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Automorphism Groups of Certain Orbifold Vertex Operator Algebras Arising from Coinvariant Lattices Associated with the Leech Lattice

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Abstract. We determine the automorphism groups of the orbifold vertex operator algebras associated with the coinvariant lattices of isometries of the Leech lattice in the conjugacy classes $3C$, $5C$, $11A$, and $23A$. These orbifold vertex operator algebras appear in a classification given by Lam and Shimakura.

1. Introduction

The *orbifold* of a vertex operator algebra (VOA) for an automorphism group is the fixed-point subVOA. It is natural to ask if all automorphisms of the orbifold VOA can be obtained from the original VOA. Let us explain it precisely. For a VOA V and an automorphism group G of V, V^G is defined by the set of all fixed-points. We naturally obtain a group homomorphism from the normalizer $N_{\mathrm{Aut}(V)}(G)$ to $\mathrm{Aut}(V^G)$. The main question is if the automorphism group $\mathrm{Aut}(V^G)$ is isomorphic to a quotient group of $N_{\mathrm{Aut}(V)}(G)$ or not. An automorphism of V^G is called *extra* if it cannot be obtained from the image of the normalizer $N_{\mathrm{Aut}(V)}(G)$. It is important to determine when the orbifold VOA has extra automorphisms.

The lattice VOA V_L associated with a positive-definite even lattice L is a significant example of VOA. Let g be a fixed-point free isometry of L. Then g can be lifted to an element $\widehat{g}$ of $\mathrm{Aut}(V_L)$. Let $V_L^{\widehat{g}}$ denote the orbifold VOA $V_L^{\langle \widehat{g} \rangle}$. The case where the order of g is 2, that is, g is the -1-isometry and that L is rootless, has been studied in [27]; $V_L^{\widehat{g}}$ has extra automorphisms if and only if L can be constructed by Construction B from a doubly even binary code. In [24], the case where g is a fixed-point free isometry of odd prime order p was treated and Lam and Shimakura classified all rootless even lattices such that $V_L^{\widehat{g}}$ has extra automorphisms. As for the above, Lam and Shimakura also classified rootless even lattices L such that the τ-conjugate $V_L(1) \circ \tau$ is of twisted type for some $\tau \in \mathrm{Aut}(V_L^{\widehat{g}})$ (see Definition 2.7 for the definition of $V_L(1) \circ \tau$). The following is the classification.

Theorem 1.1. [24, Main Corollary 2] *Let L be a positive-definite rootless even lattice. Let g be a fixed-point free isometry of L of prime order p and let $\widehat{g}$ be a standard lift of g. Then the following are equivalent:*

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- *There exists an automorphism τ of $V_L^{\widehat{g}}$ such that $V_L(1) \circ \tau$ is of twisted type;*
- *L is isometric to the coinvariant lattice Λ_{pX} of the Leech lattice Λ associated with the conjugacy class $pX \in \{2A, -2A, 3B, 3C, 5B, 5C, 7B, 11A, 23A\}$.*

On the other hand, the orbifold VOA $V_{\Lambda_g}^{\widehat{g}}$ associated with the coinvariant lattice Λ_g is related to the classification of holomorphic VOAs of central charge 24. Here, Λ denotes the Leech lattice and g is some isometry of the Leech lattice, and $\widehat{g}$ is a lift of g. In [18], Höhn suggested that a holomorphic VOA V of central charge 24 with $V_1 \neq 0$ can be viewed as a simple current extension of the tensor product VOA $V_{L_{\mathfrak{g}}} \otimes V_{\Lambda_g}^{\widehat{g}}$, where $V_{L_{\mathfrak{g}}}$ is the lattice VOA related to the root lattice of $\mathfrak{g} = V_1$. Additionally, Höhn described the possible isometries g in [18, Table 4].

The orbifold VOA $V_{\Lambda_g}^{\widehat{g}}$ appears in the table above, where $pX \in \{2A, 2C, 3B, 5B, 7B\}$ and $g \in pX$. Such VOAs are useful to analyze the structures of holomorphic VOAs of central charge 24. Indeed, the automorphism groups of the five cases are important in the study of holomorphic VOAs of central charge 24 (see [2]). Therefore, it will be useful to determine the automorphism groups of the orbifold VOAs having the rich symmetry in Theorem 1.1. In [8, 21, 27], the automorphism groups associated with $2A$, $-2A$, $3B$, $5B$, and $7B$ were determined. So far the others have not been determined.

In this paper, we determine all of the remaining automorphism groups of the orbifold VOAs $V_{\Lambda_{pX}}^{\widehat{g}}$, where $pX \in \{3C, 5C, 11A, 23A\}$.

For $pX \in \{3C, 5C, 11A, 23A\}$, we describe how to determine the automorphism group of $V_L^{\widehat{g}}$, where $L = \Lambda_{pX}$. The automorphism group $\mathrm{Aut}(V_L^{\widehat{g}})$ acts on the set $\mathrm{Irr}(V_L^{\widehat{g}})$ of all isomorphism classes of irreducible $V_L^{\widehat{g}}$-modules. Moreover, under some assumptions, this action preserves a non-degenerate quadratic form q of $\mathrm{Irr}(V_L^{\widehat{g}})$ (see Theorem 3.6 and the beginning of Section 4). Hence, we obtain a group homomorphism μ from $\mathrm{Aut}(V_L^{\widehat{g}})$ to the orthogonal group $O(\mathrm{Irr}(V_L^{\widehat{g}}), q)$

$$\mu\colon\ \mathrm{Aut}(V_L^{\widehat{g}}) \longrightarrow O(\mathrm{Irr}(V_L^{\widehat{g}}), q).$$

To determine the group structure of the automorphism group $\mathrm{Aut}(V_L^{\widehat{g}})$, we determine the group structures of $\mathrm{Ker}\,\mu$ and $\mathrm{Im}\,\mu$. To determine the group structure of $\mathrm{Ker}\,\mu$, we prove a generalization of [27, Lemma 3.7] (see Theorem 4.1).

Theorem 1.2. *Let L be a positive-definite rootless even lattice and let p be an odd prime. Let g be a fixed-point free isometry of L of order p and let $\widehat{g}$ be a standard lift of g. If $(1-g)L^* \subset L$ and the conformal weight $\varepsilon(V_L[\widehat{g}])$ is in $(1/p)\mathbb{Z}$ (for the definition of $\varepsilon(V_L[\widehat{g}])$, see Subsections 2.3 and 3.2), then we have the following exact sequence:*

$$1 \longrightarrow A \longrightarrow \mathrm{Ker}\,\mu \xrightarrow{\widetilde{\varphi}} B \longrightarrow 1,$$

where $A = \mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_p)$*,* $B = \{h \in C_{O(L)}(g) \mid h = 1$ *on* $L^*/L\}/\langle g\rangle$*, and the group homomorphism* μ *is as in* (4.1)*.*

To determine the group structure of $\mathrm{Im}\,\mu$, we determine the index of $\mathrm{Im}\,\mu$ in the orthogonal group of the quadratic space $(\mathrm{Irr}(V_L^{\widehat{g}}), q)$ and the group structure of the stabilizer $\mathrm{Stab}_{\mathrm{Im}\,\mu}(V_L(1))$. To do this, we examine the orbit $\mathrm{Orbit}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))$ of $V_L(1)$ and the group structure of the stabilizer $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))$. In the cases of $11A$ and $23A$, to narrow down the possibilities for $\mathrm{Im}\,\mu$, we use the classifications of the maximal subgroups of $\Omega_4^-(11)$ and $\Omega_3(23)$, respectively.

The following theorem describes the group structures of the remaining automorphism groups of the orbifold VOAs:

Theorem 1.3. *Let* $g \in pX$*, where* $pX \in \{3C, 5C, 11A, 23A\}$*. Let* L *be the coinvariant lattice* Λ_{pX} *associated with* g *and let* $\widehat{g}$ *be a standard lift of* g *as in Subsection* 2.4*. Then the automorphism groups of the orbifold VOAs* $V_L^{\widehat{g}}$ *are the following:*

- $\mathrm{Aut}(V_{\Lambda_{3C}}^{\widehat{g}}) \cong (3^4\,.\,(3^4:2))\,.\,(\Omega_7(3)\,.\,2)$*;*
- $\mathrm{Aut}(V_{\Lambda_{5C}}^{\widehat{g}}) \cong (5^2\,.\,(5^2:2))\,.\,(2\times\Omega_5(5))$*;*
- $\mathrm{Aut}(V_{\Lambda_{11A}}^{\widehat{g}}) \cong \Omega_4^-(11)\,.\,2$*;*
- $\mathrm{Aut}(V_{\Lambda_{23A}}^{\widehat{g}}) \cong \Omega_3(23)\,.\,2$*.*

2. Preliminaries

2.1. Lattices

A *lattice* means a free abelian group of finite rank with a rational valued, positive-definite symmetric bilinear form $(\cdot|\cdot)$. The *rank* of a lattice L means the rank as a free abelian group, which is denoted by $\mathrm{rank}\,L$. Let L be a lattice with a positive-definite bilinear form $(\cdot|\cdot)$. The symbol $O(L)$ denotes the isometry group of L. The *dual lattice* is defined by $L^* = \{\alpha \in \mathbb{Q}\otimes_{\mathbb{Z}} L \mid (\alpha|L) \subset \mathbb{Z}\}$. A lattice is said to be *integral* if $(\alpha|\beta) \in \mathbb{Z}$ for any $\alpha, \beta \in L$. A lattice is said to be *even* if $(\alpha|\alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$. If L is even, then L is integral and $L \subset L^*$.

Let $g \in O(L)$. We define the *fixed-point sublattice* L^g of g and the *coinvariant lattice* L_g of L associated with g by $L^g = \{\alpha \in L \mid g\alpha = \alpha\}$ and $L_g = \{\alpha \in L \mid (\alpha|L^g) = 0\}$, respectively.

Let L be an even lattice. The *discriminant group* $\mathcal{D}(L)$ is defined by L^*/L, which is a finite abelian group. Let

$$q_L\colon \mathcal{D}(L) \to \mathbb{Q}/\mathbb{Z}, \quad \alpha + L \mapsto \frac{(\alpha|\alpha)}{2} + \mathbb{Z}.$$

Then q_L defines a quadratic form on $\mathcal{D}(L)$. The associated bilinear form on $\mathcal{D}(L)$ given by $(\alpha + L|\beta + L) = (\alpha|\beta) + \mathbb{Z}$ is non-degenerate. Hence $(\mathcal{D}(L), q_L)$ is a non-degenerate quadratic space. An element x of a quadratic space (V, q) is called *singular* if $x \neq 0$ and $q(x) = 0$.

Definition 2.1. Let p be a prime. An integral lattice L is said to be *p-elementary* if $pL^* \subset L$. An integral lattice L is said to be *unimodular* if $L^* = L$.

We recall results from [17, 22].

Lemma 2.2. [17, Lemma A.1] *Let L be a lattice and let g be a fixed-point free isometry of L of prime order p. Then we have*

$$L/(1-g)L \cong \mathbb{Z}_p^{\operatorname{rank} L/(p-1)}.$$

Lemma 2.3. [22, Lemma 4.2] *Let L be an even unimodular lattice and let $g \in O(L)$. Then we have $(1-g)L_g^* \subset L_g$.*

The following lemmas will be used later.

Lemma 2.4. *Let L be an even lattice and let g be a fixed-point free isometry of L of order n. If $(1-g)L^* \subset L$, then we have $nL^* \subset L$.*

Proof. Since g is fixed-point free, we see that $1-g$ is non-singular. Moreover, since $(1-g)(1+g+\cdots+g^{n-1}) = 0$, we have $1+g+\cdots+g^{n-1} = 0$. By the assumption $(1-g)L^* \subset L$, we see that $nL^* = -(1-g)\big(\sum_{k=1}^{n-1} kg^k\big)L^* \subset (1-g)L^* \subset L$. □

Lemma 2.5. *Let L be an even lattice and let g be a fixed-point free isometry of L of prime order p. If $(1-g)L^* \subset L$, then we have*

(1) $\mathcal{D}(L) \cong \mathbb{Z}_p^k$ *for some k.*

(2) $L/(1-g)L^* \cong \mathbb{Z}_p^{\operatorname{rank} L/(p-1)-k}$.

