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Characterization of Modules by Fitting Ideals

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Abstract. Let R be a commutative Noetherian ring and M be a finitely generated R-module. In this paper we characterize modules whose first nonzero Fitting ideal has a maximal radical or is a prime ideal. Moreover, a number of examples concerning this characterization are given.

1. Introduction and preliminaries

Throughout this paper R denotes a Noetherian commutative ring with identity and all modules are unital.

Many invariants in algebraic geometry and commutative algebra may be defined in terms of free resolutions. As a consequence of the uniqueness of minimal free resolutions theorem (see [3, Theorem 20.2]), we can make many invariants of a module out of a free resolution of a module. Some of these invariants are Fitting ideals.

In this paper we study Fitting invariants, which generalize the structure theory of modules over a principal ideal domain and, in general, give a way of expressing features of a module in terms of ideals.

Let M be a finitely generated R-module and $F \xrightarrow{\varphi} G \xrightarrow{\psi} M \to 0$ be a free presentation of M with G and F, free modules of rank r and s, respectively. Let $A \in M_{r\times s}(R)$ be a matrix presenting of φ and $I_j(\varphi)$ be an ideal of R generated by the minors of size j of matrix A. We make the convention that the determinant of the 0×0 matrix is 1. We set $I_j(\varphi) = R$ if $j \leq 0$.

By the Fitting's lemma [3, Corollary 20.4], these ideals are independent of the choice of free presentation of M. So we define the jth Fitting ideal of M to be the ideal $\mathrm{Fitt}_j(M) = \mathrm{I}_{r-j}(\varphi)$. The most important Fitting ideal of M is the first one, which is nonzero. We denote this Fitting ideal simply $\mathrm{I}(M)$. Thus $\mathrm{I}(M) = \mathrm{I}_{\mathrm{rank}\,\varphi}(\varphi)$. Hence we have $R = \mathrm{I}_0(\varphi) \supseteq \mathrm{I}_1(\varphi) \supseteq \cdots \supseteq \mathrm{I}_{\mathrm{rank}\,\varphi}(\varphi) \supsetneq 0$. Note that if $\mathrm{I}(M)$ contains a nonzerodivisor, then $\mathrm{rank}(\varphi) = \mathrm{rank}(\varphi_P)$ and so $\mathrm{I}(M_P) = \mathrm{I}(M)_P$ for every prime ideal P of R.

A partial list of important contributors to the theory of Fitting ideals includes the mathematicians: Fitting, Buchsbaum, Lipman, Huneke, Katz, Northcott, and Eisenbud

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(for references for each author, see [2–4, 11, 12]). Some recent works on Fitting ideals, due to author are [5–10].

Throughout this paper, an element of R is called regular if it is a nonzerodivisor and an ideal of R is regular if it contains a regular element. Let M be a finitely generated R-module. $\mathrm{T}(M)$, the torsion submodule of M, is the submodule of M consisting of all elements of M that are annihilated by a regular element of R.

2. Some characterizations of modules using Fitting ideals

Fitting ideals are strong tools to characterize modules and to recognize some properties of them.

A lemma of Lipman asserts that if R is a quasilocal ring and $M = R^n/K$, where K is a submodule of R^n and $\mathrm{I}(M)$ is $(n-q)$th Fitting ideal of M, then $\mathrm{I}(M)$ is regular principal if and only if K is finitely generated free and $\frac{M}{\mathrm{T}(M)}$ is free of rank $n-q$ (see [12]) and Ohm generalized this result to global case [13]. Also Buchsbaum and Eisenbud have shown in [2] that if R is a Noetherian ring, then M is a finitely generated projective R-module of constant rank if and only if $\mathrm{I}(M) = R$. At this point, a natural question arises: if $\mathrm{I}(M)$ is a certain ideal, for example a maximal ideal, a prime ideal and a primary ideal, what can we say about the structure of M? In [5–10] these questions are answered in some cases.

Let R be a Noetherian UFD and Q be a maximal ideal of R. Let M be a finitely generated non-torsionfree R-module. In [7], it is shown that $\mathrm{I}(M) = Q$ implies that $M \cong \frac{R}{Q} \oplus N$ for some projective R-module N of constant rank. In [8] we have shown that this result is true for every finitely generated module over a Noetherian ring if and only if $\mathrm{T}(M) \nsubseteq QM$. As a result, it is shown that if M is an Artinian R-module and $\mathrm{I}(M) = Q$ is a regular maximal ideal of R, then $M \cong \frac{R}{Q}$.

