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A Rigidity Property of the Sum of Weighted Differentiation-composition Operators

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Abstract. The single weighted differentiation-composition operator $W_{u,\varphi}^{(k)}$ on various spaces of analytic functions has been investigated for decades, i.e., $W_{u,\varphi}^{(k)} f = u \cdot f^{(k)} \circ \varphi$. However, the study of the finite sum of weighted differentiation-composition operators with different orders are far from complete, i.e.,

$$T_{\mathbf{u},\varphi}^{(n)} f = u_0 \cdot f \circ \varphi + u_1 \cdot f' \circ \varphi + \cdots + u_n \cdot f^{(n)} \circ \varphi.$$

In this paper, we completely solve the problem about boundedness and compactness of the operator $T_{\mathbf{u},\varphi}^{(n)}$ between two Bergman spaces over the upper half-plane. We show a rigidity property of $T_{\mathbf{u},\varphi}^{(n)}$. Specifically, the boundedness and compactness of the sum $T_{\mathbf{u},\varphi}^{(n)}$ is equivalent to those of each $W_{u_k,\varphi}^{(k)}$, $0 \le k \le n$.

1. Introduction

Let $\Pi^+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ be the upper half-plane in $\mathbb{C}$ and $H(\Pi^+)$ be the space of all analytic functions on Π^+. For $0 < p < \infty$, the classical Bergman space over Π^+ is defined by

$$A^p(\Pi^+) = \left\{ f \in H(\Pi^+) : \|f\|_p^p = \int_{\Pi^+} |f(z)|^p \, dA(z) < \infty \right\},$$

where dA is the Lebesgue area measure on Π^+.

Let $S(\Pi^+)$ be the set of all analytic self-maps of Π^+. For given $\varphi \in S(\Pi^+)$ and $u \in H(\Pi^+)$, the weighted composition operator $W_{u,\varphi}$ on $H(\Pi^+)$ is defined by

$$W_{u,\varphi} f = u \cdot f \circ \varphi,$$

which plays an important role in the isometry theory of analytic function spaces. When $u = 1$, it reduces to the composition operator C_φ and when $\varphi(z) = z$, it reduces to the

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multiplication operator M_u. The relationship between the operator-theoretic properties of $W_{u,\varphi}$ and the function-theoretic properties of u and φ has been studied extensively during the past several decades. We refer to the standard references [6, 16] for the theory of composition operators on various analytic function spaces. For the investigations of $W_{u,\varphi}$ on Bergman spaces over the upper half-plane, one can refer to [2, 3, 8–10, 15, 19] and references therein.

Let $Df = f'$ be the differentiation operator on $H(\Pi^+)$ and D^n be the nth iterate of D. Note that D is typically unbounded on many familiar spaces of analytic functions. The differential operator plays an important role in various fields such as dynamical system theory and operator theory. The product of D^n and $W_{u,\varphi}$, i.e., $W_{u,\varphi}D^n f = u \cdot f^{(n)} \circ \varphi$, named the generalized weighted composition operator or weighted differentiation-composition operator, was studied by Zhu in [25, 26] on classical weighted Bergman spaces.

There are six different products of C_φ, M_u and D: $M_uC_\varphi D$, $C_\varphi M_u D$, $M_u DC_\varphi$, DM_uC_φ, $C_\varphi DM_u$ and $DC_\varphi M_u$. Recently, there has been a great interest in studying these product-type operators between various analytic function spaces. In order to treat these product-type operators in a unified manner, Stević et al. [17, 18] introduced the so-called Stević–Sharma operator, that is,

$$T_{u_1,u_2,\varphi} f = u_1 \cdot f \circ \varphi + u_2 \cdot f' \circ \varphi.$$

Under some additional conditions, Stević et al. characterized the boundedness and compactness of the operator $T_{u_1,u_2,\varphi}$ on weighted Bergman spaces. Liu et al. [11,22,23] studied the operator $T_{u_1,u_2,\varphi}$ from several specific analytic function spaces to the Bloch-type spaces and weighted-type spaces. Wang et al. [20] characterized the boundedness, compactness and order boundedness of the difference of two Stević–Sharma operators $T_{u_1,v_1,\varphi_1} - T_{u_2,v_2,\varphi_2}$ between some Banach spaces of analytic functions.

The Stević–Sharma operator is the special case $n = 1$ of the following sum operator

$$T^{(n)}_{\mathbf{u},\varphi} = W^{(0)}_{u_0,\varphi} + W^{(1)}_{u_1,\varphi} + \cdots + W^{(n)}_{u_n,\varphi},$$

where $\mathbf{u} = (u_0, u_1, \ldots, u_n)$ with $u_i \in H(\Pi^+)$ and $W^{(i)}_{u_i,\varphi} = W_{u_i,\varphi} D^i$ for $i = 0, 1, \ldots, n$. Recently, the boundedness and compactness of this sum operator from the bounded analytic function space H^∞, Hardy space and weighted Bergman space to the Bloch-type space or kth weighted-type space were characterized in [21], one can also see [1, 13].

As far as we know, in the study about the sum operator $T^{(n)}_{\mathbf{u},\varphi} \colon X \to Y$, where X, Y are two Banach spaces of analytic functions, the space Y is always restricted to the weighted-type spaces or growth-type spaces. In this paper, we are interested in the boundedness and compactness of the sum operator $T^{(n)}_{\mathbf{u},\varphi}$ between two different Bergman spaces over the upper half-plane, which has never been considered so far. We show a rigidity property of the operator $T^{(n)}_{\mathbf{u},\varphi}$.

Our main results read as follows.

Theorem 1.1. *If $0 < p \le q < \infty$, then the operator $T^{(n)}_{\mathbf{u},\varphi}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ is bounded (resp. compact) if and only if each weighted differentiation-composition operator $W^{(k)}_{u_k,\varphi}\colon A^p(\Pi^+) \to A^q(\Pi^+)$, $0 \le k \le n$, is bounded (resp. compact).*

Theorem 1.2. *If $0 < q < p < \infty$, then the following conditions are equivalent:*

(i) $T^{(n)}_{\mathbf{u},\varphi}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ *is bounded;*

(ii) $T^{(n)}_{\mathbf{u},\varphi}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ *is compact;*

(iii) $W^{(k)}_{u_k,\varphi}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ *is bounded for each* $0 \le k \le n$*;*

(iv) $W^{(k)}_{u_k,\varphi}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ *is compact for each* $0 \le k \le n$*.*

The paper is organized as follows. In Section 2, we collect some well-known facts and prove a key lemma that will be needed later. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

Throughout the paper, we write $A \lesssim B$ or $B \gtrsim A$ if there exists an absolute constant $C > 0$ such that $A \le CB$. As usual, $A \asymp B$ means both $A \lesssim B$ and $B \lesssim A$. We will be more specific if the dependence of such constants on certain parameters becomes critical.

