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Eigenvalue Ratios and Gaps of Sturm–Liouville Problems

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Abstract. In this paper, we will study the eigenvalue ratios $\frac{\lambda_n}{\lambda_m}$ and the eigenvalue gaps $\lambda_n - \lambda_m$ of the Sturm–Liouville problem

$$-(p(x)y')' + q(x)y = \lambda w(x)y.$$

We prove that if $q \leq 0$ and $\theta'' \geq 0$, then $\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2$ and if $q \geq 0$ and $\theta'' \leq 0$, then $\frac{\lambda_n}{\lambda_m} \geq \left(\lfloor \frac{n}{m} \rfloor\right)^2$. In the case where $q \equiv 0$ and $p = 1$, we prove that if $4ww'' \geq 5(w')^2$ then $\lambda_n - \lambda_m \leq \left[\left(\frac{n}{m}\right)^2 - 1\right] \frac{(m\pi)^2}{w_m}$ and if $4ww'' \leq 5(w')^2$, then $\lambda_n - \lambda_m \geq \left[\left(\lfloor \frac{n}{m} \rfloor\right)^2 - 1\right] \frac{(m\pi)^2}{w_M}$, where $\theta(x) = (p(x)w(x))^{1/4}$, $w_m = \inf_{x \in [0,1]} w(x)$ and $w_M = \max_{w \in [0,1]} w(x)$.

1. Introduction

We consider the Sturm–Liouville equation acting on $[0,1]$:

$$-(p(x)y')' + q(x)y = \lambda w(x)y, \tag{1.1}$$

with Dirichlet boundary conditions

$$y(0) = y(1) = 0. \tag{1.2}$$

Here $p > 0$, $w > 0$ and twice differentiable on $[0,1]$ and $q \in L_1(0,1)$.

It is known (see [15]) that the eigenvalues of problem (1.1) with (1.2) form a strictly increasing sequence of positive numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots < \infty$. The issues of optimal estimates for the eigenvalue ratios $\frac{\lambda_n}{\lambda_m}$ and eigenvalue gaps $\lambda_n - \lambda_n$ have attracted a lot of attention (see [1–4, 6–13]) and references therein. In particular, Ashbaugh and Benguria in [3] proved that if $q \geq 0$ and $0 < k \leq pw(x) \leq K$, then the eigenvalues of (1.1) with (1.2) satisfy

$$\frac{\lambda_n}{\lambda_1} \leq \frac{Kn^2}{k}.$$

They also established the following ratio estimate (of two arbitrary eigenvalues)

$$\frac{\lambda_n}{\lambda_m} \leq \frac{Kn^2}{km^2}, \quad n > m \geq 1$$

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with $q \equiv 0$ and $0 < k \leq pw(x) \leq K$. Later, Huang and Law [13] proved that

$$\frac{\lambda_{n+1}}{\lambda_{m+1}} \geq \frac{1}{1+\xi}\frac{n^2}{m^2}\frac{K}{k},$$

where $\xi = \frac{K\max\{pq\}}{kp_1^2 n^2\pi^2}$ and $p_1 = \left(\int_0^1 \frac{1}{p(x)}\,dx\right)^{-1}$. They also examined the gap of two arbitrary eigenvalues for problem (1.1) with $p = w = 1$, and found

$$\lambda_n - \lambda_m \geq (m\pi)^2\left[\left(\frac{(n\pi)^2}{(m\pi)^2+q_0}\right)^2 - 1\right],$$

where $q_0 = \max q - \min q$. In 2022, Hedhly [6] showed that

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \quad n > m \geq 1 \tag{1.3}$$

for single-barrier potential q and single-well pw. He also established the same estimate (1.3) for the string equation

$$-y'' = \lambda w(x) y \tag{1.4}$$

with Dirichlet boundary conditions (1.2) and single-well density w. Recently, the same author [7] showed that the eigenvalues of (1.1) with (1.2) satisfy

$$\frac{\lambda_n}{\lambda_1} \leq n^2$$

for symmetric single-well potential q and symmetric single-barrier function pw.

We say that f is a single-barrier (resp. single-well) function on $[0,1]$ if there is a point $x_0 \in [0,1]$ such that f is increasing (resp. decreasing) on $[0,x_0]$ and decreasing (resp. increasing) on $[x_0,1]$ (see [2]).