The zeroth Fitting ideal of an R-module M, $\mathrm{Fitt}_0(M)$, has the same radical as the annihilator of M. More specific, we have the following proposition.

Proposition 2.1. *If M is a finitely generated module over a Noetherian ring R which can be generated by n elements, then*

$$(\mathrm{Ann}(M))^n \subseteq \mathrm{Fitt}_0(M) \subseteq \mathrm{Ann}(M).$$

Proof. See [3, Proposition 20.7]. □

This basic theorem is generalized in [10, Lemma 2.5], as follows:

Lemma 2.2. *Let M be a finitely generated module over a Noetherian ring R. Then*

$$\mathrm{I}(M) \subseteq \mathrm{Ann}(\mathrm{T}(M)).$$

Let (R, Q) be a local ring and M be a finitely generated R-module. It is known that all minimal generator sets of M have the same cardinal. We will denote the minimal number of generators of M by $\mu(M)$.

In the following theorem we characterize modules whose first nonzero Fitting ideal has a maximal radical. In what follows $\mathrm{S}(Q)$ is the set of all elements x of M where $\mathrm{Ann}(x) = Q$.

Theorem 2.3. *Let (R, Q) be a Noetherian local ring and let M be a finitely generated R-module such that $\sqrt{\mathrm{I}(M)} = Q$ is a regular ideal. Then one of the following holds.*

(1) $M \cong \mathrm{T}(M) \oplus \frac{M}{\mathrm{T}(M)}$ *if* $\mathrm{S}(Q) \cap QM = \emptyset$;

(2) $Q^k M \cong \left(\frac{R}{Q}\right)^n \oplus N$, *for some positive integers k and n and some torsionfree R-submodule N of M, if* $\mathrm{S}(Q) \cap QM \neq \emptyset$.

Proof. By Lemma 2.2, $\mathrm{I}(M) \subseteq \mathrm{Ann}(\mathrm{T}(M))$. So

$$\sqrt{\mathrm{I}(M)} = Q \subseteq \sqrt{\mathrm{Ann}(\mathrm{T}(M))} = \bigcap_{\mathrm{Ann}(\mathrm{T}(M)) \subseteq P} P,$$

where $P \in \mathrm{Ass}(\mathrm{T}(M))$. Since Q is a maximal ideal, $\sqrt{\mathrm{Ann}(\mathrm{T}(M))} = R$ or $\sqrt{\mathrm{Ann}(\mathrm{T}(M))} = Q$. Hence $\mathrm{T}(M) = 0$ or $\mathrm{Ass}(\mathrm{T}(M)) = \{Q\}$. If $\mathrm{T}(M) = 0$, the proof is complete. Assume that $\mathrm{T}(M) \neq 0$, so $\mathrm{Ass}(\mathrm{T}(M)) = \{Q\}$ and therefore there exists an element $x_1 \in \mathrm{T}(M)$ such that $\mathrm{Ann}(x_1) = Q$. We consider two cases:

Case (1). Assume that $\mathrm{S}(Q) \cap QM = \emptyset$, so $x_1 \notin QM$. By induction on $\mu(M)$, we show that $\mathrm{T}(M)$ splits off. If $\mu(M) = 1$, then M is cyclic. Hence $M = \langle x \rangle \cong \frac{R}{\mathrm{Ann}(x)}$ for some $x \in M$. If $\mathrm{Ann}(x) = 0$, Then $M \cong R$ and so $\mathrm{T}(M) = 0$. Suppose that $\mathrm{Ann}(x) \neq 0$. By Proposition 2.1, $\mathrm{I}(M) = \mathrm{Fitt}_0(M) = \mathrm{Ann}(M)$ and since $\mathrm{I}(M)$ is a regular ideal, we have $M = \mathrm{T}(M)$.

