrow}_{++}},$$

that is,

$$\begin{aligned} \mathbf{x} \preceq \mathbf{y} \quad &\Longleftrightarrow \quad x_1 \le y_1,\ x_2 - x_1 \le y_2 - y_1,\ \ldots,\ x_m - x_{m-1} \le y_m - y_{m-1} \\ &\Longleftrightarrow \quad L^{-1}(\mathbf{x}) \le L^{-1}(\mathbf{y}). \end{aligned} \tag{2.3}$$

Then the lower triangular matrix $L\colon (\mathbb{R}^m, \le) \to (\mathbb{R}^m, \preceq)$ can be viewed as an order isomorphism between partially ordered spaces.

Recall the Riemannian metric δ on $\mathbb{R}^m_{++}$ given by $\delta(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{k=1}^m \log^2 \frac{x_k}{y_k}}$. Equipped with the complete metric on $\mathbb{R}^{m\uparrow}_{++}$ defined by

$$d(\mathbf{x}, \mathbf{y}) := \delta(L^{-1}(\mathbf{x}), L^{-1}(\mathbf{y})) = \sqrt{\log^2 \frac{x_1}{y_1} + \sum_{k=2}^m \log^2 \frac{x_k - x_{k-1}}{y_k - y_{k-1}}},$$

the restriction of L to the cone $\mathbb{R}^m_{++}$ is an isometry between complete metric spaces $(\mathbb{R}^m_{++}, \delta)$ and $(\mathbb{R}^{m\uparrow}_{++}, d)$. (The restricted metric δ on $\mathbb{R}^{m\uparrow}_{++}$ is not complete because it is not closed in $\mathbb{R}^m_{++}$.)

Next, consider the linear maps $M, H\colon \mathbb{R}^m \to \mathbb{S}_m$ defined for all $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ by

$$M(\mathbf{x}) = \begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_2 & x_2 & \cdots & x_2 \\ x_1 & x_2 & x_3 & \cdots & x_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & x_3 & \cdots & x_m \end{bmatrix}$$

and

$$H(\mathbf{x}) = \begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_1 + x_2 & x_1 + x_2 & \cdots & x_1 + x_2 \\ x_1 & x_1 + x_2 & x_1 + x_2 + x_3 & \cdots & x_1 + x_2 + x_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1 & x_1 + x_2 & x_1 + x_2 + x_3 & \cdots & x_1 + \cdots + x_m \end{bmatrix}.$$

Note that

$$H = M \circ L, \quad M = H \circ L^{-1}. \tag{2.4}$$

Remark 2.1. Two matrices $M(\mathbf{x})$ and $M(\mathbf{y})$ commute if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Similarly for $H(\mathbf{x})$ and $H(\mathbf{y})$. The proof is left for the reader.

The symmetric matrix $H(\mathbf{x})$ can be written as the following linear combination of rank one matrices:

$$H(\mathbf{x}) = \sum_{k=1}^{m} x_k \mathbf{w}_k \mathbf{w}_k^T, \tag{2.5}$$

where

$$\mathbf{w}_1 = \begin{bmatrix}1\\1\\1\\1\\\vdots\\1\end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix}0\\1\\1\\1\\\vdots\\1\end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix}0\\0\\1\\1\\\vdots\\1\end{bmatrix}, \quad \mathbf{w}_4 = \begin{bmatrix}0\\0\\0\\1\\\vdots\\1\end{bmatrix}, \quad \ldots, \quad \mathbf{w}_m = \begin{bmatrix}0\\0\\0\\0\\\vdots\\1\end{bmatrix}.$$

It is straightforward to see that

$$\det H(\mathbf{x}) = \prod_{k=1}^{m} x_k. \tag{2.6}$$

Let $\leq$ be the Löwner order on $\mathbb{S}_m$. $X \leq Y$ if $Y - X$ is positive semidefinite. Define $X < Y$ if $Y - X$ is positive definite.