Proof. By the assumption $(1-g)L^* \subset L$, we have a group homomorphism $f\colon L/(1-g)L \to L/(1-g)L^*$ defined by $x + (1-g)L \mapsto x + (1-g)L^*$. By Lemma 2.2, we have $L/(1-g)L \cong \mathbb{Z}_p^{\operatorname{rank} L/(p-1)}$, which implies $\operatorname{Ker} f = (1-g)L^*/(1-g)L \cong \mathcal{D}(L) \cong \mathbb{Z}_p^k$ for some k. Since $(L/(1-g)L)/\operatorname{Ker} f \cong L/(1-g)L^*$, we have $L/(1-g)L^* \cong \mathbb{Z}_p^{\operatorname{rank} L/(p-1)-k}$. □

2.2. Groups

In this subsection, we describe groups used in this paper. Let G be a group. For a subgroup H of G, the symbols $N_G(H)$ and $C_G(H)$ denote the normalizer of H in G and the centralizer of H in G, respectively. Let n denote the cyclic group with order n. The symbol D_n denotes the dihedral group with order n. Notations of extraspecial groups follow [9].

According to [5], we describe subgroups of orthogonal groups over finite fields. Let F be a finite field with $\mathrm{Char}(F) \neq 2$ and let (V, q_V) be a non-degenerate quadratic space over F. A subspace U of V is called a *totally isotropic subspace* of V if the quadratic form vanishes on U. The dimension of a maximal totally isotropic subspace of V is called the *Witt index* of V, which is denoted by $m(V)$. Then V is expressed as a form $V = U \oplus W$, where U is a certain $2m(V)$-dimensional subspace of V called hyperbolic and W is a subspace which contains no singular vector. Let $\dim(V)$ be even. If the above W vanishes, then the quadratic space V is called $(+)$*-type*. Otherwise, V is called $(-)$*-type*.

We denote the orthogonal group and the special orthogonal group of the quadratic space V by $\mathrm{GO}(V)$ and $\mathrm{SO}(V)$, respectively. For a non-singular vector $v \in V \setminus \{0\}$, we define the reflection $r_v \colon V \to V$ by $x \mapsto x - (v|x)v/q_V(v)$, where $(x|y) = q_V(x+y) - q_V(x) - q_V(y)$ for $x, y \in V$. Note that $\mathrm{GO}(V)$ is generated by the set of reflections in non-singular vectors. For $g \in \mathrm{GO}(V)$, if $g = r_{v_1} r_{v_2} \cdots r_{v_k}$, where each v_i $(\neq 0)$ is non-singular vector, then the *spinor norm* of g is 1 if $q_V(v_1)q_V(v_2)\cdots q_V(v_k)$ is square in $F \setminus \{0\}$ and -1 if it is non-square. The symbol $\Omega(V)$ denotes the kernel of the spinor norm on $\mathrm{SO}(V)$. Set $\#F = q$. If $\dim(V) = 2m+1$, then $\mathrm{GO}(V)$, $\mathrm{SO}(V)$, and $\Omega(V)$ are also written by $\mathrm{GO}_{2m+1}(q)$, $\mathrm{SO}_{2m+1}(q)$, and $\Omega_{2m+1}(q)$, respectively.

Definition 2.6. The symbol $P_{2m+1}(q)$ denotes the subgroup $\Omega_{2m+1}(q) \cup (-1)\Omega_{2m+1}(q)$ of $\mathrm{GO}_{2m+1}(q)$, and the symbol $Q_{2m+1}(q)$ denotes the subgroup $\Omega_{2m+1}(q) \cup (-\sigma)\Omega_{2m+1}(q)$ of $\mathrm{GO}_{2m+1}(q)$, where σ is an element of $\mathrm{SO}_{2m+1}(q) \setminus \Omega_{2m+1}(q)$.

Similarly, if $\dim(V) = 2m$, then $\mathrm{GO}(V)$, $\mathrm{SO}(V)$, and $\Omega(V)$ are written by $\mathrm{GO}^{\varepsilon}_{2m}(q)$, $\mathrm{SO}^{\varepsilon}_{2m}(q)$, and $\Omega^{\varepsilon}_{2m}(q)$, respectively, where ε is $+$ if V is of $(+)$-type and $-$ if V is of $(-)$-type.

2.3. VOAs, modules and automorphisms

A *vertex operator algebra* (VOA) $(V, Y, \mathbb{1}, \omega)$ is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i$ over the complex field $\mathbb{C}$ equipped with a linear map for $a \in V$,

$$Y(a, z) = \sum_{i \in \mathbb{Z}} a_i z^{-i-1} \in (\mathrm{End}\, V)[[z, z^{-1}]],$$

the vacuum vector $\mathbb{1} \in V_0$, and the conformal vector $\omega \in V_2$ satisfying some axioms (see [16]).

Let V be a VOA. A linear automorphism τ of V is called an *automorphism* of V if the linear automorphism satisfies

(1) $\tau\omega = \omega$;

(2) $\tau Y(v, z) = Y(\tau v, z)\tau$ for all $v \in V$.

The symbol $\mathrm{Aut}(V)$ denotes the group of all automorphisms of V. For an automorphism τ of V, let $V^\tau = \{v \in V \mid \tau v = v\}$, which is called the *orbifold VOA*.

For a VOA V, a *V-module* (M, Y_M) is a $\mathbb{C}$-graded vector space $M = \bigoplus_{i \in \mathbb{C}} M_i$ equipped with a linear map for $a \in V$,

$$Y_M(a, z) = \sum_{i \in \mathbb{Z}} a_i z^{-i-1} \in (\mathrm{End}\, M)[[z, z^{-1}]]$$

satisfying some conditions (see [15]). We often denote a V-module (M, Y_M) by M.

Let V be a VOA and let M be an irreducible V-module. Then there exists $\varepsilon(M) \in \mathbb{C}$ such that $M = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} M_{\varepsilon(M)+m}$ and $M_{\varepsilon(M)}$ is not trivial. The number $\varepsilon(M)$ is called the *conformal weight* of M. For a VOA V, let $\mathrm{Irr}(V)$ denote the set of all isomorphism classes of irreducible V-modules. We often identify an element in $\mathrm{Irr}(V)$ with its representative. For irreducible V-modules M_1 and M_2, under certain assumptions, the fusion product $M_1 \boxtimes M_2$ is defined as in [19].

Definition 2.7. [11] Let V be a VOA and $\tau \in \mathrm{Aut}(V)$. For a V-module M, the *τ-conjugate* $(M \circ \tau, Y_{M\circ\tau}(\,\cdot\,, z))$ of M is defined as follows:

(1) $M \circ \tau = M$ as a vector space;

(2) $Y_{M\circ\tau}(a, z) = Y_M(\tau a, z)$ for any $a \in V$.

The τ-conjugation defines an action of $\mathrm{Aut}(V)$ on $\mathrm{Irr}(V)$. The following lemma follows by definition.

Lemma 2.8. *Let M, M^1, M^2 be V-modules and let $\tau \in \mathrm{Aut}(V)$.*

(1) *If M is irreducible, then so is $M \circ \tau$. In addition, both M and $M \circ \tau$ have the same conformal weight.*

(2) *Assume that the fusion product is defined on V-modules. Then $(M^1 \circ \tau) \boxtimes (M^2 \circ \tau) \cong (M^1 \boxtimes M^2) \circ \tau$. The τ-conjugation preserves the fusion product.*

2.4. Lattice VOAs and their automorphism groups

In this subsection, we review some facts about lattice VOAs and their automorphism groups from [12, 16, 25].

Let L be an even lattice with a bilinear form $(\,\cdot\,|\,\cdot\,)$. We consider the central extension $\widehat{L}$ of L with the commutator map defined by $(\,\cdot\,|\,\cdot\,) \bmod 2$. Let $\mathrm{Aut}(\widehat{L})$ be the automorphism group of $\widehat{L}$. For $\varphi \in \mathrm{Aut}(\widehat{L})$, we define the element $\overline{\varphi} \in \mathrm{Aut}(L)$ by $\overline{\varphi}(\alpha) = \overline{\varphi(e^\alpha)}$ for $\alpha \in L$. We denote $\{\varphi \in \mathrm{Aut}(\widehat{L}) \mid \overline{\varphi} \in O(L)\}$ by $O(\widehat{L})$.

Let $M(1)$ be the Heisenberg VOA associated with $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. Let $\mathbb{C}\{L\} = \bigoplus_{\alpha \in L} \mathbb{C}e^\alpha$ be the twisted group algebra such that $e^\alpha e^\beta = (-1)^{(\alpha|\beta)} e^\beta e^\alpha$ for $\alpha, \beta \in L$. The *lattice VOA* V_L associated with L is defined by $M(1) \otimes \mathbb{C}\{L\}$ (see [16]).

For $x \in \mathfrak{h}$, set $\sigma_x = \exp\big(-2\pi\sqrt{-1}x_0\big) \in \mathrm{Aut}(V_L)$ and let $N(V_L) = \big\langle \exp(a_0) \mid a \in (V_L)_1 \big\rangle$. Then σ_x is in $N(V_L)$ for $x \in \mathfrak{h}$. We remark that $\mathrm{Aut}(V_L) = N(V_L)\, O(\widehat{L})$ (see [12]).

An element $\phi \in O(\widehat{L})$ is called a *standard lift* of $g \in O(L)$ if $\overline{\phi} = g$ and $\phi(e^\alpha) = e^\alpha$ for $\alpha \in L^g$. The following lemma can be found in [14, 23].

Lemma 2.9. *Let $g \in O(L)$ and let $\widehat{g} \in O(\widehat{L})$ be a standard lift of g.*

(1) *Any standard lift of g is conjugate to $\widehat{g}$ by an element in* $\mathrm{Aut}(V_L)$.

(2) *If g is fixed-point free or has odd order, then $|\widehat{g}| = |g|$.*

We have the following exact sequences, which will be used later.

Theorem 2.10. [25, Theorem 5.15] *Let L be a rootless even lattice. Let g be a fixed-point free isometry of L of prime order p and let $\widehat{g}$ be a standard lift of g in $O(\widehat{L})$. Then we have the following exact sequences:*

$$1 \longrightarrow \mathrm{Hom}(L/(1-g)L, \mathbb{Z}_p) \longrightarrow N_{\mathrm{Aut}(V_L)}(\langle \widehat{g} \rangle) \longrightarrow N_{O(L)}(\langle g \rangle) \longrightarrow 1;$$
$$1 \longrightarrow \mathrm{Hom}(L/(1-g)L, \mathbb{Z}_p) \longrightarrow C_{\mathrm{Aut}(V_L)}(\widehat{g}) \xrightarrow{\varphi} C_{O(L)}(g) \longrightarrow 1.$$

3. Irreducible $V_L^{\widehat{g}}$-modules and their conjugates

Let L be an even lattice and let g be a fixed-point free isometry of L of prime order p. Let $\widehat{g} \in O(\widehat{L})$ be a standard lift of g. By Lemma 2.9, $\widehat{g}$ also has order p.

In [7, 26], it was proved that the orbifold VOA $V_L^{\widehat{g}}$ is rational, C_2-cofinite, self-dual, and of CFT-type. Under these conditions, the fusion product can be defined on $V_L^{\widehat{g}}$-modules.

By [13], any irreducible $V_L^{\widehat{g}}$-module is a submodule of an irreducible $\widehat{g}^s$-twisted V_L-module for some $0 \le s \le p-1$. Regarding the definition of twisted module, refer to [11].

Definition 3.1. [24, Definition 3.1] An irreducible $V_L^{\widehat{g}}$-module is said to be of *$\widehat{g}^s$-type* if it is a $V_L^{\widehat{g}}$-submodule of an irreducible $\widehat{g}^s$-twisted V_L-module. Additionally, it is said to be of *untwisted type* (resp. *twisted type*) if it is of $\widehat{g}^0$-type (resp. of $\widehat{g}^s$-type for some $1 \le s \le p-1$).

From here, throughout this section, we assume $(1-g)L^* \subset L$.

3.1. Irreducible $V_L^{\widehat{g}}$-modules of untwisted type

In this subsection, we discuss the irreducible $V_L^{\widehat{g}}$-modules of untwisted type.

Let $\lambda + L \in \mathcal{D}(L)$ and $V_{\lambda+L} = M(1) \otimes \mathrm{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in \lambda + L\}$. Then $V_{\lambda+L}$ has an irreducible V_L-module structure (see [16]). Since $(1-g)L^* \subset L$, we see that $g(\lambda) + L = \lambda + L$. This implies that $V_{\lambda+L}$ is $\widehat{g}$-*stable*, that is $V_{\lambda+L} \circ \widehat{g} \cong V_{\lambda+L}$. Let $\widehat{g}_{\lambda+L}$ be a $\widehat{g}$-module isomorphism of $V_{\lambda+L}$ of order p. For $0 \le j \le p-1$, set

$$V_{\lambda+L}(j) = \left\{x \in V_{\lambda+L} \mid \widehat{g}_{\lambda+L}(x) = \exp\left(2\pi\sqrt{-1}j/p\right)x\right\}.$$

We recall some lemmas, which will be used later.