2. Preliminaries

In this section, we collect some preliminary facts and auxiliary lemmas which will be used later. The pseudo-hyperbolic metric on Π^+ is defined by

$$\rho(z,w) = \left|\frac{w-z}{\overline{w}-z}\right|, \quad z, w \in \Pi^+.$$

For any $z \in \Pi^+$ and $0 < r < 1$, we write $E_r(z) = \{w \in \Pi^+ : \rho(z,w) < r\}$ for the pseudo-hyperbolic metric disk centered at z with radius r. It is known that

$$(\mathrm{Im}\, z)^2 \asymp (\mathrm{Im}\, w)^2 \asymp |\overline{w} - z|^2 \asymp |E_r(z)| \tag{2.1}$$

for all $z \in \Pi^+$ and $w \in E_r(z)$. Here $|E_r(z)|$ denotes the Lebesgue area of $E_r(z)$.

We start our discussions with the following pointwise estimate for functions in Bergman spaces over the upper half-plane. Similar results can be found in [2, (3.8)] and [14, (2.3)]. For completeness, we include a brief proof.

Lemma 2.1. *Let k be a nonnegative integer, $0 < p < \infty$ and $0 < r < 1$. Then*

$$|f^{(k)}(z)|^p \lesssim \frac{1}{(\mathrm{Im}\, z)^{2+kp}} \int_{E_r(z)} |f(w)|^p \, dA(w)$$

for all $f \in H(\Pi^+)$ *and* $z \in \Pi^+$*. As a consequence,*

$$|f^{(k)}(z)|^p \lesssim \frac{\|f\|_p^p}{(\operatorname{Im} z)^{2+pk}}$$

for all $f \in A^p(\Pi^+)$ *and* $z \in \Pi^+$*.*

Proof. Let $f \in H(\Pi^+)$, we use the sub-mean value inequality to get

$$|f(w)|^p \lesssim \frac{1}{(\operatorname{Im} w)^2} \int_{E_{r/2}(w)} |f(\zeta)|^p \, dA(\zeta)$$

for any $w \in \Pi^+$. Choose $0 < t < r/4$. Note that for $z \in \Pi^+$, we have

$$2 \operatorname{Im} z = |z - \overline{z}| \leq |z - w| + |\overline{w} - z| \leq 2|\overline{w} - z|.$$

A direct calculation shows that $\rho(z, w) < r/2$ whenever $|w - z| < 2t \operatorname{Im} z$. Hence, by Cauchy's estimates and (2.1), we have

$$\begin{aligned} |f^{(k)}(z)|^p &\lesssim \frac{1}{(\operatorname{Im} z)^{kp}} \sup_{|w-z|=2t \operatorname{Im} z} |f(w)| \leq \frac{1}{(\operatorname{Im} z)^{kp}} \sup_{w \in E_{r/2}(z)} |f(w)| \\ &\lesssim \frac{1}{(\operatorname{Im} z)^{kp+2}} \sup_{w \in E_{r/2}(z)} \int_{E_{r/2}(w)} |f(\zeta)|^p \, dA(\zeta) \\ &\leq \frac{1}{(\operatorname{Im} z)^{kp+2}} \int_{E_r(z)} |f(\zeta)|^p \, dA(\zeta). \end{aligned}$$

The proof is complete. □

A sequence $\{a_j\}$ in Π^+ is called an r-lattice in the pseudo-hyperbolic metric if

$$\bigcup_{j=1}^{\infty} E_r(a_j) = \Pi^+ \text{ and } \{E_{r/2}(a_j)\}_{j=1}^{\infty} \text{ are pairwise disjoint.}$$

For any $0 < r < 1$, the r-lattice can be explicitly constructed through the same arguments as [24, Lemma 4.8]. And by [5, Lemma 4.2], we know that if $\{a_j\}$ is an r-lattice with $0 < r < 2/5$, then there exists a positive integer $N = N_r$ such that every point in Π^+ belongs to at most N of the sets $E_{2r}(a_j)$. The following lemma is given by Pang and Wang in [12, Theorem 2.8], which is a part of the atomic decomposition of $A^p(\Pi^+)$.

Lemma 2.2. *Suppose* $0 < p < \infty$ *and* $\{a_j\}$ *is an* r*-lattice in the pseudo-hyperbolic metric. Define an operator* S *on complex sequences by*

$$S(\{\lambda_j\}) = \sum_j \lambda_j \frac{(\operatorname{Im} a_j)^{M-2/p}}{(\overline{a}_j - z)^M},$$

where $M > \max\{1, 1/p\} + 1/p$*. Then* S *is bounded from* l^p *to* $A^p(\Pi^+)$*.*

Let $\widehat{\Pi^+} = \overline{\Pi^+} \cup \{\infty\}$. In the sequel, the statement "$z \to \partial\widehat{\Pi^+}$" means that "$\operatorname{Im} z \to 0^+$" or "$|z| \to \infty$". For each $w \in \Pi^+$, let $K^{[i]}_{w,\gamma}(z) = \frac{1}{(\overline{w}-z)^{\gamma+i}}$, where $\gamma > 2/p+1$ is sufficiently large. Set $k^{[i]}_{p,w,\gamma} := K^{[i]}_{w,\gamma}/\|K^{[i]}_{w,\gamma}\|_p$. Then we have the following estimates for the norm of $K^{[i]}_{w,\gamma}$.

Lemma 2.3. [12, Lemma 2.2] *Let $i \in \mathbb{N}$. Then*

$$\|K^{[i]}_{w,\gamma}\|_p \asymp \frac{1}{(\operatorname{Im} w)^{\gamma+i-2/p}}$$

for all $w \in \Pi^+$. Moreover, $k^{[i]}_{p,w,\gamma}$ converges to 0 as $w \to \partial\widehat{\Pi^+}$ uniformly on compact subsets of Π^+.

Recall that a positive Borel measure μ on Π^+ is called a (k,p,q)-Carleson measure for $A^p(\Pi^+)$ if there exists a constant $C > 0$ such that

$$\int_{\Pi^+} |f^{(k)}(z)|^q \, d\mu(z) \le C\|f\|_p^q$$

for all $f \in A^p(\Pi^+)$. Similarly, μ is called a vanishing (k,p,q)-Carleson measure if

$$\lim_{j\to\infty} \int_{\Pi^+} |f_j^{(k)}(z)|^q \, d\mu(z) = 0$$

whenever $\{f_j\}$ is a bounded sequence in $A^p(\Pi^+)$ that converges to 0 uniformly on compact subsets of Π^+.

