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Competing Finsler Double Phase Equation

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Abstract. In this paper, we consider a representation of the competing Finsler double phase equation, i.e., a representation of the competing Finsler double phase equation in the Heisenberg groups $\mathbb{H}^n$. We prove the existence of a generalized variational solution for the nonlinear Dirichlet problem driven by the Finsler–Laplacian competing operator in the Heisenberg group. This solution differs from the concept of the weak solution. To achieve this, we employ a Galerkin type procedure as part of our methodology.

1. Introduction

In the early 1920s, Werner Heisenberg, alongside physicists such as Erwin Schrödinger and Max Born, played a pivotal role in the development of matrix mechanics, a fundamental aspect of quantum theory. This innovative approach marked a departure from classical mechanics by utilizing matrices and operators to elucidate the behavior of quantum systems. The shift to quantum mechanics posed significant challenges to the traditional viewpoint, triggering a transformative paradigm shift in the field of physics. At the forefront of this revolution was Werner Heisenberg's groundbreaking formulation of the Uncertainty Principle. Mathematically, the Uncertainty Principle is represented by the relationship between the uncertainties in position (Δx) and momentum (Δp) being greater than or equal to the Planck constant ($\hbar$). Within the Heisenberg groups $\mathbb{H}^n$, which serve as a basic example of a noncommutative nilpotent Lie group, we encounter an abstract representation of the commutation relations governing quantum-mechanical position and momentum operators (see [8, 16] for more details).

In this article, we study a formulation of the Finsler double phase equation that involves competition. Consequently, we revisit certain aspects pertaining to the Finsler manifold.

Definition 1.1. $F \in C^2(\mathbb{R}^N \setminus \{\mathbf{0}\})$ is called (possibly asymmetric) Finsler norm on $\mathbb{R}^N$ if it satisfies the following three conditions:

(1) (positive definiteness) $F(x) > 0$ unless $x = 0$;

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(2) (positive homogeneity) $F(\lambda x) = \lambda F(x)$ for all $\lambda \geq 0$;

(3) (F is positively homogeneous of degree one and convex) $F(x+y) \leq F(x) + F(y)$ for every two vector $x, y \in \mathbb{R}^N$,

such that

$$F(x) > 0 \quad \text{for any } x \in \mathbb{R}^N \setminus \{\mathbf{0}\}.$$

We notice that $\mathbb{R}^N$ endowed with the norm F can be viewed as a Finsler manifold, which reflects into a rich geometric structure to explore since it breaks the usual symmetry properties coming from the peculiarities of the Euclidean norm, like the directional independence (see [1, 3, 10, 17, 22, 27] for more details.

Hence, it is natural to inquire whether the Finsler framework maintains a certain form of symmetry, and one can attempt to characterize the resulting behaviors of solutions to equations within this framework. Specifically, when considering the Liouville equation, it becomes intriguing to examine the operators that arise in the Euler–Lagrange equations for Wulff-type functionals:

$$\int F^N(\nabla u)\, dx,$$

where the gradient of a function u is measured by the norm F. Such a generalization to anisotropic (or Finsler) PDEs is also of interest in its own right and from the applied point of view, as it is motivated by concrete applications in many fields, such as in the study of sharp geometric inequalities and capacity, blowup analysis, digital image processing, crystalline mean curvature flow and crystalline fracture theory (see [5–7] and the references therein).

The aim of this paper is to study the Dirichlet problem

$$\begin{cases} -\mathcal{Q}_p^{g_1} u + \mathcal{Q}_q^{g_2} u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on a bounded domain Ω in $\mathbb{H}^n$, $n \geq 3$ and a Lipschitz boundary $\partial\Omega$. Problem (1.1) is driven by the competing Finsler (p,q)-Laplacian type operator $-\mathcal{Q}_p^{g_1} + \mathcal{Q}_q^{g_2}$, with $1 < q < p < Q = 2n+2$, that we now describe.

Let $F\colon \mathbb{R}^{2n} \to [0, +\infty)$ be a convex function of class $C^2(\mathbb{R}^{2n} \setminus \{\mathbf{0}\})$, which is even and satisfies $F(\xi) > 0$ for each $\xi \neq 0$, and $F(t\xi) = |t|F(\xi)$ for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^{2n}$. Given $p \in (1, +\infty)$ we assume that there exists a constant $\gamma > 0$ such that

$$\sum_{i,j=1}^{N} \nabla^2(F^p)(\eta)\zeta_i\zeta_j \geq \gamma|\eta|^{p-2}|\zeta|^2$$

with some positive constant γ for all $\eta \in \mathbb{R}^{2n} \setminus \{0\}$ and $\zeta \in \mathbb{R}^{2n}$.