Proposition 2.2. *The linear maps M and H are order preserving for $\preceq$ and $\leq$, respectively. Moreover $M(\mathbf{x})$ (resp. $H(\mathbf{x})$) is positive definite if and only if $\mathbf{x} \in \mathbb{R}^{m\uparrow}_{++}$ (resp. $\mathbf{x} \in \mathbb{R}^m_{++}$).*

Proof. Positive semidefiniteness of $\mathbf{w}_k \mathbf{w}_k^T$ and (2.5) ensure the $\leq$-monotonicity of H. This together with (2.3) yields the $\preceq$-monotonicity of M: for $\mathbf{x} \preceq \mathbf{y}$,

$$M(\mathbf{x}) = (H \circ L^{-1})(\mathbf{x}) = H(L^{-1}(\mathbf{x})) \leq H(L^{-1}(\mathbf{y})) = M(\mathbf{y}).$$

By (2.5) and (2.6), $H(\mathbf{x})$ is positive definite for every $\mathbf{x} \in \mathbb{R}^m_{++}$. Conversely, if $M(\mathbf{x})$ is positive definite, then $x_k > 0$ for all k by considering its 2×2 principal submatrices. For $M(\mathbf{x})$, we have that $M(\mathbf{x})$ $(= H(L^{-1}(\mathbf{x})))$ is positive definite if and only if $L^{-1}(\mathbf{x}) \in \mathbb{R}^m_{++}$ if and only if $\mathbf{x} \in \mathbb{R}^{m\uparrow}_{++}$. □

Denote by $\mathcal{M}$ the set of positive definite min matrices. Then

$$\mathcal{M} = \left\{M(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^{m\uparrow}_{++}\right\} = \left\{H(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m_{++}\right\} \subset \mathbb{P}_m. \tag{2.7}$$

The second equality follows from (2.2) and $H = M \circ L$. By the determinant formula (2.6), $\mathcal{M}$ is a closed subset of $\mathbb{P}_m$ and is closed under positive scalar multiplications (subcone). Its geodesic convexity, that is, closedness under geodesics, and an explicit formula for the Karcher mean of members of $\mathcal{M}$ are one of the main concerns of this paper. We deal with these problems in the next section where the order isomorphism L plays a key role in the proofs.

Next, we describe the set

$$\mathcal{M}^{-1} := \{A^{-1} : A \in \mathcal{M}\}.$$

For $\mathbf{x} = (x_1, \dots, x_m)$ with $x_k \neq 0$, $k = 1, \dots, m$, let $\mathbf{x}^{-1} = (x_1^{-1}, \dots, x_m^{-1})$. One can verify that

$$H(\mathbf{x}^{-1})^{-1} = \begin{bmatrix} x_1 + x_2 & -x_2 & 0 & 0 & \cdots\cdots & & & 0 \\ -x_2 & x_2 + x_3 & -x_3 & 0 & \cdots\cdots & & & 0 \\ 0 & -x_3 & x_3 + x_4 & -x_4 & \cdots\cdots & & & 0 \\ 0 & 0 & -x_4 & x_4 + x_5 & -x_5 & \cdots\cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -x_{m-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & -x_{m-1} & x_{m-1} + x_m & -x_m \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_m & x_m \end{bmatrix}$$

a special type of tridiagonal matrices. Decompose $H(\mathbf{x}^{-1})^{-1}$ as

$$H(\mathbf{x}^{-1})^{-1} = x_1 \mathbf{w}\mathbf{w}^T + \underbrace{\left(H(\mathbf{x}^{-1})^{-1} - x_1 \mathbf{w}\mathbf{w}^T\right)}_{:= T(x_2, \dots, x_m)}, \tag{2.8}$$

where $\mathbf{w} = (1, 0, \dots, 0)^T$. We note that the second term is independent of x_1 and the symmetric matrix $T(x_2, \dots, x_m)$ is defined over $(x_2, \dots, x_m) \in \mathbb{R}^{m-1}$. This induces a linear mapping

$$T \colon \mathbb{R}^{m-1} \to \mathbb{S}_m, \quad (x_1, \dots, x_{m-1}) \mapsto T(x_1, \dots, x_{m-1}), \tag{2.9}$$

which is injective. It is not difficult to see that for $(x_1, \dots, x_{m-1}) \in \mathbb{R}_{++}^{m-1}$, the tridiagonal matrix $T(x_1, \dots, x_{m-1})$ is a positive semidefinite tridiagonal matrix with

(T) positive diagonal entries, vanishing determinant, column sums, and strictly negative non-zero off-diagonal entries.

Conversely, such a matrix arises in this form. Denote by $\mathcal{T} = T(\mathbb{R}_{++}^{m-1})$ the set of $m \times m$ positive semidefinite tridiagonal matrices satisfying (T). Note that $\mathcal{T}$ is a convex subset of $\mathbb{S}_m$.