In this paper, we prove that the eigenvalues of problem (1.4) with (1.2) satisfy $\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2$ and $\lambda_n - \lambda_m \leq \left[\left(\frac{n}{m}\right)^2 - 1\right]\frac{m\pi^2}{w_m}$ (resp. $\frac{\lambda_n}{\lambda_m} \geq \left(\left\lfloor\frac{n}{m}\right\rfloor\right)^2$ and $\lambda_n - \lambda_m \geq \left[\left(\left\lfloor\frac{n}{m}\right\rfloor\right)^2 - 1\right]\frac{m\pi^2}{w_M}$) if $4ww'' \geq 5(w')^2$ (resp. $4ww'' \leq 5(w')^2$), $w_m = \inf_{x\in[0,1]} w(x)$ and $w_M = \max_{x\in[0,1]} w(x)$. We also prove the same results for the Dirichlet Sturm–Liouville problems (1.1) with (1.2). More precisely, we prove that if $q \leq 0$ and $\theta'' \geq 0$, then $\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2$ and if $q \geq 0$ and $\theta'' \leq 0$; then $\frac{\lambda_n}{\lambda_m} \geq \left(\left\lfloor\frac{n}{m}\right\rfloor\right)^2$, here $\theta(x) = (p(x)w(x))^{1/4}$.

2. Eigenvalue ratio for the vibrating string equations

We are now in position to state our main result.

Theorem 2.1. *Let the Dirichlet eigenvalue problem* (1.4) *and* (1.2) *with* w *twice differentiable.*

(a) *If* $4w(x)w''(x) \geq 5(w'(x))^2$ *for all* $x \in (0,1)$*, then*

$$\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2.$$

Equality holds if and only if $4w(x)w''(x) = 5(w'(x))^2$*.*

(b) *If* $4w(x)w''(x) \leq 5(w'(x))^2$ *for all* $x \in (0,1)$*, then*

$$\frac{\lambda_n}{\lambda_m} \geq \left(\left\lfloor\frac{n}{m}\right\rfloor\right)^2. \tag{2.1}$$

Equality holds if and only if $4w(x)w''(x) = 5(w'(x))^2$ *and* n *is a multiple of* m*.*

For the proof of Theorem 2.1 we need the following results.

Lemma 2.2. [6, Corollary 3] *If* q *is nonnegative* ($p = w = 1$)*, then*

$$\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2.$$

Equality holds if and only if $q \equiv 0$ *on* $[0,1]$*.*

Lemma 2.3. [5] *For the Schrodinger operator* (1.4) ($p = w = 1$)*, if* $q \in L_1(0,1)$*,* $q \leq 0$ *and the Dirichlet eigenvalue* λ_1*'s are positive, then for any* $n > m \geq 1$*,*

$$\frac{\lambda_n}{\lambda_m} \geq \left(\left\lfloor\frac{n}{m}\right\rfloor\right)^2.$$

Equality holds if and only if $q \equiv 0$ *and* n *is a multiple of* m*.*

Proof of Theorem 2.1. (a) We introduce the Liouville substitution (e.g., see [15, p. 2])

$$t(x) = \frac{1}{c}\int_0^x \sqrt{w(s)}\,ds \quad \text{with } c = \int_0^1 \sqrt{w(s)}\,ds,$$

which transforms problem (1.4) with (1.2) into the system with respect to new variables

$$\begin{cases} -\ddot{u} + \widetilde{q}(t)u = \widetilde{\lambda}u, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $\widetilde{\lambda} = c^2\lambda$, $u = w^{1/4}y$ and $\widetilde{q} = c^2\left(\frac{4ww''-5(w')^2}{16w^3}\right)$. Using the assumption $4ww'' \geq 5(w')^2$, one gets $\widetilde{q} \geq 0$. From this and Lemma 2.2, the eigenvalues $(\widetilde{\lambda}_n)_{n\geq 1}$ of problem (3.4) satisfy the upper bound estimate

$$\frac{\widetilde{\lambda}_n}{\widetilde{\lambda}_m} \leq \frac{n^2}{m^2}.$$

Since $\widetilde{\lambda}_n = c^2\lambda_n$, hence

$$\frac{\lambda_n}{\lambda_m} = \frac{\widetilde{\lambda}_n}{\widetilde{\lambda}_m} \leq \frac{n^2}{m^2}.$$

According to Lemma 2.2, the equality holds if and only if $\widetilde{q} = 0$, then we have

$$4ww'' = 5(w')^2.$$

(b) The proof of (2.1) proceeds in the same way. □

The above result provides an easy method to compute lower estimate for arbitrary Dirichlet eigenvalue gaps.