We end this section with the geometric characterization for (k,p,q)-Carleson measures, which is a generalization of $(0,p,q)$-Carleson measures in [12, p. 2302] (see also [4, Theorem 5.4]). For the sake of completeness, we provide the proof in detail. Before that, we make a short review of the classical Khinchine's inequality, which is an important tool in complex and functional analysis. An introduction to this topic can be found in Appendix A of [7].

The Rademacher function $r_k\colon [0,1] \to [-1,1]$ are defined as

$$r_k(t) = \operatorname{sgn}(\sin(2^k \pi t)).$$

If $0 < p < \infty$, the Khinchine's inequality gives that

$$\left(\sum_k |c_k|^2\right)^{p/2} \asymp \int_0^1 \left|\sum_k c_k r_k(t)\right|^p dt$$

for any complex sequence $\{c_k\}$.

Lemma 2.4. *Let $0 < p, q < \infty$ and μ be a positive Borel measure on Π^+.*

(i) *If* $p \le q$, *then* μ *is a* (k,p,q)*-Carleson measure if and only if*

$$\sup_{a\in\Pi^+} \frac{\mu(E_r(a))^{1/q}}{(\operatorname{Im} a)^{k+2/p}} < \infty \tag{2.2}$$

for some (or all) $0 < r < 1$, *and* μ *is a vanishing* (k,p,q)*-Carleson measure if and only if*

$$\lim_{a\to\widehat{\partial\Pi^+}} \frac{\mu(E_r(a))^{1/q}}{(\operatorname{Im} a)^{k+2/p}} = 0 \tag{2.3}$$

for some (or all) $0 < r < 1$.

(ii) *If* $q < p$, *then* μ *is a* (k,p,q)*-Carleson measure if and only if* μ *is a vanishing* (k,p,q)*-Carleson measure if and only if*

$$\frac{\mu(E_r(z))}{(\operatorname{Im} z)^{2+kq}} \in L^{p/(p-q)}$$

for some $0 < r < 2/5$.

Proof. (i) Firstly, assume μ is a (k,p,q)-Carleson measure, then by (2.1) and Lemma 2.3, we have

$$\begin{aligned} \frac{\mu(E_r(a))}{(\operatorname{Im} a)^{(k+2/p)q}} &= \int_{E_r(a)} \frac{1}{(\operatorname{Im} a)^{(k+2/p)q}}\, d\mu(z) \asymp \int_{E_r(a)} \left|(k^{[0]}_{p,a,\gamma})^{(k)}(z)\right|^q d\mu(z) \\ &\lesssim \int_{\Pi^+} \left|(k^{[0]}_{p,a,\gamma})^{(k)}(z)\right|^q d\mu(z) \lesssim \left(\int_{\Pi^+} \left|(k^{[0]}_{p,a,\gamma})(z)\right|^p dA(z)\right)^{q/p} \lesssim 1. \end{aligned} \tag{2.4}$$

This is exactly (2.2).

Conversely, assume (2.2) holds, then for any $f \in A^p(\Pi^+)$, according to Lemma 2.1 and Fubini's theorem, we get

$$\begin{aligned} \int_{\Pi^+} |f^{(k)}(z)|^q\, d\mu(z) &\lesssim \int_{\Pi^+} \left(\frac{1}{(\operatorname{Im} z)^{2+kp}} \int_{E_r(z)} |f(w)|^p\, dA(w)\right) |f^{(k)}(z)|^{q-p}\, d\mu(z) \\ &\lesssim \|f\|_p^{q-p} \int_{\Pi^+} \left(\frac{1}{(\operatorname{Im} z)^{(2/p+k)q}} \int_{E_r(z)} |f(w)|^p\, dA(w)\right) d\mu(z) \\ &\lesssim \|f\|_p^{q-p} \int_{\Pi^+} |f(w)|^p\, dA(w) \int_{E_r(w)} \frac{1}{(\operatorname{Im} z)^{(2/p+k)q}}\, d\mu(z) \\ &\lesssim \|f\|_p^{q-p} \int_{\Pi^+} |f(w)|^p \frac{\mu(E_r(w))}{(\operatorname{Im} w)^{(2/p+k)q}}\, dA(w) \\ &\lesssim \|f\|_p^q. \end{aligned} \tag{2.5}$$

This shows that μ is a (k,p,q)-Carleson measure.

Now assume μ is a vanishing (k,p,q)-Carleson measure, then for any fixed $r \in (0,1)$, the procedure in (2.4) yields

$$\lim_{a\to\partial\widehat{\Pi^+}} \frac{\mu(E_r(a))}{(\operatorname{Im} a)^{(k+2/p)q}} \lesssim \lim_{a\to\partial\widehat{\Pi^+}} \int_{\Pi^+} \left|(k^{[0]}_{p,a,\gamma})^{(k)}(z)\right|^q d\mu(z) = 0,$$

since $k^{[0]}_{p,a,\gamma}$ converges to 0 uniformly on compact subsets of Π^+ as $a \to \partial\widehat{\Pi^+}$. This is exactly (2.3).

Conversely, assume (2.3) holds, then for any $\varepsilon > 0$, there exists a compact subset $K \subset \Pi^+$ such that

$$\sup_{z\in\Pi^+\setminus K} \frac{\mu(E_r(z))^{1/q}}{(\operatorname{Im} z)^{k+2/p}} < \varepsilon.$$

Let $\{f_j\}$ be any sequence in $A^p(\Pi^+)$ that converges to 0 uniformly on compact subsets of Π^+ and $\|f_j\|_p \le M$ for all j. Proceeding as in (2.5) yields

$$\begin{aligned}\int_{\Pi^+} |f_j^{(k)}(z)|^q \, d\mu(z) &\lesssim \|f_j\|_p^{q-p} \int_{\Pi^+} |f_j(w)|^p \frac{\mu(E_r(w))}{(\operatorname{Im} w)^{(2/p+k)q}} \, dA(w) \\ &\lesssim M^{q-p} \left(\int_K + \int_{\Pi^+\setminus K} \right) |f_j(w)|^p \frac{\mu(E_r(w))}{(\operatorname{Im} w)^{(2/p+k)q}} \, dA(w).\end{aligned}$$

It is clear that

$$\lim_{j\to\infty} \int_K |f_j(w)|^p \frac{\mu(E_r(w))}{(\operatorname{Im} w)^{(2/p+k)q}} \, dA(w) = 0$$

and

$$\int_{\Pi^+\setminus K} |f_j(w)|^p \frac{\mu(E_r(w))}{(\operatorname{Im} w)^{(2/p+k)q}} \, dA(w) \le \varepsilon^q \|f_j\|_p^p \le \varepsilon^q M^p.$$

Hence,

$$\limsup_{j\to\infty} \int_{\Pi^+} |f_j^{(k)}(z)|^q \, d\mu(z) \lesssim \varepsilon^q M^q.$$

It follows immediately from the arbitrariness of ε that μ is a vanishing (k,p,q)-Carleson measure.

