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An Efficient Compact Difference Method for the Fourth-order Nonlocal Subdiffusion Problem

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Abstract. In this paper, a compact finite difference scheme is constructed and studied for the fourth-order subdiffusion equation with the Riemann–Liouville fractional integral. The Caputo time-fractional derivative term and the Riemann–Liouville fractional integral term are discretized by L1-2 discrete formula and second order convolution quadrature rule, respectively. By using the discrete energy method, the Cholesky decomposition method and the reduced-order method, the stability and convergence are attained. And the convergence orders are reached second-order in time and fourth-order in space. Numerical examples verify the theoretical analysis.

1. Introduction

In the paper, we consider a compact finite difference scheme for the fourth-order subdiffusion equation with the Riemann–Liouville fractional integral

$$D_t^{\alpha} u(x,t) - I^{\beta} u_{xx}(x,t) + \mathcal{L}u(x,t) = f(x,t), \quad (x,t) \in \Omega, \tag{1.1}$$

with boundary and initial conditions

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad 0 < t \le T, \tag{1.2}$$

$$u(x,0) = u^0(x), \quad 0 \le x \le L, \tag{1.3}$$

where $\mathcal{L}u = -u_{xx} + u_{xxxx} + pu$, $p \ge 0$, $\Omega = (0,L) \times (0,T)$, $0 < \alpha, \beta < 1$, $f(x,t)$ is the source term and $u^0(x)$ is given smooth function. In (1.1), D_t^{α} denotes Caputo fractional derivative (cf. [17]) as follows:

$$D_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, & 0 < \alpha < 1, \\ \frac{\partial u(x,t)}{\partial t}, & \alpha = 1, \end{cases}$$

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and I^β denotes the R–L operator (cf. [17]) for $0 < \beta < 1$ as

$$I^\beta u_{xx}(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u_{xx}(x,s)\, ds, \quad t > 0.$$

Fractional partial differential equations (FPDEs) are increasingly used in science and physical, and the fields of application are covered viscoelastic materials [1, 11, 20], control theory [2, 8], electroanalytical chemistry, medicine and so forth. Many numerical methods are used to solve some FPDEs, such as, finite element methods [10, 13, 27], finite difference methods [12, 19, 23, 35], quasi-wavelet methods [28, 29], finite volume method [31–33], discontinuous Galerkin methods [15, 24, 26].

During the last decade, the fourth-order integro-differential equations have received great attention and are studied by many scholars. For (1.1), when $p = 0$, Xu et al. [25] conducted a compact finite difference scheme, which can reach order 4 in space and order $\min\{2-\alpha, 1+\beta\}$ in time. For (1.1) without the R–L fractional integral term, Liu et al. [13] proposed a mixed finite element method, the stability and convergence were confirmed. Yang et al. [28,29] proposed Quasi-wavelet based numerical method for fourth-order partial integro-differential equations with a weakly singular kernel, the stability and convergence are derived. Wei et al. [24] presented and analyzed a finite difference/local discontinuous Galerkin method for the fractional diffusion-wave equation. Qiu et al. [18] considered and studied a Sinc–Galerkin method for solving the fourth-order partial integro-differential equation with a weakly singular kernel. Ji et al. [9] conducted a compact algorithm with first Dirichlet boundary conditions. Chen et al. [3, 5] concerned with a compact difference scheme and a second-order accurate numerical method with graded meshes for solving an evolution equation with a weakly singular kernel, and on this basis, chen et al. [4] used a backward Euler alternating direction implicit difference scheme to analyze the three-dimensional fractional evolution equation. Guo et al. [7] proposed a finite difference scheme for solving the nonlinear time-fractional integro-differential equation. Yang and Zhang [34] studied the orthogonal Gauss collocation method (OGCM) with an arbitrary polynomial degree for the numerical solution of a two-dimensional (2D) fourth-order subdiffusion model. Yang and Zhang [30] prove the theory that the L1 scheme for solving time fractional partial differential equations with nonsmooth data has the uniform l(1) optimal order error estimate.

In this paper, we construct a compact finite difference scheme for the fourth-order subdiffusion equation with the Riemann–Liouville fractional integral. In time direction, we use the L1-2 formula [14] to discretize the Caputo time-fractional derivative and second-order finite difference method proposed by Diethelm et al. [6] to approximate the integral term. In the spatial direction, we employ compact difference scheme to obtain a fully discrete scheme. By using the reduced-order method, discrete energy method, we prove the stability and convergence. And the convergence are obtained order 2 in time and order 4 in space. In addition, we improve and compare our scheme with the scheme proposed in Xu [25], where the convergence were reached $\mathcal{O}(k^{\min\{2-\alpha,1+\beta\}}, h^4)$, k and h are temporal and spatial step, respectively.

The rest of this paper has the following organization. In Section 2, the preliminaries are introduced. In Section 3, the construction of the compact difference scheme is presented for the problems (1.1)–(1.3). In Section 4, the stability and convergence analysis are derived. In Section 5, some numerical examples and comparative results are given. Finally, in Section 6 the main conclusions are presented.

2. Preliminaries

Take two positive integers J and N, and write $h = \frac{L}{J}$, $k = \frac{T}{N}$, $x_j = jh$, $t_n = nk$, $0 \le j \le J$, $0 \le n \le N$. h and k are spatial step and time step, respectively.

Assume that $\mathcal{W} = \{w_j^n \mid 0 \le j \le J, 0 \le n \le N\}$ is a grid function. For $1 \le n \le N$, define the following notations:

$$\delta_t w_j^n = \frac{1}{k}(w_j^n - w_j^{n-1}),\ \delta_x w_j^n = \frac{1}{h}(w_j^n - w_j^{n-1}),\ \delta_x^2 w_j^n = \frac{1}{h}(\delta_x w_{j+1}^n - \delta_x w_j^n),\ 1 \le j \le J-1,$$

$$\mathscr{A} w_j^n = \begin{cases} w_j^n, & j = 0, \\ \frac{1}{12}(w_{j+1}^n + 10 w_j^n + w_{j-1}^n) = \left(1 + \frac{h^2}{12}\delta_x^2\right) w_j^n, & 1 \le j \le J, \\ w_j^n, & j = J. \end{cases}$$

Let $W_h = \{w \mid w = (w_1, w_2, \ldots, w_{J-1})^T, w_0 = w_J = 0\}$. For any $w, v \in W_h$, we introduce the following discrete inner product and norms:

$$\langle w, v\rangle = h\sum_{j=1}^{J-1} w_j v_j, \quad \|w\| = \sqrt{\langle w, w\rangle}, \quad \|w\|_\infty = \max_{1\le j\le J-1} |w_j|,$$

$$\|\delta_x w\|^2 = h\sum_{j=0}^{J-1}(\delta_x w_{j+1})^2, \quad \|\delta_x^2 w\|^2 = h\sum_{j=1}^{J-1}(\delta_x^2 w_j)^2.$$

First, using the L1-2 discrete formula [14], we can define the following discrete operator $\mathscr{D}$ to approximate the Caputo fractional derivative D_t^α $(0 < \alpha < 1)$

$$\mathscr{D} u(\,\cdot, t_n) = \begin{cases} \widetilde{\alpha}_0^{-1}[u(\,\cdot, t_1) - u(\,\cdot, t_0)], & n = 1, \\ \alpha_0^{-1}\beta_0\left[u(\,\cdot, t_n) - \sum_{i=1}^n d_{n-i}^n u(\,\cdot, t_{n-i})\right], & 2 \le n \le N, \end{cases} \tag{2.1}$$

where for $n = 1$,

$$\widetilde{\alpha}_0 = \Gamma(2-\alpha) k^\alpha, \quad \widetilde{\alpha}_0^{-1} = \frac{k^{-\alpha}}{\Gamma(2-\alpha)},$$

for $n = 2, 3$,

$$\alpha_0 = \Gamma(3-\alpha)k^{\alpha}, \quad \alpha_0^{-1} = \frac{k^{-\alpha}}{\Gamma(3-\alpha)}, \quad \beta_0 = \left(1+\frac{\alpha}{2}\right)2^{1-\alpha}, \quad 0 < \alpha_0\beta_0^{-1} < \widetilde{\alpha}_0,$$

$$d_{2-i}^2 = -\beta_0^{-1} \times \begin{cases} b_1 - 2, & i=1, \\ a_1 + \frac{\alpha}{2}, & i=2, \end{cases} \quad d_{3-i}^3 = -\beta_0^{-1} \times \begin{cases} b_1 + c_2 - 2, & i=1, \\ a_1 + b_2 + \frac{\alpha}{2}, & i=2, \\ a_2, & i=3, \end{cases}$$

and, for $n \geq 4$

$$d_{n-i}^n = -\beta_0^{-1} \times \begin{cases} b_1 + c_2 - 2, & i=1, \\ a_1 + b_2 + c_3 + \frac{\alpha}{2}, & i=2, \\ a_{i-1} + b_i + c_{i+1}, & 3 \leq i \leq n-2, \\ a_{n-2} + b_{n-1}, & i = n-1, \\ a_{n-1}, & i=n, \end{cases}$$

here, the coefficient a_j, b_j, c_j are

$$\begin{aligned} a_j &= -\frac{3}{2}(2-\alpha)(j+1)^{1-\alpha} + \frac{1}{2}(2-\alpha)j^{1-\alpha} + (j+1)^{2-\alpha} - j^{2-\alpha}, \quad j \geq 0, \\ b_j &= 2(2-\alpha)(j+1)^{1-\alpha} - 2(j+1)^{2-\alpha} + 2j^{2-\alpha}, \quad j \geq 0, \\ c_j &= -\frac{1}{2}(2-\alpha)((j+1)^{1-\alpha} + j^{1-\alpha}) + (j+1)^{2-\alpha} - j^{2-\alpha}, \quad j \geq 0. \end{aligned}$$

From [14], when $i = 1, 2, \ldots, n$, $n = 2, 3, \ldots, N$, we have

$$\sum_{i=1}^{n} |d_{n-i}^n| \leq 2 \quad \text{and} \quad |d_0^n| < 1. \tag{2.2}$$

From [7], for any $0 < \alpha < 1$, $n \geq 4$, the coefficients satisfy

$$\sum_{i=1}^{n} d_{n-i}^n = 1; \quad d_{n-i}^n > 0, \ i = 3, 4, \ldots, n; \quad d_{n-1}^n > 0; \quad -\frac{1}{2} < d_{n-2}^n < \frac{1}{3}. \tag{2.3}$$

From [14], we can get the following truncation error. Suppose that $u(t) \in C^3[0,T]$, for any α $(0 < \alpha < 1)$, it holds that

$$\begin{aligned} &|D_t^{\alpha} u(\cdot, t_n) - \mathscr{D}u(\cdot, t_n)| \\ &= \begin{cases} \left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} \frac{u_t(\cdot, t_1)}{(t_1 - s)^{\alpha}}\, ds - \mathscr{D}u(\cdot, t_1) \right| \leq c_1 k^{2-\alpha}, & n = 1, \\ \left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{u_t(\cdot, t_n)}{(t_n - s)^{\alpha}}\, ds - \mathscr{D}u(\cdot, t_n) \right| \leq c_2 k^{3-\alpha}, & 2 \leq n \leq N, \end{cases} \end{aligned} \tag{2.4}$$

where c_1, c_2 are positive constants depending only on α.