Corollary 2.4. *Consider equation* (1.4) *with Dirichlet boundary conditions and w twice differentiable.*

(a) *If $4ww'' \geq 5(w')^2$, then*

$$\lambda_n - \lambda_m \leq \left[\left(\frac{n}{m}\right)^2 - 1\right] \frac{m\pi^2}{w_m}.$$

Equality holds if and only if $4ww'' = 5(w')^2$.

(b) *If $4ww'' \leq 5(w')^2$, then*

$$\lambda_n - \lambda_m \geq \left[\left(\left\lfloor\frac{n}{m}\right\rfloor\right)^2 - 1\right] \frac{m\pi^2}{w_M}. \tag{2.2}$$

Equality holds if and only if $4ww'' = 5(w')^2$ and n is a multiple of m. Here $w_m = \inf_{x\in[0,1]} w(x)$ and $w_M = \max_{x\in[0,1]} w(x)$.

Proof. (a) From Theorem 2.1, we have

$$\frac{\lambda_n(w)}{\lambda_m(w)} \leq \left(\frac{n}{m}\right)^2,$$

thus

$$\lambda_n - \lambda_m \leq \left[\left(\frac{n}{m}\right)^2 - 1\right] \lambda_m \leq \left[\left(\frac{n}{m}\right)^2 - 1\right] \frac{(m\pi)^2}{w_m}.$$

(b) The proof of (2.2) can be handled in the same way. □

3. Eigenvalue ratios for Sturm–Liouville problems

Theorem 3.1. *Consider the regular Sturm–Liouville problem* (1.1) *with Dirichlet boundary conditions* (1.2).

(a) *If $q \leq 0$ and $\theta'' \geq 0$ with both pw and $\frac{p}{w}$ increasing or decreasing on $(0,1)$, then for any $n > m \leq 1$,*

$$\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2. \tag{3.1}$$

Equality holds if and only if $q \equiv 0$ and pw is constant or $p = Cw$, $w = (C_1x + C_2)^2$ for some constants C, C_1, C_2.

(b) *If* $q \geq 0$ *and* $\theta'' \leq 0$ *with* pw *increasing and* $\frac{p}{w}$ *decreasing or vice versa, then for any* $n > m \leq 1$,

$$\frac{\lambda_n}{\lambda_m} \geq \left(\left\lfloor \frac{n}{m} \right\rfloor\right)^2. \tag{3.2}$$

Equality holds if and only if $q \equiv 0$, n *is a multiple of* m *and* pw *is a constant or* $p = Cw$, $w = (C_1 x + C_2)^2$ *for some constants* C, C_1, C_2.

Here $\theta(x) = (p(x)w(x))^{1/4}$.

At the end of this paper, we give an example for p and w to show that the conditions of Theorem 3.1 are still quite general, see Section 4.

In the next corollary, the result is stated without assumptions about the differentiability of pw.

Corollary 3.2. *If* $q(x) \geq 0$ *and* pw *is concave on* $[0, 1]$, *then*

$$\frac{\lambda_n}{\lambda_m} \geq \left(\left\lfloor \frac{n}{m} \right\rfloor\right)^2.$$

Equality holds if and only if $q \equiv 0$, pw *is a constant and* n *is a multiple of* m.

Proof of Theorem 3.1. First, if $q \equiv 0$ then using the Legendre substitution [14, pp. 227–228]

$$z(x) = \frac{1}{\sigma} \int_0^x \frac{1}{p(s)}\, ds \quad \text{with } \sigma = \int_0^1 \frac{1}{p(s)}\, ds,$$

the equation (1.1) can be rewritten in the string equation

$$-\ddot{y} = \widehat{\lambda} p(t) w(z) y.$$

Thus the estimates (3.1) and (3.2) are direct consequences of Theorem 2.1. In the sequel we suppose that $q \not\equiv 0$.