Lemma 3.2. [3, Lemma 3.1] *Let $h \in N_{O(L)}(g)$ and let $\widehat{h} \in N_{\mathrm{Aut}(V_L)}(\widehat{g})$ be a standard lift of h. Let $\lambda + L \in \mathcal{D}(L)$. As sets of isomorphism classes of irreducible $V_L^{\widehat{g}}$-modules, we have*

$$\{V_{\lambda+L}(j) \circ \widehat{h} \mid 0 \le j \le p-1\} = \{V_{h^{-1}(\lambda)+L}(j) \mid 0 \le j \le p-1\}.$$

Lemma 3.3. [3, Lemma 3.2] *Let $\alpha, \lambda \in L^*$ and let $0 \le i, j \le p-1$. If $(\alpha|\lambda) \in j/p + \mathbb{Z}$, then as $V_L^{\widehat{g}}$-modules,*

$$V_{\lambda+L}(i) \circ \sigma_{(1-g)^{-1}\alpha} \cong V_{\lambda+L}(i-j).$$

The following proposition describes the stabilizer of $V_L(1)$ and the stabilizer of $\{V_L(i) \mid 0 \le i \le p-1\}$.

Proposition 3.4. [27, Theorem 3.3]

(1) *The stabilizer of $V_L(1)$ in $\mathrm{Aut}(V_L^{\widehat{g}})$ is isomorphic to $C_{\mathrm{Aut}(V_L)}(\widehat{g})/\langle\widehat{g}\rangle$.*

(2) *The stabilizer of $\{V_L(i) \mid 0 \le i \le p-1\}$ in $\mathrm{Aut}(V_L^{\widehat{g}})$ is isomorphic to $N_{\mathrm{Aut}(V_L)}(\langle\widehat{g}\rangle)/\langle\widehat{g}\rangle$.*

3.2. Irreducible $V_L^{\widehat{g}}$-modules of twisted type

In this subsection, we discuss the irreducible $V_L^{\widehat{g}}$-modules of twisted type. We use the descriptions in [1, Sections 3.2 and 3.3]. Let $1 \le i \le p-1$. The irreducible $\widehat{g}^i$-twisted module $V_L^T[\widehat{g}^i]$ is given by

$$V_L^T[\widehat{g}^i] = M(1)[g^i] \otimes T,$$

where $M(1)[g^i]$ is the g^i-twisted free bosonic space and T is an irreducible module for a certain $\widehat{g}^i$-twisted central extension of L. By [10, (6.28)], the conformal weight of $V_L^T[\widehat{g}^i]$ is given as follows:

$$\varepsilon(V_L^T[\widehat{g}^i]) = \frac{m(p+1)}{24p}, \tag{3.1}$$

where m is the rank of the lattice L.

By [11], all irreducible $\widehat{g}^i$-twisted modules M are $\widehat{g}$-*stable*, that is $M \circ \widehat{g} \cong M$. Then $\widehat{g}$ acts on the module M. We denote

$$V_L^T[\widehat{g}^i](j) = \{x \in V_L^T[\widehat{g}^i] \mid \widehat{g}x = \exp\left(2\pi\sqrt{-1}j/p\right)x\},$$

where $1 \le i \le p-1$ and $0 \le j \le p-1$.

By [10], under the assumption $(1-g)L^* \subset L$, the above module T is labeled by $(1-g^i)L^*/(1-g^i)L \cong \mathcal{D}(L)$. We use the following notation as in [20]:

$$V_{\lambda+L}[\widehat{g}^i] = M(1)[g^i] \otimes T_{\lambda+L}, \quad \lambda + L \in \mathcal{D}(L).$$

3.3. Irreducible $V_L^{\widehat{g}}$-modules and the quadratic form

In this subsection, we summarize the classification of irreducible $V_L^{\widehat{g}}$-modules and review the quadratic form on $\mathrm{Irr}(V_L^{\widehat{g}})$. For the irreducible $V_L^{\widehat{g}}$-modules, we adopt the notation in [20].

The following theorem describes $\mathrm{Irr}(V_L^{\widehat{g}})$.

Theorem 3.5. [20, Theorem 5.3] *Let L be an even lattice and let g be a fixed-point free isometry of L of odd prime order p. Let $\widehat{g}$ be a standard lift of g. Assume that $(1-g)L^* \subset L$ and the conformal weight of $V_L[\widehat{g}]$ is in $(1/p)\mathbb{Z}$. Then*

$$\mathrm{Irr}(V_L^{\widehat{g}}) = \{V_{\lambda+L}[\widehat{g}^i](j) \mid \lambda + L \in \mathcal{D}(L), 0 \le i, j \le p-1\},$$

where we identify $V_{\lambda+L}(j)$ with $V_{\lambda+L}[\widehat{g}^0](j)$. Furthermore, under the fusion product, we have $\mathrm{Irr}(V_L^{\widehat{g}}) \cong \mathcal{D}(L) \times (\mathbb{Z}/p\mathbb{Z})^2$ as abelian groups.

We choose the labeling of irreducible $V_L^{\widehat{g}}$-submodules of $V_{\lambda+L}[\widehat{g}^s]$ as in [20] so that

$$\varepsilon(V_{\lambda+L}[\widehat{g}^i](j)) \equiv \frac{ij}{p} + \frac{(\lambda|\lambda)}{2} \mod \mathbb{Z}. \tag{3.2}$$

Under the assumptions $(1-g)L^* \subset L$ and $\varepsilon(V_L[\widehat{g}]) \in (1/p)\mathbb{Z}$, the explicit fusion product is given as follows (see [24, (3.9)]):

$$V_{\lambda_1+L}[\widehat{g}^{i_1}](j_1) \boxtimes V_{\lambda_2+L}[\widehat{g}^{i_2}](j_2) = V_{\lambda_1+\lambda_2+L}[\widehat{g}^{i_1+i_2}](j_1+j_2). \tag{3.3}$$

By the theorem above, $\mathrm{Irr}(V_L^{\widehat{g}})$ forms a finite abelian group under the fusion product. By (3.2), we have $\varepsilon(M) \in \mathbb{Q}$ for $M \in \mathrm{Irr}(V_L^{\widehat{g}})$. Let

$$q\colon \mathrm{Irr}(V_L^{\widehat{g}}) \to \mathbb{Q}/\mathbb{Z}, \quad M \mapsto \varepsilon(M) \mod \mathbb{Z}.$$

Then the form $\langle \cdot | \cdot \rangle\colon \mathrm{Irr}(V_L^{\widehat{g}}) \times \mathrm{Irr}(V_L^{\widehat{g}}) \to \mathbb{Q}/\mathbb{Z}$ associated with q is defined by

$$\langle M^1 | M^2 \rangle \equiv q(M^1 \boxtimes M^2) - q(M^1) - q(M^2) \mod \mathbb{Z}. \tag{3.4}$$

The following theorem describes the quadratic space $(\mathrm{Irr}(V_L^{\widehat{g}}), q)$.

Theorem 3.6. [14, Theorem 3.4] $(\mathrm{Irr}(V_L^{\widehat{g}}), q)$ *is a non-degenerate quadratic space, that is, the form* $\langle \cdot | \cdot \rangle$ *is non-degenerate and bilinear.*

To examine certain orbits, we will use the following proposition later.

Proposition 3.7. [24, Lemma 3.5] *Let* $\alpha \in L^*$ *and let* $1 \leq i \leq p-1$. *For* $0 \leq j \leq p-1$,

$$V_L[\widehat{g}^i](j) \circ \sigma_{(1-g^i)^{-1}\alpha} \cong V_{\alpha+L}[\widehat{g}^i](j'),$$

where j' *is determined by* $ij \equiv p(\alpha|\alpha)/2 + ij' \mod p$. *In particular, all irreducible* $V_L^{\widehat{g}}$*-modules of* $\widehat{g}^i$*-type with the same conformal weight are conjugate under the action of* $\mathrm{Aut}(V_L^{\widehat{g}})$.

4. The structure of the kernel associated with the action of $\mathrm{Aut}(V_L^{\widehat{g}})$

Let L be an even lattice and let g be a fixed-point free isometry of L of odd prime order p. Let $\widehat{g}$ be a standard lift of g. In this section, we consider the kernel of the action of $\mathrm{Aut}(V_L^{\widehat{g}})$ on $\mathrm{Irr}(V_L^{\widehat{g}})$.

We review the action of $\mathrm{Aut}(V_L^{\widehat{g}})$ on $\mathrm{Irr}(V_L^{\widehat{g}})$. By Subsection 2.3, $\mathrm{Aut}(V_L^{\widehat{g}})$ acts on $\mathrm{Irr}(V_L^{\widehat{g}})$ by the conjugate action. Moreover, the action preserves the quadratic form of $\mathrm{Irr}(V_L^{\widehat{g}})$ as described in Theorem 3.6. Hence we can consider the group homomorphism

$$\mu\colon \mathrm{Aut}(V_L^{\widehat{g}}) \longrightarrow O(\mathrm{Irr}(V_L^{\widehat{g}}), q). \tag{4.1}$$

We determine the group structure of the kernel of μ. The following is a generalization of [27, Lemma 3.7].

Theorem 4.1. *Let L be a rootless even lattice and let g be a fixed-point free isometry of L of odd prime order p. If $(1-g)L^* \subset L$ and the conformal weight $\varepsilon(V_L[\widehat{g}])$ is in $(1/p)\mathbb{Z}$, then we have the following exact sequence:*

$$1 \longrightarrow A \longrightarrow \mathrm{Ker}\,\mu \xrightarrow{\widetilde{\varphi}} B \longrightarrow 1,$$

where $A = \mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_p)$ *and* $B = \{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$.

Proof. By Theorem 2.10, we have the following exact sequence:

$$1 \longrightarrow \mathrm{Hom}(L/(1-g)L, \mathbb{Z}_p) \longrightarrow C_{\mathrm{Aut}(V_L)}(\widehat{g})/\langle \widehat{g}\rangle \xrightarrow{\widetilde{\varphi}} C_{O(L)}(g)/\langle g\rangle \longrightarrow 1. \tag{4.2}$$

We first determine $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_p) \cap \mathrm{Ker}\,\mu$. By Lemma 2.4, since L is p-elementary, for any $\alpha, \lambda \in L^*$, we have $(\alpha|\lambda) + \mathbb{Z} \in \{i/p + \mathbb{Z} \mid 0 \leq i \leq p-1\}$. Hence, by Lemma 3.3, if α is an element of $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_p) \cap \mathrm{Ker}\,\mu$, then we have $\alpha \in (L^*)^* = L$. This means that $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_p) \cap \mathrm{Ker}\,\mu \subset \{\sigma_\alpha \mid \alpha \in (1-g)^{-1}L/L^*\}$. Conversely, by Lemma 3.3 and Proposition 3.7, σ_α ($\alpha \in (1-g)^{-1}L/L^*$) fixes the all elements of $\{V_{\lambda+L}(i), V_L[\widehat{g}](0) \mid \lambda \in L^*, 0 \le i \le p-1\}$. Note that the above set generates $\mathrm{Irr}(V_L^{\widehat{g}})$ under the fusion product. Since the conjugate action by $\mathrm{Aut}(V_L^{\widehat{g}})$ preserves the fusion product, we have $\sigma_\alpha \in \mathrm{Ker}\,\mu$ for $\alpha \in (1-g)^{-1}L/L^*$. Hence, we have $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_p) \cap \mathrm{Ker}\,\mu = \{\sigma_\alpha \mid \alpha \in (1-g)^{-1}L/L^*\}$. Note that $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_p) \cong \{\sigma_\alpha \mid \alpha \in (1-g)^{-1}L/L^*\}$.

Next we prove that the map $\widetilde{\varphi}$ from $\mathrm{Ker}\,\mu$ to B is surjective. By Lemma 3.2, for any $\lambda \in L^*$ and $0 \le j \le p-1$, there exists $0 \le j' \le p-1$ such that

$$V_{\lambda+L}(j) \circ \widehat{h} \cong V_{h^{-1}(\lambda)+L}(j'),$$

where $\widehat{h} \in C_{\mathrm{Aut}(V_L)}(\widehat{g})/\langle \widehat{g} \rangle$ and $\widetilde{\varphi}(\widehat{h}) = h$. Hence, if $\tau \in \mathrm{Ker}\,\mu$, then for any $\lambda \in \mathcal{D}(L)$, we have $\varphi(\tau)^{-1}(\lambda) - \lambda \in L$, which means that $\varphi(\tau)$ is 1 on $\mathcal{D}(L)$. Thus we have $\widetilde{\varphi}(\mathrm{Ker}\,\mu) \subset B$.