Now let $\mu(M) = 2$. We have $x_1 \notin QM$, so there exists an element $x_2 \in M$ such that $\{x_1, x_2\}$ is a minimal generating set for M and since $\mathrm{Ann}(x_1) = Q$, hence

$$M = \langle x_1 \rangle \oplus \langle x_2 \rangle \cong \frac{R}{\mathrm{Ann}(x_1)} \oplus \frac{R}{\mathrm{Ann}(x_2)}.$$

If $\mathrm{Ann}(x_2) = 0$, then $\langle x_2 \rangle \cong R$, so

$$M = \langle x_1 \rangle \oplus \langle x_2 \rangle \cong \mathrm{T}(M) \oplus \frac{M}{\mathrm{T}(M)} \cong \frac{R}{Q} \oplus R.$$

If $\mathrm{Ann}(x_2) \neq 0$, then by [1, page 174], $\mathrm{I}(M) = Q\,\mathrm{Ann}(x_2)$. Therefore $Q = Q \cap \sqrt{\mathrm{Ann}(x_2)} \subseteq \sqrt{\mathrm{Ann}(x_2)}$. This implies that $\mathrm{Ann}(x_2) = R$ or $\sqrt{\mathrm{Ann}(x_2)} = Q$. Since $x_2 \neq 0$, hence

$\sqrt{\mathrm{Ann}(x_2)} = Q$. Since Q is finitely generated, there exists some $m \in \mathbb{N}$ such that $Q^m \subseteq \mathrm{Ann}(x_2)$ and since Q is regular, $\mathrm{Ann}(x_2)$ is a regular ideal. So,

$$M = \langle x_1 \rangle \oplus \langle x_2 \rangle = \mathrm{T}(M).$$

Assume that $\mu(M) = n$ and for every R-module M' with $\sqrt{\mathrm{I}(M')} = Q$ and $\mu(M') < n$, in which $\mathrm{S}(Q) \neq \emptyset$ and $\mathrm{S}(Q) \cap QM' = \emptyset$, $\frac{M'}{\mathrm{T}(M')}$ is a free R-module.

We have $x_1 \notin QM$. So $\{x_1\}$ can be extended to a minimal generating set for M. Therefore there exists a submodule N of M such that $M = \langle x_1 \rangle \oplus N$. Hence $\mu(N) = n-1$. We have $\mathrm{I}(M) = Q\mathrm{I}(N)$. So $Q = Q \cap \sqrt{\mathrm{I}(N)} \subseteq \sqrt{\mathrm{I}(N)}$. This implies that $\mathrm{I}(N) = R$ or $\sqrt{\mathrm{I}(N)} = Q$. If $\mathrm{I}(N) = R$, then by [2, Lemma 1], N is free and we are done. If $\sqrt{\mathrm{I}(N)} = Q$, then similar to above (replacing N by M at the first of the proof) we have $\mathrm{T}(N) = 0$, or there exists an element $y \in N$ such that $Q = \mathrm{Ann}(y)$.

If N is torsionfree, we have

$$M = \langle x_1 \rangle \oplus N = \mathrm{T}(M) \oplus \frac{M}{\mathrm{T}(M)},$$

so we are done. Otherwise, there exists an element $y \in N$ such that $Q = \mathrm{Ann}(y)$. Since $\mathrm{S}(Q) \cap QM = \emptyset$, hence $\mathrm{S}(Q) \cap QN = \emptyset$. By induction hypothesis $\frac{N}{\mathrm{T}(N)}$ is free. So $N = \mathrm{T}(N) \oplus \frac{N}{\mathrm{T}(N)}$. Because $M = \langle x_1 \rangle \oplus N$, it is easily seen that $\frac{M}{\mathrm{T}(M)} \cong \frac{N}{\mathrm{T}(N)}$. So $\mathrm{T}(M)$ splits off in this case.

Case (2). Assume that $\mathrm{S}(Q) \cap QM \neq \emptyset$. Without loss of generality we assume that $x_1 \in QM$. By Krull's Intersection Theorem, there exists some positive integer k_1 such that $x_1 \in Q^{k_1}M \setminus Q^{k_1+1}M$. Thus $\{x_1\}$ can be extended to a minimal generating set for $Q^{k_1}M$. Since $\mathrm{Ann}(x_1) = Q$, it is easily seen that

$$Q^{k_1}M = \langle x_1 \rangle \oplus N_1 \cong \frac{R}{Q} \oplus N_1 \tag{2.1}$$