(a) We introduce the Liouville substitution (e.g., see [15, p. 2])

$$t(x) = \frac{1}{\gamma} \int_0^x \sqrt{\frac{w(s)}{p(s)}}\, ds \quad \text{with } \gamma = \int_0^1 \sqrt{\frac{w(s)}{p(s)}}\, ds \tag{3.3}$$

to transform problem (1.1) with (1.2) into the system with respect to new variables

$$\begin{cases} -\ddot{u} + Q(t)u = \widetilde{\lambda} u, & t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{3.4}$$

where $\widetilde{\lambda} = \gamma^2 \lambda$, $u = (p(x)w(x))^{1/4} y$ and $Q(t) = \frac{\ddot{\theta}}{\theta} - \gamma^2 \frac{q}{w}$, with $\theta' = \frac{1}{4}(pw)^{-3/4}(pw)'$ and $\ddot{\theta} = \frac{\theta'' pw + \theta'(p/w)'}{w^2}$. Using the assumption $q \leq 0$ and $\theta'' \geq 0$ with pw increasing and $\frac{p}{w}$

increasing, we have $\ddot{\theta} \geq 0$, then $Q(t) \geq 0$. So, from this and Lemma 2.3, the eigenvalues $(\widetilde{\lambda}_n)_{n\geq 1}$ of problem (3.4) satisfy the upper bound estimate

$$\frac{\widetilde{\lambda}_n}{\widetilde{\lambda}_m} \leq \frac{n^2}{m^2},$$

and hence

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}.$$

According to Lemma 2.3, the equality $\frac{\lambda_n}{\lambda_m} = \frac{n^2}{m^2}$ holds if and only if $Q \equiv 0$, thus $\frac{\ddot{\theta}}{\theta} = \gamma^2 \frac{q}{w}$. Furthermore, we have $q \leq 0$ and $\ddot{\theta} \geq 0$ which is possible if and only if $q \equiv 0$ and $\ddot{\theta} \equiv 0$ implies that $\theta'' pw + \theta'(p/w)' = 0$, or we have $\theta'' pw \geq 0$ and $\theta'(p/w)' \geq 0$, this implies that $\theta'' pw = \theta'(p/w)' = 0$. If $\theta' = 0$ then pw is a constant. But since $(p/w)' = 0$, we have $p = Cw$ and $\theta = C_0 w^{1/2}$. Hence $\theta'' = 0$ implies that $2ww'' - (w')^2 = 0$, which can be integrated to obtain $w = (C_1 x + C_2)^2$.

(b) The proof of (3.2) is similar and is omitted here. □

In order to prove Corollary 3.2, we need the following result from Chung-Chuan Chen et al. [5, Lemma 4.1].

Lemma 3.3. *If $f \in C(0,1)$ is positive and a concave function on $[0,1]$, then for all $\epsilon > 0$ there exists a positive C^∞ function $\widetilde{f}_\epsilon$ on $[0,1]$ such that $\widetilde{f}_\epsilon \to f$ in $L^1(0,1)$. Furthermore, each $\widetilde{f}_\epsilon$ satisfies $\widetilde{f}''_\epsilon \leq 0$ except possibly at two points in $[0,1]$.*

Proof of Corollary 3.2. Using the Legendre substitution (3.3) with respect to new variables, equation (1.1) becomes

$$-\ddot{y} + p(z)q(z)y = \sigma^2 \lambda p(z)w(z)y.$$

Since pw is concave, then according to Lemma 3.3, there exists $\widetilde{w}_\epsilon > 0$ in $[0,1]$ such that $\widetilde{w}_\epsilon \to pw$. Now, let us pose the following problem

$$-\ddot{y} + p(z)q(z)y = \widehat{\lambda}\widetilde{w}_\epsilon y, \tag{3.5}$$

where $\widehat{\lambda} = \sigma^2\lambda$. Then by the Liouville substitution (e.g., see [15, p. 2])

$$\xi(x) = \frac{1}{c}\int_0^x \sqrt{\widetilde{w}_\epsilon}\, ds \quad \text{with } c = \int_0^1 \sqrt{\widetilde{w}_\epsilon}\, ds,$$

which transforms problem (3.5) with (1.2) into the system with respect to new variables

$$\begin{cases} -\partial_\xi^2 u + Q(\xi)u = \widetilde{\lambda}u, & \xi \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{3.6}$$

where $\widetilde{\lambda} = c^2\widehat{\lambda}$, $u = (\widetilde{w}_\epsilon)^{1/4}y$ and