(ii) In the case $0 < q < p < \infty$, it is trivial that μ is a (k,p,q)-Carleson measure if μ is a vanishing (k,p,q)-Carleson measure. Now we prove that μ is a vanishing (k,p,q)-Carleson measure if $\frac{\mu(E_r(z))}{(\operatorname{Im} z)^{2+kq}} \in L^{p/(p-q)}$.

For $0 < r < 2/5$ and $z \in \Pi^+$, it is known that $E_r(z)$ is a Euclidean disk centered at $\operatorname{Re} z + i\frac{1+r^2}{1-r^2} \operatorname{Im} z$ with radius $\frac{2r}{1-r^2} \operatorname{Im} z$. Let $0 < t < \infty$. If $|z| > t$ and $w \in E_r(z)$, then

$$|w| > \sqrt{(\operatorname{Re} z)^2 + \left(\frac{1+r^2}{1-r^2} \operatorname{Im} z\right)^2} - \frac{2r}{1-r^2} \operatorname{Im} z > \frac{1-r}{1+r} t.$$

And if $\frac{1+r^2}{1-r^2} \operatorname{Im} z < \frac{1}{t}$ and $w \in E_r(z)$, then

$$\frac{1+r^2}{1-r^2} \operatorname{Im} w < \frac{1+r^2}{1-r^2} \left(\frac{1+r^2}{1-r^2} \operatorname{Im} z + \frac{2r}{1-r^2} \operatorname{Im} z \right) < \frac{1}{\frac{1-r}{1+r} t}.$$

Let $A_{r,t} = \{z \in \Pi^+ : |z| > t \text{ or } \frac{1+r^2}{1-r^2}\operatorname{Im} z < \frac{1}{t}\}$. From above, we know that $w \in A_{r,\frac{1-r}{1+r}t}$ whenever $z \in A_{r,t}$ and $w \in E_r(z)$.

Let $\{f_j\}$ be any bounded sequence in $A^p(\Pi^+)$ that converges to 0 uniformly on every compact subset of Π^+, then by Lemma 2.1 and Fubini's theorem, we have

$$\begin{aligned}\int_{A_{r,t}} |f_j^{(k)}(z)|^q \, d\mu(z) &\lesssim \int_{A_{r,t}} \left(\frac{1}{(\operatorname{Im} z)^{2+kq}} \int_{E_r(z)} |f_j(w)|^q \, dA(w) \right) d\mu(z) \\ &\lesssim \int_{\Pi^+} |f_j(w)|^q \, dA(w) \int_{A_{r,t}} \frac{\chi_{E_r(w)}(z)}{(\operatorname{Im} z)^{2+kq}} \, d\mu(z).\end{aligned} \tag{2.6}$$

Note that if $w \notin A_{r,\frac{1-r}{1+r}t}$, then $\chi_{E_r(w)\cap A_{r,t}} = 0$ for every $z \in \Pi^+$. And using (2.1), we have

$$\begin{aligned}&\int_{\Pi^+} |f_j(w)|^q \, dA(w) \int_{A_{r,t}} \frac{\chi_{E_r(w)}(z)}{(\operatorname{Im} z)^{2+kq}} \, d\mu(z) \\ &= \int_{A_{r,\frac{1-r}{1+r}t}} |f_j(w)|^q \, dA(w) \int_{A_{r,t}} \frac{\chi_{E_r(w)}(z)}{(\operatorname{Im} z)^{2+kq}} \, d\mu(z) \\ &\lesssim \int_{A_{r,\frac{1-r}{1+r}t}} |f_j(w)|^q \frac{\mu(E_r(w))}{(\operatorname{Im} w)^{2+kq}} \, dA(w).\end{aligned} \tag{2.7}$$

Then we can apply Hölder's inequality to obtain

$$\begin{aligned}&\int_{A_{r,\frac{1-r}{1+r}t}} |f_j(w)|^q \frac{\mu(E_r(w))}{(\operatorname{Im} w)^{2+kq}} \, dA(w) \\ &\le \left(\int_{\Pi^+} |f_j(w)|^p \, dA(w) \right)^{q/p} \left(\int_{A_{r,\frac{1-r}{1+r}t}} \left(\frac{\mu(E_r(w))}{(\operatorname{Im} w)^{2+kq}} \right)^{p/(p-q)} dA(w) \right)^{(p-q)/p} \\ &\lesssim \|f_j\|_p^q \left(\int_{A_{r,\frac{1-r}{1+r}t}} \left(\frac{\mu(E_r(w))}{(\operatorname{Im} w)^{2+kq}} \right)^{p/(p-q)} dA(w) \right)^{(p-q)/p}.\end{aligned} \tag{2.8}$$

Since the function $w \mapsto \frac{\mu(E_r(w))}{(\operatorname{Im} w)^{2+kq}} \in L^{p/(p-q)}$, for any given $\varepsilon > 0$, we can choose $t_0 \in (0,\infty)$ large enough such that

$$\left(\int_{A_{r,\frac{1-r}{1+r}t_0}} \left(\frac{\mu(E_r(w))}{(\operatorname{Im} w)^{2+kq}} \right)^{p/(p-q)} dA(w) \right)^{(p-q)/p} < \varepsilon. \tag{2.9}$$

Combining (2.6), (2.7), (2.8) and (2.9), we obtain

$$\int_{A_{r,t_0}} |f_j^{(k)}(z)|^q \, d\mu(z) \lesssim \varepsilon.$$

On the other hand, it is easy to check that

$$\lim_{j\to\infty}\int_{\Pi^+\setminus A_{r,t_0}} |f_j^{(k)}(z)|^q\,d\mu(z) = 0.$$

Therefore,

$$\limsup_{j\to\infty}\int_{\Pi^+} |f_j^{(k)}(z)|^q\,d\mu(z) \lesssim \varepsilon$$

for any $\varepsilon > 0$. This implies that μ is a vanishing (k, p, q)-Carleson measure.