for some submodule N_1 of M. If $\mathrm{T}(N_1) = 0$, then we are done. If $\mathrm{T}(N_1) \neq 0$, again Lemma 2.2 implies that

$$\sqrt{\mathrm{I}(M)} = Q = \sqrt{\mathrm{Ann}(\mathrm{T}(M))} \subseteq \sqrt{\mathrm{Ann}(\mathrm{T}(N_1))} = \bigcap_{\mathrm{Ann}(\mathrm{T}(N_1)) \subseteq P} P,$$

where $P \in \mathrm{Ass}(\mathrm{T}(N_1))$. So there exists an element $x_2 \in \mathrm{T}(N_1)$ such that $\mathrm{Ann}(x_2) = Q$. If $x_2 \notin QN_1$, then $N_1 = \langle x_2 \rangle \oplus N_2$ for some R-submodule N_2 of M. Replacing in (2.1), we have

$$Q^{k_1}M = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus N_2 \cong \left(\frac{R}{Q}\right)^2 \oplus N_2.$$

If $x_2 \in QN_1$, again by Krull's Intersection Theorem, there exists some positive integer k_2 such that $x_2 \in Q^{k_2}N_1 \setminus Q^{k_2+1}N_1$. Therefore $Q^{k_2}N_1 = \langle x_2 \rangle \oplus N_2$ for some R-submodule

N_2 of M. So (2.1) implies

$$Q^{k_1+k_2}M = Q^{k_2}\langle x_1\rangle \oplus Q^{k_2}N_1 = \langle x_2\rangle \oplus N_2.$$

If $\mathrm{T}(N_2) = 0$, the proof is complete in this case and if $\mathrm{T}(N_2) \neq 0$ again we use Lemma 2.2 for N_2.

Continuing this process, we obtain

$$Q^{k_s}M = \langle x_1\rangle \oplus \cdots \oplus \langle x_s\rangle \oplus N_s$$

for some positive integer k_s, some $x_i \in M$ where $\mathrm{Ann}(x_i) = Q$, $i = 1, \dots, s$ and some submodule N_s of M.

We have an increasing chain

$$\langle x_1\rangle \subseteq \langle x_1, x_2\rangle \subseteq \cdots \subseteq \langle x_1, \dots, x_n\rangle \subseteq \cdots .$$

Since M is Noetherian, this chain stops. This means that there exists some positive integer i such that N_i is torsionfree. So we are done in this case. $\square$

Example 2.4. Let $R = k[x, y]$ be the ring of polynomials over a field k. Set

$$Q := \langle x, y\rangle, \quad A := \begin{bmatrix} y & y & y & xy & 0 \\ x & x & x & x^2 & 0 \\ 0 & y & x & 0 & xy \\ 0 & 0 & 0 & y & x \end{bmatrix} \quad \text{and} \quad M := \frac{R^4}{\langle A\rangle}.$$

We have $\mathrm{I}(M) = Q^3$. Assume that A_i is the ith column of the matrix A, $1 \leq i \leq 5$. Let $(a, b, c, d)^t + \langle A\rangle \in QM$, where t denotes transpose. Thus there exist some $r_{ij} \in R$, $1 \leq i, j \leq 2$ such that $a = r_{11}x + r_{12}y$, $b = r_{21}x + r_{22}y$, $c = r_{31}x + r_{32}y$, $d = r_{41}x + r_{42}y$. It is easily seen that $(0, 0, c, d)^t = (-r_{32} - r_{31} + r_{41}y - r_{42}x)A_1 + r_{32}A_2 + (r_{31} - r_{41}y)A_3 + r_{42}A_4 + r_{41}A_5$. Thus $(a, b, c, d)^t + \langle A\rangle = (a, b, 0, 0)^t + \langle A\rangle$.