The Finsler p-Laplacian operator $\mathcal{Q}_p\colon HW_0^{1,p}(\Omega) \to HW_0^{-1,p'}(\Omega)$ is defined as

$$\mathcal{Q}_p u := \operatorname{div}_{\mathbb{H}^n} \left(F^{p-1}(D_{\mathbb{H}^n} u)\, D_{\mathbb{H}^n} F(D_{\mathbb{H}^n} u)\right) \quad \text{for all } u \in HW_0^{1,p}(\Omega). \tag{1.2}$$

If $F(\xi) = |\xi|$ (the Euclidean norm) and $p = 2$, then it becomes the Kohn–Spencer Laplacian (see [26, 28]).

We denote by λ_1 the first eigenvalue of $-\mathcal{Q}_p$, that is,

$$\lambda_1 = \min_{\varphi \in HW_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega F^p(D_{\mathbb{H}^n}\varphi)\, d\xi}{\int_\Omega |\varphi|^p\, d\xi}. \tag{1.3}$$

For more details on the operator $-\mathcal{Q}_p$ we refer to [9, 33]. As a real life application we mention Wulff's work [34] on crystal shapes.

Finally, we assume the following condition:

(H1) There exists a nonnegative function $\sigma \in L^{(p^*)'}(\Omega)$ and a constant $b \geq 0$ such that

$$|h(\xi, t)| \leq \sigma(\xi) + b|t|^{p^*-1}$$

for a.e. $\xi \in \Omega$ and all $t \in \mathbb{R}$. Denote by p^* the critical Sobolev exponent, that is, $p^* := \frac{pQ}{Q-p}$ where $Q = 2n + 2$ and $Q \geq p$.

In addition, we formulate the condition:

(H2) Given the positive constant α, it holds

$$H(\xi, t) := \int_0^t h(\xi, s)\, ds \leq c_1(|t|^p + 1)$$

for a.e. $\xi \in \Omega$ and all $t \in \mathbb{R}$ with a positive constant $c_1 < \frac{\lambda_1 a_{g_1}}{p}$.

Here, we present an example of a nonlinear term satisfying the assumptions (H1)–(H2) (see [13]).

Example 1.2. The function $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by $h(\xi, t) := \Theta_1 |t|^{\beta-2} t + \Theta_2(\xi)$ for all $(\xi, t) \in \Omega \times \mathbb{R}$, with constants $\beta \in [1, p)$ and $\Theta_1 > 0$ and a function $\Theta_2 \in L^\infty(\Omega)$, satisfies conditions (H1)–(H2).

In (1.1) we have a weighted version of the Finsler φ-Laplacian $\mathcal{Q}_\varphi$ extending (1.2), where $\varphi = p, q$. Specifically, corresponding to a continuous function $g_i\colon \mathbb{R} \to \mathbb{R}$ for which there exist constants $a_{g_i} > 0$ and $b_{g_i} > 0$ such that $a_{g_i} \leq g_i(t) \leq b_{g_i}$ for all $t \geq 0$ and $i = 1, 2$, one sets

$$\mathcal{Q}_\varphi^{g_i} u = D_{\mathbb{H}^n} \left(g_i \left(\frac{1}{\varphi} F^\varphi(D_{\mathbb{H}^n} u) \right) F^{\varphi-1}(D_{\mathbb{H}^n} u)\, D_{\mathbb{H}^n} F(D_{\mathbb{H}^n} u) \right)$$

for all $u \in HW_0^{1,\varphi}(\Omega)$.

We are in a position to state our result regarding problem (1.1).

Theorem 1.3. *Suppose* $1 < q < p < Q$. *Assume that* $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function for which the conditions* (H1) *and* (H2) *hold. Then there exists a generalized variational solution to problem* (1.1) *in the sense of Definition* 3.1.

The rest of the paper is organized as follows: In Section 2 we briefly recall the relevant definitions and notations related to the Heisenberg group functional setting and Sobolev embedding theorems in this setting. In Section 3 we review the concept of a generalized variational solution. Notice that the generalized variational solution given in Definition 3.1 is completely different from the weak solution in (3.2). In Section 4 we present the proof of Theorem 1.3 which shows the existence of a generalized variational solution (in the sense of Definition 3.1) for problem (1.1) by applying the Galerkin type procedure. In Section 5, we generalize problem (1.1) and consider a system version of the Finsler double phase equation. Then we prove the existence of a generalized variational solution for problem (5.1) by applying the Galerkin type procedure, i.e., we prove Theorem 5.1.