Next, to complete the construction and analysis of the scheme, we also give the following lemmas.

Lemma 2.1. *Set*

$$a_{i,n}^n = \frac{k^\beta}{\beta(\beta+1)} \times \begin{cases} (n-1)^{\beta+1} - (n-1-\beta)n^\beta, & i=0, \\ (n-i+1)^{\beta+1} - 2(n-i)^{\beta+1} + (n-i-1)^{\beta+1}, & 1 \le i \le n-1, \\ 1, & i=n. \end{cases}$$

From [7], *for any β $(0<\beta<1)$ and $a_{i,n}^n$ $(0 \le i \le n,\ 1 \le n \le N)$, it holds that*

$$\begin{cases} 0 < a_{0,n}^n < \frac{k^\beta}{\beta+1} < \frac{T^\beta}{\beta}, & 1 \le n \le N, \\ a_{i,n}^n > 0, & 0 \le i \le n,\ 1 \le n \le N, \\ \sum_{i=1}^{n-1} |a_{i,n}^n| \le \frac{T^\beta}{\beta}, & 2 \le n \le N. \end{cases} \tag{2.5}$$

From [6], *assume $z(t) \in C^2[0,T]$, for any β $(0<\beta<1)$, it holds that*

$$\left| \int_0^{t_n} (t_n - s)^{\beta-1} z(s)\, ds - \sum_{i=0}^{n} a_{n-i,n}^n z(t_{n-i}) \right| \le c_3 k^2, \quad 1 \le n \le N,$$

where a positive constant c_3 is dependent only on β.

Lemma 2.2. *Suppose that $u(x,t) \in C_{x,t}^{6,2}([0,T]\times(0,T])$, then*

$$\left| \mathcal{I}^{(\beta)} \mathscr{A} u_{xx}(x_j,t_n) - \frac{1}{\Gamma(\beta)} \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 U_j^{n-i} \right| \le c_4(k^2+h^4), \quad 1 \le j \le J-1,\ 1 \le n \le N,$$

where $c_4 = \max\left\{c_3, \frac{1}{\Gamma(\beta)} \sum_{i=0}^n a_{n-i,n}^n \frac{h^4}{240} \max_{1\le i\le n} \left|\frac{\partial^6 u}{\partial x^6}(\xi_{1,j},t_{n-i})\right|\right\}$, $\xi_{1,j} \in (x_{j-1},x_{j+1})$.

Proof. Applying Lemma 2.1 and the Taylor expansion, it follows

$$\begin{aligned} \mathcal{I}^{(\beta)} \mathscr{A} u_{xx}(x_j,t_n) &= \frac{1}{\Gamma(\beta)} \int_0^{t_n} (t_n-s)^{\beta-1} \mathscr{A} u_{xx}(x_j,s)\, ds \\ &\le \frac{1}{\Gamma(\beta)} \sum_{i=0}^{n} a_{n-i,n}^n \mathscr{A} u_{xx}(x_j,t_{n-i}) + c_3 k^2 \\ &\le \frac{1}{\Gamma(\beta)} \sum_{i=0}^{n} a_{n-i,n}^n \left\{ \delta_x^2 u(x_j,t_{n-i}) + \frac{h^4}{240} \frac{\partial^6 u}{\partial x^6}(\xi_{1,j},t_{n-i}) \right\} + c_3 k^2 \\ &\le \frac{1}{\Gamma(\beta)} \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 U_j^{n-i} + c_4(k^2+h^4). \end{aligned}$$

The proof is finished. □

Lemma 2.3. [22] *Suppose that $y(x) \in C_x^6([0,L])$. Then, it holds that*

$$\begin{aligned} &\frac{1}{12}[y''(x_{j+1}) + 10y''(x_j) + y''(x_{j-1})] - \frac{1}{h^2}[y(x_{j+1}) - 2y(x_j) + y(x_{j-1})] \\ &= \frac{h^4}{240} y^{(6)}(\xi_j), \quad \xi_j \in (x_{j-1},x_{j+1}),\ 1 \le j \le J-1. \end{aligned}$$

3. Construction of the compact difference scheme

In this section, the compact difference scheme will be formulated for the problem (1.1)–(1.3).

In fact, problem (1.1)–(1.3) is equivalent to

$$
\begin{cases}
D_t^{\alpha} u(x,t) - u_{xx}(x,t) - I^{\beta} u_{xx}(x,t) & \\
\quad + v_{xx}(x,t) + pu(x,t) = f(x,t), & (x,t) \in \Omega, \\
v(x,t) = u_{xx}(x,t), & 0 < x < L,\ 0 < t \le T, \\
u(x,0) = u^0(x), & 0 \le x \le L, \\
u(0,t) = u(L,t) = v(0,t) = v(L,t) = 0, & 0 < t \le T.
\end{cases} \tag{3.1}
$$

Define the following grid functions

$$
U_j^n = u(x_j, t_n), \quad V_j^n = v(x_j, t_n), \quad f_j^n = f(x_j, t_n), \quad 0 \le j \le J,\ 0 \le n \le N.
$$

Considering (3.1) at the point (x_j, t_n), we have for $n = 1$,

$$
\begin{cases}
D_t^{\alpha} u(x_j,t_1) - u_{xx}(x_j,t_1) - I^{\beta} u_{xx}(x_j,t_1) & \\
\quad + v_{xx}(x_j,t_1) + pu(x_j,t_1) = f(x_j,t_1), & 1 \le j \le J-1, \\
v(x_j,t_1) = u_{xx}(x_j,t_1), & 1 \le j \le J-1,
\end{cases} \tag{3.2}
$$

for $n \ge 2$,

$$
\begin{cases}
D_t^{\alpha} u(x_j,t_n) - u_{xx}(x_j,t_n) - I^{\beta} u_{xx}(x_j,t_n) & \\
\quad + v_{xx}(x_j,t_n) + pu(x_j,t_n) = f(x_j,t_n), & 1 \le j \le J-1,\ 2 \le n \le N, \\
v(x_j,t_n) = u_{xx}(x_j,t_n), & 1 \le j \le J-1,\ 2 \le n \le N.
\end{cases} \tag{3.3}
$$

Applying the operator $\mathscr{A}$ to both sides of (3.2) and (3.3), then it holds that

$$
\begin{cases}
\mathscr{A} D_t^{\alpha} u(x_j,t_1) - \mathscr{A} u_{xx}(x_j,t_1) - I^{\beta} \mathscr{A} u_{xx}(x_j,t_1) & \\
\quad + \mathscr{A} v_{xx}(x_j,t_1) + p\mathscr{A} u(x_j,t_1) = \mathscr{A} f(x_j,t_1), & 1 \le j \le J-1, \\
\mathscr{A} v(x_j,t_1) = \mathscr{A} u_{xx}(x_j,t_1), & 1 \le j \le J-1,
\end{cases} \tag{3.4}
$$

$$
\begin{cases}
\mathscr{A} D_t^{\alpha} u(x_j,t_n) - \mathscr{A} u_{xx}(x_j,t_n) - I^{\beta} \mathscr{A} u_{xx}(x_j,t_n) & \\
\quad + \mathscr{A} v_{xx}(x_j,t_n) + p\mathscr{A} u(x_j,t_n) = \mathscr{A} f(x_j,t_n), & 1 \le j \le J-1,\ 2 \le n \le N, \\
\mathscr{A} v(x_j,t_n) = \mathscr{A} u_{xx}(x_j,t_n), & 1 \le j \le J-1,\ 2 \le n \le N.
\end{cases} \tag{3.5}
$$

Define

$$
(3.6)\qquad \begin{aligned}
(R_1)_j^n &= \mathcal{I}^{(\beta)}\mathscr{A}u_{xx}(x_j,t_n) - \frac{1}{\Gamma(\beta)}\sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 U_j,\\
(R_2)_j^n &= \mathscr{A}u_{xx}(x_j,t_n) - \delta_x^2 U_j^n,\\
(R_3)_j^n &= \mathscr{A}v_{xx}(x_j,t_n) - \delta_x^2 V_j^n.
\end{aligned}
$$

Using Lemmas 2.1–2.3, we have

$$
(3.7)\qquad \begin{cases}
|(R_1)_j^n| \le c_4(k^2+h^4), & 1\le j\le J,\ 1\le n\le N,\\
(R_2)_j^n = \frac{h^4}{240}\frac{\partial^6 u}{\partial x^6}(\xi_{2,j},t_n), & 1\le j\le J,\ 1\le n\le N,\\
(R_3)_j^n = \frac{h^4}{240}\frac{\partial^6 v}{\partial x^6}(\xi_{3,j},t_n), & \xi_{2,j},\xi_{3,j}\in(x_{j-1},x_{j+1}),\ 1\le j\le J,\ 1\le n\le N.
\end{cases}
$$

Define

$$
(3.8)\qquad (R_4)_j^n = \mathscr{A}D_t^\alpha u(x_j,t_n) - \mathscr{A}\mathscr{D}U_j^n.
$$

Using (2.1) and (2.4), we obtain

$$
\begin{aligned}
|(R_4)_j^n| &\le c_2\|\mathscr{A}\|_\infty k^{3-\alpha} \le c_2 k^{3-\alpha}, \quad 1\le j\le J-1,\ 2\le n\le N,\\
|(R_4)_j^1| &\le c_1\|\mathscr{A}\|_\infty k^{2-\alpha} \le c_1 k^{2-\alpha}, \quad 1\le j\le J-1.
\end{aligned}
$$

