TAIWANESE JOURNAL OF MATHEMATICS
Vol. xx, No. x, pp. 1–16, xx 20xx
DOI: 10.11650/tjm/241003

On Disruptions of Semigroups and Points Strongly Focusing Chaos

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Abstract. In the mathematical literature, there are various approaches to the concept of chaos, also in the local aspects. Points of chaos (connected with homoclinic points of Poincare), points of distributional chaos and points focusing entropy are known. It can be shown in [14], by using topological tools that these concepts are significantly different. In this paper we join them, considering new kind of points (strongly focusing chaos) combinig them with problem of disruptions of semigroups of functions. The paper ends with open problems.

1. Introduction

Issues related to the theory of dynamical systems have also been widely studied in the context of groups and semigroups generated by finite families of functions (e.g., [1,8]), also in relation to the local aspects of chaos theory (e.g., [5, 13]).

The concept of chaos has many meanings, often nonequivalent (e.g., [15]). Also in the case of points of chaos, different approaches to this issue can be found. There are considered points focusing entropy, DC1 points and chaotic points.

The importance of these concepts and their diversity makes it worth to consider the new notion: point strongly focusing chaos, which is a combination of earlier notions. In this paper we will combine this type of points with the disruptions of semigroups. The starting point for our considerations is the observation that there exists a finite family $\mathcal{A}$ of functions defined on the unit interval and a point x_0 such that x_0 is a point strongly focusing chaos of any function from the family $\mathcal{A}$ and there exists a function τ belonging to the semigroup generated by $\mathcal{A}$ such that x_0 is not a point strongly focusing chaos of τ.

Therefore, the question arises: if we have a finite family of functions $\mathcal{A}$ and a certain point x_0, can this family be disturbed in such a way that the point x_0 becomes a strongly focusing point of any function from the semigroup generated by $\mathcal{A}$? This paper contains the answer to this question. At the end we formulate two open problems.

Received April 7, 2024; Accepted October 4, 2024.
Communicated by Cheng-Hsiung Hsu.
2020 *Mathematics Subject Classification.* 37B40, 37C85, 54H15, 37D45, 54C70.
Key words and phrases. chaos, distributional chaos, entropy, semigroup, disruptions, point focusing entropy (chaos, distributional chaos), homoclinic point, point strongly focusing chaos.
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2. Preliminaries

We use standard symbols and notations. By $\mathbb{N}_+$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{I}$ we denote the set of all positive integers, nonnegative integers, real numbers and the interval $[0,1]$, respectively. To simplify the notation we use the same letters $\mathbb{R}$ and $\mathbb{I}^l$ for metric spaces equipped with the natural metric (we denote these metrics with a common symbol d_e).

By $\mathrm{Int}(A)$, $\mathrm{Fr}(A)$, $\mathrm{card}(A)$ and $\mathrm{diam}(A)$ we denote interior, boundary, cardinality and diameter of the set A. Now let $x \in \mathbb{I}^l$, $r > 0$. Symbol $B(x,r)$ ($\overline{B}(x,r)$) stands for open (closed) ball with the centre at x and radius r.

Many issues in this paper refer to the topics analyzed in [5,6] considered, among others, in relation to topological manifolds. In order to reduce the length of the paper and simplify the notation, we will only consider space $\mathbb{I}^l$ for $l \in \mathbb{N}_+$, although with a slight refinement of the proofs, the presented theorems will remain true for topological manifolds.

Let $a, b \in \mathbb{R}^l$. We denote the closed interval with endpoints a, b by $\mathbb{I}(a,b)$ and an arbitrary arc with endpoints $a, b \in \mathbb{I}^l$ by $\mathcal{L}(a,b)$. Note that there can be more than one arc with the same endpoints. If we consider space $\mathbb{R}$ and $a < b$ then for closed (right-hand open, etc.) intervals we use the standard notation $[a,b]$ ($[a,b)$, etc.). Moreover, symbol $[\![a,b]\!]$ stands for a set $[a,b] \cap \mathbb{N}_0$. By log we mean the logarithm to base 2.

Similar to [6, 13], we are going to use a family of intervals. Let us begin with the following definition. The cube $\mathfrak{K} \subset \mathbb{R}^l$, where the length of the edge is denoted by $\mathfrak{s}(\mathfrak{K}) > 0$, is the set $[a_1, b_1] \times \cdots \times [a_l, b_l]$, where $b_i - a_i = \mathfrak{s}(\mathfrak{K})$ for $a_i, b_i \in \mathbb{R}$ and $i \in [\![1, l]\!]$.

Fix $x_0 = (x_1, \ldots, x_l) \in \mathbb{I}^l$ and $n \in \mathbb{N}_+$. Let $\mathfrak{K}_{x_0,n} = [a_1, b_1] \times \cdots \times [a_l, b_l]$ be the cube such that $\mathfrak{s}(\mathfrak{K}_{x_0,n}) = \frac{1}{n}$ and x_i is a midpoint of the interval $[a_i, b_i]$ for $i \in [\![1, l]\!]$. By symbol $\mathbb{K}^l(x_0)$ we denote the sequence $\{K_n\}_{n=1}^{\infty} \subset \mathbb{I}^l$ such that $K_n = \mathfrak{K}_{x_0,n} \cap \mathbb{I}^l$ for $n \in \mathbb{N}_+$. Obviously, the sequence $\mathbb{K}^l(x_0)$ satisfies the following conditions:

(K1) $x_0 \in \mathrm{Int}(K_n)$ for $n \in \mathbb{N}_+$;

(K2) $K_{n+1} \subset \mathrm{Int}(K_n)$ for $n \in \mathbb{N}_+$;

(K3) $\lim_{n\to\infty} \mathrm{diam}(K_n) = 0$;

(K4) the sequence $\{K_n\}_{n=1}^{\infty}$ has the extension property i.e., for any $i, j \in \mathbb{N}_+$ and any continuous function $f\colon A \to K_j$, where $A \subset K_i$ is a closed set, there exists a continuous function $f^*\colon K_i \to K_j$ being an extension of f, that is $f^*_{|A} = f$.

In this paper we will consider only continuous functions and we will denote the set of all continuous functions $f\colon \mathbb{I}^l \to \mathbb{I}^l$ ($l \in \mathbb{N}_+$) by $\mathcal{C}(\mathbb{I}^l)$. By f^{-1} we denote the inverse function or preimage, depending on the context. Symbol f^0 stands for the identity function. Let X be a topological space. Consider the cover $\{A_s\}_{s\in S}$ of X and the family $\{f_s\}_{s\in S}$ of continuous functions $f_s\colon A_s \to Y$ where Y is a topological space. We say that functions f_s are compatible if for any $s_1, s_2 \in S$ we have $f_{s_1}(x) = f_{s_2}(x)$ for $x \in A_{s_1} \cap A_{s_2}$. By the combination of functions (see [4]) we will call the function $\nabla_{s\in S} f_s\colon X \to Y$ given by the formula $\nabla_{s\in S} f_s(x) = f_s(x)$ for $x \in A_s$. In case of finite number of functions $f_1, f_2, \ldots, f_k$ we will also write $f_1 \nabla f_2 \nabla \cdots \nabla f_k$.

Lemma 2.1. [4] *If $\{A_s\}_{s\in S}$ is a locally finite closed cover (i.e., A_s are closed sets for $s \in S$ and each point $x \in X$ has an open neighbourhood U such that the set $\{s \in S : A_s \cap U \neq \emptyset\}$ is finite) of a topological space X, $\{f_s\}_{s\in S}$, ($f_s\colon A_s \to Y$, Y is a metric space) is a family of continuous compatible functions then also $\nabla_{s\in S} f_s\colon X \to Y$ is a continuous function.*

Let $\mathcal{A} = \{f_0, f_1, \ldots, f_k\} \subset \mathcal{C}(\mathbb{I}^l)$ be a family of functions. We say that a family $\mathcal{A}$ is uniformly nowhere constant at x_0 (briefly u.n.c. at x_0) if there exists an interval $\mathbb{I}(x_0, a) \subset \mathbb{I}^l$ such that $f_i^{-1}(\{f_i(x_0)\}) \cap \mathbb{I}(x_0, a) = \{x_0\}$ for $i \in [\![0, k]\!]$. Obviously, if a family $\mathcal{A}$ is u.n.c. at x_0 then for any open neighborhood V of x_0 there exists a nondegenerate interval $\mathbb{I}(x_0, y_0) \subset V$ such that the equality $f_i^{-1}(\{f_i(x_0)\}) \cap \mathbb{I}(x_0, y_0) = \{x_0\}$ holds for $i \in [\![0, k]\!]$. In the context of the above definition, let us note a useful statement.