The diagonal vector of $T(x_1, \ldots, x_{m-1}) \in \mathcal{T}$,

$$(x_1, x_1 + x_2, x_2 + x_3, \ldots, x_{m-2} + x_{m-1}, x_{m-1}) \in \mathbb{R}^m_{++}$$

satisfies the following *alternating sum positive property.*

Definition 2.3. A vector $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_{++}$ is said to be *alternating sum positive* if

$$a_2 - a_1 > 0, \quad a_3 - a_2 + a_1 > 0, \quad \ldots, \quad a_{m-1} - a_{m-2} + \cdots + (-1)^m a_1 > 0$$

and

$$a_m - a_{m-1} + \cdots + (-1)^{m+1} a_1 = 0.$$

Conversely, for such a vector $(a_1, \ldots, a_m)$,

$$\begin{bmatrix} a_1 & -a_1 & 0 & 0 & \ldots. & 0 \\ -a_1 & a_2 & -(a_2 - a_1) & 0 & \ldots. & 0 \\ 0 & -(a_2 - a_1) & a_3 & -(a_3 - a_2 - a_1) & \ldots. & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & * \\ 0 & 0 & 0 & 0 & 0 \;\; * & a_m \end{bmatrix}$$

belongs to $\mathcal{T}$, where $* := -\sum_{k=1}^{m-1} (-1)^{k+1} a_{m-k} = -a_m$.

Proposition 2.4. *We have*

$$\mathcal{M}^{-1} = \mathbb{R}_{++}(\mathbf{w}\mathbf{w}^T) + \mathcal{T}.$$

Moreover, the diagonal vector map $\operatorname{diag}(\cdot)\colon \mathbb{S}_m \to \mathbb{R}^m$ *sends* $\mathcal{T}$ *bijectively onto the convex subset of all alternating sum positive vectors in* $\mathbb{R}^m$*.*

Proof. By (2.7) and (2.8), $\mathcal{M}^{-1} = \left\{ H(\mathbf{x}^{-1})^{-1} : \mathbf{x} \in \mathbb{R}^m_{++} \right\} \subset \mathbb{R}_{++}(\mathbf{w}_1\mathbf{w}_1^T) + \mathcal{T}$. The reverse inclusion and the remaining part of the proof are immediate from the construction of $\mathcal{T}$ and the alternating sum positive property. □

The set of alternating sum positive vectors and the convex subset $\mathcal{T}$ of the Euclidean space $\mathbb{S}_m$, which lies in the boundary of the open convex cone $\mathbb{P}_m$, are obtained from min matrices and seem to be new in the context of matrix analysis. We will briefly discuss the geometric mean of two matrices in $\mathcal{T}$ in the next section.

3. Proof of Theorem 1.1

We let $\operatorname{diag}(\mathbf{x})$ be the $m \times m$ diagonal matrix whose (i,i)-entry is x_i for $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$.

The Cholesky decomposition of the min matrix $M_{\min} = LL^T$ motivates the following LDL^T factorizations of $M(\mathbf{x})$ and $H(\mathbf{x})$, which, though straightforward, are crucial for our purpose:

$$\begin{aligned} M(\mathbf{x}) &= \begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_2 & x_2 & \cdots & x_2 \\ x_1 & x_2 & x_3 & \cdots & x_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & x_3 & \cdots & x_m \end{bmatrix} \\ &= L \operatorname{diag}(x_1, x_2 - x_1, x_3 - x_2, \ldots, x_m - x_{m-1})\, L^T \\ &= L \operatorname{diag}(L^{-1}(\mathbf{x}))\, L^T \end{aligned}$$

and

$$H(\mathbf{x}) = \begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_1 + x_2 & x_1 + x_2 & \cdots & x_1 + x_2 \\ x_1 & x_1 + x_2 & x_1 + x_2 + x_3 & \cdots & x_1 + x_2 + x_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1 & x_1 + x_2 & x_1 + x_2 + x_3 & \cdots & x_1 + \cdots + x_m \end{bmatrix} = L \operatorname{diag}(\mathbf{x})\, L^T. \tag{3.1}$$

The following result is immediate from (3.1) and invariancy of weighted geometric means under congruence transformations

$$C(A \#_t B) C^T = (CAC^T) \#_t (CBC^T), \quad C \in \mathrm{GL}(m, \mathbb{R}).$$