Substituting (3.6)–(3.8) into (3.4) and (3.5), we have

$$
(3.9)\qquad \begin{aligned}
\mathscr{A}\mathscr{D}U_j^1 = \delta_x^2 U_j^1 &+ \frac{1}{\Gamma(\beta)}\sum_{i=0}^{1} a_{1-i,1}^1 \delta_x^2 U_j^{1-i} - \delta_x^2 V_j^1\\
&- p\mathscr{A}U_j^1 + \mathscr{A}f_j^1 + (R_5)_j^1, \quad 1\le j\le J-1,
\end{aligned}
$$

$$
(3.10)\qquad \mathscr{A}V_j^1 = \delta_x^2 U_j^1 + (R_2)_j^1, \quad 1\le j\le J-1,
$$

$$
(3.11)\qquad \begin{aligned}
\mathscr{A}\mathscr{D}U_j^n = \delta_x^2 U_j^n &+ \frac{1}{\Gamma(\beta)}\sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 U_j^{n-i} - \delta_x^2 V_j^n\\
&- p\mathscr{A}U_j^n + \mathscr{A}f_j^n + (R_5)_j^n, \quad 1\le j\le J-1,\ 2\le n\le N,
\end{aligned}
$$

$$
(3.12)\qquad \mathscr{A}V_j^n = \delta_x^2 U_j^n + (R_2)_j^n, \quad 1\le j\le J-1,\ 2\le n\le N,
$$

where

$$
(3.13)\qquad \begin{aligned}
|(R_5)_j^n| &= |(R_1)_j^n + (R_2)_j^n - (R_3)_j^n - (R_4)_j^n|\\
&\le c_5(k^{3-\alpha}+k^2+h^4) \le c_5(k^2+h^4), \quad 1\le j\le J-1,\ 2\le n\le N,\\
|(R_5)_j^1| &= |(R_1)_j^1 + (R_2)_j^1 - (R_3)_j^1 - (R_4)_j^1|\\
&\le \overline{c_5}(k^{2-\alpha}+k^2+h^4) \le \overline{c_5}(k^{2-\alpha}+h^4), \quad 1\le j\le J-1,
\end{aligned}
$$

here,

$$
\begin{aligned}
c_5 &= \max\left\{c_2, c_4, \frac{1}{240}\max\left\{\left|\frac{\partial^6 u}{\partial x^6}(\xi_{2,j}, t_n)\right|\right\}, \frac{1}{240}\max\left\{\left|\frac{\partial^6 v}{\partial x^6}(\xi_{3,j}, t_n)\right|\right\}\right\},\\
\overline{c_5} &= \max\left\{c_1, c_4, \frac{1}{240}\max\left\{\left|\frac{\partial^6 u}{\partial x^6}(\xi_{2,j}, t_n)\right|\right\}, \frac{1}{240}\max\left\{\left|\frac{\partial^6 v}{\partial x^6}(\xi_{3,j}, t_n)\right|\right\}\right\}.
\end{aligned}
$$

In addition, we give the initial and boundary value conditions as follows:

$$
\begin{cases}
U_0^n = U_J^n = 0, \quad V_0^n = V_J^n = 0, & 1 \le n \le N,\\
U_j^0 = u^0(x_j), & 0 \le j \le J.
\end{cases} \tag{3.14}
$$

Omitting the small terms $(R_5)_j^n$ in (3.11), $(R_5)_j^1$ in (3.9), $(R_2)_j^1$ in (3.10), and $(R_2)_j^n$ in (3.12) and replacing the function U_j^n and V_j^n with its numerical approximation u_j^n and v_j^n, respectively, the compact finite difference scheme can be obtained as follows:

$$
\begin{cases}
\mathscr{A}\mathscr{D}u_j^1 = \delta_x^2 u_j^1 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{1}\delta_x^2 u_j^{1-i} - \delta_x^2 v_j^1 - p\mathscr{A}u_j^1 + \mathscr{A}f_j^1, & 1 \le j \le J-1,\\
\mathscr{A}v_j^1 = \delta_x^2 u_j^1, & 1 \le j \le J-1,\\
u_0^1 = u_J^1 = 0, \quad v_0^1 = v_J^1 = 0, & \\
u_j^0 = u^0(x_j), & 0 \le j \le J,
\end{cases} \tag{3.15}
$$

$$
\begin{cases}
\mathscr{A}\mathscr{D}u_j^n = \delta_x^2 u_j^n + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{n}\delta_x^2 u_j^{n-i} - \delta_x^2 v_j^n & \\
\quad -p\mathscr{A}u_j^n + \mathscr{A}f_j^n, & 1 \le j \le J-1,\ 2 \le n \le N,\\
\mathscr{A}v_j^n = \delta_x^2 u_j^n, & 1 \le j \le J-1,\ 2 \le n \le N,\\
u_0^n = u_J^n = 0, \quad v_0^n = v_J^n = 0, & 2 \le n \le N,\\
u_j^0 = u^0(x_j), & 0 \le j \le J.
\end{cases} \tag{3.16}
$$

Remark 3.1. By the properties of the function $v(x,t)$ defined in (3.1), we see that $v_j^0 = V_j^0 = \frac{d^2u^0(x_j)}{dx^2}$ is known and smooth, which will be used for the theoretical analysis in Section 4.

4. Analysis of the compact finite difference scheme

For the analysis of the stability and convergence, we give some notations and lemmas as follows:

Lemma 4.1. [4] *For any $w, v \in W_h$, we have*

$$
\langle \delta_x^2 w, v\rangle = -\sum_{j=0}^{J-1} h(\delta_x w_{j+1})(\delta_x v_{j+1}), \quad \langle \delta_x^2 v, w\rangle \le \frac{4}{h^2}\|v\|\|w\|.
$$

Lemma 4.2. [5,25] *For any $w, v \in W_h$, we have*

$$\langle \mathcal{A}w, v\rangle = \langle w, \mathcal{A}v\rangle, \quad \langle \mathcal{A}w, \delta_x^2 v\rangle = \langle \delta_x^2 w, \mathcal{A}v\rangle,$$

where

$$\mathcal{A} = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & \cdots & 0 \\ 1 & 10 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 10 & 1 \\ 0 & \cdots & 0 & 1 & 10 \end{pmatrix}_{(J-1)\times(J-1)},$$

it is obvious that $\mathcal{A}$ is a real symmetric positive definite matrix. Moreover, we have

$$\mathcal{A}\delta_x^2 w = \delta_x^2 \mathcal{A}w.$$

Lemma 4.3. [25] *For $w, v \in W_h$, it holds that*

$$\langle \mathcal{A}w, v\rangle = \langle \mathcal{B}w, \mathcal{B}v\rangle, \quad \|\mathcal{A}w\| \le \|w\|, \quad \frac{\sqrt{6}}{3}\|w\| \le \|\mathcal{B}w\| \le \|w\|,$$

by the Cholesky decomposition (square root method), there exists a positive definite upper triangular matrix $\mathcal{B}$, which satisfies

$$\mathcal{A} = \mathcal{B}^T\mathcal{B}.$$

Lemma 4.4. [22] *For any $w \in W_h$, then it holds that*

$$\|w\| \le \sqrt{L}\|w\|_\infty, \quad \|w\|_\infty \le \frac{\sqrt{L}}{2}\|\delta_x w\|,$$

from [16]*, we have*

$$\|\delta_x w\| \le \frac{1}{\pi}\|\delta_x^2 w\|.$$

Lemma 4.5 (Discrete Gronwall's inequality)**.** [21] *w_n is a nonnegative real numbers sequence, which satisfies*

$$w_n \le \sum_{i=0}^{n-1}\eta_i w_i + b_n, \quad n \ge 0,$$

where b_n is a nondecreasing sequence of nonnegative numbers, and $\eta \ge 0$. Then

$$w_n \le b_n \exp\left(\sum_{i=0}^{n-1}\eta_i\right), \quad n \ge 0.$$

Based on Lemmas 4.1–4.5, the stability and convergence can be constructed. First, we shall discuss the stability of our scheme.

4.1. Stability

Theorem 4.6. *Suppose u_j^n is the solution of the compact finite difference schemes* (3.15) *and* (3.16), *if $k^\alpha \le C_1 h^2$, it holds that*

$$\|u^n\| \le C\left(\|u^0\| + \max_{1\le i\le n}\|f^i\|\right), \quad 1 \le n \le N, \tag{4.1}$$

where C_1 is a positive constant and

$$C = \max\left\{\frac{2}{3} + \frac{8\Gamma(2-\alpha)T^\beta}{3\beta}C_1, \Gamma(2-\alpha)k^\alpha \exp\left(\frac{4}{3} + \frac{8\Gamma(2-\alpha)T^\beta}{3\beta}C_1\right)\right\}.$$

Proof. The first and second equations in (3.15) and (3.16) can be written as the following form, respectively

$$\begin{aligned} \alpha_0^{-1}\beta_0\left(\mathscr{A}u^n - \sum_{i=0}^{n} d_{n-i}^n \mathscr{A}u^{n-i}\right) &= \delta_x^2 u_j^n + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{n}\delta_x^2 u_j^{n-i} - \delta_x^2 v_j^n \\ &\quad - p\mathscr{A}u_j^n + \mathscr{A}f_j^n, \quad 1 \le j \le J-1,\ 2 \le n \le N, \end{aligned} \tag{4.2}$$

$$\mathscr{A}v_j^n = \delta_x^2 u_j^n, \quad 1 \le j \le J-1,\ 2 \le n \le N, \tag{4.3}$$

$$\begin{aligned} \widetilde{\alpha}_0^{-1}(\mathscr{A}u^1 - \mathscr{A}u^0) &= \delta_x^2 u_j^1 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{1}\delta_x^2 u_j^{1-i} - \delta_x^2 v_j^1 \\ &\quad - p\mathscr{A}u_j^1 + \mathscr{A}f_j^1, \quad 1 \le j \le J-1, \\ \mathscr{A}v_j^1 &= \delta_x^2 u_j^1, \quad 1 \le j \le J-1. \end{aligned} \tag{4.4}$$

(I) Taking the inner product of (4.2) with u^n, we obtain

$$\begin{aligned} &\alpha_0^{-1}\beta_0\langle \mathcal{A}u^n, u^n\rangle + \langle \delta_x^2 v^n, u^n\rangle + p\langle \mathcal{A}u^n, u^n\rangle \\ &= \alpha_0^{-1}\beta_0\left\langle \sum_{i=1}^{n} d_{n-i}^n \mathcal{A}u_{n-i}, u_n\right\rangle + \left\langle \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 u^{n-i}, u^n\right\rangle \\ &\quad + \langle \delta_x^2 u^n, u^n\rangle + \langle \mathcal{A}f^n, u^n\rangle, \quad 2 \le n \le N. \end{aligned}$$

Taking the inner product of (4.3) with u^n, we get

$$\langle \mathcal{A}v^n, v^n\rangle = \langle \delta_x^2 u^n, v^n\rangle = \langle \delta_x^2 v^n, u^n\rangle, \quad 2 \le n \le N.$$