Now let $(a, b, c, d)^t + \langle A\rangle = (a, b, 0, 0)^t + \langle A\rangle \in \mathrm{S}(Q)$. Therefore $\mathrm{Ann}((a, b, 0, 0)^t + \langle A\rangle) = Q$. Let q be a regular element of Q. Then $q(a, b, 0, 0)^t \in \langle A\rangle$. So $q(a, b)^t = s(y, x)^t$ for some $s \in R$. Hence $qa = sy$ and $qb = sx$. If $s = 0$, then $a = b = 0$, a contradiction, because $\mathrm{Ann}((a, b, 0, 0)^t + \langle A\rangle) = Q$. Otherwise $qab = syb = sxa$ and so $yb = ax$. We have $x \mid yb$ and $y \mid ax$. Since $\mathrm{GCD}(x, y) = 1$, $x \mid b$ and $y \mid a$. So there exist $t_1, t_2 \in R$ such that $b = t_1x$ and $a = t_2y$. Since $yb = cx$, we imply that $t_1 = t_2$. Hence $(a, b)^t = t_1(y, x)^t$. Thus $(a, b, 0, 0)^t = t_1(y, x, 0, 0)^t \in \langle A\rangle$. This is a contradiction because $\mathrm{Ann}((a, b, 0, 0)^t + \langle A\rangle) = Q$. So $\mathrm{S}(Q) \cap QM = \emptyset$ and Theorem 2.3 implies that $M \cong \mathrm{T}(M) \oplus \frac{M}{\mathrm{T}(M)}$. In fact $M \cong \left(\frac{R}{Q}\right)^2 \oplus \frac{R^2}{\left\langle \binom{x}{y} \right\rangle}$.

Example 2.5. Let $R = k[x, y]$ be the ring of polynomials over a field k. Set $Q := \langle x, y\rangle$ and $M := \frac{R}{\langle x^2,y\rangle} \oplus \frac{R}{\langle x,y^2\rangle}$. We have $\langle x + \langle x^2, y\rangle, 0\rangle \in \mathrm{S}(Q) \cap QM$. It is clear that

$$QM = \frac{\langle x, y\rangle}{\langle x^2, y\rangle} \oplus \frac{\langle x, y\rangle}{\langle x, y^2\rangle} \cong \frac{R}{Q} \oplus \frac{R}{Q}.$$

In [8, Theorem 2.1] we showed when (R, P) is a Noetherian local ring and M is a finitely generated R-module with $\mathrm{I}(M) = P$, then

(1) $M \cong \frac{R}{P} \oplus R^n$ for some nonnegative integer n, if $\mathrm{T}(M) \nsubseteq PM$.

(2) $P^kM \cong \frac{R}{P} \oplus N$ for some positive integer k and some torsionfree R-submodule N of M, if $0 \neq \mathrm{T}(M) \subseteq PM$.

In the following theorem we generalize this theorem and characterize modules whose first nonzero Fitting ideal is a prime ideal.

Theorem 2.6. *Let (R, Q) be a Noetherian local ring and M be a finitely generated R-module. Let $\mathrm{I}(M) = P$ be a regular prime ideal of R. Then*

(1) $M \cong \frac{R}{P} \oplus R^n$ *for some nonnegative integer n, if* $\mathrm{T}(M) \nsubseteq PM$.

(2) $P^kM \cong \frac{R}{P} \oplus N$ *for some positive integer k and some R-submodule N of M, if* $0 \neq \mathrm{T}(M) \subseteq PM$.

Proof. We have $\mathrm{I}(M_P) = PR_P$. If $\mathrm{T}(M_P) \nsubseteq PM_P$, then by [8, Theorem 2.1] $M_P \cong \frac{R_P}{PR_P} \oplus R_P^{m-1}$ for some positive integer m. Let $M_P = \langle \frac{x_1}{1}\rangle \oplus \langle \frac{x_2}{1}, \dots, \frac{x_m}{1}\rangle$ for some $x_i \in M$, where $\mathrm{Ann}\left(\frac{x_1}{1}\right) = PR_P$. Let $P = \langle a_1, \dots, a_s\rangle$, so $\frac{a_i}{1}\frac{x_1}{1} = \frac{0}{1}$, $1 \leq i \leq s$. Therefore there exist some $t_i \in R \setminus P$ such that $a_it_ix_1 = 0$, $1 \leq i \leq s$. Put $t = t_1 \cdots t_s$. So $Ptx_1 = 0$. Now let $a \in \mathrm{Ann}(tx_1)$. So $\frac{a}{1} \in PR_P$. This implies that $a \in P$. Hence $P = \mathrm{Ann}(tx_1)$. Replacing x_1 by tx_1, we can assume that $\mathrm{Ann}(x_1) = P$. Let $M = \langle x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k}\rangle$ for some $x_{m+i} \in M$, $1 \leq i \leq k$, and some positive integer k. Let $R^n \xrightarrow{\varphi} R^{m+k} \to M \to 0$ be a free presentation of M and $\varphi = (a_{ij})$ for some $a_{ij} \in R$, $1 \leq i \leq m+k$, $1 \leq j \leq n$. So $R_P^n \xrightarrow{\varphi_P} R_P^{m+k} \to M_P \to 0$ is a free presentation of M_P. On the other hand, since $M_P \cong \frac{R_P}{PR_P} \oplus R_P^{m-1}$, we have the minimal free presentation $R_P^s \xrightarrow{\varphi'} R_P^m \to M_P \to 0$ for M_P, where