Proposition 3.1. *For* $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_{++}$ *and* $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}^m_{++}$,

$$H(\mathbf{a}) \#_t H(\mathbf{b}) = H(\mathbf{a}^{1-t} \mathbf{b}^t), \quad t \in \mathbb{R}, \tag{3.2}$$

where $\mathbf{ab} = (a_1 b_1, a_2 b_2, \ldots, a_m b_m)$ *and* $\mathbf{a}^t = (a_1^t, \ldots, a_m^t)$. *In particular,* $\mathcal{M}$ *is geodesically convex.*

This together with (2.4) leads to

$$M(\mathbf{a}) \#_t M(\mathbf{b}) = M\big(L\big(L^{-1}(\mathbf{a})^{1-t} L^{-1}(\mathbf{b})^t\big)\big).$$

Let $G\colon \mathbb{P}_m^N \to \mathbb{P}_m$ be a map satisfying the congruence invariance

$$G(CA_1C^T, \dots, CA_NC^T) = CG(A_1, \dots, A_N)C^T, \quad C \in \mathrm{GL}(m, \mathbb{R}),$$

and the consistency with scalars

$$G(A_1, \dots, A_N) = A_1^{1/N} \cdots A_N^{1/N}$$

for commuting A_i's. Then for $\mathbf{a}_j \in \mathbb{R}_{++}^m$, $j = 1, \dots, N$,

$$\begin{aligned} G(H(\mathbf{a}_1), \dots, H(\mathbf{a}_N)) &= G\big(L\,\mathrm{diag}(\mathbf{a}_1)L^T, \dots, L\,\mathrm{diag}(\mathbf{a}_N)L^T\big) \\ &= L\,G\big(\,\mathrm{diag}(\mathbf{a}_1), \dots, \mathrm{diag}(\mathbf{a}_N)\big)L^T \\ &= L\,\mathrm{diag}\,\big(\sqrt[N]{\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_N}\big)L^T \\ &= H\big(\,\mathrm{diag}\,\big(\sqrt[N]{\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_N}\big)\big). \end{aligned}$$

Applying to the Karcher mean, we have

Theorem 3.2. *For* $\mathbf{a}_k \in \mathbb{R}_{++}^m$, $k = 1, \dots, N$,

$$\Lambda(H(\mathbf{a}_1), \dots, H(\mathbf{a}_N)) = H\big(\sqrt[N]{\mathbf{a}_1\cdots\mathbf{a}_N}\big). \tag{3.3}$$

Proof of Theorem 1.1. Applying the preceding theorem with (2.4) and $\mathbf{a}_k = \big(\lambda_1^{(k)}, \lambda_2^{(k)}, \dots, \lambda_m^{(N)}\big)$, $1 \le k \le N$, yields that

$$\begin{aligned} \Lambda\big(M(\mathbf{a}_1), M(\mathbf{a}_2), \dots, M(\mathbf{a}_N)\big) &= \Lambda\big(H(L^{-1}(\mathbf{a}_1)), H(L^{-1}(\mathbf{a}_2)), \dots, H(L^{-1}(\mathbf{a}_N))\big) \\ &= H\big(\sqrt[N]{L^{-1}(\mathbf{a}_1)L^{-1}(\mathbf{a}_2)\cdots L^{-1}(\mathbf{a}_N)}\big) \\ &= M\big(L\big(\sqrt[N]{L^{-1}(\mathbf{a}_1)L^{-1}(\mathbf{a}_2)\cdots L^{-1}(\mathbf{a}_N)}\big)\big). \end{aligned}$$

The proof is completed from the fact that the vector $L^{-1}(\mathbf{a}_1)L^{-1}(\mathbf{a}_2)\cdots L^{-1}(\mathbf{a}_N)$ is equal to

$$\left(\prod_{k=1}^N \lambda_1^{(k)}, \prod_{k=1}^N (\lambda_2^{(k)} - \lambda_1^{(k)}), \dots, \prod_{k=1}^N (\lambda_m^{(k)} - \lambda_{m-1}^{(k)})\right). \qquad \square$$

Example 3.3. Let

$$A_k = \begin{bmatrix} k & k & k & \cdots & k \\ k & k+1 & k+1 & \cdots & k+1 \\ k & k+1 & k+2 & \cdots & k+2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k & k+1 & k+2 & \cdots & k+m-1 \end{bmatrix}, \quad k = 1, \dots, N.$$