Using the Cauchy–Schwarz inequality, Lemmas 4.2 and 4.3, it is easy to obtain that

$$\begin{aligned} &\alpha_0^{-1}\beta_0\|\mathcal{B}u^n\|^2 + \|\mathcal{B}v^n\|^2 + p\|\mathcal{B}u^n\|^2 \\ &\le \alpha_0^{-1}\beta_0\left\|\sum_{i=1}^{n} d_{n-i}^n \mathcal{B}u^{n-i}\right\| \|\mathcal{B}u^n\| + \frac{4}{h^2}\left\|\sum_{i=1}^{n} a_{n-i,n}^n u^{n-i}\right\| \|u^n\| \\ &\quad - \|\delta_x u^n\|^2 + \|\mathcal{B}f^n\|\|\mathcal{B}u^n\|, \quad 2 \le n \le N. \end{aligned}$$

Using Lemma 4.3, we get

$$\frac{2}{3}\|u^n\|^2 \leq \|\mathcal{B}u^n\|^2 \leq \sum_{i=1}^{n} |d_{n-i}^n|\|u^{n-i}\|\|u^n\| + \frac{4\alpha_0\beta_0^{-1}}{h^2}\left|\sum_{i=1}^{n} a_{n-i,n}^n\right|\|u^{n-i}\|\|u^n\| \\ + \alpha_0^{-1}\beta_0\|f^n\|\|u^n\|, \quad 2 \leq n \leq N.$$

After elimination of one factor u^n, we get

$$\|u^n\| \leq \left(\frac{2}{3}|d_0^n| + \frac{8\Gamma(2-\alpha)k^\alpha}{3h^2}|a_{0,n}^n|\right)\|u^0\| \\ + \frac{2}{3}\sum_{i=1}^{n-1}\left(|d_{n-i}^n| + \frac{4\Gamma(2-\alpha)k^\alpha}{h^2}|a_{n-i,n}^n|\right)\|u^{n-i}\| + \Gamma(2-\alpha)k^\alpha\|f^n\|, \quad 2 \leq n \leq N.$$

According to (2.3) and (2.5), we obtain

$$\|u^n\| \leq \frac{2}{3}\sum_{i=1}^{n-1}\left(|d_{n-i}^n| + \frac{4\Gamma(2-\alpha)k^\alpha}{h^2}|a_{n-i,n}^n|\right)\|u^{n-i}\| + \left(\frac{2}{3} + \frac{8\Gamma(2-\alpha)k^\alpha T^\beta}{3\beta h^2}\right)\|u^0\| \\ + \Gamma(2-\alpha)k^\alpha \max_{1\leq i\leq N}\|f^i\|, \quad 2 \leq n \leq N$$

for $k^\alpha \leq C_1h^2$, $2 \leq n \leq N$, using coefficients conditions (2.2), (2.5) and discrete Gronwall's inequality (see Lemma 4.5)

$$\|u^n\| \leq \left(\left(\frac{2}{3} + \frac{8\Gamma(2-\alpha)T^\beta C_1}{3\beta}\right)\|u^0\| + \Gamma(2-\alpha)k^\alpha \max_{1\leq i\leq N}\|f^i\|\right) \\ \times \exp\left(\frac{4}{3} + \frac{8\Gamma(2-\alpha)T^\beta C_1}{3\beta}\right). \tag{4.5}$$

(II) Taking the inner product of (4.4) with u^1, we obtain

$$\widetilde{\alpha}_0^{-1}\beta_0\langle\mathcal{A}u^1, u^1\rangle + \langle\delta_x^2 v^1, u^1\rangle + p\langle\mathcal{A}u^1, u^1\rangle \\ = \widetilde{\alpha}_0^{-1}\beta_0\langle\mathcal{A}u^0, u^1\rangle + \left\langle\sum_{i=0}^{n} a_{1-i,1}^1\delta_x^2 u^{1-i}, u^1\right\rangle + \langle\delta_x^2 u^1, u^1\rangle + \langle\mathcal{A}f^1, u^1\rangle.$$

Similar to the derivative process of (4.5), and when $k^\alpha \leq C_1h^2$, we get

$$\|u^1\| \leq \left(\frac{2}{3} + \frac{8\Gamma(2-\alpha)k^\alpha T^\beta}{3\beta h^2}\right)\|u^0\| + \Gamma(2-\alpha)k^\alpha\|f^1\| \\ \leq \left(\frac{2}{3} + \frac{8\Gamma(2-\alpha)T^\beta}{3\beta}\right)\|u^0\| + \Gamma(2-\alpha)k^\alpha\|f^1\|. \tag{4.6}$$

Using (4.5) and (4.6) get (4.1). Therefore, the proof of the theorem is finished. □

4.2. Convergence

Next, we conduct the convergence of the compact finite difference scheme (3.15) and (3.16). Let

$$e_j^n = U_j^n - u_j^n, \quad \eta_j^n = V_j^n - v_j^n, \quad 0 \le j \le J,\ 0 \le n \le N.$$

Subtracting (3.16) from (3.11)–(3.12) and (3.14), and subtracting (3.15) from (3.9)–(3.10) and (3.14), then we get the following error equations

$$\text{(4.7)} \qquad \begin{cases} \mathscr{A}\mathscr{D}e_j^n = \delta_x^2 e_j^n + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 e_j^{n-i} & \\ \qquad\qquad -\delta_x^2\eta_j^n - p\mathscr{A}e_j^n + (R_5)_j^n, & 1 \le j \le J-1,\ 2 \le n \le N, \\ \mathscr{A}\eta_j^n = \delta_x^2 e_j^n + (R_2)_j^n, & 1 \le j \le J-1,\ 2 \le n \le N, \\ e_0^n = e_J^n = 0, & 2 \le n \le N, \\ \eta_j^n = \eta_J^n = 0, & 2 \le n \le N, \\ e_j^0 = 0, & 0 \le j \le J, \end{cases}$$

$$\text{(4.8)} \qquad \begin{cases} \mathscr{A}\mathscr{D}e_j^1 = \delta_x^2 e_j^1 + \frac{1}{\Gamma(\beta)}\sum_{i=0}^{1} a_{1-i,1}^1 \delta_x^2 e_j^{1-i} & \\ \qquad\qquad -\delta_x^2\eta_j^1 - p\mathscr{A}e_j^1 + (R_5)_j^1, & 1 \le j \le J-1, \\ \mathscr{A}\eta_j^1 = \delta_x^2 e_j^1 + (R_2)_j^1, & 1 \le j \le J-1, \\ e_0^1 = e_J^1 = 0, \quad \eta_j^1 = \eta_J^1 = 0, & \\ e_j^0 = 0, & 0 \le j \le J. \end{cases}$$

Theorem 4.7. *Assume that the problem* (1.1)–(1.3) *has the smooth solution* $U^n = (U_1^n, U_2^n, \ldots, U_{J-1}^n)^T$, $u = (u_1^n, u_2^n, \ldots, u_{J-1}^n)^T$ *is the solution of the compact difference scheme* (3.15), (3.16) *and* $v^n = (v_1^n, v_2^n, \ldots, v_{J-1}^n)^T$, $1 \le n \le N$. *Then it holds that*

$$\max_{1\le n\le N} \|U^n - u^n\|_\infty = \mathcal{O}(k^2 + h^4), \quad 1 \le n \le N,$$

and

$$\max_{1\le n\le N} \|U^n - u^n\| = \mathcal{O}(k^2 + h^4), \quad 1 \le n \le N.$$

Proof. Applying the operator $\mathscr{D}$ to the both sides of the second equality of (3.16), we have

$$\text{(4.9)} \qquad \begin{cases} \mathscr{A}\mathscr{D}v_j^n = \mathscr{D}\delta_x^2 u_j^n, \quad 1 \le j \le J-1,\ 2 \le n \le N, \\ \mathscr{A}\mathscr{D}v_j^1 = \mathscr{D}\delta_x^2 u_j^1, \quad 1 \le j \le J-1. \end{cases}$$

Using (2.4) and Lemma 2.3, we obtain

$$\text{(4.10)} \qquad \begin{cases} \mathscr{A}\mathscr{D}V_j^n = \mathscr{D}\delta_x^2 U_j^n + \mathcal{O}(k^{3-\alpha} + h^4), \quad 1 \le j \le J-1,\ 2 \le n \le N, \\ \mathscr{A}\mathscr{D}V_j^1 = \mathscr{D}\delta_x^2 U_j^1 + \mathcal{O}(k^{2-\alpha} + h^4), \quad 1 \le j \le J-1. \end{cases}$$

Subtracting (4.9) from (4.10), we have

$$\begin{cases} \mathscr{A}\mathscr{D}\eta_j^n = \mathscr{D}\delta_x^2 e_j^n + (R_6)_j^n, & 1 \le j \le J-1,\ 2 \le n \le N, \\ \mathscr{A}\mathscr{D}\eta_j^1 = \mathscr{D}\delta_x^2 e_j^1 + (R_6)_j^1, & 1 \le j \le J-1, \end{cases} \tag{4.11}$$

where

$$|(R_6)_j^n| \le c_5(k^{3-\alpha} + h^4), \quad |(R_6)_j^1| \le \overline{c_5}(k^{2-\alpha} + h^4). \tag{4.12}$$

First, thinking about $2 \le n \le N$ situation. Taking the inner product of the first equality of (4.7) with $\mathcal{A}\delta_x^2\eta^n$ and taking the inner product of (4.11) with $\mathcal{A}^2\eta^n$, we obtain

$$\begin{aligned} \langle \mathcal{A}\mathscr{D}e^n, \mathcal{A}\delta_x^2\eta^n\rangle = \langle \delta_x^2 e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \frac{1}{\Gamma(\beta)}\left\langle \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 e^{n-i}, \mathcal{A}\delta_x^2\eta^n \right\rangle \\ - \langle \delta_x^2\eta^n, \mathcal{A}\delta_x^2\eta^n\rangle - p\langle \mathcal{A}e^n, \mathcal{A}\delta_x^2\eta^n\rangle \\ + \langle (R_5)^n, \mathcal{A}\delta_x^2\eta^n\rangle, \quad 2 \le n \le N, \end{aligned} \tag{4.13}$$

$$\langle \mathcal{A}\mathscr{D}\eta^n, \mathcal{A}^2\eta^n\rangle = \langle \mathscr{D}\delta_x^2 e^n, \mathcal{A}^2\eta^n\rangle + \langle (R_6)^n, \mathcal{A}^2\eta^n\rangle, \quad 2 \le n \le N. \tag{4.14}$$