2. Function space

The underlying space for problem (1.1) is $HW_0^{1,p}(\Omega)$. Thus we briefly recall the relevant definitions and notations related to the Heisenberg group functional setting. For a complete treatment, we refer to [4, 11, 14, 18–20, 24, 26, 28–32].

Let $\mathbb{H}^n$ be the Heisenberg group of topological dimension $2n+1$, that is, the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, endowed with the non-Abelian group law

$$\psi \circ \psi' = \left(z + z', t + t' + 2\sum_{i=1}^{n} (y_i x_i' - x_i y_i') \right)$$

for all $\psi, \psi' \in \mathbb{H}^n$ with

$$\psi = (z,t) = (x_1, \dots, x_n, y_1, \dots, y_n, t) \quad \text{and} \quad \psi' = (z',t') = (x_1', \dots, x_n', y_1', \dots, y_n', t').$$

In $\mathbb{H}^n$ the natural origin is denoted by $O = (0,0)$. Define

$$r(\psi) = R(z,t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } \psi = (z,t) \in \mathbb{H}^n,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^{2n}$. The Korányi norm is homogeneous of degree 1 with respect to the dilations

$$\delta_R\colon (z,t) \mapsto (Rz, R^2 t), \quad R > 0.$$

Indeed, for all $\psi = (z,t) \in \mathbb{H}^n$,

$$r(\delta_R(\psi)) = r(Rz, R^2 t) = (|Rz|^4 + R^4 t^2)^{1/4} = Rr(\psi).$$

The Korányi distance is

$$d_K(\psi, \psi') = r(\psi^{-1} \circ \psi') \quad \text{for all } (\psi, \psi') \in \mathbb{H}^n \times \mathbb{H}^n,$$

and the Korányi open ball of radius R centered at ψ_0 is

$$B_R(\psi_0) = \{\psi \in \mathbb{H}^n : d_K(\psi, \psi_0) < R\},$$

where B_R denotes the ball of radius R centered at $\psi_0 = O$.

The Jacobian determinant of δ_R is R^{2n+2}. The natural number $Q = 2n + 2$, which is the so-called homogeneous dimension of $\mathbb{H}^n$, plays a role analogous to the topological dimension in the Euclidean context, see [19, 32] and the references therein.

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. It is invariant under left translations and Q-homogeneous with respect to dilations. Hence, as noted in [18], the topological dimension $2n+1$ of $\mathbb{H}^n$ is strictly less than its Hausdorff dimension $Q = 2n + 2$. We denote by $|E|$ the $(2n + 1)$-dimensional Lebesgue measure of any measurable set $E \subset \mathbb{H}^n$. Then,

$$|\delta_R(E)| = R^Q |E|, \quad d(\delta_R \psi) = R^Q \, d\psi.$$

In particular, if $E = B_R$, then $|B_R| = |B_1| R^Q$.

The vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$

constitute a basis for the real Lie algebra of invariant vector fields on $\mathbb{H}^n$. This basis satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk} \partial/\partial t$, all other commutators being zero. A vector field in the span of $\{X_j, Y_j\}_{j=1}^n$ will be called horizontal.

Let $u \in C^1(\mathbb{H}^n)$ be fixed. The horizontal gradient $D_{\mathbb{H}^n} u$ is

$$D_{\mathbb{H}^n} u = \sum_{j=1}^{n} [(X_j u) X_j + (Y_j u) Y_j],$$

that is, an element of the span of $\{X_j, Y_j\}_{j=1}^n$ is the basis of the horizontal left invariant vector fields. In other words, the vector

$$D_{\mathbb{H}^n} u = (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u)$$

denotes the horizontal gradient of u. Furthermore, if $f \in C^1(\mathbb{R})$, then $D_{\mathbb{H}^n} f(u) = f'(u) D_{\mathbb{H}^n} u$.

The natural inner product in the span of $\{X_j, Y_j\}_{j=1}^n$

$$(W, Z)_{\mathbb{H}^n} = \sum_{j=1}^{n} (w^j z^j + \widetilde{w}^j \widetilde{z}^j)$$

for $W = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ and $Z = \{z^j X_j + \widetilde{z}^j Y_j\}_{j=1}^n$ produces the Hilbertian norm

$$|D_{\mathbb{H}^n} u|_{\mathbb{H}^n} = \sqrt{(D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} u)_{\mathbb{H}^n}}$$

for the horizontal vector field $D_{\mathbb{H}^n} u$.