Lemma 2.2. *Let $\mathfrak{K} \subset \mathbb{R}^l$ be an l-dimensional cube, $\mathcal{A} = \{f_0, f_1, \ldots, f_k\}$ be a family of functions on $\mathfrak{K}$ and $\mathbb{I}(x_0, y_0) \subset \mathfrak{K}$ be an interval such that $f_i(x_0) \neq f_i(y_0)$ for $i \in [\![0, k]\!]$. Then there exists $q_0 \in \mathbb{I}(x_0, y_0) \setminus \{x_0\}$ such that for any $x \in \mathbb{I}(x_0, q_0) \setminus \{q_0\}$ we have $f_i(x) \neq f_i(y_0)$ for $i \in [\![0, k]\!]$.*

Let $\mathcal{A} = \{f_0, f_1, \ldots, f_k\} \subset \mathcal{C}(\mathbb{I}^l)$ and $x_0 \in \mathbb{I}^l$. We say that x_0 is a fixed point of the family $\mathcal{A}$ (briefly $x_0 \in \mathrm{Fix}(\mathcal{A})$) if $x_0 = f(x_0)$ for any $f \in \mathcal{A}$. If $\mathcal{A} = \{f\}$ we write $x_0 \in \mathrm{Fix}(f)$. Moreover $\mathrm{FIX}(x_0) = \{f \in \mathcal{C}(\mathbb{I}^l) : x_0 \in \mathrm{Fix}(f)\}$. We will write $x_0 \in \overline{\mathrm{Fix}}(\mathcal{A})$ if $\mathcal{A}$ is simultaneously an u.n.c. at x_0 and $x_0 \in \mathrm{Fix}(\mathcal{A})$.

Based on [5] we define the set $\approx (f, g) = \{x : f(x) \neq g(x)\}$ for $f, g\colon \mathbb{I}^l \to \mathbb{I}^l$. Fix $x_0 \in \mathbb{I}^l$ and $\varepsilon > 0$. Let us consider the equivalence relation $f \underset{x_0}{\overset{\varepsilon}{-}} g$ defined as

$$f \underset{x_0}{\overset{\varepsilon}{-}} g \quad \Longleftrightarrow \quad \big(\big(\approx (f, g) \cup f(\approx (f, g)) \cup g(\approx (f, g))\big) \subset \mathrm{B}(x_0, \varepsilon) \wedge f, g \in \mathrm{FIX}(x_0)\big).$$

By $[f]^{\varepsilon}_{x_0}$ we denote an equivalence class generated by the relation $\underset{x_0}{\overset{\varepsilon}{-}}$ for f.

We will use the concept of (autonomous) dynamical system following [2,15]. Let X be a compact space. A dynamical system (X, f) (denoted by (f)) is given by a continuous function $f\colon X \to X$. The evolution of the system is given by the successive iterations of the function i.e., $f^0(x) = x$ and $f^m(x) = f \circ f^{m-1}(x)$ for $x \in X$ and $n \in \mathbb{N}_+$.

We will adopt the classical definition of entropy (see [2, 16]). Let us consider a dynamical system (f). Fix $n \in \mathbb{N}_+$, $\varepsilon > 0$ and $Y \subset \mathbb{I}^l$. We say that a set $E \subseteq Y$ is

(n,ε)-separated in Y if for any distinct points $x,y \in E$ there exists $j \in [\![0,n-1]\!]$ such that $d_e(f^j(x),f^j(y)) > \varepsilon$. By $s_n(f,Y,\varepsilon)$ we denote the maximal cardinality of (n,ε)-separated set in Y. The topological entropy of a system (f) on $Y \subset \mathbb{I}^l$ is the number $h(f,Y) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(f,Y,\varepsilon)$. If $Y = \mathbb{I}^l$, then we use symbols $s_n(f,\varepsilon)$ and $h(f)$, respectively.

In our considerations we use various well known theorems to calculate or estimate the value of entropy. Let us begin with the results contained in [11]. We will not introduce all the concepts from [11], but adapt some of them to conform to the terminology used in this paper.

Let $P = \{A_1,\dots,A_k\}$ be a nonsingleton family of sets for $k > 1$. We will say that a family $V_1,\dots,V_k$ of open sets weakly separates P if $A_i \subset V_i$ for $i \in [\![1,k]\!]$ and $A_i \cap V_j = \emptyset$ for distinct $i,j \in [\![1,k]\!]$.

We will say that dynamical system (f) is k-turbulent if there exists a family $P = \{A_1,\dots,A_k\}$ consisting of nonempty sets such that $\cup P$ is a closed set, topology of space $\mathbb{I}^l$ weakly separates P and, moreover, $\bigcup_{j=1}^k A_j \subset \bigcap_{j=1}^k f_i(A_j)$ for $i \in \mathbb{N}_+$.

Lemma 2.3. [11] *If (f) is a k-turbulent dynamical system, then $h(f) \geq \log k$.*

In some considerations it would be more convenient to use the term n-horseshoe. Let $f\colon \mathbb{I} \to \mathbb{I}$. If $J_1,\dots,J_n$ are nondegenerate closed intervals with pairwise disjoint interiors such that $\bigcup_{k=1}^n J_k \subset f(J_i)$ for any $i \in [\![1,n]\!]$, then $(J_1,\dots,J_n)$ is called an n-horseshoe.

Lemma 2.4. [15] *Let $f\colon \mathbb{I} \to \mathbb{I}$. If f has an n-horseshoe, then $h(f) \geq \log n$.*

Our main considerations connected with "entropy points" will be with those of points focusing entropy. Let (f) be a dynamical system. We say that $x_0 \in \mathbb{I}^l$ is a point focusing entropy of a system (f) if the equality $h(f,U) = h(f)$ holds for any open neighbourhood U of x_0. This definition is analogous to the one introduced in [17].

During the study of the local aspects of dynamical systems, the chaotic points were analyzed, among others. In this paper we will base on concepts from [12, 13]. Let (f) be a dynamical system on $\mathbb{I}^l$ and $x_0 \in \mathrm{Fix}(f)$. By $\mathcal{W}(x_0,f)$ we denote the set of all points $t \in \mathbb{I}^l$ such that there exist sequences $\{y_n\}_{n=1}^\infty \subset \mathbb{I}^l$ and $\{k_n\}_{n=1}^\infty \subset \mathbb{N}_0$ such that $y_n \to x_0$ and $f^{k_n}(y_n) = t$. A point $t \in \mathbb{I}^l$ is called an (x_0,f)-homoclinic point if $x_0 \neq t \in \mathcal{W}(x_0,f)$ and x_0 is a limit of $\{f^{m_k}(t)\}_{k=0}^\infty$ for some sequence of positive integers $\{m_k\}_{k=0}^\infty$.

We say that a point x_0 is a chaotic point of a system (f) if for each neighbourhood of x_0 there exists an (x_0,f)-homoclinic point.

In 1994, Schweizer and Smítal have introduced the concept of distributional chaos (see [16]). In this paper we will base on this concept. Due to restriction of our considerations to $\mathbb{I}^l$, the following definitions are formulated only for this space.

Let (f) be a dynamical system on $\mathbb{I}^l$, fix $t > 0$ and $x, y \in \mathbb{I}^l$. Consider the functions given by the formulas

$$\Phi_{x,y}^{(f)}(t) = \liminf_{n\to\infty} \frac{1}{n} \operatorname{card}\big(\{j \in [\![0, n-1]\!] : d_e(f^j(x), f^j(y)) < t\}\big),$$
$$\Phi_{x,y}^{*(f)}(t) = \limsup_{n\to\infty} \frac{1}{n} \operatorname{card}\big(\{j \in [\![0, n-1]\!] : d_e(f^j(x), f^j(y)) < t\}\big).$$

Let $x, y \in \mathbb{I}^l$. We say that a pair (x, y) is distributionally chaotic of type 1 for a dynamical system (f) if $\Phi_{x,y}^{*(f)}(t) = 1$ for any $t > 0$ and there exists $t_0 > 0$ such that $\Phi_{x,y}^{(f)}(t_0) = 0$. We say that $A \subset \mathbb{I}^l$ is a distributionally scrambled set of type 1 (briefly DS-set) for a dynamical system (f) if $\operatorname{card}(A) > 1$ and for each $x, y \in A$ such that $x \neq y$ the pair (x, y) is distributionally chaotic of type 1 for this system. A dynamical system (f) is distributionally chaotic of type 1 if there exists an uncountable DS-set for this system. We say that $x_0 \in \mathbb{I}^l$ is a DC1 point (distributionally chaotic point) of a dynamical system (f) if for any $\varepsilon > 0$, there exists an uncountable set S being a DS-set for the dynamical system (f) such that there are $n \in \mathbb{N}_+$ and a closed set $A \supset S$ such that $A \subset f^{i\cdot n}(A) \subset \mathrm{B}(x_0, \varepsilon)$ for $i \in \mathbb{N}_+$. The set A described above is called (n, ε)-envelope of the set S (see [13]).