Employing Lemma 4.2, we have

$$\langle \mathcal{A}\mathscr{D}e^n, \mathcal{A}\delta_x^2\eta^n\rangle = \langle \mathcal{A}\mathscr{D}e^n, \delta_x^2\mathcal{A}\eta^n\rangle = \langle \mathscr{D}\delta_x^2 e^n, \mathcal{A}^2\eta^n\rangle, \quad 2 \le n \le N,$$

then, adding (4.13) and (4.14) together, we obtain

$$\begin{aligned} &\langle \delta_x^2\eta^n, \mathcal{A}\delta_x^2\eta^n\rangle + \langle \mathcal{A}\mathscr{D}\eta^n, \mathcal{A}^2\eta^n\rangle \\ &= \langle \delta_x^2 e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \frac{1}{\Gamma(\beta)}\left\langle \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 e^{n-i}, \mathcal{A}\delta_x^2\eta^n \right\rangle - p\langle \mathcal{A}e^n, \mathcal{A}\delta_x^2\eta^n\rangle \\ &\quad + \langle (R_5)^n, \mathcal{A}\delta_x^2\eta^n\rangle + \langle (R_6)^n, \mathcal{A}^2\eta^n\rangle, \quad 2 \le n \le N. \end{aligned}$$

Using Lemma 4.3, it holds that

$$\begin{aligned} &\|\mathcal{B}\delta_x^2\eta^n\|^2 + \langle \mathscr{D}\mathcal{B}\mathcal{A}\eta^n, \mathcal{B}\mathcal{A}\eta^n\rangle \\ &= \langle \delta_x^2 e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \frac{1}{\Gamma(\beta)}\left\langle \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 e^{n-i}, \mathcal{A}\delta_x^2\eta^n \right\rangle \\ &\quad - p\langle \mathcal{A}e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \langle \mathcal{B}(R_5)^n, \mathcal{B}\delta_x^2\eta^n\rangle + \langle \mathcal{B}(R_6)^n, \mathcal{B}^2\eta^n\rangle, \end{aligned} \tag{4.15}$$

$$\begin{aligned} \langle \mathscr{D}\mathcal{B}\mathcal{A}\eta^n, \mathcal{B}\mathcal{A}\eta^n\rangle &= \alpha_0^{-1}\beta_0\|\mathcal{B}\mathcal{A}\eta^n\|^2 \\ &\quad - \alpha_0^{-1}\beta_0\left\langle \sum_{i=1}^{n} d_{n-i}^n \mathcal{B}\mathcal{A}\eta^{n-i}, \mathcal{B}\mathcal{A}\eta^n \right\rangle, \quad 2 \le n \le N. \end{aligned} \tag{4.16}$$

Substituting (4.16) into (4.15), we have

$$\begin{aligned}
&\|\mathcal{B}\delta_x^2\eta^n\|^2 + \alpha_0^{-1}\beta_0\|\mathcal{B}\mathcal{A}\eta^n\|^2 \\
&= \langle \delta_x^2 e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \alpha_0^{-1}\beta_0 \left\langle \sum_{i=1}^{n} d_{n-i}^n \mathcal{B}\mathcal{A}\eta^{n-i}, \mathcal{B}\mathcal{A}\eta^n \right\rangle + \frac{1}{\Gamma(\beta)} \left\langle \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 e^{n-i}, \mathcal{A}\delta_x^2\eta^n \right\rangle \\
&\quad - p\langle \mathcal{A}e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \langle \mathcal{B}(R_5)^n, \mathcal{B}\delta_x^2\eta^n\rangle + \langle \mathcal{B}(R_6)^n, \mathcal{B}^2\eta^n\rangle.
\end{aligned}$$

Multiplying both sides of the above equation by $\alpha_0\beta_0^{-1}$, then

$$\begin{aligned}
&\alpha_0\beta_0^{-1}\|\mathcal{B}\delta_x^2\eta^n\|^2 + \|\mathcal{B}\mathcal{A}\eta^n\|^2 \\
&= \alpha_0\beta_0^{-1}\langle \delta_x^2 e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \left\langle \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 e^{n-i}, \mathcal{A}\delta_x^2\eta^n \right\rangle \\
&\quad - p\alpha_0\beta_0^{-1}\langle \mathcal{A}e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \alpha_0\beta_0^{-1}\langle \mathcal{B}(R_5)^n, \mathcal{B}\delta_x^2\eta^n\rangle \\
&\quad + \alpha_0\beta_0^{-1}\langle \mathcal{B}(R_6)^n, \mathcal{B}^2\eta^n\rangle + \left\langle \sum_{i=1}^{n} d_{n-i}^n \mathcal{B}\mathcal{A}\eta^{n-i}, \mathcal{B}\mathcal{A}\eta^n \right\rangle.
\end{aligned} \tag{4.17}$$

Each term in (4.17) will be estimated. First, for the first term of the right-hand side of (4.17), we take the inner product of the second equation of (4.7) with $\mathcal{A}\delta_x^2\eta^n$,

$$\langle \mathcal{A}\eta^n, \mathcal{A}\delta_x^2\eta^n\rangle = \langle \delta_x^2 e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \langle (R_2)^n, \mathcal{A}\delta_x^2\eta^n\rangle, \quad 2 \le n \le N.$$

Using Lemmas 4.1, 4.3 and Young inequality $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, $a, b \in \mathbb{R}$, $\varepsilon > 0$, we have

$$\begin{aligned}
\alpha_0\beta_0^{-1}\langle \delta_x^2 e^n, \mathcal{A}\delta_x^2\eta^n\rangle &= \alpha_0\beta_0^{-1}(\langle \mathcal{A}\eta^n, \mathcal{A}\delta_x^2\eta^n\rangle - \langle (R_2)^n, \mathcal{A}\delta_x^2\eta^n\rangle) \\
&= \alpha_0\beta_0^{-1}(\langle \mathcal{A}\eta^n, \delta_x^2\mathcal{A}\eta^n\rangle - \langle \mathcal{B}(R_2)^n, \mathcal{B}\delta_x^2\eta^n\rangle) \\
&= \alpha_0\beta_0^{-1}(-\|\delta_x\mathcal{A}\eta^n\|^2 - \langle \mathcal{B}(R_2)^n, \mathcal{B}\delta_x^2\eta^n\rangle) \\
&\le \alpha_0\beta_0^{-1}\langle \mathcal{B}(R_2)^n, \mathcal{B}\delta_x^2\eta^n\rangle \\
&\le 4\alpha_0\beta_0^{-1}\|\mathcal{B}(R_2)^n\|^2 + \frac{\alpha_0\beta_0^{-1}}{16}\|\mathcal{B}\delta_x^2\eta^n\|^2, \quad 2 \le n \le N.
\end{aligned} \tag{4.18}$$

Second, for the second term on the right-hand side of (4.17), taking the inner product of the second equality of (4.7) with $\mathcal{A}\delta_x^2\eta^n$, we have

$$\langle \mathcal{A}\eta^{n-i}, \mathcal{A}\delta_x^2\eta^n\rangle = \langle \delta_x^2 e^{n-i}, \mathcal{A}\delta_x^2\eta^n\rangle + \langle (R_2)^{n-i}, \mathcal{A}\delta_x^2\eta^n\rangle, \quad 2 \le n \le N.$$

Then we get

$$\begin{aligned}
&\frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \left\langle \sum_{i=0}^{n} a_{n-i,n}^n \delta_x^2 e^{n-i}, \mathcal{A}\delta_x^2\eta^n \right\rangle \\
&= \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \sum_{i=0}^{n} |a_{n-i,n}^n| \langle \mathcal{A}\eta^{n-i}, \mathcal{A}\delta_x^2\eta^n\rangle - \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \sum_{i=0}^{n} |a_{n-i,n}^n| \langle (R_2)^{n-i}, \mathcal{A}\delta_x^2\eta^n\rangle \\
&= \Phi_1 + \Phi_2.
\end{aligned}$$

Using Young inequality and Lemma 4.3, (2.5) and noting $\eta^0 = 0$ (see Remark 3.1), and $0 < \alpha_0\beta_0^{-1} < \widetilde{\alpha}_0$, it holds that

$$
\begin{aligned}
\Phi_1 &= \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \sum_{i=0}^{n} |a_{n-i,n}^n| \langle \mathcal{B}\delta_x^2 \eta^{n-i}, \mathcal{B}\mathcal{A}\eta^n \rangle + \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} a_{n,n}^n \langle \mathcal{B}\delta_x^2\eta^n, \mathcal{B}\mathcal{A}\eta^n\rangle \\
&\le \frac{5}{4}\left(\frac{\alpha_0\beta_0^{-1}T^\beta}{\Gamma(\beta+1)}\right)^2 \|\mathcal{B}\delta_x^2\eta^{n-i}\|^2 + \frac{1}{5}\|\mathcal{B}\mathcal{A}\eta^n\|^2 \\
&\quad + \frac{5}{4}\left(\frac{\alpha_0\beta_0^{-1}T^\beta}{\Gamma(\beta+1)}\right)^2 \|\mathcal{B}\delta_x^2\eta^{n}\|^2 + \frac{1}{5}\|\mathcal{B}\mathcal{A}\eta^n\|^2 \\
&\le \frac{5}{4}\left(\frac{\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right)^2 \|\mathcal{B}\delta_x^2\eta^{n-i}\|^2 + \frac{1}{5}\|\mathcal{B}\mathcal{A}\eta^n\|^2 \\
&\quad + \frac{5}{4}\left(\frac{\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right)^2 \|\mathcal{B}\delta_x^2\eta^{n}\|^2 + \frac{1}{5}\|\mathcal{B}\mathcal{A}\eta^n\|^2.
\end{aligned} \tag{4.19}
$$

Taking the suitable m so that

$$
\|\mathcal{B}(R_2)^m\| = \max_{1\le i\le n} \|\mathcal{B}(R_2)^{n-i}\|.
$$

Equation (2.5) indicates $a_{i,n}^n > 0$, $0 < \alpha_0\beta_0^{-1} < \widetilde{\alpha}_0$. using (2.5), the Cauchy–Schwarz inequality and Young inequality, we obtain

$$
\begin{aligned}
\Phi_2 &= -\frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \sum_{i=0}^{n} a_{n-i,n}^n \langle \mathcal{B}(R_2)^{n-i}, \mathcal{B}\delta_x^2\eta^n\rangle \\
&\le \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \sum_{i=0}^{n} |a_{n-i,n}^n| \|\mathcal{B}(R_2)^{n-i}\| \|\mathcal{B}\delta_x^2\eta^n\| \\
&\le \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \sum_{i=0}^{n} |a_{n-i,n}^n| \|\mathcal{B}(R_2)^{m}\| \|\mathcal{B}\delta_x^2\eta^n\| \\
&= \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \left(\sum_{i=1}^{n-1} |a_{n-i,n}^n| + a_{n,n}^n + a_{0,n}^n\right) \|\mathcal{B}(R_2)^{m}\| \|\mathcal{B}\delta_x^2\eta^n\| \\
&\le \frac{\alpha_0\beta_0^{-1}}{\Gamma(\beta)} \left(\frac{T^\beta}{\beta} + \frac{T^\beta}{\beta} + \frac{T^\beta}{\beta}\right) \|\mathcal{B}(R_2)^{m}\| \|\mathcal{B}\delta_x^2\eta^n\| \\
&\le \frac{12T^\beta\alpha_0\beta_0^{-1}}{\Gamma(\beta+1)} \|\mathcal{B}(R_2)^m\|^2 + \frac{3T^\beta\alpha_0\beta_0^{-1}}{16\Gamma(\beta+1)} \|\mathcal{B}\delta_x^2\eta^n\|^2 \\
&\le \frac{12T^\beta\Gamma(2-\alpha)k^\alpha}{\Gamma(\beta+1)} \|\mathcal{B}(R_2)^m\|^2 + \frac{3T^\beta\Gamma(2-\alpha)k^\alpha}{16\Gamma(\beta+1)} \|\mathcal{B}\delta_x^2\eta^n\|^2.
\end{aligned} \tag{4.20}
$$