Conversely, let $h \in B$. By Theorem 2.10, we obtain an element $\widehat{h}$ of $C_{\mathrm{Aut}(V_L^{\widehat{g}})}(\widehat{g})$ such that $\varphi(\widehat{h}) = h$. By Lemma 3.2, for any $\lambda + L \in \mathcal{D}(L)$, there exists $0 \le j \le p-1$ such that

$$V_{\lambda+L}(0) \circ \widehat{h} \cong V_{\lambda+L}(j).$$

For $\lambda + L \in \mathcal{D}(L)$, we denote the above j by $j_{\lambda+L}$. By Lemma 2.8(2) and (3.3), we can define a group homomorphism f from $\mathcal{D}(L)$ to $\mathbb{Z}/p\mathbb{Z}$ by $\lambda + L \mapsto j_{\lambda+L}$. Since L is p-elementary, we can consider the following group isomorphism:

$$\mathcal{D}(L) \longrightarrow \mathrm{Hom}(L^*/L, \mathbb{Z}_p), \quad \alpha \mapsto p(\alpha \,|\, \cdot\,) \mod p.$$

Hence, there exists an element β of $\mathcal{D}(L)$ such that $f = -p(\beta \,|\, \cdot\,) \mod p$. Define $\widehat{h}_0 = \sigma_{(1-g)^{-1}\beta}\widehat{h}$. By Lemma 3.2, for any $\lambda + L \in \mathcal{D}(L)$, we have

$$V_{\lambda+L}(0) \circ \widehat{h}_0 \cong V_{\lambda+L}(0).$$

The orthogonal complement of the subspace $\{V_{\lambda+L}(0) \mid \lambda + L \in \mathcal{D}(L)\}$ in $\mathrm{Irr}(V_L^{\widehat{g}})$ with respect to the bilinear form in (3.4) is $\{V_L[\widehat{g}^i](j) \mid 0 \le i, j \le p-1\}$, which is preserved by the action of $\widehat{h}_0$. Since $\widehat{h}$ and $\widehat{g}$ are commutative, $\widehat{h}_0$-conjugates of $\widehat{g}$-twisted modules are also $\widehat{g}$-twisted modules. Since $\varphi(\widehat{h}) = h \in B$, by Proposition 3.7, we have

$$V_L[\widehat{g}](0) \circ \widehat{h}_0 \cong V_L[\widehat{g}](0).$$

By Theorem 3.5, $\mathrm{Irr}(V_L^{\widehat{g}})$ is generated by $\{V_L(j), V_{\lambda+L}(0), V_L[\widehat{g}](0) \mid 0 \le j \le p-1, \lambda \in \mathcal{D}(L)\}$. Since the above generators are preserved by the action of $\widehat{h}_0$, by Lemma 2.8(2), all elements of $\mathrm{Irr}(V_L^{\widehat{g}})$ are preserved by the action of $\widehat{h}_0$. This means that $\widehat{h}_0 \in \mathrm{Ker}\,\mu$ and $\varphi(\widehat{h}_0) = h$, as desired. □

By Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let L be a rootless even lattice and let g be a fixed-point free isometry of L of odd prime order p. Let μ be as in (4.1). If $(1-g)L^* \subset L$ and the conformal weight $\varepsilon(V_L[\widehat{g}])$ is in $(1/p)\mathbb{Z}$, then we have the following exact sequence:*

$$1 \longrightarrow H_1/H_2 \longrightarrow \mathrm{Stab}_{\mathrm{Im}\,\mu}(V_L(1)) \xrightarrow{\widetilde{\varphi}'} K_1/K_2 \longrightarrow 1,$$

where $H_1 = \mathrm{Hom}(L/(1-g)L, \mathbb{Z}_p)$, $H_2 = \mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_p)$, $K_1 = C_{O(L)}(g)/\langle g\rangle$, and $K_2 = \{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$.

Proof. By Proposition 3.4, we can identify $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))$ with $C_{\mathrm{Aut}(V_L)}(\widehat{g})/\langle\widehat{g}\rangle$. Let π_1 be the natural map from $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))$ to $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))/\operatorname{Ker}\mu$ and let π_2 be the natural map from K_1 to K_1/K_2. Let i be the inclusion in (4.2) and let $\widetilde{\varphi}$ be the surjective map in (4.2). By Theorem 4.1, we see that $\mathrm{Ker}(\pi_1 \circ i) = H_2$, which means that we have the injective map i' from H_1/H_2 to $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))/\operatorname{Ker}\mu$. Moreover, we see that $\mathrm{Ker}(\pi_2 \circ \widetilde{\varphi}) \supset \operatorname{Ker}\mu$, which implies that we have the surjective map $\widetilde{\varphi}'$ from $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))/\operatorname{Ker}\mu$ to K_1/K_2. Hence we have the following sequence:

$$1 \longrightarrow H_1/H_2 \xrightarrow{i'} \mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))/\operatorname{Ker}\mu \xrightarrow{\widetilde{\varphi}'} K_1/K_2 \longrightarrow 1.$$

Finally, we verify that $\operatorname{Ker}\widetilde{\varphi}' = i'(H_1/H_2)$. Note that $i'(H_1/H_2) \subset \operatorname{Ker}\widetilde{\varphi}'$. Conversely, let $x\operatorname{Ker}\mu \in \operatorname{Ker}\widetilde{\varphi}'$, where $x \in \mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))$. Since $\widetilde{\varphi}(x) \in K_2$, by Theorem 4.1, there exists an element $x' \in \operatorname{Ker}\mu$ such that $\widetilde{\varphi}(x') = \widetilde{\varphi}(x)$. Since $xx'^{-1} \in \operatorname{Ker}\widetilde{\varphi}$, by (4.2), there exists an element $h \in H_1$ such that $i'(hH_2) = (xx'^{-1})\operatorname{Ker}\mu = x\operatorname{Ker}\mu$, which means that $\operatorname{Ker}\widetilde{\varphi}' \subset i'(H_1/H_2)$. Since $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))/\operatorname{Ker}\mu \cong \mathrm{Stab}_{\mathrm{Im}\,\mu}(V_L(1))$, we have the desired result. ☐

5. Structure of $\mathrm{Aut}(V_L^{\widehat{g}})$ in the case $L = \Lambda_{3C}$, Λ_{5C}, Λ_{11A}, or Λ_{23A}

Throughout this section, we consider the group structures of the automorphism groups of orbifold vertex operator algebras in the cases $L = \Lambda_{3C}$, Λ_{5C}, Λ_{11A}, or Λ_{23A}. To compute the automorphism groups, we use MAGMA (see [4]). We explain how to determine the group structures of the automorphism groups of the orbifold VOAs. Since the Leech lattice Λ is even unimodular, by Lemma 2.3, we have $(1-g)\Lambda_g \subset \Lambda_g$ for $g \in O(\Lambda)$. Moreover, for $pX \in \{3C, 5C, 11A, 23A\}$ and $g \in pX$, we verify that the coinvariant lattice $L = \Lambda_{pX}$ satisfies the condition that the conformal weight of $V_L[\widehat{g}]$ is in $(1/p)\mathbb{Z}$, where $\widehat{g}$ is a standard lift of g. In this situation, we can apply Theorem 3.5 to $V_L^{\widehat{g}}$. As in (4.1), we obtain a group homomorphism

$$\mu\colon\ \mathrm{Aut}(V_L^{\widehat{g}}) \longrightarrow O(\mathrm{Irr}(V_L^{\widehat{g}}), q).$$

To determine the group structure of $\operatorname{Ker}\mu$, we examine $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_p)$ and $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$ in Theorem 4.1.

To determine the group structure of $\mathrm{Im}\,\mu$, we determine the index of $\mathrm{Im}\,\mu$ in the orthogonal group of the quadratic space $(\mathrm{Irr}(V_L^{\hat{g}}), q)$ and $\mathrm{Stab}_{\mathrm{Im}\,\mu}(V_L(1))$. To do this, we examine the orbit $\mathrm{Orbit}_{\mathrm{Aut}(V_L^{\hat{g}})}(V_L(1))$ of $V_L(1)$ and the group structure of the stabilizer $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\hat{g}})}(V_L(1))$.

5.1. The case $L = \Lambda_{3C}$

Let $g \in 3C$ and let $L = \Lambda_g$. Note that g is a fixed-point free isometry of L of order 3. Let $\hat{g}$ be a standard lift of g. By [24, Table 1], we have the following lemma.

Lemma 5.1. [24, Table 1] *The rank of L is 18 and $\mathcal{D}(L) \cong 3^5$ as groups.*

Combining Lemma 5.1 with Lemmas 2.2 and 2.5, we have the following lemma.

Lemma 5.2. $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_3) \cong 3^9$ *and* $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_3) \cong 3^4$.

By (3.1) and Lemma 5.1, we have

$$\varepsilon(V_L[\hat{g}]) = \frac{18 \cdot 4}{24 \cdot 3} = 1 \in \frac{1}{3}\mathbb{Z}.$$

Next we determine the group structure of $\mathrm{Ker}\,\mu$ in (4.1). By Theorem 4.1, we examine $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_3)$ and $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$. By using MAGMA, we see that $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\} \cong 3_+^{1+4} : 2$ and $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle \cong 3^4 : 2$. Combining these results with Theorem 4.1 and Lemma 5.1, we have the following proposition.

Proposition 5.3. $\{h \in C_{O(L)}(g) \mid h = 1$ *on* $L^*/L\}/\langle g\rangle \cong 3^4 : 2$ *and* $\mathrm{Ker}\,\mu \cong 3^4 \,.\, (3^4 : 2)$.

Next we determine the group structure of $\mathrm{Im}\,\mu$ in (4.1). By using MAGMA, we have $\#C_{O(L)}(g)/\langle g\rangle = 2^8 \cdot 3^8 \cdot 5$. Hence we have the following lemma.

Lemma 5.4. $\#C_{O(L)}(g)/\langle g\rangle = 2^8 \cdot 3^8 \cdot 5$ *and* $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1$ *on* $L^*/L\} \cong 2 \times \Omega_5(3)$.

Proof. By Proposition 5.3, we have $\#C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\} = 2^7 \cdot 3^4 \cdot 5$. Since $\#\,\mathrm{GO}_5(3) = 2^8 \cdot 3^4 \cdot 5$, $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}$ is isomorphic to a subgroup of $\mathrm{GO}_5(3)$ of index 2. There exist precisely three subgroups of $\mathrm{GO}_5(3)$ of index 2, which are $\mathrm{SO}_5(3)$, $P_5(3)$, and $Q_5(3)$ (for the definitions of $P_5(3)$ and $Q_5(3)$, see Definition 2.6). Moreover, since the center of $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}$ is not trivial and $P_5(3)$ is the only group whose center is not trivial among the above three groups of index 2, we have $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\} \cong 2 \times \Omega_5(3)$. □

Set $\mathcal{L}_{3C} = \{\lambda + L \in \mathcal{D}(L) \mid q_L(\lambda + L) = 0\} \setminus \{L\}$, then the subgroup $P_5(3)$ of $\mathrm{GO}_5(3)$ acts transitively on $\mathcal{L}_{3C}$ (see Proposition A.5 for the proof). Hence, by Lemma 5.4, we have the following proposition.

Proposition 5.5. *$C_{O(L)}(g)/\langle g\rangle$ acts transitively on $\mathcal{L}_{3C}$.*

By Theorem 3.5 and Lemma 5.1, we have $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 3^7$ as abelian groups. Let S_{3C} be the set of all singular vectors in $\mathrm{Irr}(V_L^{\widehat{g}})$. By [28, (3.27)], we have $\# S_{3C} = 3^6 - 1$. Since Λ_{3C} is realized by Construction B in [24, Section 4] and $C_{O(L)}(g)/\langle g\rangle$ acts transitively on $\mathcal{L}_{3C}$, we can imitate the proof of [3, PROPOSITION 6.5]. Hence we obtain the following proposition.

Proposition 5.6. *$\mathrm{Aut}(V_L^{\widehat{g}})$ acts transitively on S_{3C}.*

We determine the order of $\mathrm{Aut}(V_L^{\widehat{g}})$. By Theorem 2.10, Proposition 3.4, Lemmas 5.2 and 5.4, we have the following proposition.

Proposition 5.7. $\#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = 2^8 \cdot 3^{17} \cdot 5$.

Proof. $\#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = \#\,\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_3) \# C_{O(L)}(g)/\langle g\rangle = 2^8 \cdot 3^{17} \cdot 5$. □

Next, we consider the order of $\mathrm{Aut}(V_L^{\widehat{g}})$.

Proposition 5.8. $\#\,\mathrm{Aut}(V_L^{\widehat{g}}) = 2^{11} \cdot 3^{17} \cdot 5 \cdot 7 \cdot 13$ *and* $\#\,\mathrm{Im}\,\mu = 2^{10} \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$.