For any horizontal vector field $W = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$ the horizontal divergence is defined by

$$\operatorname{div}_{\mathbb{H}^n} W = \sum_{j=1}^n [X_j(w^j) + Y_j(\widetilde{w}^j)].$$

Similarly, if $u \in C^2(\mathbb{H}^n)$, then the Kohn–Spencer Laplacian, or equivalently the horizontal Laplacian, or the sub-Laplacian, in $\mathbb{H}^n$, of u is

$$\begin{aligned} \Delta_{\mathbb{H}^n} u &:= \sum_{j=1}^n (X_j^2 + Y_j^2) u \\ &= \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j^2} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial x_j \partial t}. \end{aligned}$$

A well known generalization of the Kohn–Spencer Laplacian is the horizontal p-Laplacian on the Heisenberg group $p \in (1, \infty)$, defined by

$$\Delta_{\mathbb{H}^n}^p u = \operatorname{div}_{\mathbb{H}^n}(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u) \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n).$$

Let us now review some classical facts about the first order Sobolev spaces on the Heisenberg group $\mathbb{H}^n$. We just consider the special case in which $1 \le p < Q$ and Ω is an open set in $\mathbb{H}^n$.

Definition 2.1. $HW^{1,p}(\Omega)$ denotes the horizontal Sobolev space consisting of the functions $u \in L^p(\Omega)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^p(\Omega)$. This space endowed with the natural norm

$$\|u\|_{HW^{1,p}(\Omega)}^p := \|u\|_{L^p(\Omega)}^p + \|D_{\mathbb{H}^n} u\|_{L^p(\Omega)}^p,$$

where $\|u\|_{L^p(\Omega)}^p := \int_\Omega |u|^p \, d\psi$ and $\|D_{\mathbb{H}^n} u\|_{L^p(\Omega)}^p := \int_\Omega |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^p \, d\psi$.

Thanks to [11, 12, 14, 15, 21] we know that if $1 \le p < Q$, then the embedding

$$HW^{1,p}(\Omega) \hookrightarrow L^s(\Omega) \quad \text{for all } s \in [p, p^*],\ p^* = \frac{Qp}{Q-p}$$

is continuous and it is compact, provided that $1 \le s < p^*$.

Also we have the following theorem from [25].

Theorem 2.2. *$C_0^\infty(\mathbb{H}^n)$ is dense in $HW^{1,p}(\mathbb{H}^n)$ for every p with $1 \le p < \infty$.*

Definition 2.3. The space $HW_0^{1,p}(\Omega)$ is defined by the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^p_{HW_0^{1,p}(\Omega)} := \int_\Omega |D_{\mathbb{H}^n} u|^p_{\mathbb{H}^n}\, d\psi.$$

Notice that [23, Theorem 1] implies that

$$\int_{\mathbb{H}^n} \psi^p \frac{|\phi|^p}{r^p}\, d\xi \le \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{H}^n} |D_{\mathbb{H}^n}\phi|^p_{\mathbb{H}^n}\, d\xi \tag{2.1}$$

for all $\phi \in C_0^\infty(\mathbb{H}^n \backslash \{O\})$ with $O = (0,0)$ the natural origin in $\mathbb{H}^n$. The best Hardy–Sobolev constant $H_p = H(p, Q)$ is given by

$$H_p = \inf_{\substack{u \in S^{1,p}(\mathbb{H}^n)\\ u \neq 0}} \frac{\|D_{\mathbb{H}^n} u\|_p^p}{\|u\|^p_{H_p}}, \quad \|u\|^p_{H_p} = \int_{\mathbb{H}^n} \psi^p \frac{|\phi|^p}{r^p}\, d\xi,$$

where $S^{1,p}(\mathbb{H}^n)$ is the Folland–Stein space, defined as the completion of $C_0^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|D_{\mathbb{H}^n} u\|_p^p = \int_{\mathbb{H}^n} |D_{\mathbb{H}^n} u|^p_{\mathbb{H}^n}\, d\xi.$$

Clearly, $H_p > 0$ thanks to (2.1).

Notice that the embedding

$$S^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$$

for every $s \in (1, p^*)$ is compact. However, if $s = p^*$, then the embedding is only continuous (see [2, 32]).

Before ending this section we recall the definition of pseudomonotone map.

Definition 2.4. The map $A\colon X \to X^*$ is called pseudomonotone if for each sequence $\{u_n\} \subset X$ satisfying $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$, it holds

$$\langle A(v), u - v\rangle \le \liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle \quad \text{for all } v \in X.$$

3. Generalized variational solution

Let E be a separable and reflexive Banach space. We recall that E is separable if there exists a countable dense subset $\{h_i\}_{i\ge1}$ of E. It is said that a sequence of finite dimensional subspaces $(E_n)_{n=1}^\infty \subset E$ has the approximation property if

$$E_n \subset E_{n+1} \ \text{ for } n \ge 1 \quad \text{and} \quad \overline{\bigcup_{n=1}^\infty E_n} = E.$$

There always exists such a sequence $(E_n)_{n=1}^{\infty}$ by defining E_n for $n \in \mathbb{N}$ as the linear hull of $\{h_1, \dots, h_n\}$.