Then for the third term on the right-hand side of (4.17), we first take the inner product of the second equality of (4.7) with $k\mathcal{A}^2\eta^n$, then using Lemma 4.3, Young inequality and $0 < \alpha_0\beta_0^{-1} < \widetilde{\alpha}_0$, we get

$$
\langle \mathcal{A}\eta^n, \mathcal{A}^2\eta^n\rangle = \langle \delta_x^2 e^n, \mathcal{A}^2\eta^n\rangle + \langle (R_2)^n, \mathcal{A}^2\eta^n\rangle = \langle \mathcal{A}e^n, \mathcal{A}\delta_x^2\eta^n\rangle + \langle (R_2)^n, \mathcal{A}^2\eta^n\rangle, \tag{4.21}
$$

$$\langle \mathcal{A}e^n, \mathcal{A}\delta_x^2\eta^n\rangle = \langle \mathcal{A}\eta^n, \mathcal{A}^2\eta^n\rangle - \langle (R_2)^n, \mathcal{A}^2\eta^n\rangle, \tag{4.22}$$

$$\begin{aligned} -p\alpha_0\beta_0^{-1}\langle \mathcal{A}e^n, \mathcal{A}\delta_x^2\eta^n\rangle &= -p\alpha_0\beta_0^{-1}\langle \mathcal{B}\mathcal{A}\eta^n, \mathcal{B}\mathcal{A}\eta^n\rangle + p\alpha_0\beta_0^{-1}\langle (R_2)^n, \mathcal{A}^2\eta^n\rangle \\ &= -p\alpha_0\beta_0^{-1}\|\mathcal{B}\mathcal{A}\eta^n\|^2 + p\alpha_0\beta_0^{-1}\langle (R_2)^n, \mathcal{A}^2\eta^n\rangle \\ &\le p\alpha_0\beta_0^{-1}\langle \mathcal{B}(R_2)^n, \mathcal{B}\mathcal{A}\eta^n\rangle \\ &\le \frac{1}{5}\|\mathcal{B}\mathcal{A}\eta^n\|^2 + \frac{5}{4}(p\Gamma(2-\alpha)k^\alpha)^2\|\mathcal{B}(R_2)^n\|^2. \end{aligned} \tag{4.23}$$

Next, for the fourth term on the right-hand side of (4.17), using the Cauchy–Schwarz inequality, Young inequality and Lemma 4.3, we obtain

$$\begin{aligned} \alpha_0\beta_0^{-1}\langle (R_5)^n, \mathcal{A}\delta_x^2\eta^n\rangle &\le \alpha_0\beta_0^{-1}\|\mathcal{B}(R_5)^n\|\|\mathcal{B}\delta_x^2\eta^n\| \\ &\le 4\alpha_0\beta_0^{-1}\|\mathcal{B}(R_5)^n\|^2 + \frac{\alpha_0\beta_0^{-1}}{16}\|\mathcal{B}\delta_x^2\eta^n\|^2, \quad 2 \le n \le N. \end{aligned} \tag{4.24}$$

Next, for the fifth term on the right-hand side of (4.17), using Young inequality and $0 < \alpha_0\beta_0^{-1} < \widetilde{\alpha}_0$, then

$$\begin{aligned} \alpha_0\beta_0^{-1}\langle \mathcal{B}(R_6)^n, \mathcal{B}\mathcal{A}\eta^n\rangle &\le \frac{1}{5}\|\mathcal{B}\mathcal{A}\eta^n\|^2 + \frac{5}{4}(\alpha_0\beta_0^{-1})^2\|\mathcal{B}(R_6)^n\|^2 \\ &\le \frac{1}{5}\|\mathcal{B}\mathcal{A}\eta^n\|^2 + \frac{5}{4}(\Gamma(2-\alpha)k^\alpha)^2\|\mathcal{B}(R_6)^n\|^2, \quad 2 \le n \le N. \end{aligned} \tag{4.25}$$

Finally, let's analyze the last term on the right-hand side of (4.17), using (2.3), Lemma 4.3, Cauchy–Schwarz inequality and Young inequality, we obtain

$$\begin{aligned} \left\langle \sum_{i=1}^{n} d_{n-i}^n \mathcal{B}\mathcal{A}\eta^{n-i}, \mathcal{B}\mathcal{A}\eta^n \right\rangle &\le \sum_{i=1}^{n} |d_{n-i}^n|\|\mathcal{B}\mathcal{A}\eta^{n-i}\|\|\mathcal{B}\mathcal{A}\eta^n\| \le 2\|\mathcal{A}\eta^{n-i}\|\|\mathcal{B}\mathcal{A}\eta^n\| \\ &\le 2\|\eta^{n-i}\|\|\mathcal{B}\mathcal{A}\eta^n\| \le \frac{L}{\pi}\|\delta_x^2\eta^{n-i}\|\|\mathcal{B}\mathcal{A}\eta^n\| \\ &\le \frac{5}{4}\left(\frac{L}{\pi}\right)^2\|\delta_x^2\eta^{n-i}\|^2 + \frac{1}{5}\|\mathcal{B}\mathcal{A}\eta^n\|^2. \end{aligned} \tag{4.26}$$

Substituting the inequalities (4.18)–(4.26) into (4.17), using (2.4), we have

$$\begin{aligned} &\left(\frac{14}{16}\alpha_0\beta_0^{-1} - \frac{5}{4}\left(\frac{\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right)^2 - \frac{3}{16}\frac{\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right)\|\mathcal{B}\delta_x^2\eta^n\|^2 \\ &\le \frac{5}{4}\left(\left(\frac{\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right)^2 + \left(\frac{L}{\pi}\right)^2\right)\|\mathcal{B}\delta_x^2\eta^{n-i}\|^2 \\ &\quad + \left(4\alpha_0\beta_0^{-1} + \frac{5}{4}(\Gamma(2-\alpha)k^\alpha p)^2\right)\|\mathcal{B}(R_2)^n\|^2 + 4\alpha_0\beta_0^{-1}\|\mathcal{B}(R_5)^n\|^2 \\ &\quad + \frac{5}{4}(\Gamma(2-\alpha)k^\alpha)^2\|\mathcal{B}(R_6)^n\|^2 + \frac{12\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\|\mathcal{B}(R_2)^m\|^2. \end{aligned}$$

From Lemma 4.3, utilizing (3.13) and (4.12), for k sufficiently small, there exists a positive constant $c_6 = \left[\frac{14}{16}\frac{\Gamma(3-\alpha)k^\alpha 2^\alpha}{2+\alpha} - \frac{5}{4}\left(\frac{\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right)^2 - \frac{3}{16}\frac{\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right]^{-1}$, such that

$$\begin{aligned}\|\delta_x^2\eta^n\|^2 &\le \frac{3}{2}c_6c_7\|\delta_x^2\eta^{n-i}\|^2\\ &\quad+\frac{3}{2}c_8\max_{2\le n\le N}\left(\|(R_2)^n\|^2+\|(R_5)^n\|^2+\|(R_6)^n\|^2+\|(R_2)^m\|^2\right)\\ &\le \frac{3}{2}c_6c_7\|\delta_x^2\eta^{n-i}\|^2+\frac{3}{2}c_8(k^2+h^4)^2.\end{aligned}$$

Then we obtain

$$\|\delta_x^2\eta^n\| \le \left(\frac{3}{2}c_6c_7\right)^{1/2}\|\delta_x^2\eta^{n-i}\|+\left(\frac{3}{2}c_8\right)^{1/2}(k^2+h^4),\quad 2\le n\le N,$$

where $c_7 = \frac{5}{4}\left(\left(\frac{\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right)^2+\left(\frac{L}{\pi}\right)^2\right)$, $c_8 = \max\left\{4\frac{\Gamma(3-\alpha)k^\alpha 2^\alpha}{2+\alpha}+\frac{5}{4}(\Gamma(2-\alpha)k^\alpha p)^2, \frac{12\Gamma(2-\alpha)k^\alpha T^\beta}{\Gamma(\beta+1)}\right\}$. Utilizing Lemma 4.5, we have

$$\|\delta_x^2\eta^n\| \le \left(\frac{3}{2}c_8\right)^{1/2}(k^2+h^4)e^{\left(\frac{3}{2}c_6c_7\right)^{1/2}},\quad 2\le n\le N.$$

Further, using Lemma 4.4, we get

$$\|\eta^n\|_\infty \le \frac{\sqrt{L}}{2\pi}\|\delta_x^2\eta^n\| \le \frac{\sqrt{L}}{2\pi}\left(\frac{3}{2}c_8\right)^{1/2}e^{\left(\frac{3}{2}c_6c_7\right)^{1/2}}(k^2+h^4),\quad 2\le n\le N.$$

Using (3.13) and the second equality of (4.7), we have

$$\begin{aligned}\|\delta_x^2 e^n\|_\infty &\le \|\mathcal{A}\eta^n\|_\infty+\|(R_2)^n\|_\infty \le \|\eta^n\|_\infty+\|(R_2)^n\|_\infty\\ &\le \frac{\sqrt{L}}{2\pi}\left(\frac{3}{2}c_8\right)^{1/2}e^{\left(\frac{3}{2}c_6c_7\right)^{1/2}}(k^2+h^4)+c_5h^4 \le c_9(k^2+h^4),\quad 2\le n\le N,\end{aligned}$$

where $c_9 = \max\left\{2\frac{\sqrt{L}}{2\pi}\left(\frac{3}{2}c_8\right)^{1/2}e^{\left(\frac{3}{2}c_6c_7\right)^{1/2}}, 2c_5\right\}$. From Lemma 4.4, it holds that

$$\|e^n\|_\infty \le \frac{\sqrt{L}}{2\pi}\|\delta_x^2 e^n\| \le \frac{L}{2\pi}\|\delta_x^2 e^n\|_\infty \le \frac{Lc_9}{2\pi}(k^2+h^4) = \mathcal{O}(k^2+h^4),\quad 2\le n\le N. \tag{4.27}$$