Now let us note the statement, which will be useful for our consideration.

Lemma 2.5. [9, 15, 16] *The function $f\colon \mathbb{I} \to \mathbb{I}$ has positive entropy if and only if the dynamical system (f) is distributionally chaotic of type 1.*

Lemma 2.6. [13] *Let $\mathcal{L} \subset X$ be an arc, $\varphi\colon [0,1] \to \mathcal{L}$ be a homeomorphism, $f \in \mathcal{C}(\mathbb{I})$ and $g = \varphi \circ f \circ \varphi^{-1}$. If $S \subset [0,1]$ is an uncountable DS-set for the dynamical system (f), then $\varphi(S)$ is an uncountable DS-set for the dynamical system (g).*

In many papers the concept of chaos is combined with entropy (e.g., [7]). In line with this concept one can consider the following definitions. We say that x_0 is *a point focusing chaos* if it is simultaneously a chaotic point and a point focusing entropy. We say that x_0 is *a point focusing distributional chaos* if it is simultaneously a DC1 point and a point focusing entropy. From the point of view of the considerations contained in this paper, the most important thing is to combine all those notions. We say that x_0 is *a point strongly focusing chaos* if it is simultaneously a chaotic point, DC1 point and a point focusing entropy.

3. Semigroups, disruptions

In this chapter we will discuss the problems of semigroups related to the theory of dynamical systems. We will adopt concepts and symbols based on [1, 6]. Let $\mathcal{A} = \{f_0 = \mathrm{id}_{\mathbb{I}^l}, f_1, f_2, \ldots, f_k\}$ be a finite family of continuous functions where $f_i\colon \mathbb{I}^l \to \mathbb{I}^l$ for $i \in [\![0, k]\!]$

and $l \in \mathbb{N}_+$. Put $G_n(\mathcal{A}) = \{f_{i_1} \circ \cdots \circ f_{i_n} : f_{i_1}, \ldots, f_{i_n} \in \mathcal{A}\}$ for any $n \in \mathbb{N}_+$. The set $G(\mathcal{A}) = \bigcup_{n=1}^{\infty} G_n(\mathcal{A})$ is a semigroup of functions generated by the family $\mathcal{A}$. Then the family $\mathcal{A}$ will be called the set of generators of the semigroup $G(\mathcal{A})$.

The notion of entropy of semigroup has been introduced in [1]. We take the same approach as in [6]. Let $\mathcal{A} = \{f_0, f_1, f_2, \ldots, f_k\}$ where $f_i \colon \mathbb{I}^l \to \mathbb{I}^l$ for $i \in [\![0, k]\!]$ be the set of generators. Let $n \in \mathbb{N}_+$, $\varepsilon > 0$ and $Y \subset \mathbb{I}^l$. We say that the set $Z \subset Y$ is (n, ε)-separated by $G(\mathcal{A})$ in Y if for any distinct points $p, q \in Z$ there exists function $g \in G_n(\mathcal{A})$ such that $d_e(g(p), g(q)) > \varepsilon$. By $s(n, \varepsilon, G(\mathcal{A}), Y)$ we denote the maximal cardinality of the set (n, ε)-separated by $G(\mathcal{A})$ in Y. Then the entropy of semigroup $G(\mathcal{A})$ on the set Y is the number $h(G(\mathcal{A}), Y) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(G(\mathcal{A}), Y, \varepsilon)$.

In [13] various variants of disturbances of dynamical systems were analysed. Moreover, in [6] the disruption of semigroup was considered. Let us now introduce the necessary definitions and notation.

Let $x_0 \in \overline{\mathrm{Fix}}(\mathcal{A})$ and $\mathcal{A} = \{f_0, f_1, \ldots, f_k\}$ be the set of generators. The family $\mathcal{A}_d = \{f_0, g_1 \circ f_1, \ldots, g_k \circ f_k\}$ where $g_i \in [f_i]^{\varepsilon}_{x_0}$ for $i \in [\![1, k]\!]$ is called an ε-disruption of $\mathcal{A}$ at x_0.

We say that x_0 is a point strongly focusing chaos of the family $\mathcal{A} = \{f_0, f_1, \ldots, f_k\}$ where $f_i \colon \mathbb{I}^l \to \mathbb{I}^l$ for $i \in [\![0, k]\!]$ if for any function $g \in G(\mathcal{A}) \setminus \{f_0\}$ the point x_0 is a point focusing chaos of g and x_0 is a point strongly focusing chaos of g. It can be seen that there exist examples of the family $\mathcal{A} = \{f_0, f_1, f_2\}$ where $f_i \colon \mathbb{I} \to \mathbb{I}$ for $i \in \{1, 2, 3\}$ such that $x_0 = \frac{1}{2}$ is a point focusing chaos and focusing distributional chaos of f_1 and f_2, but x_0 is not a point strongly focusing chaos of the family $\mathcal{A}$.

Then, there arises the natural question: if we have a given family $\mathcal{A}$ and $\varepsilon > 0$ does there exist its ε-disruption $\mathcal{A}_d$, such that the fixed point x_0 becomes a point strongly focusing chaos of the family $\mathcal{A}_d$. The following theorem gives an answer to that question.

Theorem 3.1. *Let $\mathcal{A} = \{f_0, f_1, \ldots, f_k\}$ be a family of functions $f_i \colon \mathbb{I}^l \to \mathbb{I}^l$ ($i \in [\![0, k]\!]$) and let $x_0 \in \overline{\mathrm{Fix}}(\mathcal{A})$. For any $\varepsilon > 0$ there exists ε-disruption $\mathcal{A}_d$ of $\mathcal{A}$ such that x_0 is a point strongly focusing chaos of the family $\mathcal{A}_d$.*

Proof. Let us assume the symbols as in the Theorem and fix the sequence $\mathbb{K}^l(x_0)$. From (K1), we conclude that $x_0 \in \mathrm{Int}(K_n)$ for any $n \in \mathbb{N}_+$. By (K3) there exists $n_0 \in \mathbb{N}_+$ such that $K_{n_0} \subset B(x_0, \varepsilon)$.

We will construct sequences of appropriate numbers, cubes, intervals, arcs and sets. For a better explanation of the following operations, we will present very precisely the first step of the construction, and then the reasoning would be analogous to that in the first step. We will list the dependencies and properties necessary in the further part of the proof and thus the necessary changes which should be done in successive steps will be clearly visible.

Let us move on to the first step of construction. Directly by the definition of $\mathbb{K}^l(x_0)$ we can conclude that there exist $m_1, n_1 \in \mathbb{N}_+$ such that $m_1 > n_1 > n_0$ and

$$\begin{aligned} K_{m_1} \subset \operatorname{Int}(K_{n_1}) \subset K_{n_1} \subset \operatorname{Int}(K_{n_0}) \subset K_{n_0}, \\ f_i(K_{n_1}) \subset \operatorname{Int}(K_{n_0}) \quad \text{for } i \in [\![0,k]\!], \\ f_i(K_{m_1}) \subset \operatorname{Int}(K_{n_1}) \quad \text{for } i \in [\![0,k]\!]. \end{aligned} \tag{3.1}$$

Of course, there exists a nondegenerate interval

$$\mathrm{L}_{m_1} = \mathbb{I}(x_0, z_{m_1}) \subset \operatorname{Int}(K_{m_1}) \subset K_{n_1} \tag{3.2}$$

such that $f_i^{-1}(\{f_i(x_0)\}) \cap \mathbb{I}(x_0, z_{m_1}) = \{x_0\}$ for $i \in [\![0,k]\!]$.