Let $A\colon E \to E^*$ be a potential operator, which means that $A = J'$ (the differential of J) for a Gâteaux differentiable function $J\colon E \to \mathbb{R}$ called the potential of A. Note that the critical points of J coincide with the solutions to the equation

$$A(u) = 0, \tag{3.1}$$

or equivalently (in the weak sense)

$$\langle A(u), v\rangle = 0 \quad \text{for all } v \in E. \tag{3.2}$$

Taking advantage of the variational structure of problem (3.1) involving the functional J, the following definition sets forth a new type of solution.

Definition 3.1. An element $u \in E$ is said to be a generalized variational solution to problem (3.1) if there exists a sequence of finite dimensional subspaces $(E_n)_{n=1}^{\infty} \subset E$ with the approximation property and a sequence of elements $(u_n)_{n=1}^{\infty}$ with $u_n \in E_n$ such that

(a) $u_n \rightharpoonup u$ in E as $n \to \infty$;

(b) $\inf_{v\in E_n} J(v) = J(u_n)$;

(c) $A(u_n) \rightharpoonup 0$ in E^* and $\langle A(u_n), u_n - u\rangle \to 0$.

We quote the following abstract result from [13, Theorem 4] (stated here in the particular case $k = 0$).

Theorem 3.2. *Assume that the operator $A\colon E \to E^*$ is bounded (i.e., A maps bounded sets into bounded sets) with a coercive potential $J\colon E \to \mathbb{R}$ (i.e., $\lim_{\|u\|\to\infty} J(u) = +\infty$). Then problem* (3.1) *has at least one generalized variational solution in the sense of Definition* 3.1.

4. Generalized variational solutions for competing Finsler operator

In this section, we prove the existence of generalized variational solutions for problems (1.1) by Theorem 1.3, i.e., we present the proof of Theorem 1.3.

We apply Theorem 3.2 taking with $E = HW_0^{1,p}(\Omega)$ and $A\colon HW_0^{1,p}(\Omega) \to HW^{-1,p'}(\Omega)$ given by

$$\begin{aligned} Au = &-\operatorname{div}_{\mathbb{H}^n}\left(g_1\left(\frac{1}{p}F^p(D_{\mathbb{H}^n}u)\right)F^{p-1}(D_{\mathbb{H}^n}u)\,D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}u)\right)\\ &+\operatorname{div}_{\mathbb{H}^n}\left(g_2\left(\frac{1}{q}F^q(D_{\mathbb{H}^n}u)\right)F^{q-1}(D_{\mathbb{H}^n}u)\,D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}u)\right) - h(x,u) \end{aligned} \tag{4.1}$$

for all $u \in HW_0^{1,p}(\Omega)$. Observe that problem (1.1) can be written as the operator equation (3.1) with A in (4.1).

By Hölder's inequality, for all $u \in HW_0^{1,p}(\Omega)$, we have

$$\|F(D_{\mathbb{H}^n}u)\|^q_{L^q(\Omega)} \le |\Omega|^{(p-q)/p}\|F(D_{\mathbb{H}^n}u)\|^q_{L^p(\Omega)}.$$

Since for the function F there exist two constants $0 < a < b < +\infty$ such that $a|\xi| \le F(\xi) \le b|\xi|$ for all $\xi \in \mathbb{R}^{2n}$, the operator $-\mathcal{Q}_p^{g_1} + \mathcal{Q}_q^{g_2}$ is well defined, continuous and bounded on $HW_0^{1,p}(\Omega)$.

The Carathéodory function $h(\xi,t)$ entering equation (1.1) determines the Nemytskij operator $N_h\colon HW_0^{1,p}(\Omega) \to HW^{-1,p'}(\Omega)$ by

$$N_h(u) = h(\,\cdot\,, u(\cdot)) \quad \text{for all } u \in HW_0^{1,p}(\Omega).$$

Assumption (H1), Hölder's inequality and Sobolev embedding theorem imply that there is a constant $C > 0$ such that

$$\begin{aligned}\int_\Omega |h(\xi,w(\xi))v(\xi)|\,d\xi &\le \int_\Omega |\sigma(\xi)|v(\xi)\,d\xi + b\int_\Omega |w(\xi)|^{p^*-1}v(\xi)\,d\xi\\ &\le C\big(\|\sigma\|_{L^{(p^*)'}(\Omega)} + \|w\|^{p^*-1}_{L^{p^*}(\Omega)}\big)\|\nabla v\|_{L^p(\Omega)}\end{aligned}$$

for all $v, w \in HW^{1,p}(\Omega)$. Hence, for all $w \in HW_0^{1,p}(\Omega)$ we have

$$\|N_h(w)\|_{HW^{-1,p'}(\Omega)} \le C\big(\|\sigma\|_{L^{(p^*)'}(\Omega)} + \|w\|^{p^*-1}_{L^{p^*}(\Omega)}\big). \tag{4.2}$$