Moreover, utilizing Lemma 4.4, we obtain

$$\|e^n\| \le \sqrt{L}\|e^n\|_\infty \le \frac{L^{2/3}c_9}{2\pi}(k^2+h^4) = \mathcal{O}(k^2+h^4),\quad 2\le n\le N. \tag{4.28}$$

For the case of $n=1$, our approach is similar to when $2\le n\le N$, so we can obtain

$$\begin{aligned}&\langle\delta_x^2\eta^1,\mathcal{A}\delta_x^2\eta^1\rangle+\langle\mathcal{A}\mathscr{D}\eta^1,\mathcal{A}^2\eta^1\rangle\\ &=\langle\delta_x^2 e^1,\mathcal{A}\delta_x^2\eta^1\rangle+\frac{1}{\Gamma(\beta)}\left\langle\sum_{i=0}^{1}a_{1-i,1}^1\delta_x^2 e^{1-i},\mathcal{A}\delta_x^2\eta^1\right\rangle-p\langle\mathcal{A}e^1,\mathcal{A}\delta_x^2\eta^1\rangle\\ &\quad+\langle(R_5)^1,\mathcal{A}\delta_x^2\eta^1\rangle+\langle(R_6)^1,\mathcal{A}^2\eta^1\rangle.\end{aligned} \tag{4.29}$$

Afterwards, analyzing each term of (4.29) according to the practice of $2 \le n \le N$, we can get

$$\begin{aligned}
&\left[\frac{3}{4} - \left(\frac{k^\beta}{\Gamma(\beta+2)}\right)^2\right] \|\mathcal{B}\delta_x^2\eta^1\|^2 \\
&\le (4+p^2)\|\mathcal{B}(R_2)^1\|^2 + 4\|\mathcal{B}(R_5)^1\|^2 + \|\mathcal{B}(R_6)^1\|^2 + 4\left(\frac{k^\beta T^\beta(\beta+1)}{\Gamma(\beta+2)}\right)^2 \|\mathcal{B}(R_2)^m\|^2 \\
&\le (4+p^2)\|(R_2)^1\|^2 + 4\|(R_5)^1\|^2 + \|(R_6)^1\|^2 + 4\left(\frac{k^\beta T^\beta(\beta+1)}{\Gamma(\beta+2)}\right)^2 \|(R_2)^m\|^2 \\
&\le c_{10}(k^2+h^4)^2,
\end{aligned}$$

where $c_{10} = \max\left\{4+p^2, 4\left(\frac{k^\beta T^\beta(\beta+1)}{\Gamma(\beta+2)}\right)^2\right\}$.

Using Lemma 4.3, we obtain

$$\|\delta_x^2\eta^1\| \le \sqrt{\frac{3c_{10}c_{11}}{2}}(k^2+h^4),$$

where $c_{11} = \left[\frac{3}{4} - \left(\frac{k^\beta}{\Gamma(\beta+2)}\right)^2\right]^{-1}$ and

$$\|\eta^1\|_\infty \le \left(\frac{3Lc_{10}c_{11}}{8\pi^2}\right)^{1/2}(k^2+h^4).$$

Using (3.13) and the second equality of (4.8), we have

$$\begin{aligned}
\|\delta_x^2 e^1\|_\infty &\le \|\mathcal{A}\eta^1\|_\infty + \|(R_2)^1\|_\infty \le \|\eta^1\|_\infty + \|(R_2)^1\|_\infty \\
&\le \left(\frac{3Lc_{10}c_{11}}{8\pi^2}\right)^{1/2}(k^2+h^4) + \overline{c_5}h^4 \le c_{12}(k^2+h^4),
\end{aligned}$$

where $c_{12} = \max\left\{2\left(\frac{3Lc_{10}c_{11}}{8\pi^2}\right)^{1/2}, 2\overline{c_5}\right\}$.

From Lemma 4.4, it holds that

$$\|e^1\|_\infty \le \frac{\sqrt{L}}{2\pi}\|\delta_x^2 e^1\| \le \frac{L}{2\pi}\|\delta_x^2 e^1\|_\infty \le \frac{Lc_{12}}{2\pi}(k^2+h^4) = \mathcal{O}(k^2+h^4), \tag{4.30}$$

and

$$\|e^1\| \le \sqrt{L}\|e^1\|_\infty \le \frac{L^{2/3}c_{12}}{2\pi}(k^2+h^4) = \mathcal{O}(k^2+h^4). \tag{4.31}$$

Using (4.27), (4.28), (4.30) and (4.31), the proof is finished. □

5. Numerical results

In this section, we consider the problem (1.1)–(1.3) with the parameters $L = T = 1$, $p = 0$, $p = 1$, $p = 10^{-2}$ and use the compact difference scheme (3.15) and (3.16) to simulate this problem.

Define the maximum-norm errors, L_2-norm errors, and corresponding space-time convergence orders as follows

$$E_1(h,k)=\max_{1\leq n\leq N}\|u(jh,nk)-u_j^n\|_\infty,\quad E_2(h,k)=\|u(jh,Nk)-u_j^N\|_\infty,$$

and

$$\mathrm{Rate}_i^t=\log_2\left(\frac{E_i(h,2k)}{E_i(h,k)}\right),\quad \mathrm{Rate}_i^x=\log_2\left(\frac{E_i(2h,k)}{E_i(h,k)}\right),\quad i=1,2.$$

Example 5.1. In the example, $p=0$, the initial condition is $u^0(x)=0$, and the source term is

$$f(x,t)=\left(\frac{\Gamma(3+\alpha+\beta)}{\Gamma(3+\beta)}t^{-\alpha}+\frac{\pi^2\Gamma(3+\alpha+\beta)}{\Gamma(3+\alpha+2\beta)}t^{\beta}+\pi^2+\pi^4\right)t^{2+\alpha+\beta}\sin\pi x.$$

The corresponding exact solution is

$$u(x,t)=t^{2+\alpha+\beta}\sin\pi x.$$

Table 5.1: Comparisons the errors and convergence orders in time with different α, β for Example 5.1.

α, β	h	k	$E_1(h,k)$	Rate_1^t	$E_1(h,k)$ in [25]	Rate_1^t in [25]
	1/512	1/16	1.0228e-4		1.6524e-4	
$\alpha=0.65$		1/32	2.7928e-5	1.8727	9.2712e-5	0.8338
$\beta=0.35$		1/64	7.4168e-6	1.9129	4.3507e-5	1.0915
		1/128	1.7466e-6	2.0862	1.9377e-5	1.1669
	1/512	1/16	1.1178e-4		6.5244e-5	
$\alpha=0.50$		1/32	2.8967e-5	1.9482	9.4609e-6	2.7858
$\beta=0.50$		1/64	7.9579e-6	1.8340	7.4779e-6	0.3393
		1/128	2.1334e-6	1.8989	4.2173e-6	0.8263
	1/512	1/16	1.0672e-4		2.2130e-4	
$\alpha=0.35$		1/32	2.6497e-5	2.0099	5.1109e-5	2.1144
$\beta=0.75$		1/64	7.0107e-6	1.9182	1.0570e-5	2.2736
		1/128	1.6989e-6	2.0450	2.2973e-6	2.2019
	1/512	1/16	5.6730e-5		1.6659e-3	
$\alpha=0.95$		1/32	1.3984e-5	2.0203	8.9836e-4	0.8909
$\beta=0.65$		1/64	3.3878e-6	2.0454	4.5874e-5	0.9696
		1/128	8.9711e-7	2.0595	2.2772e-7	1.0104

In Tables 5.1–5.3, we compare the numerical results with Xu [25]. In Table 5.1, fixed $h = 1/512$, we give the maximum-norm errors and the corresponding temporal convergence orders of different α and β, and compare these with Xu [25].

Table 5.2: Comparisons the errors, convergence orders in time with fixed α for Example 5.1.

α	β	h	k	$E_1(h,k)$	Rate_1^x	$E_1(h,k)$ in [25]	Rate_1^x in [25]
0.25	0.45	1/512	1/16	9.6764e-5		1.5214e-4	
			1/32	2.4329e-5	1.9923	3.7364e-5	1.9968
			1/64	6.1829e-6	1.9757	8.6612e-6	2.0313
			1/128	1.3609e-6	2.1837	2.1984e-6	2.0354
	0.65	1/512	1/16	1.0437e-4		2.2246e-4	
			1/32	2.6122e-5	1.9984	5.6287e-5	1.9880
			1/64	6.4564e-6	2.0165	1.3770e-6	2.0554
			1/128	1.1369e-6	2.5057	3.3587e-7	2.1685
	0.85	1/512	1/16	1.0008e-4		2.7235e-4	
			1/32	2.4479e-5	2.0315	6.8670e-5	1.9870
			1/64	6.1170e-6	2.0007	1.6509e-6	2.0571
			1/128	1.5770e-6	1.9556	3.6705e-7	2.1691
0.80	0.45	1/512	1/16	9.3867e-5		5.7890e-4	
			1/32	2.6315e-5	1.8347	3.0245e-4	0.9366
			1/64	6.9488e-6	1.9211	1.4477e-4	1.0630
			1/128	1.3802e-6	2.3319	6.6294e-5	1.1268
	0.65	1/512	1/16	8.1720e-5		6.3253e-4	
			1/32	2.2615e-5	1.8534	3.4147e-4	0.8894
			1/64	5.9166e-6	1.9345	1.6664e-4	1.0350
			1/128	1.1576e-6	2.3536	7.6745e-5	7.6744
	0.85	1/512	1/16	5.3638e-5		7.3218e-4	
			1/32	1.6239e-5	1.7238	3.9462e-4	0.8917
			1/64	3.5454e-6	2.1955	1.9201e-4	1.0392
			1/128	5.8706e-7	2.5944	8.8980e-5	1.1097

Fixed $\alpha = 0.25$, 0.80 and $h = 1/512$, Table 5.2 presents comparisons the maximum-norm errors and the corresponding time convergence orders with Xu [25]. In Table 5.3, with different α, fixing $\beta = 0.45$, 0.75 and $h = 1/512$, comparisons the maximum-norm errors, temporal convergence orders with Xu [25] are listed.

Table 5.3: Comparisons the errors, convergence orders in time with fixed β for Example 5.1.