From Lemma 2.2 we infer that there exists a point $q_{m_1} \in \mathrm{L}_{m_1} \setminus \{x_0\}$ such that

$$f_i(x) \neq f_i(z_{m_1}), \;\; i \in [\![0,k]\!], \quad \text{for any } x \in \mathbb{I}(x_0, q_{m_1}) \setminus \{q_{m_1}\}.$$

Let us fix $x_{m_1} \in \mathbb{I}(x_0, q_{m_1}) \setminus \{x_0, q_{m_1}\}$. Moreover, we define $D_{m_1} = \mathbb{I}(x_{m_1}, z_{m_1})$ and $X_{n_1,i} = f_i(D_{m_1})$ for $i \in [\![0,k]\!]$.

From (3.1) and (3.2) we get

$$X_{n_1,i} \subset \operatorname{Int}(K_{n_1}) \quad \text{for } i \in [\![0,k]\!]. \tag{3.3}$$

Obviously, $x_0 \notin D_{m_1}$. We can also infer that $x_0 \notin X_{n_1,i}$ and $\operatorname{card}(X_{n_1,i}) > 1$ for $i \in [\![0,k]\!]$. Moreover, it is easy to see that the set $X_{n_1,i}$ for $i \in [\![0,k]\!]$ is closed and arcwise-connected.

From (3.3) and (3.1) we get $f_i(\operatorname{Fr}(K_{n_1}) \cup X_{n_1,i}) \subset K_{n_0}$ for $i \in [\![0,k]\!]$. Based on (3.3) it can also be seen that $\operatorname{Fr}(K_{n_1}) \cap X_{n_1,i} = \emptyset$ for $i \in [\![0,k]\!]$. Let us consider the arc $S_{n_1,i} = \mathcal{L}(f_i(x_{m_1}), f_i(z_{m_1})) \subset X_{n_1,i}$ for $i \in [\![0,k]\!]$. Let us choose two disjoint arcs $S^j_{n_1,i} \subset S_{n_1,i}$ ($j \in \{1,2\}$) for any $i \in [\![0,k]\!]$. It is easy to see that $x_0 \notin S^j_{n_1,i}$ for $i \in [\![0,k]\!]$, $j \in [\![1,2]\!]$.

Moreover, let $T^j_{n_1,i} = f_i^{-1}(S^j_{n_1,i}) \cap D_{m_1}$ for $i \in [\![0,k]\!]$ and $j \in \{1,2\}$. Obviously, $T^j_{n_1,i}$ is a closed set and $f_i(T^j_{n_1,i}) = S^j_{n_1,i}$ for $i \in [\![0,k]\!]$ and $j \in \{1,2\}$. By (K1) and (K3) we can conclude that

(3.4)

$$\text{there exists } \sigma_1 > 0 \text{ such that } \mathrm{B}(x_0, \sigma_1) \subset \operatorname{Int}(K_{m_1}) \text{ and } \mathrm{B}(x_0, \sigma_1) \cap \left(\bigcup_{i=0}^{k} X_{n_1,i} \right) = \emptyset.$$

Note that $\bigcup_{i=0}^k \bigcup_{j=1}^2 T^j_{n_1,i} \subset D_{m_1} = X_{n_1,0}$. Then

$$\left(\bigcup_{i=0}^{k} \bigcup_{j=1}^{2} T^j_{n_1,i} \cup \bigcup_{i=0}^{k} \bigcup_{j=1}^{2} S^j_{n_1,i} \right) \subset \bigcup_{i=0}^{k} X_{n_1,i}.$$

Thus by (3.4) one can infer that

$$\mathrm{B}(x_0,\sigma_1)\cap\left(\bigcup_{i=0}^{k}\bigcup_{j=1}^{2}T^j_{n_1,i}\cup\bigcup_{i=0}^{k}\bigcup_{j=1}^{2}S^j_{n_1,i}\right)=\emptyset.$$

Moreover, by (3.4) we conclude that there exist $w_1, v_1 \in \mathbb{N}_+$ such that $w_1 > v_1 > m_1$ and

$$\begin{gathered}K_{w_1}\subset \mathrm{Int}(K_{v_1})\subset K_{v_1}\subset \mathrm{B}(x_0,\sigma_1)\subset \mathrm{Int}(K_{m_1}),\\ f_i(K_{v_1})\subset \mathrm{B}(x_0,\sigma_1)\subset \mathrm{Int}(K_{m_1})\quad \text{for } i\in [\![0,k]\!],\\ f_i(K_{w_1})\subset \mathrm{Int}(K_{v_1})\quad \text{for } i\in [\![0,k]\!].\end{gathered}$$

The construction presented below is analogous to the one presented above. The changes will only apply to considerations related to arcs.

Let us start with the observation that there exists a nondegenerate interval $\mathrm{L}_{w_1} = \mathbb{I}(x_0, z_{w_1}) \subset \mathrm{Int}(K_{w_1}) \subset \mathrm{Int}(K_{v_1})$, such that $f_i^{-1}(\{f_i(x_0)\})\cap\mathbb{I}(x_0, z_{w_1}) = \{x_0\}$ for $i \in [\![0, k]\!]$. Fix a point $q_{w_1} \in \mathrm{L}_{w_1} \setminus \{x_0\}$ such that for any $x \in \mathbb{I}(x_0, q_{w_1}) \setminus \{q_{w_1}\}$ we get $f_i(x) \neq f_i(z_{w_1})$ for $i \in [\![0, k]\!]$. Let us choose $x_{w_1} \in \mathbb{I}(x_0, q_{w_1}) \setminus \{x_0, q_{w_1}\}$. Put $D_{w_1} = \mathbb{I}(x_{w_1}, z_{w_1})$ and $X_{v_1,i} = f_i(D_{w_1})$ for $i \in [\![0, k]\!]$. Then $X_{v_1,i} \subset \mathrm{Int}(K_{v_1})$ and $x_0 \notin X_{v_1,i}$ for $i \in [\![0, k]\!]$. Note that $X_{v_1,i}$ is non-singleton, closed and arcwise-connected set for $i \in [\![0, k]\!]$.

Let us consider an arc $A^1_{v_1,i} \subset X_{v_1,i}$ for $i \in [\![0, k]\!]$. Then $x_0 \notin A^1_{v_1,i}$ and $A^1_{v_1,i}\cap\mathrm{Fr}(K_{v_1}) = \emptyset$ for $i \in [\![0, k]\!]$. According to the fact that $x_0 \notin X_{v_1,i}$ for $i \in [\![0, k]\!]$ we can show that

(3.5)

there exists $\kappa_1 > 0$ such that $\mathrm{B}(x_0, \kappa_1) \subset \mathrm{Int}(K_{w_1})$ and $\mathrm{B}(x_0, \kappa_1) \cap \left(\bigcup_{i=0}^{k} X_{v_1,i}\right) = \emptyset.$

Indeed. We have $x_0 \in \mathbb{I}^l \setminus \left(\bigcup_{i=0}^k X_{v_1,i}\right)$. Since the set $\bigcup_{i=0}^k X_{v_1,i}$ is closed, there exists $\kappa_1 > 0$ such that $\mathrm{B}(x_0, \kappa_1) \subset \mathbb{I}^l \setminus \left(\bigcup_{i=0}^k X_{v_1,i}\right)$ and $\mathrm{B}(x_0, \kappa_1) \subset \mathrm{Int}(K_{w_1})$, which ends the proof of (3.5).

In the second step of the construction, the reasoning can be carried out analogous to the one above, "approaching the point x_0" and increasing the number of arcs considered (details can be read from the general description presented below).