$$\varphi' = \begin{bmatrix} \frac{a_1}{1} & \cdots & \frac{a_s}{1} \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

By [3, Theorem 20.2], φ_P may be put in the form of

$$\varphi_P = \begin{bmatrix} \varphi' & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

where 1 is the $k \times k$ identity matrix. Now let $(a_{1j}, a_{2j}, \ldots, a_{(m+k)j})^t$ be a column of the matrix $\varphi = (a_{ij})$ for $1 \leq j \leq n$, where t denotes transpose. Thus $\left(\frac{a_{1j}}{1}, \ldots, \frac{a_{(m+k)j}}{1}\right)^t$ is a column of the matrix $\varphi_P = \left[\begin{smallmatrix} \varphi' & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]$. So $\frac{a_{1j}}{1} \in PR_P$ and hence $a_{1j} \in P$. Since $P = \mathrm{Ann}(x_1)$, so $a_{1j}x_1 = 0$. This means that $\langle x_1 \rangle \cap \langle x_2, \ldots, x_{m+k} \rangle = 0$. Hence $M = \langle x_1 \rangle \oplus \langle x_2, \ldots, x_{m+k} \rangle \cong \frac{R}{P} \oplus \langle x_2, \ldots, x_{m+k} \rangle$. By [1, page 174], we have $P = PI(\langle x_2, \ldots, x_{m+k} \rangle)$. So there is an element $a \in I(\langle x_2, \ldots, x_{m+k} \rangle)$ such that $(1 + a)P = 0$. Since P is a regular ideal, so $(1 + a) = 0$, thus $I(\langle x_2, \ldots, x_{m+k} \rangle) = R$. Therefore by [2, Lemma 1], $\langle x_2, \ldots, x_{m+k} \rangle$ is free. Thus $M = \frac{R}{P} \oplus R^{m+k-1}$.

Now let $0 \neq \mathrm{T}(M_P) \subseteq PM_P$, again [8, Theorem 2.1] implies that there exists some positive integer k' such that $P^{k'}M_P \cong \frac{R_P}{PR_P} \oplus N_P$ for some torsionfree R_P-submodule N_P of M_P. Similar to the proof of case $\mathrm{T}(M_P) \nsubseteq PM_P$, $P^{k'}M = \langle x_1, \ldots, x_m, \ldots, x_{m+k} \rangle$; where $P^{k'}M_P = \langle \frac{x_1}{1} \rangle \oplus \langle \frac{x_2}{1}, \ldots, \frac{x_m}{1} \rangle$ and $\mathrm{Ann}(x_1) = P$. Let $R^t \xrightarrow{\varphi} R^{m+k} \to P^{k'}M \to 0$ be a free presentation of $P^{k'}M$ and $\varphi = (a_{ij})$ for some $a_{ij} \in R$, $1 \leq i \leq m + k$. We have the minimal free presentation $R_P^t \xrightarrow{\varphi'} R_P^m \to M_P \to 0$ for M_P, where $\varphi' = \begin{bmatrix} \frac{a_1}{1} & \cdots & \frac{a_t}{1} \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$.