The Karcher mean $\Lambda_N := \Lambda(A_1, \dots, A_N)$ is

$$\Lambda_N = \begin{bmatrix} \sqrt[N]{N!} & \sqrt[N]{N!} & \sqrt[N]{N!} & \cdots & \sqrt[N]{N!} \\ \sqrt[N]{N!} & \sqrt[N]{N!}+1 & \sqrt[N]{N!}+1 & \cdots & \sqrt[N]{N!}+1 \\ \sqrt[N]{N!} & \sqrt[N]{N!}+1 & \sqrt[N]{N!}+2 & \cdots & \sqrt[N]{N!}+2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sqrt[N]{N!} & \sqrt[N]{N!}+1 & \sqrt[N]{N!}+2 & \cdots & \sqrt[N]{N!}+m-1 \end{bmatrix} = \sqrt[N]{N!}\,\mathbf{w}_1\mathbf{w}_1^T + \begin{bmatrix} 0 & 0 \\ 0 & M_{\min} \end{bmatrix},$$

where $M_{\min}$ is the classical min matrix of size $m-1$. Since $\sqrt[N]{N!} < \sqrt[N+1]{(N+1)!}$, we have that

$$\Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_N \le \Lambda_{N+1} \le \cdots .$$

Recall the tridiagonal map $T\colon \mathbb{R}^{m-1}_{++} \to \mathcal{T} \subset \mathbb{S}_m$ from (2.9).

Corollary 3.4. *The map T is monotonic on $(\mathbb{R}^{m-1}_{++}, \le)$ and for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m-1}_{++}$,*

$$T(\mathbf{x})\#T(\mathbf{y}) \le T\big(\sqrt{\mathbf{x}\mathbf{y}}\big). \tag{3.4}$$

Proof. Let $\mathbf{x} := (x_1, \dots, x_{m-1}) \le \mathbf{y} := (y_1, \dots, y_{m-1})$ in $\mathbb{R}^{m-1}_{++}$. For every $\epsilon > 0$,

$$\mathbf{a} := (\epsilon, x_1, \dots, x_{m-1}) \le \mathbf{b} := (\epsilon, y_1, \dots, y_{m-1})$$

and hence $\mathbf{b}^{-1} \le \mathbf{a}^{-1}$. It follows from monotonicity of H (see Proposition 2.2) that $H(\mathbf{b}^{-1}) \le H(\mathbf{a}^{-1})$ and hence $H(\mathbf{a}^{-1})^{-1} \le H(\mathbf{b}^{-1})^{-1}$. By (2.8),

$$\epsilon\mathbf{w}\mathbf{w}^T + T(x_1, \dots, x_{m-1}) \le \epsilon\mathbf{w}\mathbf{w}^T + T(y_1, \dots, y_{m-1}).$$

By (2.8) and (3.2), and by monotonicity and self-duality properties of the geometric $A\#B$,

$$\begin{aligned} T(\mathbf{x})\#T(\mathbf{y}) &= \big(H(\mathbf{a}^{-1})^{-1} - \epsilon\mathbf{w}\mathbf{w}^T\big)\#\big(H(\mathbf{b}^{-1})^{-1} - \epsilon\mathbf{w}\mathbf{w}^T\big) \\ &\le H(\mathbf{a}^{-1})^{-1}\#H(\mathbf{b}^{-1})^{-1} = \big[H(\mathbf{a}^{-1})\#H(\mathbf{b}^{-1})\big]^{-1} \\ &= H\Big(\sqrt{\mathbf{a}\mathbf{b}^{-1}}\Big)^{-1} = T\big(\sqrt{\mathbf{x}\mathbf{y}}\big) + \epsilon\mathbf{w}\mathbf{w}^T. \end{aligned}$$

As $\epsilon \to 0^+$, we have the desired conclusion. □

Remark 3.5. The diagonal embedding $D\colon \mathbb{R}^m_{++} \to \mathbb{P}_m$, $D(x_1, \dots, x_m) = \mathrm{diag}(x_1, \dots, x_m)$ satisfies (3.3). To the best of our knowledge, the map H (resp. T) appears firstly among such maps (resp. maps satisfying (3.4)) on $\mathbb{R}^m_{++}$ with *noncommuting* values (see Remark 2.1). We further note that (3.3) holds true for every matrix geometric mean (see [1] for matrix geometric means).