It remains to show that

$$\frac{\mu(E_r(z))}{(\operatorname{Im} z)^{2+kq}} \in L^{p/(p-q)}$$

when μ is a (k, p, q)-Carleson measure. To this end, let $\{w_j\}$ be any r-lattice in the pseudo-hyperbolic metric with $0 < r < 2/5$ and $\{c_j\} \in l^p$. We set

$$h_s(z) = \sum_j c_j \frac{(\operatorname{Im} w_j)^{s-2/p}}{(\overline{w}_j - z)^s},$$

where $s > 2/p + 1$. Then $h \in A^p(\Pi^+)$ and $\|h_s\|_p \lesssim \|\{c_j\}\|_{l^p}$ by Lemma 2.2. If μ is a (k, p, q)-Carleson measure, we have

$$\int_{\Pi^+} \left|\sum_j c_j \frac{(\operatorname{Im} w_j)^{s-2/p}}{(\overline{w}_j - z)^{s+k}}\right|^q d\mu(z) \lesssim \int_{\Pi^+} |h_s^{(k)}(z)|^q\,d\mu(z) \lesssim \|\{c_j\}\|_{l^p}^q. \tag{2.10}$$

In (2.10), we replace c_j by $r_j(t)c_j$, in which case the right-hand side does not change. Then we integrate both sides with respect to t from 0 to 1 to obtain

$$\int_0^1\int_{\Pi^+} \left|\sum_j c_j r_j(t) \frac{(\operatorname{Im} w_j)^{s-2/p}}{(\overline{w}_j - z)^{s+k}}\right|^q d\mu(z)\,dt \lesssim \|\{c_j\}\|_{l^p}^q.$$

By Fubini's theorem and Khinchine's inequality,

$$\begin{aligned}
&\int_{\Pi^+} \left(\sum_j |c_j|^2 \left(\frac{(\operatorname{Im} w_j)^{s-2/p}}{|\overline{w}_j - z|^{s+k}}\right)^2\right)^{q/2} d\mu(z)\\
&\lesssim \int_{\Pi^+}\int_0^1 \left|\sum_j c_j r_j(t) \frac{(\operatorname{Im} w_j)^{s-2/p}}{(\overline{w}_j - z)^{s+k}}\right|^q dt\,d\mu(z)\\
&\lesssim \int_0^1\int_{\Pi^+} \left|\sum_j c_j r_j(t) \frac{(\operatorname{Im} w_j)^{s-2/p}}{(\overline{w}_j - z)^{s+k}}\right|^q d\mu(z)\,dt\\
&\lesssim \|\{c_j\}\|_{l^p}^q.
\end{aligned}$$

On the other hand, recall that there is a positive integer N such that each point $z \in \Pi^+$ belongs to at most N of the disks $E_{2r}(w_j)$. Applying Minkowski's inequality if $2/q \le 1$ and Hölder's inequality if $2/q > 1$ on the sum, which contains at most N nonzero terms, we obtain

$$\begin{aligned}\sum_j |c_j|^q \frac{\mu(E_{2r}(w_j))}{(\operatorname{Im} w_j)^{(k+2/p)q}} &\asymp \sum_j |c_j|^q \int_{E_{2r}(w_j)} \left(\frac{(\operatorname{Im} w_j)^{s-2/p}}{|\overline{w}_j - z|^{s+k}} \right)^q d\mu(z) \\ &= \int_{\Pi^+} \sum_j \chi_{E_{2r}(w_j)}(z) |c_j|^q \left(\frac{(\operatorname{Im} w_j)^{s-2/p}}{|\overline{w}_j - z|^{s+k}} \right)^q d\mu(z) \\ &\le C \int_{\Pi^+} \left(\sum_j |c_j|^2 \left(\frac{(\operatorname{Im} w_j)^{s-2/p}}{|\overline{w}_j - z|^{s+k}} \right)^2 \right)^{q/2} d\mu(z),\end{aligned}$$

where $C = \max\{1, N^{(2-q)/2}\}$. Therefore, $\sum_j |c_j|^q \frac{\mu(E_{2r}(w_j))}{(\operatorname{Im} w_j)^{(k+2/p)q}} \lesssim \|\{c_j\}\|_{l^p}^q$ for any $\{c_j\} \in l^p$. This implies that

$$\frac{\mu(E_{2r}(w_j))}{(\operatorname{Im} w_j)^{(k+2/p)q}} \in (l^{p/q})^* = l^{p/(p-q)}.$$

Note that $E_r(z) \subset E_{2r}(w_j)$ whenever $z \in E_r(w_j)$, hence

$$\begin{aligned}\int_{\Pi^+} \left(\frac{\mu(E_r(z))}{(\operatorname{Im} z)^{2+kq}} \right)^{p/(p-q)} dA(z) &\le \sum_{j=1}^{\infty} \int_{E_r(w_j)} \left(\frac{\mu(E_r(z))}{(\operatorname{Im} z)^{2+kq}} \right)^{p/(p-q)} dA(z) \\ &\lesssim \sum_{j=1}^{\infty} \left(\frac{\mu(E_{2r}(w_j))}{(\operatorname{Im} w_j)^{2+kq}} \right)^{p/(p-q)} (\operatorname{Im} w_j)^2 \\ &\asymp \sum_{j=1}^{\infty} \left(\frac{\mu(E_{2r}(w_j))}{(\operatorname{Im} w_j)^{(k+2/p)q}} \right)^{p/(p-q)} < \infty.\end{aligned} \tag{2.11}$$

The proof is complete. □

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. We first characterize the boundedness and compactness of the weighted differentiation-composition operator $W_{u,\varphi}^{(k)}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ via (k,p,q)-Carleson measure for $0 < p \le q < \infty$.

For $\varphi \in S(\Pi^+)$ and $u \in H(\Pi^+)$, we define the weighted pull-back measure by

$$\mu_{\varphi,u^q}(E) = \int_{\varphi^{-1}(E)} |u|^q \, dA,$$

where E is any Borel subset of Π^+.