So $\varphi_P = \left[\begin{smallmatrix} \varphi' & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]$, where 1 is the $k \times k$ identity matrix. We have $a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{(m+k)j}x_{m+k} = 0$ for every j, $1 \leq j \leq t$. So $\left(\frac{a_{1j}}{1}, \ldots, \frac{a_{(m+k)j}}{1}\right)^t$ is a column of φ_P. Thus $a_{1j} \in P$ for every j, $1 \leq j \leq t$. So $P^{k'}M_P = \langle x_1 \rangle \oplus \langle x_2, \ldots, x_{m+k} \rangle \cong \frac{R}{P} \oplus \langle x_2, \ldots, x_{m+k} \rangle$. □

Corollary 2.7. *Let (R, Q) be a Noetherian local domain and M be a finitely generated R-module. Let $\mathrm{I}(M) = P$ be a regular prime ideal of R. Then $P^kM \cong \frac{R}{P} \oplus N$ for some positive integer k and some torsionfree R-submodule N of M, if $0 \neq \mathrm{T}(M) \subseteq PM$.*

Proof. It is sufficient to show that $\langle x_2, \ldots, x_{m+k} \rangle$ is torsionfree in the proof of Theorem 2.6. Let $r_2x_2 + \cdots + r_{m+k}x_{m+k} \in \mathrm{T}(\langle x_2, \ldots, x_{m+k} \rangle)$ for some $r_i \in R$, $2 \leq i \leq m + k$. By Lemma 2.2, $P \subseteq \mathrm{Ann}(\mathrm{T}(M)$, so for some regular element $q \in P$, we have $q(r_2x_2 + \cdots + r_{m+k}x_{m+k}) = 0$. Thus $\left(\frac{0}{1}, \frac{qr_2}{1}, \ldots, \frac{qr_{m+k}}{1}\right)^t$ is a column of matrix φ_P. Because $q \in P$, $\left(\frac{0}{1}, \frac{qr_2}{1}, \ldots, \frac{qr_{m+k}}{1}\right)^t$ can not be a column of identity matrix, therefore $\frac{qr_2}{1} = \cdots = \frac{qr_{m+1}}{1} = \cdots = \frac{qr_{m+k}}{1} = \frac{0}{1}$ and since q is regular,

$$\frac{r_2}{1} = \cdots = \frac{r_{m+1}}{1} = \cdots = \frac{r_{m+k}}{1} = \frac{0}{1}.$$

Since R is an integral domain then $r_2 = \cdots = r_{m+k} = 0$ and so

$$\mathrm{T}(\langle x_2, \ldots, x_{m+k} \rangle) = 0.$$

□

References

[1] W. C. Brown, *Matrices over Commutative Rings*, Monographs and Textbooks in Pure and Applied Mathematics **169**, Marcel Dekker, New York, 1993.

[2] D. A. Buchsbaum and D. Eisenbud, *What makes a complex exact?*, J. Algebra **25** (1973), no. 2, 259–268.

[3] D. Eisenbud, *Commutative Algebra: With a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.

[4] H. Fitting, *Die Determinantenideale eines Moduls*, Jahresber. Dtsch. Math.-Ver. **46** (1936), 195–228.

[5] S. Hadjirezaei, *On finitely generated module whose first nonzero fitting ideal is prime*, Bull. Iranian Math. Soc. **46** (2020), no. 6, 1675–1681.

[6] ———, *On the relation between torsion submodule and Fitting ideals*, J. Algebra Appl. **21** (2022), no. 9, Paper No. 2250173, 9 pp.

[7] S. Hadjirezaei and S. Hedayat, *On the first nonzero fitting ideal of a module over a UFD*, Comm. Algebra **41** (2013), no. 1, 361–366.

[8] ———, *On finitely generated module whose first nonzero fitting ideal is maximal*, Comm. Algebra **46** (2018), no. 2, 610–614.

[9] ———, *Decomposition of finitely generated modules using Fitting ideals*, Czechoslovak Math. J. **70** (2020), no. 4, 1179–1190.

[10] S. Hadjirezaei and S. Karimzadeh, *On the first nonzero Fitting ideal of a module over a UFD (II)*, Comm. Algebra **46** (2018), no. 12, 5427–5432.

[11] C. Huneke, D. A. Jorgensen and D. Katz, *Fitting ideals and finite projective dimension*, Math. Proc. Cambridge Philos. Soc. **138** (2005), no. 1, 41–54.

[12] J. Lipman, *On the Jacobian ideal of the module of differentials*, Proc. Amer. Math. Soc. **21** (1969), 422–426.

[13] J. Ohm, *On the first nonzero Fitting ideal of a module*, J. Algebra **320** (2008), no. 1, 417–425.

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