Proof. By Theorem 3.5 and Lemma 5.1, we have $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 3^7$ as abelian groups. Combining this with Propositions 5.6 and 5.7, we have

$$\#\,\mathrm{Aut}(V_L^{\widehat{g}}) = \# S_{3C} \#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = (3^6 - 1) \cdot 2^8 \cdot 3^{17} \cdot 5 = 2^{11} \cdot 3^{17} \cdot 5 \cdot 7 \cdot 13.$$

Since $\#\,\mathrm{Ker}\,\mu = 2 \cdot 3^8$, we have $\mathrm{Im}\,\mu = \#\,\mathrm{Aut}(V_L^{\widehat{g}})/\#\,\mathrm{Ker}\,\mu = 2^{10} \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$. □

To determine the group structure of $\mathrm{Im}\,\mu$, we first study the group structure of $\mathrm{Stab}_{\mathrm{Im}\,\mu}(V_L(1))$. By Lemmas 5.2 and 5.4, we see that $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_3)/\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_3) \cong 3^5$ and $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\} \cong 2 \times \Omega_5(3)$. Combining these with Corollary 4.2, we have the following proposition.

Proposition 5.9. $\mathrm{Stab}_{\mathrm{Im}\,\mu}(V_L(1)) \cong 3^5 \,.\, (2 \times \Omega_5(3))$.

Finally, we determine the group structure of $\mathrm{Im}\,\mu$. Since $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 3^7$ as abelian groups, $\mathrm{Im}\,\mu$ is a subgroup of $\mathrm{GO}_7(3)$. By Proposition 5.8, we see that $|\,\mathrm{GO}_7(3) : \mathrm{Im}\,\mu| = 2$. The group $\mathrm{Im}\,\mu$ is a subgroup of $\mathrm{GO}_7(3)$ satisfying the following:

(1) $|\,\mathrm{GO}_7(3) : \mathrm{Im}\,\mu| = 2$;

(2) The stabilizer of an element in S_{3C} in $\operatorname{Im}\mu$ is isomorphic to $3^5 . (2 \times \Omega_5(3))$.

There exist precisely three subgroups of $\mathrm{GO}_7(3)$ of index 2. These three groups are $\mathrm{SO}_7(3)$, $P_7(3)$, and $Q_7(3)$, where $P_7(3)$ and $Q_7(3)$ are the groups in Definition 2.6. By the above condition (1), $\operatorname{Im}\mu$ is one of the three groups. By the above condition (2), we see that $\operatorname{Im}\mu = Q_7(3) \cong \Omega_7(3) . 2$ (see Proposition A.11 for the proof). Combining this with Proposition 5.3, we have the following theorem.

Theorem 5.10. $\mathrm{Aut}(V_{\Lambda_{3C}}^{\widehat{g}}) \cong (3^4 . (3^4 : 2)) . (\Omega_7(3) . 2)$.

5.2. The case $L = \Lambda_{5C}$

This case is similar to the case $L = \Lambda_{3C}$. Let $g \in 5C$ and let $L = \Lambda_g$. Note that g is a fixed-point free isometry of L of order 5. Let $\widehat{g}$ be a standard lift of g. By [24, Table 1], we have the following lemma.

Lemma 5.11. [24, Table 1] *The rank of L is 20 and $\mathcal{D}(L) \cong 5^3$ as groups.*

Combining Lemma 5.11 with Lemmas 2.2 and 2.5, we have the following lemma.

Lemma 5.12. $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_5) \cong 5^5$ *and* $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_5) \cong 5^2$.

By (3.1) and Lemma 5.11, we have

$$\varepsilon(V_L[\widehat{g}]) = \frac{20 \cdot 6}{24 \cdot 5} = 1 \in \frac{1}{5}\mathbb{Z}.$$

Next, we determine the group structure of $\operatorname{Ker}\mu$ in (4.1). By Theorem 4.1, we examine $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_5)$ and $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$.

By using MAGMA, we see that $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\} \cong 5_+^{1+2} : 2$ and $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle \cong 5^2 : 2$. Combining these results with Theorem 4.1 and Lemma 5.12, we have the following proposition.

Proposition 5.13. $\{h \in C_{O(L)}(g) \mid h = 1$ *on* $L^*/L\}/\langle g\rangle \cong 5^2 : 2$ *and* $\operatorname{Ker}\mu \cong 5^2 . (5^2 : 2)$.

Next we determine the group structure of $\operatorname{Im}\mu$ in (4.1). By using MAGMA, we have $\# C_{O(L)}(g)/\langle g\rangle = 2^4 \cdot 3 \cdot 5^3$. Since $\#\,\mathrm{GO}_3(5) = 2^4 \cdot 3 \cdot 5$ and $\# C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\} = 2^3 \cdot 3 \cdot 5$, we see that $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}$ is isomorphic to a subgroup of $\mathrm{GO}_3(5)$ of index 2. Since the center of $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}$ is not trivial, we have the following lemma.

Lemma 5.14. $\# C_{O(L)}(g)/\langle g\rangle = 2^4 \cdot 3 \cdot 5^3$ *and* $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1$ *on* $L^*/L\} \cong 2 \times \Omega_3(5)$.

Set $\mathcal{L}_{5C} = \{\lambda + L \in \mathcal{D}(L) \mid q_L(\lambda + L) = 0\} \setminus \{L\}$. By using MAGMA, we have $N_{O(L)}(\langle g\rangle)/\{h \in N_{O(L)}(\langle g\rangle) \mid h = 1 \text{ on } L^*/L\} \cong \mathrm{GO}_5(3)$. Hence we have the following proposition.

Proposition 5.15. *$N_{O(L)}(\langle g\rangle)/\langle g\rangle$ acts transitively on $\mathcal{L}_{5C}$.*

Remark 5.16. We see that $C_{O(L)}(g)/\langle g\rangle$ does not act transitively on $\mathcal{L}_{5C}$. More precisely, the number of orbits is two.

By Theorem 3.5 and Lemma 5.11, we have $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 5^5$ as abelian groups. Let S_{5C} be the set of all singular vectors in $\mathrm{Irr}(V_L^{\widehat{g}})$. By [28, (3.27)], we have $\#S_{5C} = 5^4 - 1$. Since Λ_{5C} is realized by Construction B in [24, Section 4] and $N_{O(L)}(\langle g\rangle)/\langle g\rangle$ acts transitively on $\mathcal{L}_{5C}$, we can imitate the proof of [3, PROPOSITION 6.5]. Hence we obtain the following proposition.

Proposition 5.17. *$\mathrm{Aut}(V_L^{\widehat{g}})$ acts transitively on S_{5C}.*

We determine the order of $\mathrm{Aut}(V_L^{\widehat{g}})$. By Theorem 2.10, Proposition 3.4, Lemmas 5.12 and 5.14, we have the following proposition.

Proposition 5.18. $\#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = 2^4 \cdot 3 \cdot 5^8$.

Imitating the proof of Proposition 5.8, we have the following proposition.

Proposition 5.19. $\#\,\mathrm{Aut}(V_L^{\widehat{g}}) = 2^8 \cdot 3^2 \cdot 5^8 \cdot 13$ *and* $\#\,\mathrm{Im}\,\mu = 2^7 \cdot 3^2 \cdot 5^4 \cdot 13$.

Proof. By Theorem 3.5 and Lemma 5.11, we have $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 5^5$ as abelian groups. Combining this with Propositions 5.17 and 5.18, we have

$$\#\,\mathrm{Aut}(V_L^{\widehat{g}}) = \#S_{5C}\#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = (5^4 - 1) \cdot 2^4 \cdot 3 \cdot 5^8 = 2^8 \cdot 3^2 \cdot 5^8 \cdot 13.$$

Since $\#\,\mathrm{Ker}\,\mu = 2 \cdot 5^4$, we have $\mathrm{Im}\,\mu = \#\,\mathrm{Aut}(V_L^{\widehat{g}})/\#\,\mathrm{Ker}\,\mu = 2^7 \cdot 3^2 \cdot 5^4 \cdot 13$. □

By Lemmas 5.12 and 5.14, we see that $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_5)/\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_5) \cong 5^3$ and $C_{O(L)}(g)/\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\} \cong 2 \times \Omega_3(5)$. Combining these with Corollary 4.2, we have the following proposition.

Proposition 5.20. $\mathrm{Stab}_{\mathrm{Im}\,\mu}(V_L(1)) \cong 5^3 \,.\, (2 \times \Omega_3(5))$.

Finally, we determine the group structure of $\mathrm{Im}\,\mu$. Since $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 5^5$ as abelian groups, $\mathrm{Im}\,\mu$ is a subgroup of $\mathrm{GO}_5(5)$. By Proposition 5.19, we see that $|\,\mathrm{GO}_5(5) : \mathrm{Im}\,\mu| = 2$. The group $\mathrm{Im}\,\mu$ is a subgroup of $\mathrm{GO}_5(5)$ satisfying the following:

(1) $|\,\mathrm{GO}_5(5) : \mathrm{Im}\,\mu| = 2$;

(2) The stabilizer of an element in S_{5C} in $\operatorname{Im}\mu$ is isomorphic to $5^3 . (2 \times \Omega_3(5))$.

There exist precisely three subgroups of $\mathrm{GO}_5(5)$ of index 2. These three groups are $\mathrm{SO}_5(5)$, $P_5(5)$, and $Q_5(5)$, where $P_5(5)$ and $Q_5(5)$ are the groups in Definition 2.6. By the above condition (1), $\operatorname{Im}\mu$ is one of the three groups. By the above condition (2), we see that $\operatorname{Im}\mu = P_5(5) \cong 2 \times \Omega_5(5)$ (see Proposition A.11 for the proof). Combining this with Proposition 5.13, we have the following theorem.

Theorem 5.21. $\mathrm{Aut}(V_{\Lambda_{5C}}^{\widehat{g}}) \cong (5^2 . (5^2 : 2)) . (2 \times \Omega_5(5))$.

5.3. The case $L = \Lambda_{11A}$

Let $g \in 11A$ and let $L = \Lambda_g$. Note that g is a fixed-point free isometry of L of order 11. Let $\widehat{g}$ be a standard lift of g. By [24, Table 2 and Lemma 7.5], we have the following lemma.

Lemma 5.22. [24, Table 2 and Lemma 7.5] *The rank of L is 20. $(\mathcal{D}(L), q_L)$ is a non-singular 2-dimensional quadratic space over $\mathbb{F}_{11}$ of $(-)$-type.*

Combining Lemma 5.22 with Lemmas 2.2 and 2.5, we have the following lemma.

Lemma 5.23. $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_{11}) \cong 11^2$ *and* $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_{11})$ *is trivial.*

By (3.1) and Lemma 5.22, we have

$$\varepsilon(V_L[\widehat{g}]) = \frac{20 \cdot 12}{24 \cdot 11} = \frac{10}{11} \in \frac{1}{11}\mathbb{Z}.$$

Next, we determine the group structure of $\operatorname{Ker}\mu$ in (4.1). By Theorem 4.1, we examine $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_{11})$ and $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$. By Lemma 5.23, $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_{11})$ is trivial. By using MAGMA, we see that $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$ is trivial. Hence we have the following proposition.

Proposition 5.24. $\operatorname{Ker}\mu$ *is trivial and the group homomorphism* μ *is injective.*

By Theorem 3.5 and Lemma 5.22, we see that $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 11^4$ as abelian groups. By Lemma 5.22, the quadratic space $(\mathrm{Irr}(V_L^{\widehat{g}}), q)$ has $(-)$-type. By [28, (3.27)], we can count the all singular vectors in $(\mathrm{Irr}(V_L^{\widehat{g}}), q)$. Hence, we have the following proposition.

Proposition 5.25. $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 11^4$ *as groups and* $(\mathrm{Irr}(V_L^{\widehat{g}}), q)$ *is of $(-)$-type. There are* 1220 *singular vectors in the quadratic space* $\mathrm{Irr}(V_L^{\widehat{g}})$.

By using MAGMA, we see that $C_{O(L)}(g)/\langle g\rangle \cong D_{12}$. By Theorem 2.10, Proposition 3.4, and Lemma 5.23, we have the following proposition.

Proposition 5.26. *$C_{O(L)}(g)/\langle g\rangle \cong D_{12}$ and $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) \cong 11^2 \,.\, D_{12}$.*

Next we consider the orbit of $V_L(1)$. By Theorem 1.1, there exists $\tau \in \mathrm{Aut}(V_L^{\widehat{g}})$ such that

$$V_L(1) \circ \tau \ \text{ is of twisted type.} \tag{5.1}$$

Let $X_{k,11A} = \{M \in \mathrm{Irr}(V_L^{\widehat{g}}) \mid q(M) = 0 \text{ and } M \text{ is of } \widehat{g}^k\text{-type}\}$. Imitating [28, (3.27), (3.28)], we can count the number of elements of $X_{k,11A}$ for $0 \le k \le 10$. Hence we have the following lemma.