We infer from (4.2) that the operator $N_h\colon HW_0^{1,p}(\Omega) \to HW^{-1,p'}(\Omega)$ is well defined in bounded. Taking into account (4.1), it follows that the operator $A = -\mathcal{Q}_p^{g_1} + \mathcal{Q}_q^{g_2} - N_h$ is well defined and bounded from $HW_0^{1,p}(\Omega)$ to $HW^{-1,p'}(\Omega)$.

We are going to show that the operator A in (4.1) is potential. To this end, we define the functional $J\colon HW_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega G_1\left(\frac{1}{p}F^p(\nabla u)\right)d\xi - \int_\Omega G_2\left(\frac{1}{q}F^q(\nabla u)\right)d\xi - \int_\Omega H(x,u(\xi))\,d\xi \tag{4.3}$$

for all $u \in HW_0^{1,p}(\Omega)$, where

$$G_i(t) = \int_0^t g_i(\xi,s)\,ds$$

for a.e. $\xi \in \Omega$, $i = 1,2$ and all $t \in \mathbb{R}$.

The boundedness of g_i implies the existence of a constant $c > 0$ with $|G_i(t)| \le c(|t|+1)$ for all $t \in \mathbb{R}$. Then, arguing on the basis of assumption (H1), we can prove through

Lebesgue's dominated convergence theorem that the functional J in (4.3) is Gâteaux differentiable with the differential

$$
\begin{aligned}
\langle J'(u), v\rangle_{\mathbb{H}^n} &= \int_\Omega g_1\left(\frac{1}{p}F^p(D_{\mathbb{H}^n}u)\right)F^{p-1}(D_{\mathbb{H}^n}u)\langle D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}u), D_{\mathbb{H}^n}v\rangle_{\mathbb{H}^n}\,d\xi \\
&\quad - \int_\Omega g_2\left(\frac{1}{q}F^q(D_{\mathbb{H}^n}u)\right)F^{q-1}(D_{\mathbb{H}^n}u)\langle D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}u), D_{\mathbb{H}^n}v\rangle_{\mathbb{H}^n}\,d\xi \\
&\quad - \int_\Omega h(\xi, u(\xi))v(\xi)\,d\xi
\end{aligned} \tag{4.4}
$$

for all $v \in HW_0^{1,p}(\Omega)$.

By (4.1) and (4.4) we note that $Au = J'(u)$ for all $u \in HW_0^{1,p}(\Omega)$. As a consequence, we can infer that A in (4.1) is a potential operator with the potential J given by (4.3).

Now we focus on the coerciveness of the functional J in (4.3). Assumption (H2) and (1.3) imply that

$$\int_\Omega H(\xi, u(\xi))\,d\xi \le c_1\big(\lambda_1^{-1}\|F^p(D_{\mathbb{H}^n}u)\|^p_{L^p(\Omega)} + |\Omega|\big).$$

Then, in view of (4.3), $G_1(t) \ge a_{g_1}t$ and $G_2(t) \le b_{g_2}t$ for all $t \ge 0$, we are led to

$$J(u) \ge \left(\frac{a_{g_1}}{p} - c_1\lambda_1^{-1}\right)\|F(D_{\mathbb{H}^n}u)\|^p_{L^p(\Omega)} - \frac{b_{g_2}}{q}|\Omega|^{(p-q)/p}\|F(D_{\mathbb{H}^n}u)\|^q_{L^p(\Omega)} - c_1|\Omega|$$

for all $u \in HW_0^{1,p}(\Omega)$. Since $q < p$ and $c_1 < \frac{a_{g_1}\lambda_1}{p}$, we obtain that J is coercive.

All the hypotheses required to apply Theorem 3.2 to the functional J in (4.3) are fulfilled. Then the existence of a generalized variational solution to problem $Au = 0$ with A given in (4.1) is established. This completes the proof concerning the original problem (1.1).