Proposition 3.1. *If $0 < p \le q < \infty$, then the following conditions are equivalent.*

(i) $W_{u,\varphi}^{(k)}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ *is bounded;*

(ii) μ_{φ,u^q} *is a* (k,p,q)*-Carleson measure;*

(iii) $\displaystyle\sup_{w\in\Pi^+}\int_{\Pi^+}|u(z)|^q\frac{(\operatorname{Im} w)^{2q}}{|\overline{w}-\varphi(z)|^{(k+2+2/p)q}}\,dA(z)<\infty.$

Proof. For any $f \in A^p(\Pi^+)$,

$$\|W_{u,\varphi}^{(k)}f\|_q^q=\int_{\Pi^+}|u(z)|^q|f^{(k)}\circ\varphi(z)|^q\,dA(z)=\int_{\Pi^+}|f^{(k)}(w)|^q\,d\mu_{\varphi,u^q}(w).$$

Thus $W_{u,\varphi}^{(k)}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ is bounded if and only if μ_{φ,u^q} is a (k,p,q)-Carleson measure, i.e., (i) $\Leftrightarrow$ (ii) holds.

Now we prove (i) $\Rightarrow$ (iii). Suppose $W_{u,\varphi}^{(k)}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ is bounded. Then for any $w \in \Pi^+$,

$$\|W_{u,\varphi}^{(k)}\|\geq\|W_{u,\varphi}^{(k)}k_{p,w,\gamma}^{[0]}\|_q\asymp\left(\int_{\Pi^+}|u(z)|^q\frac{(\operatorname{Im} w)^{(\gamma-2/p)q}}{|\overline{w}-\varphi(z)|^{(\gamma+k)q}}\,dA(z)\right)^{1/q}.$$

In particular, letting $\gamma = 2/p + 2$, we obtain

$$\sup_{w\in\Pi^+}\int_{\Pi^+}|u(z)|^q\frac{(\operatorname{Im} w)^{2q}}{|\overline{w}-\varphi(z)|^{(k+2+2/p)q}}\,dA(z)<\infty.$$

It remains to show (iii) $\Rightarrow$ (ii). Suppose (iii) holds, then by (2.1), $|\overline{w}-\varphi(z)| \asymp \operatorname{Im} w$ whenever $z \in \varphi^{-1}(E_r(w))$. It follows that

$$\begin{aligned}\sup_{w\in\Pi^+}\frac{\mu_{\varphi,u^q}(E_r(w))}{(\operatorname{Im} w)^{(k+2/p)q}}&\asymp\sup_{w\in\Pi^+}\int_{\varphi^{-1}(E_r(w))}\frac{|u(z)|^q}{|\overline{w}-\varphi(z)|^{(k+2/p)q}}\,dA(z)\\&\lesssim\sup_{w\in\Pi^+}\int_{\Pi^+}|u(z)|^q\frac{(\operatorname{Im} w)^{2q}}{|\overline{w}-\varphi(z)|^{(2/p+k+2)q}}\,dA(z)<\infty.\end{aligned}$$

Hence μ_{φ,u^q} is a (k,p,q)-Carleson measure by Lemma 2.4. The proof is complete. □

By routine arguments of taking the limit, we could obtain the following results for the compactness of $W_{u,\varphi}^{(k)}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ for $0 < p \leq q < \infty$ and we omit the details of the proof.

Proposition 3.2. *If* $0 < p \leq q < \infty$*, then the following conditions are equivalent.*

(i) $W_{u,\varphi}^{(k)}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ *is compact;*

(ii) μ_{φ,u^q} *is a vanishing* (k,p,q)*-Carleson measure;*

(iii) $\displaystyle\lim_{w\to\partial\widehat{\Pi^+}}\int_{\Pi^+}|u(z)|^q\frac{(\operatorname{Im} w)^{2q}}{|\overline{w}-\varphi(z)|^{(k+2+2/p)q}}\,dA(z)=0.$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The "if part" is clear from the triangle inequality. Now we prove the "only if part". Assume $T^{(n)}_{\mathbf{u},\varphi}\colon A^p(\Pi^+)\to A^q(\Pi^+)$ is bounded, then

$$\|T^{(n)}_{\mathbf{u},\varphi}\| \geq \|T^{(n)}_{\mathbf{u},\varphi}k^{[i]}_{p,w,\gamma}\|_q$$

for all $w\in\Pi^+$ and $0\leq i\leq n$. It follows that

$$\sup_{w\in\Pi^+}\int_{\Pi^+}\left|\frac{u_0(z)(\operatorname{Im} w)^{\gamma+i-2/p}}{(\overline{w}-\varphi(z))^{\gamma+i}}+\sum_{l=1}^{n}C_{\gamma,i,l}\frac{u_l(z)(\operatorname{Im} w)^{\gamma+i-2/p}}{(\overline{w}-\varphi(z))^{\gamma+i+l}}\right|^q dA(z)\lesssim \|T^{(n)}_{\mathbf{u},\varphi}\|^q$$

for each $0\leq i\leq n$, where $C_{\gamma,i,l}=\prod_{m=0}^{l-1}(\gamma+i+m)$. Set

$$L^{[0]}_i(z)=u_0(z)+\sum_{l=1}^{n}C_{\gamma,i,l}\frac{u_l(z)}{(\overline{w}-\varphi(z))^l},\quad 0\leq i\leq n.$$

For any fixed $0<r<1$, it follows from (2.1) that

$$\begin{aligned}
&\frac{1}{(\operatorname{Im} w)^{2q/p}}\int_{\varphi^{-1}(E_r(w))}|L^{[0]}_i(z)|^q\,dA(z)\\
&\lesssim\int_{\varphi^{-1}(E_r(w))}\left|\frac{u_0(z)(\operatorname{Im} w)^{\gamma+i-2/p}}{(\overline{w}-\varphi(z))^{\gamma+i}}+\sum_{l=1}^{n}C_{\gamma,i,l}\frac{u_l(z)(\operatorname{Im} w)^{\gamma+i-2/p}}{(\overline{w}-\varphi(z))^{\gamma+i+l}}\right|^q dA(z)\\
&\lesssim\int_{\Pi^+}\left|\frac{u_0(z)(\operatorname{Im} w)^{\gamma+i-2/p}}{(\overline{w}-\varphi(z))^{\gamma+i}}+\sum_{l=1}^{n}C_{\gamma,i,l}\frac{u_l(z)(\operatorname{Im} w)^{\gamma+i-2/p}}{(\overline{w}-\varphi(z))^{\gamma+i+l}}\right|^q dA(z)<\infty
\end{aligned}\tag{3.1}$$

for all $w\in\Pi^+$ and $0\leq i\leq n$. Let

$$\begin{aligned}
L^{[1]}_i(z)&=L^{[0]}_i(z)-L^{[0]}_{i-1}(z)\\
&=\frac{u_1(z)}{\overline{w}-\varphi(z)}+\sum_{l=2}^{n}l(\gamma+i)\cdots(\gamma+i+l-2)\frac{u_l(z)}{(\overline{w}-\varphi(z))^l},\quad 1\leq i\leq n.
\end{aligned}$$