4. Power min matrices

In this section, we consider the *power min matrix* whose (i,j) entry is $x^{\min\{i,j\}}$, $x \geq 0$:

$$P_x := [x^{\min\{i,j\}}]_{m\times m} = \begin{bmatrix} x & x & x & \cdots & x \\ x & x^2 & x^2 & \cdots & x^2 \\ x & x^2 & x^3 & \cdots & x^3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x & x^2 & x^3 & \cdots & x^m \end{bmatrix}.$$

This defines a one-parameter family of symmetric matrices

$$P\colon [0,\infty) \to \mathbb{S}_m, \quad x \mapsto P_x.$$

It is continuous (and analytic on $x > 0$), starting at the zero matrix (when $x = 0$) and passing the positive semidefinite matrix (when $x = 1$):

$$P_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T.$$

By (1.1),

$$\det P_x = (x-1)^{m-1} x^{(m^2-m+2)/2}$$

and hence P_x is invertible if and only if $x \neq 0, 1$. Moreover, $\det P_x > 0$ if and only if $x > 1$.

If $x > 1$, then P_x is a generalized min matrix because $x^{\min\{i,j\}} = \min\{x^i, x^j\}$:

$$P_x = M(x, x^2, \ldots, x^m), \quad x > 1,$$

and hence P_x is positive definite. If $0 < x < 1$, then $x^{\min\{i,j\}} = \max\{x^i, x^j\}$ and hence P_x can be viewed as a generalized *max* matrix with strictly negative determinant. However, the generalized min matrix of $x^m < x^{m-1} < \cdots < x$ is

$$M(x^m, x^{m-1}, \ldots, x) = \begin{bmatrix} x^m & x^m & x^m & \cdots & x^m \\ x^m & x^{m-1} & x^{m-1} & \cdots & x^{m-1} \\ x^m & x^{m-1} & x^{m-2} & \cdots & x^{m-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x^m & x^{m-1} & x^{m-2} & \cdots & x \end{bmatrix}$$

which is positive definite.

By the determinant formula (1.1), we have

Proposition 4.1 (Positivity). *$P_x \geq 0$ if and only if $x \geq 1$, and $P_x > 0$ if and only if $x > 1$.*

Note that $P_x = xS_x$, where

$$S_x := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & x & x & \cdots & x \\ 1 & x & x^2 & \cdots & x^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x & x^2 & \cdots & x^{m-1} \end{bmatrix}.$$

The symmetric matrix S_x is invertible for $x \neq 1$ and is positive definite for $x > 1$. Moreover

$$\det S_x = x^{-m} \det(P_x) = (x-1)^{m-1} x^{(m-1)(m-2)/2}.$$

The curves $P, S \colon [0,\infty) \to \mathbb{S}_m$ lie in the convex cone of positive definite (resp. semidefinite) matrices for $x \in (1,\infty)$ (resp. $[1,\infty)$).

Proposition 4.2. *The maps P, S are monotonic increasing on $[1,\infty)$.*

Proof. Let $1 \leq x \leq y$. Then

$$(x, x^2, \dots, x^m) \preceq (y, y^2, \dots, y^m) \quad \text{and} \quad (1, x, \dots, x^{m-1}) \preceq (1, y, \dots, y^{m-1}).$$

Monotonic property of M (see Proposition 2.2) ensures

$$P_x = M(x, x^2, \dots, x^m) \leq M(y, y^2, \dots, y^m) = P_y.$$

Similarly, $S_x \leq S_y$. □

The following expression of S_x is direct:

$$S_x = H\big(1, x-1, x(x-1), x^2(x-1), \dots, x^{m-2}(x-1)\big).$$

We then have from (3.2) that for $x, y > 1$,

$$S_x \# S_y = H\Big(1, \sqrt{(x-1)(y-1)}, \sqrt{xy(x-1)(y-1)}, \dots, \sqrt{x^{m-2}y^{m-2}(x-1)(y-1)}\Big),$$

and

$$\begin{aligned} & P_x \# P_y \\ &= (xS_x)\#(yS_y) = \sqrt{xy} S_x \# S_y \\ &= \sqrt{xy} H\Big(1, \sqrt{(x-1)(y-1)}, \sqrt{xy(x-1)(y-1)}, \dots, \sqrt{x^{m-2}y^{m-2}(x-1)(y-1)}\Big) \\ &= H\Big(\sqrt{xy}, \sqrt{xy(x-1)(y-1)}, \sqrt{x^2y^2(x-1)(y-1)}, \dots, \sqrt{x^{m-1}y^{m-1}(x-1)(y-1)}\Big). \end{aligned}$$