Continuing the reasoning in the same way as described earlier, we will establish a sequence of cubes $\{K_{n_s}\}_{s=1}^{\infty}$ and $\{K_{m_s}\}_{s=1}^{\infty}$ satisfying the conditions:

(3.6)

$$\begin{cases}\text{If } s=1 \text{ and } i\in[\![0,k]\!], \text{ then } K_{m_1}\subset \mathrm{Int}(K_{n_1})\subset K_{n_1}\subset \mathrm{Int}(K_{n_0})\subset K_{n_0}\subset \mathrm{B}(x_0,\varepsilon),\\ \quad f_i(K_{n_1})\subset \mathrm{Int}(K_{n_0}),\quad f_i(K_{m_1})\subset \mathrm{Int}(K_{n_1}).\\ \text{If } s\geq 2 \text{ and } i\in[\![0,k]\!], \text{ then } K_{m_s}\subset \mathrm{Int}(K_{n_s})\subset K_{n_s}\subset \mathrm{B}(x_0,\kappa_{s-1})\subset \mathrm{Int}(K_{w_{s-1}})\subset K_{n_0},\\ \quad f_i(K_{n_s})\subset \mathrm{B}(x_0,\kappa_{s-1}),\quad f_i(K_{m_s})\subset \mathrm{Int}(K_{n_s}).\end{cases}$$

Let us fix an interval $\mathrm{L}_{m_s} = \mathbb{I}(x_0, z_{m_s}) \subset \mathrm{Int}(K_{m_s})$ for $s \in \mathbb{N}_+$, such that $x_0 \notin f_i(\mathbb{I}(x_0, z_{m_s}) \setminus \{x_0\})$ and the point $x_{m_s} \in \mathrm{L}_{m_s}$ such that $f_i(x_{m_s}) \neq f_i(z_{m_s})$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$. Then denote $D_{m_s} = \mathbb{I}(x_{m_s}, z_{m_s}) \subset \mathrm{L}_{m_s}$ (of course $x_0 \notin D_{m_s}$) and $X_{n_s,i} = f_i(D_{m_s})$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$. Obviously,

$$X_{n_s,i} \subset \mathrm{Int}(K_{n_s}) \quad \text{for } s \in \mathbb{N}_+,\ i \in [\![0, k]\!]. \tag{3.7}$$

Note that $x_0 \notin X_{n_s,i}$, $f_i(\mathrm{Fr}(K_{n_s}) \cup X_{n_s,i}) \subset K_{n_0}$ and $\mathrm{Fr}(K_{n_s}) \cap X_{n_s,i} = \emptyset$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$. Next, let us fix the arcs $S_{n_s,i} = \mathcal{L}(f_i(x_{m_s}), f_i(z_{m_s})) \subset X_{n_s,i}$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$ and consider 2^s pairwise disjoint arcs $S^j_{n_s,i} \subset S_{n_s,i}$ for $j \in [\![1, 2^s]\!]$. Of course $x_0 \notin S^j_{n_s,i}$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$, $j \in [\![1, 2^s]\!]$.

Put $T^j_{n_s,i} = f_i^{-1}(S^j_{n_s,i}) \cap D_{m_s}$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$, $j \in [\![1, 2^s]\!]$. Then

$$f_i(T^j_{n_s,i}) = S^j_{n_s,i} \quad \text{for } s \in \mathbb{N}_+,\ i \in [\![0, k]\!],\ j \in [\![1, 2^s]\!]. \tag{3.8}$$

Next, fix the sequences of cubes $\{K_{w_s}\}_{s=1}^{\infty}$ and $\{K_{v_s}\}_{s=1}^{\infty}$ such that

$$\begin{aligned}
&\mathrm{B}(x_0, \sigma_s) \subset \mathrm{Int}(K_{m_s}), \quad \mathrm{B}(x_0, \sigma_s) \cap \left(\bigcup_{i=0}^{k} X_{n_s,i} \right) = \emptyset, \\
&K_{w_s} \subset \mathrm{Int}(K_{v_s}) \subset K_{v_s} \subset \mathrm{B}(x_0, \sigma_s) \subset \mathrm{Int}(K_{m_s}), \\
&f_i(K_{v_s}) \subset \mathrm{B}(x_0, \sigma_1) \subset \mathrm{Int}(K_{m_s}) \subset K_{n_0}, \quad f_i(K_{w_s}) \subset \mathrm{Int}(K_{v_s}).
\end{aligned} \tag{3.9}$$

Then there exists an interval

$$\mathrm{L}_{w_s} = \mathbb{I}(x_0, z_{w_s}) \subset \mathrm{Int}(K_{w_s}) \quad \text{for } s \in \mathbb{N}_+ \tag{3.10}$$

such that $x_0 \notin f_i(\mathbb{I}(x_0, z_{w_s}) \setminus \{x_0\})$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$ and the point $x_{w_s} \in \mathrm{L}_{w_s}$ such that $f_i(x_{w_s}) \neq f_i(z_{w_s})$. Put $D_{w_s} = \mathbb{I}(x_{w_s}, z_{w_s}) \subset \mathrm{L}_{w_s}$ and $X_{v_s,i} = f_i(D_{w_s})$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$. Obviously, $x_0 \notin D_{w_s}$ and $x_0 \notin X_{v_s,i}$ for $s \in \mathbb{N}_+$, $i \in [\![0, k]\!]$.

Now we will define a finite sequence of arcs. If $s = 1$ then, according to previous arrangements, $A^1_{v_1,i} \subset X_{v_1,i}$, where $i \in [\![0, k]\!]$. If $s \geq 2$, then $A^r_{v_s,i} \subset X_{v_s,i}$, where $r \in [\![1, 2]\!]$ are pairwise disjoint for $i \in [\![0, k]\!]$. Hence

$$\begin{aligned}
&x_0 \notin A^r_{v_s,i} && \text{for } s \in \mathbb{N}_+,\ r \in [\![1, 2]\!],\ i \in [\![0, k]\!], \\
&A^r_{v_s,i} \cap \mathrm{Fr}(K_{v_s}) = \emptyset && \text{for } i \in [\![0, k]\!],\ s \in \mathbb{N}_+,\ r \in [\![1, 2]\!].
\end{aligned} \tag{3.11}$$

There exists $\kappa_s > 0$, $s \in \mathbb{N}_+$ such that

$$\mathrm{B}(x_0, \kappa_s) \subset \mathrm{Int}(K_{w_s}) \quad \text{and} \quad \mathrm{B}(x_0, \kappa_s) \cap \left(\bigcup_{i=0}^{k} X_{v_s,i} \right) = \emptyset. \tag{3.12}$$

Note that by (3.6), (3.9), (3.12) for $s>1$, the following inclusions hold:

$$\begin{aligned} K_{w_s} &\subset \operatorname{Int}(K_{v_s}) \subset K_{v_s} \subset \mathrm{B}(x_0,\sigma_s) \subset \operatorname{Int}(K_{m_s}) \subset K_{m_s} \subset \operatorname{Int}(K_{n_s}) \\ &\subset K_{n_s} \subset \mathrm{B}(x_0,\kappa_{s-1}) \subset \operatorname{Int}(K_{w_{s-1}}) \subset K_{w_{s-1}} \subset \operatorname{Int}(K_{v_{s-1}}) \subset \cdots \\ &\subset K_{w_1} \subset \operatorname{Int}(K_{v_1}) \subset K_{v_1} \subset \mathrm{B}(x_0,\sigma_1) \subset \operatorname{Int}(K_{m_1}) \subset K_{m_1} \subset \operatorname{Int}(K_{n_1}) \\ &\subset K_{n_1} \subset \operatorname{Int}(K_{n_0}) \subset K_{n_0} \subset \mathrm{B}(x_0,\varepsilon). \end{aligned} \tag{3.13}$$

Let us move to the definitions of the functions essential in the further part of the proof. Consider the family of homeomorphisms $\lambda^j_{n_s,i}\colon S^j_{n_s,i} \xrightarrow{\text{on}} D_{m_s}$ for $s\in\mathbb{N}_+$, $j\in[\![1,2^s]\!]$, $i\in[\![0,k]\!]$ and a family of functions $\lambda_{n_s,i}\colon \bigcup_{j=1}^{2^s} S^j_{n_s,i} \xrightarrow{\text{on}} D_{m_s}$ defined by the formula $\lambda_{n_s,i}(x)=\nabla_{r=1}^{2^s}\lambda^r_{n_s,i}(x)$ for $s\in\mathbb{N}_+$, $i\in[\![0,k]\!]$.

For any $s\in\mathbb{N}_+$ and $i\in[\![0,k]\!]$ there exists a continuous function $\tau_{n_s,i}\colon X_{n_s,i} \xrightarrow{\text{on}} D_{m_s}$ which is an extension of the function $\lambda_{n_s,i}\colon \bigcup_{j=1}^{2^s} S^j_{n_s,i} \xrightarrow{\text{on}} D_{m_s}$.

Let us now define continuous functions $\psi_{n_s,i}\colon \operatorname{Fr}(K_{n_s})\cup X_{n_s,i}\to K_{n_{s-1}}$ for $s\in\mathbb{N}_+$, $i\in[\![0,k]\!]$ in the following way

$$\psi_{n_s,i}(x)=\begin{cases}\tau_{n_s,i}(x) & \text{for } x\in X_{n_s,i},\\ f_i(x) & \text{for } x\in\operatorname{Fr}(K_{n_s}).\end{cases} \tag{3.14}$$

By (K4) one can consider the continuous function $\varphi_{n_s,i}\colon K_{n_s}\to K_{n_{s-1}}$ for $s\in\mathbb{N}_+$ and $i\in[\![0,k]\!]$ which is an extension of $\psi_{n_s,i}$.