5. System version of double phase problem

In this section we consider the Dirichlet problem

$$
\begin{cases}
-\mathcal{Q}^{g_1}_{p_1}u + \mathcal{Q}^{g_2}_{q_1}u = h_u(x,u,v) & \text{in } \Omega,\\
-\mathcal{Q}^{g_1}_{p_2}v + \mathcal{Q}^{g_2}_{q_2}v = h_v(x,u,v) & \text{in } \Omega,\\
u = v = 0 & \text{on } \partial\Omega,
\end{cases} \tag{5.1}
$$

on a bounded domain Ω in $\mathbb{H}^n$, $n \ge 3$ and a Lipschitz boundary $\partial\Omega$. Also, we assume $1 < \min\{q_1, q_2, p_2\} < \max\{q_1, q_2, p_2\} < p_1$ and

(H3) The function $H \in C^1(\Omega)$;

(H4) There exists a positive constant $M < \frac{\lambda_1 a_{g_1}}{p_1}$ and $\sigma \in L^{(p_1^*)'}(\Omega)$ with $p_1^* = \frac{Qp_1}{Q-p_1}$ such that

$$|H_\zeta(x,\zeta,\eta)\zeta + H_\eta(x,\zeta,\eta)\eta| \le M(1 + |\zeta|^{p_1^*} + |\eta|^{p_1^*})$$

for a.e. $x \in \Omega$ and for all $\zeta, \eta \in \mathbb{R}$;

(H5) There exists a positive constant M such that

$$|H(x,\zeta,\eta)| \le M(1 + |\zeta|^{p_1} + |\eta|^{p_1})$$

for a.e. $x \in \Omega$ and for all $\zeta, \eta \in \mathbb{R}$;

(H6) $H_\zeta(x,\zeta,\eta)\zeta + H_\eta(x,\zeta,\eta)\eta \ge p_1 H(x,\zeta,\eta)$.

Now we can state our second result regarding problem (5.1).

Theorem 5.1. *Suppose* $1 < \min\{q_1, q_2, p_2\} < \max\{q_1, q_2, p_2\} < p_1 < Q$. *Assume that* $h\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function for which the conditions* (H3)–(H6) *hold. Then there exists a generalized variational solution to problem* (5.1).

Proof. We set $X := HW_0^{1,p_1}(\Omega) \times HW_0^{1,p_2}(\Omega)$ and define the functional $J\colon X \to \mathbb{R}$ by

$$\begin{aligned} J(u,v) &= \int_\Omega G_1\left(\frac{1}{p_1}F^{p_1}(\nabla u)\right) d\xi - \int_\Omega G_2\left(\frac{1}{q_1}F^{q_1}(\nabla u)\right) d\xi \\ &\quad + \int_\Omega G_1\left(\frac{1}{p_2}F^{p_2}(\nabla v)\right) d\xi - \int_\Omega G_2\left(\frac{1}{q_2}F^{q_2}(\nabla v)\right) d\xi \\ &\quad - \int_\Omega H(x,u(\xi),v(\xi))\, d\xi \end{aligned} \tag{5.2}$$

for all $(u,v) \in X$.

By the same argument as before we can show that the functional J in (5.2) is Gâteaux differentiable with the differential

$$\begin{aligned} &\langle J'(u,v),(\phi,\psi)\rangle_{\mathbb{H}^n} \\ &= \int_\Omega g_1\left(\frac{1}{p_1}F^{p_1}(D_{\mathbb{H}^n}u)\right) F^{p_1-1}(D_{\mathbb{H}^n}u)\langle D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}u), D_{\mathbb{H}^n}\phi\rangle_{\mathbb{H}^n}\, d\xi \\ &\quad - \int_\Omega g_2\left(\frac{1}{q_1}F^{q_1}(D_{\mathbb{H}^n}u)\right) F^{q_1-1}(D_{\mathbb{H}^n}u)\langle D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}u), D_{\mathbb{H}^n}\phi\rangle_{\mathbb{H}^n}\, d\xi \\ &\quad + \int_\Omega g_1\left(\frac{1}{p_2}F^{p_2}(D_{\mathbb{H}^n}v)\right) F^{p_2-1}(D_{\mathbb{H}^n}v)\langle D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}v), D_{\mathbb{H}^n}\psi\rangle_{\mathbb{H}^n}\, d\xi \\ &\quad - \int_\Omega g_2\left(\frac{1}{q_2}F^{q_2}(D_{\mathbb{H}^n}v)\right) F^{q_2-1}(D_{\mathbb{H}^n}v)\langle D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}v), D_{\mathbb{H}^n}\psi\rangle_{\mathbb{H}^n}\, d\xi \\ &\quad - \int_\Omega H_u(\xi,u(\xi),v(\xi))\phi(\xi) + H_v(\xi,u(\xi),v(\xi))\psi(\xi)\, d\xi \end{aligned} \tag{5.3}$$

for all $(\phi, \psi) \in X$.