Lemma 5.27. *$\#X_{0,11A} = 11$ and $\#X_{k,11A} = 121$ for $1 \le k \le 10$.*

Combining Lemma 5.27 with Proposition 3.7 and (5.1), we have the following lemma.

Lemma 5.28. *Under the action of $\mathrm{Im}\,\mu$, $\#\,\mathrm{Orbit}(V_L(1)) \ge 122$ and $2 \le |\,\mathrm{GO}_4^-(11) : \mathrm{Im}\,\mu| \le 20$.*

Proof. By (5.1), we see that $V_L(1)\circ\tau$ is of twisted type for some $\tau \in \mathrm{Aut}(V_L^{\widehat{g}})$. Let $V_L(1)\circ\tau$ be of $\widehat{g}^s$-type, where s is a positive integer such that $1 \le s \le 10$. By Proposition 3.7, all irreducible $V_L^{\widehat{g}}$-modules of $\widehat{g}^s$-type with the same conformal weight are conjugate under the action of $\mathrm{Aut}(V_L^{\widehat{g}})$. By Lemma 5.27, since $\#X_{s,11A} = 121$, we have $\#\,\mathrm{Orbit}(V_L(1)) \ge 122$. Hence, we see that $\#\,\mathrm{Im}\,\mu \ge 122 \cdot \#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1))$. By Proposition 5.26, we have $\#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = 2^2 \cdot 3 \cdot 11^2$. Since $\#\,\mathrm{GO}_4^-(11) = 2^5 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$, we have $|\,\mathrm{GO}_4^-(11) : \mathrm{Im}\,\mu| \le 20$.

By Proposition 5.25, since $\mathrm{Irr}(V_L^{\widehat{g}})$ has 1220 singular vectors, we see that $\#\,\mathrm{Im}\,\mu \le 1220 \cdot \#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = 1220 \cdot 2^2 \cdot 3 \cdot 11^2 = \#\,\mathrm{GO}_4^-(11)/2$. This implies that $2 \le |\,\mathrm{GO}_4^-(11) : \mathrm{Im}\,\mu|$. □

Hence $\mathrm{Im}\,\mu$ is a subgroup of $\mathrm{GO}_4^-(11)$ satisfying the following:

(1) $2 \le |\,\mathrm{GO}_4^-(11) : \mathrm{Im}\,\mu| \le 20$;

(2) The stabilizer of $V_L(1)$ in $\mathrm{Im}\,\mu$ is isomorphic to $11^2 \,.\, D_{12}$.

By the list of the maximal subgroups of $\Omega_4^-(11)$ (see [5, Table 8.17]), for any subgroup of $\mathrm{GO}_4^-(11)$ which does not contain $\Omega_4^-(11)$, the index in $\mathrm{GO}_4^-(11)$ is greater than or equal to 122. By the condition (1), since $\mathrm{Im}\,\mu$ contains $\Omega_4^-(11)$, there are precisely four subgroups of $\mathrm{GO}_4^-(11)$ satisfying the condition (1). These four groups are $\Omega_4^-(11)$, $\mathrm{SO}_4^-(11)$, $\Omega_4^-(11) \cup \sigma_2\Omega_4^-(11)$, and $\Omega_4^-(11) \cup (\sigma_1\sigma_2)\Omega_4^-(11)$, where σ_1 is an element of $\mathrm{SO}_4^-(11) \setminus \Omega_4^-(11)$ and σ_2 is an element of $\mathrm{GO}_4^-(11) \setminus \mathrm{SO}_4^-(11)$. By the above condition (2), we have $\mathrm{Im}\,\mu \cong \Omega_4^-(11) \cup \sigma_2\Omega_4^-(11) \cong \Omega_4^-(11) \cup (\sigma_1\sigma_2)\Omega_4^-(11)$ as groups (see Proposition A.25 for the proof). Note that $\mathrm{Im}\,\mu \cong \Omega_4^-(11) \,.\, 2$. Combining this with Proposition 5.24, we have the following theorem.

Theorem 5.29. $\mathrm{Aut}(V_{\Lambda_{11A}}^{\widehat{g}}) \cong \Omega_4^-(11)\,.\,2$. $\mathrm{Aut}(V_{\Lambda_{11A}}^{\widehat{g}})$ *acts transitively on all singular vectors in* $\mathrm{Irr}(V_{\Lambda_{11A}}^{\widehat{g}})$.

5.4. The case $L = \Lambda_{23A}$

Let $g \in 23A$ and let $L = \Lambda_g$. Note that g is a fixed-point free isometry of L of order 23. Let $\widehat{g}$ be a standard lift of g. By [24, Table 2], we have the following lemma.

Lemma 5.30. [24, Table 2] *The rank of* L *is* 22 *and* $\mathcal{D}(L) \cong 23$.

Combining Lemma 5.30 with Lemmas 2.2 and 2.5, we have the following lemma.

Lemma 5.31. $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_{23}) \cong 23$ *and* $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_{23})$ *is trivial.*

By (3.1) and Lemma 5.30, we have

$$\varepsilon(V_L[\widehat{g}]) = \frac{22 \cdot 24}{24 \cdot 23} = \frac{22}{23} \in \frac{1}{23}\mathbb{Z}.$$

Next, we determine the group structure of $\mathrm{Ker}\,\mu$ in (4.1). By Theorem 4.1, we examine $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_{23})$ and $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$. By Lemma 5.31, $\mathrm{Hom}(L/(1-g)L^*, \mathbb{Z}_{23})$ is trivial. By using MAGMA, we see that $\{h \in C_{O(L)}(g) \mid h = 1 \text{ on } L^*/L\}/\langle g\rangle$ is trivial. Hence we have the following proposition.

Proposition 5.32. $\mathrm{Ker}\,\mu$ *is trivial and the group homomorphism* μ *is injective.*

By Theorem 3.5 and Lemma 5.30, we see that $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 23^3$ as abelian groups. Hence, we have the following proposition.

Proposition 5.33. $\mathrm{Irr}(V_L^{\widehat{g}}) \cong 23^3$ *as groups.*

By using MAGMA, we see that $C_{O(L)}(g)/\langle g\rangle \cong 2$. By Theorem 2.10, Proposition 3.4, and Lemma 5.31, we have the following proposition.

Proposition 5.34. $C_{O(L)}(g)/\langle g\rangle \cong 2$ *and* $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) \cong 23\,.\,2$.

Next we consider the orbit of $V_L(1)$. By Theorem 1.1, there exists $\tau \in \mathrm{Aut}(V_L^{\widehat{g}})$ such that

$$V_L(1) \circ \tau \ \text{ is of twisted type.}$$

Let $X_{k,23A} = \{M \in \mathrm{Irr}(V_L^{\widehat{g}}) \mid q(M) = 0 \text{ and } M \text{ is of } \widehat{g}^k\text{-type}\}$. Imitating [28, (3.27), (3.28)], we can count the number of elements of $X_{k,23A}$ for $0 \le k \le 22$. Hence we have the following lemma.

Lemma 5.35. $\#X_{k,23A} = 23$ *for* $0 \le k \le 22$.

By imitating the discussion in the case $11A$, we have the following lemma.

Lemma 5.36. $\#\,\mathrm{Orbit}(V_L(1)) \geq 24$ *and* $|\,\mathrm{GO}_3(23) : \mathrm{Im}\,\mu| \leq 22$.

Proof. Imitating the proof of Lemma 5.28, we have $\#\,\mathrm{Orbit}(V_L(1)) \geq 24$. Proposition 5.34 shows $\#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = 2 \cdot 23$, and thus $\#\,\mathrm{Im}\,\mu \geq 24 \cdot \#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) = 2^4 \cdot 3 \cdot 23$. Since $\#\,\mathrm{GO}_3(23) = 2^5 \cdot 3 \cdot 11 \cdot 23$, we have $|\,\mathrm{GO}_3(23) : \mathrm{Im}\,\mu| \leq \#\,\mathrm{GO}_3(23)/(2^4 \cdot 3 \cdot 23) = 22$. □

Hence $\mathrm{Aut}(V_L^{\widehat{g}})$ is a subgroup of $\mathrm{GO}_3(23)$ satisfying the following:

(1) $|\,\mathrm{GO}_3(23) : \mathrm{Im}\,\mu| \leq 22$;

(2) The stabilizer of $V_L(1)$ in $\mathrm{Im}\,\mu$ is isomorphic to $23\,.\,2$.

By the list of the maximal subgroups of $\Omega_3(23)$ (see [5, Table 8.7]), for any subgroup of $\mathrm{GO}_3(23)$ which does not contain $\Omega_3(23)$, the index in $\mathrm{GO}_3(23)$ is greater than or equal to 24. By the condition (1), since $\mathrm{Im}\,\mu$ contains $\Omega_3(23)$, there are precisely five subgroups of $\mathrm{GO}_3(23)$ satisfying the condition (1). These five groups are $\Omega_3(23)$, $\mathrm{SO}_3(23)$, $P_3(23)$, $Q_3(23)$, and $\mathrm{GO}_3(23)$, where $P_3(23)$ and $Q_3(23)$ are the groups in Definition 2.6. By the above condition (2), we have $\mathrm{Im}\,\mu = Q_3(23)$ or $\mathrm{GO}_3(23)$ (see Lemma A.13 for the proof).

Remark 5.37. For $Q_3(23)$ and $\mathrm{GO}_3(23)$, we can count the number of the orbits on 23^3. The number of the orbits of $Q_3(23)$ (resp. $\mathrm{GO}_3(23)$) is 2 (resp. 1). For the group $Q_3(23)$, the number of elements of each orbit is 264.

We determine which the group $\mathrm{Aut}(V_L^{\widehat{g}})$ is. To determine, we first examine the group structure of the normalizer $N_{\mathrm{Aut}(V_L)}(\langle\widehat{g}\rangle)/\langle\widehat{g}\rangle$. By using MAGMA, we see that $N_{O(L)}(\langle g\rangle)/\langle g\rangle \cong 22$. Moreover, by Lemma 5.31, we see $\mathrm{Hom}(L/(1-g)L, \mathbb{Z}_{23}) \cong 23$. Combining these with Theorem 2.10 and Proposition 3.4, we have the following lemma.

Lemma 5.38. $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(\{V_L(i) \mid 0 \leq i \leq 22\}) \cong 23\,.\,22$.

By using Lemma 5.38, we have the following lemma.

Lemma 5.39. *Under the action of the stabilizer* $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(\{V_L(i) \mid 0 \leq i \leq 22\})$ *on* $\{V_L(i) \mid 0 \leq i \leq 22\}$, *we have* $\#\,\mathrm{Orbit}(V_L(1)) = 11$.

Proof. By Lemma 5.38, we have $\#\,\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(\{V_L(i) \mid 0 \leq i \leq 22\}) = 506$. Since $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(V_L(1)) \subset \mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(\{V_L(i) \mid 0 \leq i \leq 22\})$, we have $\#\,\mathrm{Orbit}(V_L(1)) = 506/46 = 11$, as desired. □

Finally, we have the following theorem.

Theorem 5.40. $\mathrm{Aut}(V_{\Lambda_{23A}}^{\widehat{g}}) \cong \Omega_3(23) \,.\, 2$. *Under the action of* $\mathrm{Aut}(V_{\Lambda_{23A}}^{\widehat{g}})$*, the set of all singular vectors in* $\mathrm{Irr}(V_{\Lambda_{23A}}^{\widehat{g}})$ *is divided into two orbits which have same number of elements.*

Proof. For a contradiction, suppose that $\mathrm{Aut}(V_L^{\widehat{g}}) = \mathrm{GO}_3(23)$. The center of $\mathrm{GO}_3(23)$ is $\{1, -1\}$, where -1 is the diagonal matrix whose all diagonal entries are -1. Since $\{V_L(i) \mid 0 \le i \le 22\}$ is a subspace of $\mathrm{Irr}(V_{\Lambda_{23A}}^{\widehat{g}})$, the matrix -1 preserves $\{V_L(i) \mid 0 \le i \le 22\}$. Hence we see that -1 is in $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(\{V_L(i) \mid 0 \le i \le 22\})$.

On the other hand, by Lemma 5.39, under the action of $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(\{V_L(i) \mid 0 \le i \le 22\})$, we have $\#\,\mathrm{Orbit}(V_L(1)) = 11$. Since the stabilizer $\mathrm{Stab}_{\mathrm{Aut}(V_L^{\widehat{g}})}(\{V_L(i) \mid 0 \le i \le 22\})$ has the element -1 and -1 is fixed-point free, its orbits which do not contain the element $V_L(0)$ have even number of elements. This is a contradiction, and we have the desired result. □

6. Conclusion

Throughout Section 5, we determined the automorphism groups of the orbifold VOAs. In conclusion, we have the following theorem.