The sets $\mathcal{P} := \{P_x\}_{x>1}$ and $\mathcal{S} := \{S_x\}_{x>1}$ are closed in $\mathbb{P}_m$ but not geodesically convex, in general. For $m = 2$,

$$\begin{aligned} S_x \#_t S_y &= H(1, x-1) \#_t H(1, y-1) = H(1, (x-1)^{1-t}(y-1)^t) \\ &= S_{1+(x-1)^{1-t}(y-1)^t} \in \mathcal{S}, \end{aligned} \tag{4.1}$$

and hence $\mathcal{S}$ is geodesically convex in $\mathbb{P}_2$. But for $x \neq y$ and $0 < t < 1$,

$$P_x \#_t P_y = x^{1-t} y^t (S_x \#_t S_y) = x^{1-t} y^t \begin{bmatrix} 1 & 1 \\ 1 & 1 + (x-1)^{1-t}(y-1)^t \end{bmatrix} \notin \mathcal{P}. \tag{4.2}$$

For $m > 2$,

$$S_x \# S_y \in \mathcal{S} \iff x = y,$$

which follows from the proceeding formula on $S_x \# S_y$:

$$\left(1 + \sqrt{(x-1)(y-1)}\right)^2 = 1 + \sqrt{(x-1)(y-1)} + \sqrt{xy(x-1)(y-1)}$$

implies that $x = y$.

We close this section with an alternative closed form for geometric means of 2×2 power min matrices.

Proposition 4.3. *Let G be a matrix geometric mean and let $m = 2$. Then for $1 < x_1 \leq x_2 \leq \cdots \leq x_N$ with $x_1 \neq x_N$,*

$$G(P_{x_1}, \dots, P_{x_N}) = \sqrt[N]{\prod_{k=1}^N x_k} \begin{bmatrix} 1 & 1 \\ 1 & 1 + (x_1 - 1)^{1-t}(x_N - 1)^t \end{bmatrix},$$

where

$$t = \frac{1}{N} \sum_{k=1}^N \frac{\log(x_k - 1) - \log(x_1 - 1)}{\log(x_N - 1) - \log(x_1 - 1)}.$$

Proof. Set

$$t_k = \frac{\log(x_k - 1) - \log(x_1 - 1)}{\log(x_N - 1) - \log(x_1 - 1)}, \quad k = 2, \dots, N-1.$$

Then $x_k - 1 = (x_1 - 1)^{1-t_k}(x_N - 1)^{t_k}$ and by (4.1),

$$S_{x_1} \#_{t_k} S_{x_N} = S_{1+(x_1-1)^{1-t_k}(x_N-1)^{t_k}} = S_{x_k}.$$

We note that

$$G(A \#_{s_1} B, \dots, A \#_{s_N} B) = A \#_{\frac{1}{N}\sum_{k=1}^N s_k} B$$

for every geometric mean G, and hence

$$G(S_{x_1}, \dots, S_{x_N}) = G(S_{x_1} \#_{t_1} S_{x_N}, \dots, S_{x_1} \#_{t_N} S_{x_N}) = S_{x_1} \#_{\frac{1}{N}\sum_{k=1}^N t_k} S_{x_N} = S_{x_1} \#_t S_{x_N}.$$

By joint homogeneity of G,

$$\begin{aligned} G(P_{x_1},\dots,P_{x_N}) &= G(x_1S_{x_1},\dots,x_NS_{x_N}) \\ &= \sqrt[N]{\prod_{k=1}^{N} x_k}\, G(S_{x_1},\dots,S_{x_N}) = \sqrt[N]{\prod_{k=1}^{N} x_k}\,(S_{x_1}\#_t S_{x_N}) \\ &= \sqrt[N]{\prod_{k=1}^{N} x_k}\,(x_1^{-1}P_{x_1})\#_t(x_N^{-1}P_{x_N}) = \sqrt[N]{\prod_{k=1}^{N} x_k}\, x_1^{t-1}x_N^{-t}(P_{x_1}\#_t P_{x_N}). \end{aligned}$$

This together with (4.2) completes the proof. □

By permutation invariance of geometric means, it holds the following.