If $s\geq 1$ for $i\in[\![0,k]\!]$, then $\xi^1_{v_s,i}\colon A^1_{v_s,i}\to\{x_0\}$ will be defined as $\xi^1_{v_s,i}(x)=x_0$. If $s>1$ we will fix homeomorphisms $\xi^2_{v_s,i}\colon A^2_{v_s,i}\xrightarrow{\text{on}} D_{w_{s-1}}$ for $i\in[\![0,k]\!]$.

In order to simplify further entries assume that $A^2_{v_1,i}=\emptyset$ for $i\in[\![0,k]\!]$ and $D_{w_0}=\emptyset$. Let us define continuous functions $\xi_{v_s,i}\colon \bigcup_{r=1}^2 A^r_{v_s,i}\to\{x_0\}\cup D_{w_{s-1}}$ for $s\in\mathbb{N}_+$ and $i\in[\![0,k]\!]$ as

$$\xi_{v_s,i}(x)=\begin{cases}\xi^1_{v_s,i}(x) & \text{for } x\in A^1_{v_s,i},\\ \xi^2_{v_s,i}(x) & \text{for } x\in A^2_{v_s,i}\end{cases}$$

and $\pi_{v_s,i}\colon \bigcup_{r=1}^2 A^r_{v_s,i}\cup\operatorname{Fr}(K_{v_s})\cup\operatorname{Fr}(K_{n_{s+1}})\to K_{n_{s-1}}$ for $s\in\mathbb{N}_+$, $i\in[\![0,k]\!]$ in the following way

$$\pi_{v_s,i}(x)=\begin{cases}\xi_{v_s,i}(x) & \text{for } x\in\bigcup_{r=1}^2 A^r_{v_s,i},\\ \varphi_{n_s,i}(x) & \text{for } x\in\operatorname{Fr}(K_{v_s}),\\ f_i(x) & \text{for } x\in\operatorname{Fr}(K_{n_{s+1}}).\end{cases}$$

Of course $\operatorname{Fr}(K_{v_s})\subset K_{n_s}$. Note that by (3.11), (3.6) and (3.12) the sets $A^1_{v_s,i}\cup A^2_{v_s,i}$, $\operatorname{Fr}(K_{v_s})$ and $\operatorname{Fr}(K_{n_{s+1}})$ are pairwise disjoint. According to (3.13), (3.11) and (3.10) it is easy to show that $A^1_{v_s,i}\cup A^2_{v_s,i}\cup\operatorname{Fr}(K_{v_s})\cup\operatorname{Fr}(K_{n_{s+1}})\subset K_{v_s}$.

Applying (K4) we can consider the function $\varphi_{v_s,i}\colon K_{v_s} \to K_{n_{s-1}}$ which is a continuous extension of $\pi_{v_s,i}$ for $s \in \mathbb{N}_+$ and $i \in [\![0,k]\!]$. Finally, let us define a continuous function $\varphi_i\colon \mathbb{I}^l \to \mathbb{I}^l$ for $i \in [\![0,k]\!]$ in the following way

$$\varphi_i(x) = \begin{cases} f_i(x) & \text{for } x \notin K_{n_1}, \\ x_0 & \text{for } x = x_0, \\ \varphi_{n_s,i}(x) & \text{for } x \in K_{n_s} \setminus K_{v_s}, \text{ where } s \in \mathbb{N}_+, \\ \varphi_{v_s,i}(x) & \text{for } x \in K_{v_s} \setminus K_{n_{s+1}}, \text{ where } s \in \mathbb{N}_+. \end{cases}$$

Now one can consider the family $\mathcal{A}_d = \{f_0, \varphi_1 \circ f_1, \varphi_2 \circ f_2, \dots, \varphi_k \circ f_k\}$.

We will show that the family $\mathcal{A}_d$ is an ε-disruption of $\mathcal{A}$ at x_0. For this purpose we will prove that $\varphi_i \in [f_i]^{\varepsilon}_{x_0}$ for $i \in [\![1,k]\!]$. So let us fix $i \in [\![1,k]\!]$.

Notice that we have $\varphi_i(x) = f_i(x)$ for $x \in \mathbb{I}^l \setminus K_{n_1}$, so $\approx (f_i,\varphi_i) \subset K_{n_1} \subset \mathrm{B}(x_0,\varepsilon)$. Based on (3.6) and (3.13) we can conclude that $f_i(\approx (f_i,\varphi_i)) \subset f_i(K_{n_1}) \subset K_{n_0} \subset \mathrm{B}(x_0,\varepsilon)$. Now we will show that $\varphi_i(\approx (f_i,\varphi_i)) \subset \mathrm{B}(x_0,\varepsilon)$. For $x \in\approx (f_i,\varphi_i) \subset K_{n_1}$ by virtue of the definition of the function φ_i ($i \in \mathbb{N}_+$) and (3.13) we get $x \in K_{n_{s-1}} \subset \mathrm{B}(x_0,\varepsilon)$. Thus we have proved that $\approx (f_i,\varphi_i) \cup f_i(\approx (f_i,\varphi_i)) \cup \varphi_i(\approx (f_i,\varphi_i)) \subset \mathrm{B}(x_0,\varepsilon)$. We also know that $f_i \in \mathrm{FIX}(x_0)$ and, by virtue of the definition of the function φ_i, we get $\varphi_i(x_0) = x_0$. Hence $\varphi_i \in \mathrm{FIX}(x_0)$. The above considerations prove that $f_i \frac{\varepsilon}{x_0} \varphi_i$.

Now we will show that x_0 is a point strongly focusing chaos of the family $\mathcal{A}_d$, so we will prove that for any function $g \in G(\mathcal{A}_d) \setminus \{f_0\}$ the point x_0 fulfils the assertion of the Theorem. Fix $\mu \in \mathbb{N}_+$ and let us consider the function $g \in G(\mathcal{A}_d) \setminus \{f_0\}$ i.e.,

$$g = t_\mu \circ t_{\mu-1} \circ \dots \circ t_1 \quad \text{where } t_p \in \mathcal{A}_d \text{ for } p \in [\![1,\mu]\!]. \tag{3.15}$$

There is no loss of generality in assuming that $t_p \neq f_0$ for $p \in [\![1,\mu]\!]$. Hence $t_p = \varphi_{i_p} \circ f_{i_p}$, where $p \in [\![1,\mu]\!]$ and $i_p \in [\![1,k]\!]$. Obviously

$$t_p(x_0) = x_0 \quad \text{for } p \in [\![1,\mu]\!]. \tag{3.16}$$

Note that

$$t_p(D_{m_s}) = D_{m_s} \quad \text{for } s \in \mathbb{N}_+ \text{ and } p \in [\![1,\mu]\!]. \tag{3.17}$$

Indeed. Fix $s \in \mathbb{N}_+$ and $p \in [\![1,\mu]\!]$. Then according to the definition of $X_{n_s,i}$ we have

$$t_p(D_{m_s}) = \varphi_{i_p} \circ f_{i_p}(D_{m_s}) = \varphi_{i_p}(X_{n_s,i_p}).$$

By (3.9) we get $X_{n_s,i_p} \cap K_{v_s} = \emptyset$ and by (3.7) we have $X_{n_s,i_p} \subset \mathrm{Int}(K_{n_s})$. Then $X_{n_s,i_p} \subset K_{n_s} \setminus K_{v_s}$, hence by (3.14) we gain $\varphi_{i_p}(X_{n_s,i_p}) = \psi_{n_s,i_p}(X_{n_s,i_p}) = \tau_{n_s,i_p}(X_{n_s,i_p}) = D_{m_s}$, which ends the proof of (3.17). From (3.17) and (3.15) it is easy to see that

$$g(D_{m_s}) = D_{m_s} \quad \text{for } s \in \mathbb{N}_+. \tag{3.18}$$

Now we will show that

$$g(T^j_{n_s,i_1}) = D_{m_s} \quad \text{for } s \in \mathbb{N}_+ \text{ and } j \in [\![1, 2^s]\!]. \tag{3.19}$$