$$Q_\epsilon = \frac{\partial_\epsilon^2 \theta_\epsilon}{\theta_\epsilon} - c^2 \frac{pq}{\widetilde{w}_\epsilon},$$

with $\theta_\epsilon = (\widetilde{w}_\epsilon)^{1/4}$. By a simple calculation, we have

$$\dot{\theta}_\epsilon = \frac{1}{4}\widetilde{w}_\epsilon^{-3/4}\dot{\widetilde{w}}_\epsilon \quad \text{and} \quad \ddot{\theta}_\epsilon = \frac{4\widetilde{w}_\epsilon \ddot{\widetilde{w}}_\epsilon - 3(\dot{\widetilde{w}})^2}{16(\widetilde{w}_\epsilon)^{7/4}}.$$

Since $\widetilde{w}_\epsilon$ is concave, then $\ddot{\widetilde{w}}_\epsilon \leq 0$, hence $\dot{\theta}_\epsilon \leq 0$.

Now we will show that $\partial_\xi^2 \theta_\epsilon \leq 0$, we have

$$\partial_\xi \theta_\epsilon = \dot{\theta}_\epsilon \frac{dz}{d\xi} \dot{\theta}_\epsilon \frac{c}{\sqrt{\widetilde{w}_\epsilon}},$$

then by a simple calculation, we have

$$\partial_\xi^2 \theta_\epsilon = \frac{c^2}{\sqrt{\widetilde{w}_\epsilon}} \left(\frac{2\ddot{\theta}_\epsilon \widetilde{w}_\epsilon - \dot{\widetilde{w}}_\epsilon \dot{\theta}_\epsilon}{2\widetilde{w}_\epsilon^{-3/2}} \right) = \frac{c^2}{\sqrt{\widetilde{w}_\epsilon}} \left(\frac{2\ddot{\theta}_\epsilon \widetilde{w}_\epsilon - \frac{1}{4}(\dot{\widetilde{w}}_\epsilon)^2 \widetilde{w}_\epsilon^{-3/4}}{2\widetilde{w}_\epsilon^{-3/2}} \right) \leq 0.$$

Hence $Q_\epsilon \leq 0$. Then by Theorem 3.1, the eigenvalues of problem (3.6) with (1.2) satisfy

$$\frac{\widetilde{\lambda}_n}{\widetilde{\lambda}_m} \geq \left(\left\lfloor \frac{n}{m} \right\rfloor \right)^2.$$

Consequently, the eigenvalues of problem (3.5) with (1.2) satisfy

$$\frac{\widehat{\lambda}_n}{\widehat{\lambda}_m} \geq \left(\left\lfloor \frac{n}{m} \right\rfloor \right)^2.$$

Furthermore, by the continuity of eigenvalues, one gets

$$\frac{\widehat{\lambda}_n}{\widehat{\lambda}_m} \to \frac{\lambda_n}{\lambda_m} \quad \text{as } \epsilon \to 0,$$

and hence

$$\frac{\lambda_n}{\lambda_m} \geq \left(\left\lfloor \frac{n}{m} \right\rfloor \right)^2. \qquad \square$$

4. Remarks

We give here an example for p and w to show that the conditions of Theorem 3.1 are still quite general.

Let $p(x) = f(x)^\alpha$ and $w(x) = f(x)^\beta$. So

$$pw(x) = f(x)^{\alpha+\beta}, \quad \frac{p}{w}(x) = f(x)^{\alpha-\beta}.$$

Let $\gamma = \frac{\alpha+\beta}{4}$. Then $\theta(x) = f(x)^\gamma$, so that

$$\theta''(x) = \gamma(\gamma-1)f(x)^{\gamma-2}(f'(x))^2 + \gamma f(x)^{\gamma-1}f''(x).$$

Hence if $\alpha > \beta > 2$, and f is positive, convex and increasing on $(0,1)$, then $\theta'' > 0$ and both pw and $\frac{p}{w}$ are increasing. If $\alpha < \beta < 0$, and f is concave, positive and increasing on $(0,1)$, then $\theta'' > 0$ but pw and $\frac{p}{w}$ are both decreasing. Therefore the conditions in Theorem 3.1(a) are quite general. Similar arguments apply to those conditions for Theorem 3.1(b).

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