We set $A\colon X \to HW^{-1,p_1'}(\Omega) \times HW^{-1,p_2'}(\Omega)$ by

$$
\begin{aligned}
A(u,v) = &- \operatorname{div}_{\mathbb{H}^n}\left(g_1\left(\frac{1}{p_1}F^{p_1}(D_{\mathbb{H}^n}u)\right)F^{p_1-1}(D_{\mathbb{H}^n}u)\,D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}u)\right)\\
&+ \operatorname{div}_{\mathbb{H}^n}\left(g_2\left(\frac{1}{q_1}F^{q_1}(D_{\mathbb{H}^n}u)\right)F^{q_1-1}(D_{\mathbb{H}^n}u)\,D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}u)\right)\\
&- \operatorname{div}_{\mathbb{H}^n}\left(g_1\left(\frac{1}{p_2}F^{p_2}(D_{\mathbb{H}^n}v)\right)F^{p_2-1}(D_{\mathbb{H}^n}v)\,D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}v)\right)\\
&+ \operatorname{div}_{\mathbb{H}^n}\left(g_2\left(\frac{1}{q_2}F^{q_2}(D_{\mathbb{H}^n}v)\right)F^{q_2-1}(D_{\mathbb{H}^n}v)\,D_{\mathbb{H}^n}F(D_{\mathbb{H}^n}v)\right)\\
&- H_u(x,u,v)u - H_v(x,u,v)v
\end{aligned}
\tag{5.4}
$$

for all $(u,v) \in X$. Then by (5.4) and (5.3) we note that $A(u,v) = J'(u,v)$ for all $(u,v) \in X$. As a consequence, we can infer that A in (5.4) is a potential operator with the potential J given by (5.2).

By Hölder's inequality, for all $v \in HW_0^{1,p_1}(\Omega) \subset HW_0^{1,p_2}(\Omega)$ $(p_2 < p_1)$, we have

$$
\|F(D_{\mathbb{H}^n}u)\|^{p_2}_{L^{p_2}(\Omega)} \le |\Omega|^{(p_1-p_2)/p_1}\|F(D_{\mathbb{H}^n}v)\|^{p_2}_{L^{p_1}(\Omega)}.
$$

Assumption (H5) and (1.3) imply that

$$
\begin{aligned}
&\int_\Omega H(\xi, u(\xi), v(\xi))\,d\xi\\
&\le M\big(\lambda_{p_1 1}^{-1}\|F^{p_1}(D_{\mathbb{H}^n}u)\|^{p_1}_{L^{p_1}(\Omega)} + \lambda_{p_2 1}^{-1}\|F^{p_2}(D_{\mathbb{H}^n}v)\|^{p_2}_{L^{p_2}(\Omega)} + |\Omega|\big)\\
&\le M\big(\lambda_{p_1 1}^{-1}\|F^{p_1}(D_{\mathbb{H}^n}u)\|^{p_1}_{L^{p_1}(\Omega)} + \lambda_{p_2 1}^{-1}|\Omega|^{(p_1-p_2)/p_1}\|F(D_{\mathbb{H}^n}v)\|^{p_2}_{L^{p_1}(\Omega)} + |\Omega|\big).
\end{aligned}
$$

Then, in view of (5.2), $G_1(t) \ge a_{g_1}t$ and $G_2(t) \le b_{g_2}t$ for all $t \ge 0$, we are led to

$$
\begin{aligned}
J(u,v) \ge &\left(\frac{a_{g_1}}{p_1} - M\lambda_{p_1 1}^{-1}\right)\|F(D_{\mathbb{H}^n}u)\|^{p_1}_{L^{p_1}(\Omega)} - \frac{b_{g_2}}{q_1}|\Omega|^{(p_1-q_1)/p_1}\|F(D_{\mathbb{H}^n}u)\|^{q_1}_{L^{p_1}(\Omega)}\\
&+ \left(\frac{a_{g_1}}{p_2} - M\lambda_{p_2 1}^{-1}\right)|\Omega|^{(p_1-p_2)/p_1}\|F(D_{\mathbb{H}^n}u)\|^{p_2}_{L^{p_1}(\Omega)}\\
&- \frac{b_{g_2}}{q_1}|\Omega|^{(p_1-q_2)/p_1}\|F(D_{\mathbb{H}^n}u)\|^{q_2}_{L^{p_1}(\Omega)} - c_1|\Omega|
\end{aligned}
$$

for all $(u,v) \in X$, where $\lambda_{p_i 1}$ is given by (1.3). Since $1 < \min\{q_1, q_2, p_2\} < \max\{q_1, q_2, p_2\} < p_1 < Q$ and $M < \frac{a_{g_1}\lambda_1}{p_1}$, we obtain that J is coercive.

The rest of proof is the same argument in the proof of Theorem 1.3. □

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