Theorem 6.1. *Let* $g \in pX$*, where* $pX \in \{3C, 5C, 11A, 23A\}$*. Let* L *be the coinvariant lattice* Λ_{pX} *associated with* g *and let* $\widehat{g}$ *be a standard lift of* g*. Then the automorphism groups of the orbifold VOAs* $V_L^{\widehat{g}}$ *are the following:*

- $\mathrm{Aut}(V_{\Lambda_{3C}}^{\widehat{g}}) \cong (3^4 \,.\, (3^4 : 2)) \,.\, (\Omega_7(3) \,.\, 2)$;
- $\mathrm{Aut}(V_{\Lambda_{5C}}^{\widehat{g}}) \cong (5^2 \,.\, (5^2 : 2)) \,.\, (2 \times \Omega_5(5))$;
- $\mathrm{Aut}(V_{\Lambda_{11A}}^{\widehat{g}}) \cong \Omega_4^-(11) \,.\, 2$;
- $\mathrm{Aut}(V_{\Lambda_{23A}}^{\widehat{g}}) \cong \Omega_3(23) \,.\, 2$.

A. Properties of subgroups of orthogonal groups

We describe some properties of subgroups of orthogonal groups over finite fields. In particular, by considering the stabilizer of a singular vector, we provide criteria for small-index subgroups of orthogonal groups. These are used to determine the automorphism groups of the orbifold VOAs in Section 5.

Throughout this section, we use the following notations. The symbol $\mathrm{diag}(a_0, \dots, a_{n-1})$ denotes the diagonal matrix whose diagonal entries are $a_0, \dots, a_{n-1}$ and the symbol J_n denotes the $n \times n$ anti-diagonal matrix whose all anti-diagonal entries are 1s. We denote $(\mathbb{F}_p^\times)^2 = \{a^2 \mid a \in \mathbb{F}_p^\times\}$.

A.1. Properties of subgroups of orthogonal groups in odd dimension

Unless otherwise described, let $n = 2m + 1$ ($m \geq 1$) and let p be an odd prime. Let V be an n-dimensional quadratic space over $\mathbb{F}_p$ with a basis $v_0, v_1, \dots, v_{n-1}$ such that the Gram matrix is J_n.

We recall a result from [28, Section 3.7].

Proposition A.1. *The following hold:*

(1) $\#\,\mathrm{GO}_n(p) = 2p^{m^2}(p^2-1)(p^4-1)\cdots(p^{2m}-1)$.

(2) $\#\Omega_n(p) = \#\,\mathrm{GO}_n(p)/4$.

Next, in order to determine the group structure of the stabilizer of a singular vector in non-degenerate quadratic space over $\mathbb{F}_p$, we consider the orbit of a singular vector under $\Omega_n(p)$. To achieve this, we describe some lemmas.

Lemma A.2. *We have*

$$\{\mathrm{diag}(a, 1, \dots, 1, a^{-1}) \mid a \in \mathbb{F}_p^\times\} \cap \Omega_n(p) = \{\mathrm{diag}(a, 1, \dots, 1, a^{-1}) \mid a \in (\mathbb{F}_p^\times)^2\}.$$

Remark A.3. Similarly, we obtain Lemma A.2 for $\mathrm{diag}(1, a, 1, \dots, 1, a^{-1}, 1)$ and so on.

By [6, Lemma 2.5.10], we have the following lemma.

Lemma A.4. [6, Lemma 2.5.10] *The group $\Omega_n(p)$ acts transitively on the set of* 1-*dimensional totally isotropic subspaces.*

Combining Lemma A.2 with Lemma A.4, we obtain the transitivity of $\Omega_n(p)$ ($m \geq 2$).

Proposition A.5. *If $m \geq 2$, then $\Omega_n(p)$ acts transitively on the set of singular vectors.*

Proof. Let $G = \mathrm{Stab}_{\Omega_n(p)}\langle v_0 \rangle$. By Lemma A.4, it suffices to prove that

$$\#\,\mathrm{Orbit}_G\, v_0 = p - 1.$$

Note that $\Omega_n(p) \lhd \mathrm{SO}_n(p)$ and $\mathrm{SO}_n(p)/\Omega_n(p) \cong \mathbb{Z}_2$. By Remark A.3, we have

$$\mathrm{diag}(a, a, 1, \dots, 1, a^{-1}, a^{-1}) \in \Omega_n(p) \quad \text{for } a \in \mathbb{F}_p^\times \setminus (\mathbb{F}_p^\times)^2.$$

Combining this with Lemma A.2, we have $\#\,\mathrm{Orbit}_G\, v_0 = p - 1$, as desired. □

Next, for subgroups of orthogonal groups, we consider the stabilizer of a singular vector. By direct calculations, we have the following lemma.

Lemma A.6. *Let $m \geq 2$. If $A \in \mathrm{Stab}_{\mathrm{GO}_n(p)}\, v_0$, then the shape of A is the following:*

$$\begin{pmatrix} 1 & x & -\frac{{}^t x J_{n-2} x}{2} \\ 0 & A' & -A' J_{n-2}{}^t x \\ 0 & 0 & 1 \end{pmatrix},$$

where $x \in \mathbb{F}_p^{n-2}$, $A' \in \mathrm{GO}_{n-2}(p)$.

We define a group homomorphism $\varphi_n \colon \mathrm{Stab}_{\mathrm{GO}_n(p)}\, v_0 \to \mathrm{GO}_{n-2}(p)$ by $A \mapsto A'$, where A' is the matrix as in Lemma A.6. By Lemma A.6, we have the following lemma.

Lemma A.7. *Let φ_n be the above group homomorphism. Then*

$$\operatorname{Ker}\varphi_n = \left\{ \begin{pmatrix} 1 & x & -\frac{{}^t x J_{n-2} x}{2} \\ 0 & E_{n-2} & -J_{n-2}{}^t x \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{F}_p^{n-2} \right\}.$$

In particular, $\operatorname{Ker}\varphi_n \cong p^{n-2}$.

By Lemmas A.6 and A.7, we have the following proposition.

Proposition A.8. *Let $m \geq 2$. Then $\mathrm{Stab}_{\mathrm{GO}_n(p)}\, v_0 \cong \operatorname{Ker}\varphi_n : \mathrm{GO}_{n-2}(p)$, where φ_n is the group homomorphism as in Lemma* A.7.

We describe the group structures of the stabilizers of a singular vector.

Proposition A.9. *Let $m \geq 2$. Then the following hold:*

(i) *If $p \equiv 1 \mod 4$,*

$$\begin{aligned} \mathrm{Stab}_{\mathrm{SO}_n(p)}\, v_0 &\cong p^{n-2} : \mathrm{SO}_{n-2}(p), \\ \mathrm{Stab}_{P_n(p)}\, v_0 &\cong p^{n-2} : P_{n-2}(p), \\ \mathrm{Stab}_{Q_n(p)}\, v_0 &\cong p^{n-2} : Q_{n-2}(p). \end{aligned}$$

(ii) *If $p \equiv 3 \mod 4$,*

$$\begin{aligned} \mathrm{Stab}_{\mathrm{SO}_n(p)}\, v_0 &\cong p^{n-2} : \mathrm{SO}_{n-2}(p), \\ \mathrm{Stab}_{P_n(p)}\, v_0 &\cong p^{n-2} : Q_{n-2}(p), \\ \mathrm{Stab}_{Q_n(p)}\, v_0 &\cong p^{n-2} : P_{n-2}(p). \end{aligned}$$

Proof. Let G be one of the groups $\mathrm{SO}_n(p)$, $P_n(p)$, and $Q_n(p)$. We first determine the order of $\mathrm{Stab}_G\, v_0$. Let $\widetilde{G} = \mathrm{Stab}_G\, v_0$. By orbit-stabilizer theorem, we have $\#G = \#\,\mathrm{Orbit}_G\, v_0 \# \widetilde{G}$. Combining this with Proposition A.5, we obtain

$$\#\widetilde{G} = p^{m^2}(p^2-1)\cdots(p^{2m-2}-1).$$

Let φ_n be the group homomorphism as in Lemma A.7. Since $\mathrm{Ker}\,\varphi_n|_{\widetilde{G}} \subset \mathrm{Ker}\,\varphi_n$, we see that $\#\,\mathrm{Ker}\,\varphi_n|_{\widetilde{G}}$ is a divisor of p^{n-2}. Moreover, since $\#\,\mathrm{Im}\,\varphi_n|_{\widetilde{G}}$ is a divisor of $\#\,\mathrm{GO}_{n-2}(p)$ and $\#\,\mathrm{GO}_{n-2}(p) = 2p^{(m-1)^2}(p^2-1)\cdots(p^{2m-2}-1)$, we have

$$\mathrm{Ker}\,\varphi_n|_{\widetilde{G}} = \mathrm{Ker}\,\varphi_n.$$

Hence $\#\,\mathrm{Im}\,\varphi_n|_{\widetilde{G}} = p^{(m-1)^2}(p^2-1)\cdots(p^{2m-2}-1)$, which means that $\mathrm{Im}\,\varphi_n|_{\widetilde{G}}$ is a subgroup of $\mathrm{GO}_{n-2}(p)$ of index 2. Since $\mathrm{Im}\,\varphi_n|_{\mathrm{Stab}_{\mathrm{SO}_n(p)}\, v_0} \subset \mathrm{SO}_{n-2}(p)$, by Proposition A.8, we have $\mathrm{Stab}_{\mathrm{SO}_n(p)}\, v_0 \cong p^{n-2} : \mathrm{SO}_{n-2}(p)$.

(i) The case $p \equiv 1 \mod 4$. By Lemma A.2, $\mathrm{diag}(-1,1,\ldots,1,-1) \in \Omega_n(p)$. Hence we have

$$-\mathrm{diag}(-1,1,\ldots,1,-1) \in P_n(p).$$

This implies that $\mathrm{Stab}_{P_n(p)}\, v_0 \cong p^{n-2} : P_{n-2}(p)$. Moreover, by Proposition A.8, we have $\mathrm{Stab}_{Q_n(p)}\, v_0 \cong p^{n-2} : Q_{n-2}(p)$.

(ii) The case $p \equiv 3 \mod 4$. By Lemma A.2, $\mathrm{diag}(-1,1,\ldots,1,-1) \notin \Omega_n(p)$. Hence we have

$$-\mathrm{diag}(-1,1,\ldots,1,-1) \in Q_n(p).$$

This implies that $\mathrm{Stab}_{Q_n(p)}\, v_0 \cong p^{n-2} : P_{n-2}(p)$. By Proposition A.8, we have $\mathrm{Stab}_{P_n(p)}\, v_0 \cong p^{n-2} : Q_{n-2}(p)$. □

Next, by considering the stabilizer of a singular vector, we provide a criterion for subgroups of orthogonal group of index 2. In order to do this, we prove the following lemma.

Lemma A.10. *Let $m \geq 2$ and let φ_n be the group homomorphism as in Lemma A.7. If $(n,p) \neq (5,3)$, $H \lhd \mathrm{Stab}_G\, v_0$, and $\#H = p^{n-2}$, then we have $H = \mathrm{Ker}\,\varphi_n$, where G is one of the groups $\mathrm{SO}_n(p)$, $P_n(p)$, and $Q_n(p)$.*

Proof. By Proposition A.8, we have $\mathrm{Stab}_G\, v_0 \cong \mathrm{Ker}\,\varphi_n : \widetilde{G}$, where $\widetilde{G}$ is one of the groups $\mathrm{SO}_{n-2}(p)$, $P_{n-2}(p)$, and $Q_{n-2}(p)$. Let $\pi\colon \mathrm{Stab}_G\, v_0 \to \widetilde{G}$ be the projection. Since $\pi\colon \mathrm{Stab}_G\, v_0 \to \widetilde{G}$ is surjective, we have $\pi(H) \lhd \widetilde{G}$. When p is an odd prime and $m \geq 2$, note that $\Omega_{n-2}(p)$ is simple except for $(n,p) = (5,3)$. Since $\#H = p^{n-2}$, by simplicity of $\Omega_{n-2}(p)$, we see that $\pi(H)$ is trivial. Since $H \subset \mathrm{Ker}\,\varphi_n$ and $\#H = \#\,\mathrm{Ker}\,\varphi_n = p^{n-2}$, we have $H = \mathrm{Ker}\,\varphi_n$. □

By Proposition A.9 and Lemma A.10, we obtain a criterion for subgroups of orthogonal group of index 2.

Proposition A.11. *Let $m \geq 2$ and let G be one of the groups $\mathrm{SO}_n(p)$, $P_n(p)$, and $Q_n(p)$. If $(n,p) \neq (5,3)$ and $\mathrm{Stab}_G\, v_0 \cong p^{n-2} . P_{n-2}(p)$, then the following hold:*

$$G = \begin{cases} P_n(p) & \text{if } p \equiv 1 \mod 4, \\ Q_n(p) & \text{if } p \equiv 3 \mod 4. \end{cases}$$

Next we consider the case $n = 3$. First, we examine the orbit of a singular vector.