Corollary 4.4. *Let G be a matrix geometric mean and let $m = 2$. Then for $1 < x_1, x_2, \dots, x_N$ with $x_{\min} := \min\{x_k\} \neq x_{\max} := \max\{x_k\}$,*

$$G(P_{x_1},\dots,P_{x_N}) = \sqrt[N]{\prod_{k=1}^{N} x_k}\begin{bmatrix} 1 & 1 \\ 1 & 1+(x_{\min}-1)^{1-t}(x_{\max}-1)^t \end{bmatrix},$$

where

$$t = \frac{1}{N}\sum_{k=1}^{N}\frac{\log(x_k-1)-\log(x_{\min}-1)}{\log(x_{\max}-1)-\log(x_{\min}-1)}. \tag{4.3}$$

In terms of the Riemannian distance $\delta(x,y) = |\log x - \log y|$ on positive reals $\mathbb{R}_{++}$, the value t in (4.3) becomes

$$t = \frac{1}{N}\sum_{k=1}^{N}\frac{\delta(x_k-1,x_{\min}-1)}{\delta(x_{\max}-1,x_{\min}-1)}.$$

This gives rise to a symmetric (permutation invariant) function from $\mathbb{R}_{++}^{N\uparrow}$ to $(0,1)$:

$$(x_1,\dots,x_N) \mapsto \frac{1}{N}\sum_{k=1}^{N}\frac{\delta(x_k,x_1)}{\delta(x_N,x_1)}, \tag{4.4}$$

which is invariant under scalar multiplications $\alpha\mathbf{x} = (\alpha x_1,\dots,\alpha x_N)$. Moreover,

$$(x_1,\dots,x_N)^* := (x_N^{-1},\dots,x_1^{-1}) \mapsto \frac{1}{N}\sum_{k=1}^{N}\frac{\delta(x_k,x_N)}{\delta(x_N,x_1)}.$$

It is of interest to have a geometric interpretation of the map (4.4). Its Euclidean version is

$$(x_1,\dots,x_N) \mapsto \frac{1}{N}\sum_{k=1}^{N}\frac{|x_k-x_1|}{x_N-x_1}$$

for $-\infty < x_1 < x_k < x_N < \infty$, $k = 2,\dots,N-1$.

5. Final remarks

A variant of the classical Cholesky factorization of positive definite matrices is known as the LDL^T factorization (square-root-free Cholesky factorization)

$$A = LDL^T,$$

where L is a lower unit triangular matrix (the main diagonal entries are all 1) and D is diagonal with positive diagonal elements. This gives a smooth diffeomorphism from $\mathcal{L}\times\mathcal{D}$ onto the Riemannian manifold $\mathbb{P}_m$, where $\mathcal{L} \equiv \mathbb{R}^{m(m-1)/2}$ and $\mathcal{D} \equiv \mathbb{R}^m_{++}$ denote the spaces of lower unit triangular matrices and positive diagonal matrices, respectively. For $L \in \mathcal{L}$, denote by $\mathcal{D}_L$ the L-orbit or L-leaf

$$\mathcal{D}_L := \{LDL^T : D \in \mathcal{D}\}.$$

For instance, $\mathcal{D}_I = \mathcal{D}$. By the congruence invariance of matrix geometric means G,

$$G(A_1,\dots,A_N) = LG(D_1,\dots,D_N)L^T = L\big(\sqrt[N]{D_1\cdots D_N}\big)L^T$$

for $A_k = LD_kL^T \in \mathcal{D}_L$, $k = 1,\dots,N$. The space $\mathcal{M}$ of min matrices coincides with the leaf $\mathcal{D}_{L_{\mathbf{1}}}$ where $L_{\mathbf{1}}$ is the lower triangular matrix in (2.1). It is natural and of interest to describe other LDL^T leaves like the geodesic submanifolds of positive diagonal matrices ($L = I = L_{\mathbf{0}}$) and positive definite min matrices ($L = L_{\mathbf{1}}$).

The problem of finding a closed form for the Karcher mean of *three matrices* is left open even for the 2×2 case, in which case it can be reduced to

$$\Lambda\big(I, \operatorname{diag}(x, x^{-1}), L_t \operatorname{diag}(y, y^{-1})L_t^T\big)$$

by the LDL^T factorization, where $L_t = \left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right]$ and $t \in \mathbb{R}$. By analyticity of the Karcher mean in each variable [11], we obtain an analytic function $l\colon \mathbb{R}^2_{++}\times\mathbb{R} \to \mathbb{R}$ which is uniquely defined by the LDL^T foliation

$$s = l(x,y,t) \quad\Longleftrightarrow\quad \Lambda\big(I, \operatorname{diag}(x, x^{-1}), L_t \operatorname{diag}(y, y^{-1})L_t^T\big) \in \mathcal{D}_{L_s}.$$

Finding an explicit form of l is an another attractive problem in the theory of matrix means.

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