Then by (3.1), we have

$$\sup_{w\in\Pi^+}\frac{1}{(\operatorname{Im} w)^{2q/p}}\int_{\varphi^{-1}(E_r(w))}|L^{[1]}_i(z)|^q\,dA(z)<\infty\tag{3.2}$$

for each $1\leq i\leq n$. Let

$$\begin{aligned}
L^{[2]}_i(z)&=L^{[1]}_i(z)-L^{[1]}_{i-1}(z)\\
&=2\frac{u_2(z)}{(\overline{w}-\varphi(z))^2}+\sum_{l=3}^{n}l(l-1)(\gamma+i)\cdots(\gamma+i+l-3)\frac{u_l(z)}{(\overline{w}-\varphi(z))^l},\quad 2\leq i\leq n.
\end{aligned}$$

Then by (3.2), we have

$$\sup_{w\in\Pi^+}\frac{1}{(\operatorname{Im} w)^{2q/p}}\int_{\varphi^{-1}(E_r(w))}|L_i^{[2]}(z)|^q\,dA(z)<\infty$$

for each $2\le i\le n$. Through a similar procedure as above, we get

$$L_i^{[n-1]}(z)=(n-1)!\frac{u_{n-1}(z)}{(\overline{w}-\varphi(z))^{n-1}}+n!(\gamma+i)\frac{u_n(z)}{(\overline{w}-\varphi(z))^n},\quad i=n-1,n$$

and

$$L_n^{[n]}(z)=L_{n-1}^{[n-1]}(z)-L_n^{[n-1]}(z)=n!\frac{u_n(z)}{(\overline{w}-\varphi(z))^n}.$$

Therefore,

$$\sup_{w\in\Pi^+}\frac{1}{(\operatorname{Im} w)^{2q/p}}\int_{\varphi^{-1}(E_r(w))}\left|\frac{u_n(z)}{(\overline{w}-\varphi(z))^n}\right|^q dA(z)<\infty.$$

This, together with (2.1), implies that

$$\sup_{w\in\Pi^+}\frac{1}{(\operatorname{Im} w)^{(n+2/p)q}}\int_{\varphi^{-1}(E_r(w))}|u_n(z)|^q\,dA(z)<\infty.$$

It follows from Proposition 3.1 that $W^{(n)}_{u_n,\varphi}\colon A^p(\Pi^+)\to A^q(\Pi^+)$ is bounded.

We have proved that $W^{(n)}_{u_n,\varphi}$ is bounded if $T^{(n)}_{\mathbf{u},\varphi}$ is bounded when $n\ge 1$. Hence their difference, $T^{(n-1)}_{\mathbf{u},\varphi}$, is bounded. By repeating this procedure and noting that $T^{(0)}_{\mathbf{u},\varphi}=W^{(0)}_{u_0,\varphi}$, we conclude that each $W^{(k)}_{u_k,\varphi}\colon A^p(\Pi^+)\to A^q(\Pi^+)$, $0\le k\le n$, is bounded.

The proof for the "compactness part" is similar and we omit the details. □

4. Proof of Theorem 1.2

We now proceed to prove Theorem 1.2. First we also characterize the boundedness and compactness of weighted differentiation-composition operator $W^{(k)}_{u,\varphi}\colon A^p(\Pi^+)\to A^q(\Pi^+)$ for $0<q<p<\infty$.

Proposition 4.1. *Let $u\in H(\Pi^+)$ and $\varphi\in S(\Pi^+)$. If $0<q<p<\infty$, then the following conditions are equivalent.*

- (i) $W^{(k)}_{u,\varphi}\colon A^p(\Pi^+)\to A^q(\Pi^+)$ *is bounded;*
- (ii) $W^{(k)}_{u,\varphi}\colon A^p(\Pi^+)\to A^q(\Pi^+)$ *is compact;*
- (iii) μ_{φ,u^q} *is a (k,p,q)-Carleson measure;*
- (iv) μ_{φ,u^q} *is a vanishing (k,p,q)-Carleson measure;*
- (v) *The function* $w\mapsto\frac{1}{(\operatorname{Im} w)^{2+kq}}\int_{\varphi^{-1}(E_r(w))}|u(z)|^q\,dA(z)$ *belongs to $L^{p/(p-q)}$ for some $0<r<2/5$.*

Proof. The equivalence of (iii), (iv) and (v) is given in Lemma 2.4. And by the proof of Proposition 3.1, we know that (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv). $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The equivalence of (iii) and (iv) was given in Proposition 4.1. It is obvious that (ii) implies (i). That (iii) implies (i) and that (iv) implies (ii) follow from the triangle inequality. Thus we only need to prove that (i) implies (iii). To this end, let $\{w_j\}$ be any r-lattice in the pseudo-hyperbolic metric with $0 < r < 2/5$ and $\{\lambda_j\} \in l^p$, we set

$$h_{i,\gamma}(z) = \sum_j \lambda_j \frac{(\operatorname{Im} w_j)^{\gamma+i-2/p}}{(z-\overline{w}_j)^{\gamma+i}}, \quad 0 \le i \le n,$$

where $\gamma > 2/p + 1$. It follows from Lemma 2.2 that

$$\|h_{i,\gamma}\|_p \lesssim \|\{\lambda_j\}\|_{l^p}, \quad 0 \le i \le n.$$

If $T^{(n)}_{\mathbf{u},\varphi}\colon A^p(\Pi^+) \to A^q(\Pi^+)$ is bounded, then

$$\begin{aligned}
\|T^{(n)}_{\mathbf{u},\varphi} h_{i,\gamma}\|_q^q &= \int_{\Pi^+} \left| \sum_{l=0}^n u_l(z) h^{(l)}_{i,\gamma} \circ \varphi(z) \right|^q dA(z) \\
&= \int_{\Pi^+} \left| \sum_{l=0} \sum_j \lambda_j C_{\gamma,i,l} \frac{(\operatorname{Im} w_j)^{\gamma+i-2/p} u_l(z)}{(\varphi(z)-\overline{w}_j)^{\gamma+i+l}} \right|^q dA(z) \\
&\lesssim \|\{\lambda_j\}\|_{l^p}^q, \quad 0 \le i \le n,
\end{aligned} \tag{4.1}$$

where $C_{\gamma,i,0} = 1$ and $C_{\gamma,i,l} = \prod_{m=0}^{l-1}(\gamma+i+m)$ for $1 \le l \le n$.