Let us fix $s \in \mathbb{N}_+$ and $j \in [\![1, 2^s]\!]$. First note that using (3.8), (3.9), (3.7), (3.14), taking into account the definition of arcs $S_{n_s,i}$ and subarcs $S^j_{n_s,i} \subset S_{n_s,i}$ we gain

$$t_1(T^j_{n_s,i_1}) = \varphi_{n_s,i_1}(S^j_{n_s,i_1}) = \tau_{n_s,i_1}(S^j_{n_s,i_1}) = \lambda_{n_s,i_1}(S^j_{n_s,i_1}) = D_{m_s}.$$

From the above considerations, by virtue of (3.17) it is easy to see that

$$g(T^j_{n_s,i_1}) = t_\mu \circ \cdots \circ t_3 \circ t_2 \circ t_1(T^j_{n_s,i_1}) = t_\mu \circ \cdots \circ t_3 \circ t_2(D_{m_s}) = \cdots = D_{m_s},$$

which ends the proof of (3.19). From (3.19) one can easily conclude that

$$\bigcap_{j=1}^{2^s} g(T^j_{n_s,i_1}) = D_{m_s} \quad \text{for } s \in \mathbb{N}_+. \tag{3.20}$$

Now we will show that

$$h(g_{|D_{m_s}}) \geq \log 2^s \quad \text{for } s \in \mathbb{N}_+. \tag{3.21}$$

Let us recall that by virtue of (3.15) we have $g = t_\mu \circ t_{\mu-1} \circ \cdots \circ t_1$ for $\mu \in \mathbb{N}_+$. Consider function $t_1 = \varphi_{i_1} \circ f_{i_1}$ for $i_1 \in [\![1, k]\!]$. Let us fix $s \in \mathbb{N}_+$. We chose 2^s pairwise disjoint closed sets $T^j_{n_s,i_1}$ contained in D_{m_s} such that $f_{i_1}(T^j_{n_s,i_1}) = S^j_{n_s,i_1}$ for $j \in [\![1, 2^s]\!]$. Moreover, we can note that the family $\{T^j_{n_s,i_1} \mid j \in [\![1, 2^s]\!]\}$ is closed and the topology of the space $\mathbb{I}^l$ weakly separates the family $\{T^j_{n_s,i_1} \mid j \in [\![1, 2^s]\!]\}$. In order to prove (3.21) at first we will show that

$$\bigcup_{j=1}^{2^s} T^j_{n_s,i_1} \subset \bigcap_{j=1}^{2^s} g_{|D_{m_s}}(T^j_{n_s,i_1}). \tag{3.22}$$

Note that for $j \in [\![1, 2^s]\!]$ from $T^j_{n_s,i_1} \subset D_{m_s}$ we have $\bigcup_{j=1}^{2^s} T^j_{n_s,i_1} \subset D_{m_s}$. By (3.20) we get $D_{m_s} = \bigcap_{j=1}^{2^s} g(T^j_{n_s,i_1})$, and therefore $\bigcup_{j=1}^{2^s} T^j_{n_s,i_1} \subset \bigcap_{j=1}^{2^s} g(T^j_{n_s,i_1})$. Then, by (3.18) and inclusion $T^j_{n_s,i_1} \subset D_{m_s}$ we have $g_{|D_{m_s}}(T^j_{n_s,i_1}) = g(T^j_{n_s,i_1})$ for $j \in [\![1, 2^s]\!]$. Consequently, by (3.20) one can infer that $\bigcup_{j=1}^{2^s} T^j_{n_s,i_1} \subset \bigcap_{j=1}^{2^s} g_{|D_{m_s}}(T^j_{n_s,i_1})$. This finishes the proof of (3.22).

In view of the above properties, we conclude that the function $g_{|D_{m_s}}$ is 2^s-turbulent. Therefore, all assumptions of the Lemma 2.3 are fulfilled. Hence $h(g_{|D_{m_s}}) \geq \log 2^s$, which ends the proof of inequality (3.21).

In the next step of the proof we will show that x_0 is a point focusing entropy of the function g. Let U be an arbitrary open neighbourhood of x_0. We will show that

$$h(g, U) = \infty.$$

Fix any $\beta_* > 0$. Let s_* be a positive integer such that $s_* > \beta_*$ and $K_{m_{s_*}} \subset U$. Let us consider the interval $D_{m_{s_*}} \subset K_{m_{s_*}} \subset U$. By (3.21) we get $h(g_{|D_{m_{s_*}}}) \geq \log 2^{s_*} > \beta_*$.

According to (3.18) one can observe that $h(g, D_{m_{s_*}}) = h(g_{|D_{m_{s_*}}}) > \beta_*$. Definition of the interval $D_{m_{s_*}}$ allows us to conclude that $D_{m_{s_*}} \subset K_{m_{s_*}} \subset U$. Hence $h(g, U) \geq h(g, D_{m_{s_*}}) > \beta_*$. By the arbitrariness of $\beta_* > 0$ we get $h(g, U) = \infty$.

Now we are going to prove that x_0 is a point of distributional chaos of the function g. Fix $\eta > 0$. By (K3) and $\lim_{s\to\infty} \kappa_s = 0$ we conclude, that there exists a number $s_\eta \geq 2$ such that $K_{n_{s_\eta}} \subset \mathrm{B}(x_0, \kappa_{s_\eta - 1}) \subset \mathrm{B}(x_0, \eta)$. So, let us consider the interval $D_{m_{s_\eta}} = \mathbb{I}(x_{m_{s_\eta}}, z_{m_{s_\eta}}) \subset \mathrm{B}(x_0, \eta)$.

For the function f_{i_1} we have chosen sets $T^j_{n_{s_\eta}, i_1} \subset D_{m_{s_\eta}}$ for $j \in [\![1, 2^{s_\eta}]\!]$. Consider two sets $T^1_{n_{s_\eta}, i_1}, T^2_{n_{s_\eta}, i_1} \subset D_{m_{s_\eta}}$. Hence, by (3.19) and (3.18), we have $g(T^j_{n_{s_\eta}, i_1}) = D_{m_{s_\eta}}$ for $j = 1, 2$, and $g(D_{m_{s_\eta}}) = D_{m_{s_\eta}}$. Let $\chi\colon [0,1] \to D_{m_{s_\eta}}$ be a homeomorphism. So, one can consider a function $\theta = \chi^{-1} \circ g_{|D_{m_{s_\eta}}} \circ \chi\colon [0,1] \to [0,1]$ and disjoint and closed sets $\chi^{-1}(T^j_{n_{s_\eta}, i_1})$ for $j = 1, 2$.

By (3.19) we get $\theta \circ \chi^{-1}(T^j_{n_{s_\eta}, i_1}) = [0,1]$. Therefore $\chi^{-1}(T^1_{n_{s_\eta}, i_1}) \cup \chi^{-1}(T^2_{n_{s_\eta}, i_1}) \subset \theta(\chi^{-1}(T^1_{n_{s_\eta}, i_1})) \cap \theta(\chi^{-1}(T^2_{n_{s_\eta}, i_1}))$. This inclusion allows us to infer that the function $\theta\colon [0,1] \to [0,1]$ has a horseshoe. Hence by Lemma 2.4 we gain $h(\theta) > 0$ and by Lemma 2.5 there exists the set $S \subset [0,1]$ which is an uncountable DS-set for the function θ. Obviously, $\chi \circ \theta \circ \chi^{-1} = g_{|D_{m_{s_\eta}}}$. Therefore, the assumptions of Lemma 2.6 are fulfilled, which means, that $\chi(S) \subset D_{m_{s_\eta}}$ is an uncountable DS-set for the function $g_{|D_{m_{s_\eta}}}$. Of course if $x \in D_{m_{s_\eta}}$ then $g_{|D_{m_{s_\eta}}}(x) = g(x)$. Consequently, due to (3.18) we have $g^z_{|D_{m_{s_\eta}}}(x) = g^z(x)$ for $x \in D_{m_{s_\eta}}$ and $z \in \mathbb{N}_+$. It means that for any distinct points $x, y \in \chi(S)$ there exists $t_0 > 0$ such that $0 = \Phi^{(g_{|D_{m_{s_\eta}}})}_{x,y}(t_0) = \liminf_{n\to\infty} \frac{1}{n} \operatorname{card}(\{z \in [\![0, n-1]\!] : |g^z(x) - g^z(y)| < t_0\}) = \Phi^{(g)}_{x,y}(t_0)$ and for any distinct points $x, y \in \chi(S)$ and any $t > 0$ we have $1 = \limsup_{n\to\infty} \frac{1}{n} \operatorname{card}(\{z \in [\![0, n-1]\!] : |g^z(x) - g^z(y)| < t\}) = \Phi^{*(g)}_{x,y}(t)$, therefore $\chi(S)$ is an uncountable DS-set for the system (g).