Proposition A.12. $\#\,\mathrm{Orbit}_{\Omega_3(p)}\, v_0 = (p^2-1)/2$.

Proof. By Lemma A.2, we have $\#\,\mathrm{Orbit}_{\Omega_3(p)}\, v_0 = (p^2-1)/2$ or p^2-1. Since $\#\Omega_3(p) = \#\,\mathrm{Orbit}_{\Omega_3(p)}\, v_0 \#\,\mathrm{Stab}_{\Omega_3(p)}\, v_0$, we have $\#\,\mathrm{Orbit}_{\Omega_3(p)}\, v_0 \mid (1/2)p(p^2-1)$. Since p is an odd prime, we have $\#\,\mathrm{Orbit}_{\Omega_3(p)}\, v_0 = (p^2-1)/2$. □

By Proposition A.12, we obtain a criterion for small-index subgroups of $\mathrm{GO}_3(p)$, where $p \equiv 3 \mod 4$.

Lemma A.13. *Let p be an odd prime such that $p \equiv 3 \mod 4$ and let G be one of the groups $\Omega_3(p)$, $\mathrm{SO}_3(p)$, $P_3(p)$, $Q_3(p)$, and $\mathrm{GO}_3(p)$. If $\mathrm{Stab}_G\, v_0 \cong p . 2$, then we have $G = Q_3(p)$ or $\mathrm{GO}_3(p)$.*

Proof. By Proposition A.12, we have $\mathrm{Stab}_{\Omega_3(p)}\, v_0 = p$. Moreover, since $p \equiv 3 \mod 4$, the groups $\mathrm{SO}_3(p)$ and $P_3(p)$ act transitively on the set of singular vectors. This implies that $\mathrm{Stab}_{\widetilde{G}}\, v_0 = p$, where $\widetilde{G}$ is one of the groups $\mathrm{SO}_3(p)$ and $P_3(p)$. Since $\#\,\mathrm{Stab}_G\, v_0 = 2p$, we can omit the cases where $G = \Omega_3(p)$, $\mathrm{SO}_3(p)$, or $P_3(p)$. Hence we have the desired result $G = Q_3(p)$ or $\mathrm{GO}_3(p)$. □

A.2. Properties of subgroups of orthogonal groups in even dimension

Throughout this subsection, we use the following notations. Let $n = 2m$ $(m \geq 2)$ and let p be an odd prime. The symbol K_n denotes the following matrix:

$$\begin{pmatrix} O & & J_{m-1} \\ & E_2 & \\ J_{m-1} & & O \end{pmatrix},$$

where E_2 is the 2×2 identity matrix. Let V be an n-dimensional non-degenerate quadratic space over $\mathbb{F}_p$ with a basis $v_0, v_1, \ldots, v_{n-1}$ such that the Gram matrix is K_n.

We recall a result from [28, Section 3.7].

Proposition A.14. *The following hold:*

(1) $\# \mathrm{GO}_n^-(p) = 2p^{m(m-1)}(p^2-1)(p^4-1)\cdots(p^{n-2}-1)(p^m+1)$.

(2) $\#\Omega_n^-(p) = \# \mathrm{GO}_n^-(p)/4$.

Next, we consider the group structure of the stabilizer of a singular vector in non-degenerate quadratic space over $\mathbb{F}_p$. To achieve this, we describe some properties of subgroups of orthogonal groups.

By direct calculations, we have the following lemma.

Lemma A.15. *We have*

$$\{\mathrm{diag}(a,1,\ldots,1,a^{-1}) \mid a \in \mathbb{F}_p^\times\} \cap \Omega_n^-(p) = \{\mathrm{diag}(a,1,\ldots,1,a^{-1}) \mid a \in (\mathbb{F}_p^\times)^2\}.$$

By [6, Lemma 2.5.10], we have the following lemma.

Lemma A.16. [6, Lemma 2.5.10] *The group $\Omega_n^-(p)$ acts transitively on the set of* 1-*dimensional totally isotropic subspaces.*

The following lemma describes the shape of the elements in the stabilizer of a singular vector.

Lemma A.17. *If $A \in \mathrm{Stab}_{\mathrm{GO}_n^-(p)} v_0$, then the shape of A is the following:*

$$\begin{pmatrix} 1 & x & -\frac{{}^t x K_{n-2} x}{2} \\ 0 & A' & -A' K_{n-2} {}^t x \\ 0 & 0 & 1 \end{pmatrix},$$

where $x \in \mathbb{F}_p^{n-2}$, $A' \in \mathrm{GO}_{n-2}^-(p)$.

We define a group homomorphism $\varphi_n \colon \mathrm{Stab}_{\mathrm{GO}_n^-(p)} v_0 \to \mathrm{GO}_{n-2}^-(p)$ by $A \mapsto A'$, where A' is the matrix as in Lemma A.17.

By Lemma A.17, we have the following lemma.

Lemma A.18. *Let φ_n be the above group homomorphism. Then*

$$\mathrm{Ker}\,\varphi_n = \left\{ \begin{pmatrix} 1 & x & -\frac{{}^t x K_{n-2} x}{2} \\ 0 & E_{n-2} & -K_{n-2} {}^t x \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{F}_p^{n-2} \right\}.$$

In particular, $\mathrm{Ker}\,\varphi_n \cong p^{n-2}$.

The following lemma describes the kernel of the restriction of φ_n to $\mathrm{Stab}_{\Omega_n^-(p)} v_0$.

Lemma A.19. *Let* $G = \mathrm{Stab}_{\Omega_n^-(p)} v_0$*. Then we have* $\mathrm{Ker}\,\varphi_n|_G = \mathrm{Ker}\,\varphi_n$.

Proof. We see that $\#\,\mathrm{Ker}\,\varphi_n|_G$ is a divisor of $\#\,\mathrm{Ker}\,\varphi_n = p^{n-2}$. By Lemmas A.15 and A.16, we have $\#\,\mathrm{Orbit}_G\, v_0 = (p^m+1)(p^{m-1}-1)/2$ or $(p^m+1)(p^{m-1}-1)$. If $\#\,\mathrm{Orbit}_G\, v_0 = (p^m+1)(p^{m-1}-1)/2$, then we have $\mathrm{Stab}_G\, v_0 = p^{m(m-1)}(p^2-1)\cdots(p^{n-4}-1)(p^{m-1}+1)$. If $\#\,\mathrm{Orbit}_G\, v_0 = (p^m+1)(p^{m-1}-1)$, then we have $\mathrm{Stab}_G\, v_0 = (1/2)p^{m(m-1)}(p^2-1)\cdots(p^{n-4}-1)(p^{m-1}+1)$. Since $\#\,\mathrm{Im}\,\varphi_n|_G$ is a divisor of $\#\,\mathrm{GO}_{n-2}^-(p)$ and since $\#\,\mathrm{GO}_{n-2}^-(p) = 2p^{(m-1)(m-2)}(p^2-1)\cdots(p^{n-4}-1)(p^{m-1}+1)$, by considering the prime factor p, we have the desired result $\mathrm{Ker}\,\varphi_n|_G = \mathrm{Ker}\,\varphi_n$. □

Next we prove the transitivity of $\Omega_n^-(p)$.

Proposition A.20. *The group* $\Omega_n^-(p)$ *acts transitively on the set of singular vectors.*

Proof. Let $G = \mathrm{Stab}_{\Omega_n^-(p)} v_0$. Since $\mathrm{Im}\,\varphi_n|_G \subset \mathrm{SO}_{n-2}^-(p)$ and $\mathrm{diag}(1, A', 1) \in G$ for any $A' \in \Omega_{n-2}^-(p)$, we see that $\mathrm{Im}\,\varphi_n|_G = \Omega_{n-2}^-(p)$ or $\mathrm{SO}_{n-2}^-(p)$. By Lemma A.19, we have $\mathrm{Im}\,\varphi_n|_G = \Omega_{n-2}^-(p)$. Hence we have $\#G = \#\,\mathrm{Ker}\,\varphi_n \#\Omega_{n-2}^-(p) = (1/2)p^{m(m-1)}(p^2-1)\cdots(p^{n-4}-1)(p^{m-1}+1)$. Thus we have $\#\,\mathrm{Orbit}_{\Omega_n^-(p)} v_0 = \#\Omega_n^-(p)/\#G = (p^m+1)(p^{m-1}-1)$, which means $\Omega_n^-(p)$ acts transitively on the set of singular vectors. □

By the definition of $\Omega_n^-(p)$, we see that $\mathrm{GO}_n^-(p)/\Omega_n^-(p) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let

$$\mathrm{GO}_n^-(p)/\Omega_n^-(p) = \{\Omega_n^-(p), \sigma_1\Omega_n^-(p), \sigma_2\Omega_n^-(p), \sigma_1\sigma_2\Omega_n^-(p)\},$$

where σ_1 is an element of $\mathrm{SO}_n^-(p)\setminus\Omega_n^-(p)$ and σ_2 is an element of $\mathrm{GO}_n^-(p)\setminus\mathrm{SO}_n^-(p)$. Note that $\mathrm{SO}_n^-(p) = \Omega_n^-(p) \cup \sigma_1\Omega_n^-(p)$. By Proposition A.20, we have the following corollary.

Corollary A.21. *Let G be one of the groups* $\mathrm{SO}_n^-(p)$, $\Omega_n^-(p)\cup\sigma_2\Omega_n^-(p)$, *and* $\Omega_n^-(p)\cup(\sigma_1\sigma_2)\Omega_n^-(p)$*. Then we have* $\mathrm{Stab}_G\, v_0 = p^{m(m-1)}(p^2-1)\cdots(p^{n-4}-1)(p^{m-1}+1)$.

Note that $\mathrm{GO}_2^-(p) \cong D_{2(p+1)}$. Hence, by Lemma A.19 and Corollary A.21, we have the following proposition.

Proposition A.22. *Let* $G_1 = \mathrm{SO}_4^-(p)$, $G_2 = \Omega_4^-(p)\cup\sigma_2\Omega_4^-(p)$, *and* $G_3 = \Omega_4^-(p)\cup(\sigma_1\sigma_2)\Omega_4^-(p)$*. Then we have*

$$\mathrm{Stab}_{G_1} v_0 \cong p^2 : \mathbb{Z}_{p+1}, \quad \mathrm{Stab}_{G_2} v_0 \cong p^2 : D_{p+1}, \quad \mathrm{Stab}_{G_3} v_0 \cong p^2 : D_{p+1}.$$

To obtain a criterion for the subgroups of $\mathrm{GO}_4^-(p)$ of index 2, we describe the following lemmas.

Lemma A.23. *Let G be one of the groups* $\mathrm{SO}_4^-(p)$, $\Omega_4^-(p)\cup\sigma_2\Omega_4^-(p)$, *and* $\Omega_4^-(p)\cup(\sigma_1\sigma_2)\Omega_4^-(p)$*. If* $H \lhd \mathrm{Stab}_G\, v_0$ *and* $\#H = p^2$*, then we have* $H = \mathrm{Ker}\,\varphi_4$*, where* φ_4 *is the group homomorphism as in Lemma* A.18.

Proof. Let $\pi\colon \mathrm{Stab}_G\, v_0 \to \widetilde{G}$ be the projection, where $\widetilde{G}$ is one of the groups $\mathbb{Z}_{p+1}$ and D_{p+1}. Since $\#\pi(H) \mid \#H = p^2$ and since $\#\pi(H) \mid \#\widetilde{G} = p+1$, we have $\pi(H) = 1$. This implies the desired result $H = \mathrm{Ker}\,\varphi_4$. □

Lemma A.24. $\Omega_n^-(p) \cup \sigma_2\Omega_n^-(p) \cong \Omega_n^-(p) \cup (\sigma_1\sigma_2)\Omega_n^-(p)$ *as groups.*

By Proposition A.22, and Lemmas A.23 and A.24, we obtain a criterion for the subgroups of $\mathrm{GO}_4^-(p)$ of index 2.

Proposition A.25. *Let G be one of the groups $\Omega_4^-(p)$, $\mathrm{SO}_4^-(p)$, $\Omega_4^-(p) \cup \sigma_2\Omega_4^-(p)$, and $\Omega_4^-(p) \cup (\sigma_1\sigma_2)\Omega_4^-(p)$. If $\mathrm{Stab}_G\, v_0 \cong p^2 \,.\, D_{p+1}$, then we have $G \cong \Omega_4^-(p) \cup \sigma_2\Omega_4^-(p) \cong \Omega_4^-(p) \cup (\sigma_1\sigma_2)\Omega_4^-(p)$ as groups.*

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