We replace λ_j by $\lambda_j r_j(t)$, in which case the last item in (4.1) remains unchanged. Then integrating both sides with respect to t from 0 to 1 and using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\Pi^+}\int_0^1 \left| \sum_{l=0} \sum_j \lambda_j r_j(t) C_{\gamma,i,l} \frac{(\operatorname{Im} w_j)^{\gamma+i-2/p} u_l(z)}{(\varphi(z)-\overline{w}_j)^{\gamma+i+l}} \right|^q dt\, dA(z) \\
&= \int_0^1\int_{\Pi^+} \left| \sum_{l=0} \sum_j \lambda_j r_j(t) C_{\gamma,i,l} \frac{(\operatorname{Im} w_j)^{\gamma+i-2/p} u_l(z)}{(\varphi(z)-\overline{w}_j)^{\gamma+i+l}} \right|^q dA(z)\, dt \\
&\lesssim \|\{\lambda_j\}\|_{l^p}^q.
\end{aligned} \tag{4.2}$$

On the other hand, by Khinchine's inequality, we have

$$\begin{aligned}
&\int_{\Pi^+} \left(\sum_j |\lambda_j|^2 \left| \sum_{l=0}^n C_{\gamma,i,l} \frac{(\operatorname{Im} w_j)^{\gamma+i-2/p} u_l(z)}{(\varphi(z)-\overline{w}_j)^{\gamma+i+l}} \right|^2 \right)^{q/2} dA(z) \\
&\lesssim \int_{\Pi^+}\int_0^1 \left| \sum_{l=0} \sum_j \lambda_j r_j(t) C_{\gamma,i,l} \frac{(\operatorname{Im} w_j)^{\gamma+i-2/p} u_l(z)}{(\varphi(z)-\overline{w}_j)^{\gamma+i+l}} \right|^q dt\, dA(z)
\end{aligned} \tag{4.3}$$

for each $0 \le i \le n$. We briefly denote by $E_j = \varphi^{-1}(E_r(w_j))$. By (2.1),

$$\chi_{E_j}(z) \lesssim \frac{\operatorname{Im} w_j}{|\varphi(z) - \overline{w}_j|}.$$

This, together with (4.2) and (4.3), shows that

$$\int_{\Pi^+} \left(\sum_j |\lambda_j|^2 \chi_{E_j}(z) \frac{1}{(\operatorname{Im} w_j)^{4/p}} \left| \sum_{l=0}^n C_{\gamma,i,l} \frac{u_l(z)}{(\varphi(z) - \overline{w}_j)^l} \right|^2 \right)^{q/2} dA(z)$$
$$\lesssim \int_{\Pi^+} \left(\sum_j |\lambda_j|^2 \left| \sum_{l=0}^n C_{\gamma,i,l} \frac{(\operatorname{Im} w_j)^{\gamma+i-2/p} u_l(z)}{(\varphi(z) - \overline{w}_j)^{\gamma+i+l}} \right|^2 \right)^{q/2} dA(z) \lesssim \|\{\lambda_j\}\|_{l^p}^q$$

for each $0 \le i \le n$.

Recall that there is a positive integer N such that each point $\varphi(z) \in \Pi^+$ belongs to at most N of the disks $E_r(w_j)$, then each point $z \in \Pi^+$ belongs to at most N of the sets E_j. Applying Minkowski's inequality if $2/q \le 1$ and Hölder's inequality if $2/q > 1$, we obtain

$$\sum_j |\lambda_j|^q \frac{1}{(\operatorname{Im} w_j)^{2q/p}} \int_{E_j} \left| \sum_{l=0}^n C_{\gamma,i,l} \frac{u_l(z)}{(\varphi(z) - \overline{w}_j)^l} \right|^q dA(z)$$
$$= \int_{\Pi^+} \sum_j |\lambda_j|^q \chi_{E_j}(z) \frac{1}{(\operatorname{Im} w_j)^{2q/p}} \left| \sum_{l=0}^n C_{\gamma,i,l} \frac{u_l(z)}{(\varphi(z) - \overline{w}_j)^l} \right|^q dA(z)$$
$$\lesssim \int_{\Pi^+} \left(\sum_j |\lambda_j|^2 \chi_{E_j}(z) \frac{1}{(\operatorname{Im} w_j)^{4/p}} \left| \sum_{l=0}^n C_{\gamma,i,l} \frac{u_l(z)}{(\varphi(z) - \overline{w}_j)^l} \right|^2 \right)^{q/2} dA(z)$$

for each $0 \le i \le n$.

Now we use the same method as in the proof of Theorem 1.1 to get

$$\sum_j |\lambda_j|^q \frac{1}{(\operatorname{Im} w_j)^{2q/p}} \int_{E_j} |L_i^{[k]}(z)|^q \, dA(z) \lesssim \|\{\lambda_j\}\|_{l^p}^q$$

for $0 \le k \le n$ and $k \le i \le n$. In particular,

$$\sum_j |\lambda_j|^q \frac{1}{(\operatorname{Im} w_j)^{(n+2/p)q}} \int_{E_j} |u_n(z)|^q \, dA(z)$$
$$\asymp \sum_j |\lambda_j|^q \frac{1}{(\operatorname{Im} w_j)^{2q/p}} \int_{E_j} |L_n^{[n]}(z)|^q \, dA(z) \lesssim \|\{\lambda_j\}\|_{l^p}^q$$

for any $\{\lambda_j\} \in l^p$. This implies that

$$\sum_j \left(\frac{\mu_{\varphi,u_n^q}(E_r(w_j))}{(\operatorname{Im} w_j)^{(n+2/p)q}} \right)^{p/(p-q)} < \infty.$$

Then by (2.11) in the proof of Lemma 2.4, we get that the function $z \mapsto \frac{\mu_{\varphi,u_n^q}(E_r(z))}{(\operatorname{Im} z)^{2+nq}}$ belongs to $L^{p/(p-q)}$. It follows that μ_{φ,u_n^q} is a (n,p,q)-Carleson measure and $W_{u_n,\varphi}^{(n)}$ is bounded from $A^p(\Pi^+)$ to $A^q(\Pi^+)$ by Proposition 4.1.

Finally, by induction, we conclude that each $W_{u_k,\varphi}^{(k)}\colon A^p(\Pi^+) \to A^q(\Pi^+)$, $0 \le k \le n$, is bounded. The proof is complete. □

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