Note that for any $z \in \mathbb{N}_+$ we have $g^z(D_{m_{s_\eta}}) = D_{m_{s_\eta}} \subset \mathrm{B}(x_0, \eta)$, hence $\chi(S) \subset D_{m_{s_\eta}} = g^z(D_{m_{s_\eta}}) \subset \mathrm{B}(x_0, \eta)$ for $z \in \mathbb{N}_+$, which means that the interval $D_{m_{s_\eta}}$ is an $(1, \eta)$-envelope of the set $\chi(S)$. The above reasoning allows us to conclude that x_0 is a point of distributional chaos of the function g.

At the end we will prove that x_0 is a point of chaos of the function g. Fix $\eta_* > 0$ and let $s_{\eta_*} \geq 2$ be a positive integer such that $K_{v_{s_{\eta_*}}} \subset \mathrm{B}(x_0, \sigma_{s_{\eta_*}}) \subset \mathrm{B}(x_0, \eta_*)$. Let us consider a point $a_1 \in A^1_{v_{s_{\eta_*}}, i_1}$. Therefore, since $A^1_{v_{s_{\eta_*}}, i_1} \subset X_{v_{s_{\eta_*}}, i_1} = f_{i_1}(D_{w_{s_{\eta_*}}})$ there exists a point $a \in D_{w_{s_{\eta_*}}}$ such that $a_1 = f_{i_1}(a)$. We will show that

$$t_1(a) = x_0. \tag{3.23}$$

Notice that by (3.11) and (3.10) one can get $A^1_{v_{s_{\eta_*}}, i_1} \subset K_{v_{s_{\eta_*}}}$. It is also known that

$a_1 \notin K_{n_{s_{\eta*}+1}}$. By (3.6) the inclusion $K_{n_{s_{\eta*}+1}} \subset \mathrm{B}(x_0, \kappa_{s_{\eta*}})$ holds. According to (3.12) and (3.11) we have $a_1 \in K_{v_{s_{\eta*}}} \setminus K_{n_{s_{\eta*}+1}}$. Consequently, it is easy do conclude that $\varphi_{i_1}(a_1) = x_0$.

From the definition of t_1 we have $t_1(a) = \varphi_{i_1} \circ f_{i_1}(a) = \varphi_{i_1}(a_1) = x_0$, which ends the proof of (3.23).

Further we will show that the point a chosen in this way is an (x_0, g)-homoclinic point.

Obviously, $a \neq x_0$ since $a \in D_{w_{s_{\eta*}}}$. First, note that according to (3.15), (3.23) and (3.16) the equalities $g(a) = t_\mu \circ \cdots \circ t_3 \circ t_2 \circ t_1(a) = t_\mu \circ \cdots \circ t_3 \circ t_2(x_0) = \cdots = x_0$ and $g(x_0) = t_\mu \circ \cdots \circ t_3 \circ t_2 \circ t_1(x_0) = t_\mu \circ \cdots \circ t_3 \circ t_2(x_0) = \cdots = x_0$ take place.

It allows us to conclude that x_0 is the limit of the sequence $\{g^w(a)\}_{w=0}^{\infty}$. Now we will show two auxiliary facts. At first we will prove that

$$\text{for any interval } D_{w_s}, \text{ any point } \alpha \in D_{w_s} \text{ and any } p \in [\![1, \mu]\!], \\ \text{there exists a point } x_* \in D_{w_{s+1}} \text{ such that } t_p(x_*) = \alpha. \tag{3.24}$$

Let us fix $s \in \mathbb{N}_+$, $\alpha \in D_{w_s}$ and $p \in [\![1, \mu]\!]$. Consider an arc $A^2_{v_s+1,i_p}$. It is not hard to see that there is a point $\beta \in A^2_{v_s+1,i_p}$ such that $\varphi_{i_p}(\beta) = \alpha$. Now, note that based on (3.11) one can infer the existence of the point $x_* \in D_{w_{s+1}}$ such that $f_{i_p}(x_*) = \beta$. Therefore $t_p(x_*) = \varphi_{i_p} \circ f_{i_p}(x_*) = \varphi_{i_p}(\beta) = \alpha$, which ends the proof of (3.24).

Using the fact proved above, we will show that

$$\text{for any interval } D_{w_s} \text{ and any point } \gamma \in D_{w_s}, \\ \text{there exists a point } \delta \in D_{w_{s+\mu}} \text{ such that } g(\delta) = \gamma. \tag{3.25}$$

Indeed. Fix $s \in \mathbb{N}_+$ and $\gamma \in D_{w_s}$. Since $\gamma \in D_{w_s}$ and by (3.24) there exists $\delta_{w_{s+1},i_\mu} \in D_{w_{s+1}}$ such that $t_\mu(\delta_{w_{s+1},i_\mu}) = \gamma$. Again, by (3.24) there exists $\delta_{w_{s+2},i_{\mu-1}} \in D_{w_{s+2}}$ such that $t_{\mu-1}(\delta_{w_{s+2},i_{\mu-1}}) = \delta_{w_{s+1},i_\mu}$. Then $t_\mu \circ t_{\mu-1}(\delta_{w_{s+2},i_{\mu-1}}) = t_\mu(\delta_{w_{s+1},i_\mu}) = \gamma$. By continuing this reasoning, we will find a point $\delta = \delta_{w_{s+\mu},i_1} \in D_{w_{s+\mu}}$ such that $t_1(\delta) = \delta_{w_{s+\mu-1},i_2}$. Then $g(\delta) = t_\mu \circ t_{\mu-1} \circ t_{\mu-2} \circ \cdots \circ t_2 \circ t_1(\delta_{w_{s+\mu},i_1}) = \gamma$, which ends the proof of (3.25).

Since $a \in D_{w_{s_{\eta*}}}$ and by (3.25) we can conclude that there exists a point $\delta^1_{w_{s_{\eta*}+\mu}} \in D_{w_{s_{\eta*}+\mu}}$ such that $g(\delta^1_{w_{s_{\eta*}+\mu}}) = a$. So let $y_1 = \delta^1_{w_{s_{\eta*}+\mu}}$ and $k_1 = 1$. Hence $g^{k_1}(y_1) = a$.

Because $\delta^1_{w_{s_{\eta*}+\mu}} \in D_{w_{s_{\eta*}+\mu}}$ and by (3.25) there exists a point $\delta^2_{w_{s_{\eta*}+2\mu}} \in D_{w_{s_{\eta*}+2\mu}}$ such that $g(\delta^2_{w_{s_{\eta*}+2\mu}}) = \delta^1_{w_{s_{\eta*}+\mu}}$. Let $y_2 = \delta^2_{w_{s_{\eta*}+2\mu}}$ and $k_2 = 2$. Then $g^{k_2}(y_2) = g^2(\delta^2_{w_{s_{\eta*}+2\mu}}) = g(\delta^1_{w_{s_{\eta*}+\mu}}) = a$.

Continuing this procedure, one can find the sequences $\{k_m\}_{m=1}^{\infty} = \{m\}_{m=1}^{\infty}$ and $\{y_m\}_{m=1}^{\infty} \subset \mathbb{I}^l$ where $y_m \in D_{w_{s_{\eta*}+m\cdot\mu}}$ for $m \in \mathbb{N}_+$ such that $g^{k_m}(y_m) = a$ for any $m \in \mathbb{N}_+$.

Using (3.10) it is easy to see that $\lim_{m\to\infty} y_m = x_0$. Finally, we can easily conclude that a is an (x_0, g)-homoclinic point. □

Problem 3.2. In the context of the theorem above and of preceding information, the following question seems interesting: With what additional assumptions about the functions creating $\mathcal{A}$ and about a fixed point x_0 it can be proved that from the fact that x_0 is the point strongly focusing chaos of each function $\xi \in \mathcal{A} \setminus \{f_0\}$ one can infer that x_0 is the point strongly focusing chaos of $\mathcal{A}$?

Problem 3.3. According to the results contained in papers [2, 3, 10] it seems natural to ask whether Theorem 3.1 will remain true if the continuity assumption would be replaced with a weaker one (e.g., quasi-continuity, almost continuity, etc.).

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