β	α	h	k	$E_1(h,k)$	Rate_1^t	$E_1(h,k)$ in [25]	Rate_1^t in [25]
0.45	0.30	1/512	1/16	1.0035e-4		1.4375e-4	
			1/32	2.5823e-5	1.9583	3.3549e-5	2.0992
			1/164	6.7356e-6	1.9388	7.3912e-6	2.1824
			1/128	1.1493e-6	2.5510	1.1845e-6	2.6415
	0.60	1/512	1/16	1.0982e-4		6.6652e-4	
			1/32	2.9659e-5	1.8886	5.2688e-5	0.3392
			1/64	7.8399e-6	1.9196	2.8073e-6	0.9083
			1/128	1.5189e-6	2.3678	1.2756e-6	1.1380
	0.65	1/512	1/16	1.0987e-4		1.4660e-4	
			1/32	2.9939e-5	1.8757	9.1681e-5	0.6772
			1/64	8.2991e-6	1.8510	4.5543e-5	1.0094
			1/128	2.1280e-6	1.9635	2.0304e-5	1.1655
0.75	0.30	1/512	1/16	1.0578e-4		2.3919e-4	
			1/32	2.7314e-5	1.9534	5.7931e-5	2.0457
			1/64	6.5497e-6	2.0601	1.3226e-5	2.1310
			1/128	1.3319e-6	2.2980	2.7920e-6	2.2440
	0.60	1/512	1/16	1.0378e-4		6.9751e-5	
			1/32	2.7712e-5	1.9050	5.4827e-5	0.3473
			1/64	6.4674e-6	2.0992	3.2373e-5	0.7601
			1/128	1.7872e-6	1.8555	1.4932e-5	1.1164
	0.80	1/512	1/16	6.9381e-5		6.7649e-4	
			1/32	1.9280e-5	1.8474	3.6665e-4	0.8837
			1/64	4.9767e-6	1.9539	1.7879e-4	1.0361
			1/128	1.2748e-6	1.9649	8.3040e-5	1.1064

In Tables 5.1–5.3, fixing space step $h = 1/512$, it can be seen that temporal convergence orders reaches order 2. Table 5.4 compares the maximum-norm errors and the convergence orders in space with Xu [25] for different α and β. Fixing the time step $k = 1/2048$ in Table 5.4, it can be seen that space convergence orders reaches order 4. Thus, the numerical results confirm the theorem results. From these numerical results, we can find that our method can achieve second-order accuracy for different α, β. But for the scheme in Xu [25], the desired order $\min\{2-\alpha, 1+\beta\}$ can't be achieved for certain values of α, β.

Table 5.4: Comparisons the errors, convergence orders in space with different α, β for Example 5.1.

α, β	k	h	$E_1(h,k)$	Rate_1^x	$E_1(h,k)$ in [25]	Rate_1^x in [25]
	1/2048	1/4	3.0180e-3		3.0180e-3	
$\alpha = 0.25$		1/8	1.8504e-4	4.0277	1.8503e-4	4.0277
$\beta = 0.75$		1/16	1.1505e-5	4.0074	1.1503e-5	4.0077
		1/32	7.1223e-7	4.0138	7.1039e-7	4.0173
	1/2048	1/4	3.0016e-3		3.0024e-3	
$\alpha = 0.65$		1/8	1.8404e-4	4.0277	1.8480e-4	4.0220
$\beta = 0.85$		1/16	1.1443e-5	4.0075	1.2213e-5	3.9195
		1/32	7.0790e-7	4.0147	1.4787e-6	3.0461
	1/2048	1/4	2.9876e-3		2.9879e-3	
$\alpha = 0.50$		1/8	1.8318e-4	4.0277	1.8330e-4	4.0268
$\beta = 0.50$		1/16	1.1389e-5	4.0075	1.1511e-5	3.9931
		1/32	7.0385e-7	4.0162	8.2600e-7	3.8007
	1/2048	1/4	3.0343e-3		3.0343e-3	
$\alpha = 0.15$		1/8	1.8604e-4	4.0277	1.8602e-4	4.0277
$\beta = 0.95$		1/16	1.1568e-5	4.0074	1.1556e-5	4.0088
		1/32	7.1683e-7	4.0124	7.0473e-7	4.0354

Figure 5.1 shows the convergence orders in time of the maximum error with different α and β. Taking $\alpha = 0.25$, 0.80, $h = 1/512$, we present the maximum error in Figure 5.2. Taking $\beta = 0.45$, 0.75, $h = 1/512$, we present the maximum error in Figure 5.3. In Figures 5.1–5.3, they show a second-order convergence in time. Figure 5.4 shows the

convergence orders in space of the maximum error with different α and β for $k = 1/2048$.

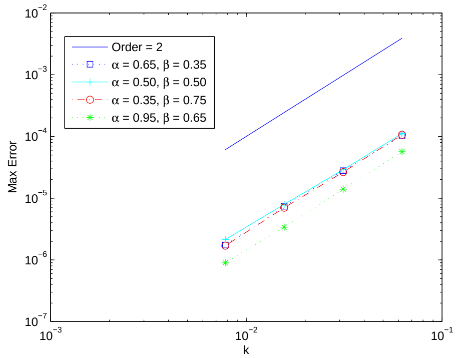

Figure 5.1: The temporal convergence orders with different α, β for Example 5.1.

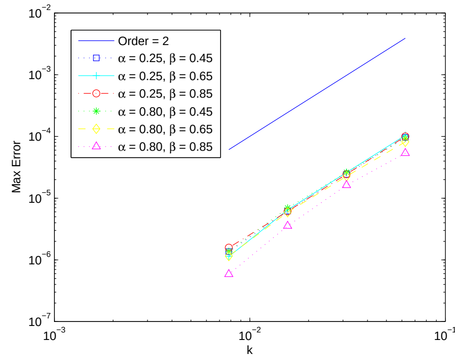

Figure 5.2: The temporal convergence orders with fixed α for Example 5.1.

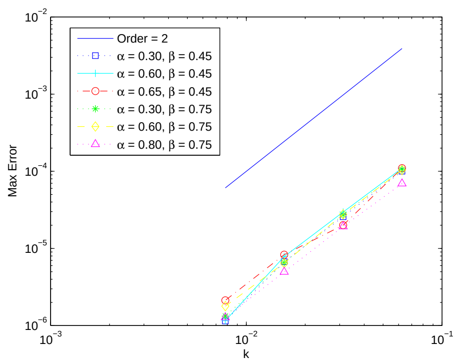

Figure 5.3: The temporal convergence orders with fixed β for Example 5.1.

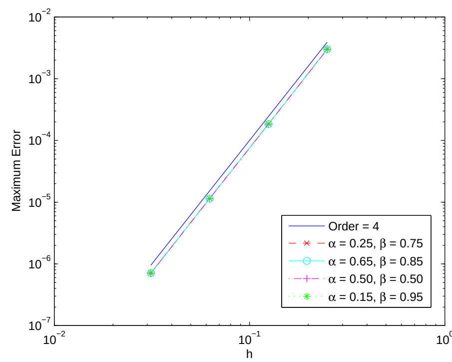

Figure 5.4: The spatial convergence orders with different α, β for Example 5.1.

Example 5.2. In the example, when $p = 1$, we consider (1.1)–(1.3), and the nonhomogeneous term f:

$$f(x,t) = \left(\frac{\Gamma(4+\beta)}{\Gamma(4-\alpha+\beta)} t^{-\alpha} + \frac{\pi^2 \Gamma(4+\beta)}{\Gamma(4+2\beta)} t^{\beta} + 4\pi^2 + \pi^4 + 1 \right) t^{3+\beta} \sin \pi x,$$

and the initial condition $u^0(x) = 0$ are chosen such that the analytical solution is

$$u(x,t) = t^{3+\beta} \sin \pi x.$$

In Table 5.5, fixing $k = 1/2048$, the L_2-norm errors and the corresponding space convergence orders with $\alpha = 0.15$, 0.75 are presented, respectively. It can be seen that the space convergence orders reach order 4. In Table 5.6 lists the L_2-norm errors and the time convergence orders with $\beta = 0.35$, 0.80, respectively. It can be seen that the time convergence orders approximate order 2. These validate our theoretical analysis.

Table 5.5: The L_2 errors and convergence orders in space with fixed α for Example 5.2.

α	β	k	h	$E_2(h,k)$	Rate_2^x
0.15	0.35	1/2048	1/4	2.9863e-3	
			1/8	1.8310e-4	4.0277
			1/16	1.1385e-5	4.0074
			1/32	7.0538e-7	4.0126
	0.55	1/2048	1/4	3.0053e-3	
			1/8	1.8426e-4	4.0277
			1/16	1.1457e-5	4.0074
			1/32	7.0939e-7	4.0135
	0.75	1/2048	1/4	3.0214e-3	
			1/8	1.8525e-4	4.0277
			1/16	1.1518e-5	4.0074
			1/32	7.1329e-7	4.0133
0.75	0.25	1/2048	1/4	2.9496e-3	
			1/8	1.8085e-4	4.0276
			1/16	1.1244e-5	4.0076
			1/64	6.9466e-7	4.0167
	0.55	1/2048	1/4	2.9754e-3	
			1/8	1.8243e-4	4.0277
			1/16	1.1342e-5	4.0076
			1/32	7.0052e-7	4.0171
	0.85	1/2048	1/4	2.9929e-3	
			1/8	1.8350e-4	4.0277
			1/16	1.1410e-5	4.0075
			1/32	7.0622e-7	4.0140

Fixing $\alpha = 0.15$, 0.75, the convergence orders in the space direction are shown intuitively in Figure 5.5. Fixing $\beta = 0.35$, 0.80, the convergence orders in the time direction are given intuitively in Figure 5.6.

Table 5.6: The L_2 errors and convergence orders in time with fixed β for Example 5.2.

β	α	h	k	$E_2(h,k)$	Rate_2^t
0.35	0.45	1/512	1/16	9.8782e-5	
			1/32	2.5656e-5	1.9450
			1/64	6.2746e-6	2.0317
			1/128	1.5275e-6	2.0384
	0.65	1/512	1/16	1.0228e-4	
			1/32	2.7929e-5	1.8727
			1/64	7.4168e-6	1.9129
			1/128	1.7466e-6	2.0863
	0.80	1/512	1/16	9.0762e-5	
			1/32	2.2562e-5	1.8451
			1/64	6.6719e-6	1.9208
			1/128	1.6954e-6	1.9765
0.80	0.45	1/512	1/16	1.0640e-4	
			1/32	2.6859e-5	1.9859
			1/64	6.7187e-6	1.9992
			1/128	1.5458e-6	2.1198
	0.65	1/512	1/16	9.5798e-5	
			1/32	2.5815e-5	1.8918
			1/64	7.3175e-6	1.8188
			1/128	1.8519e-6	1.9823
	0.80	1/512	1/16	8.8315e-5	
			1/32	2.3794e-5	1.8921
			1/64	6.5049e-6	1.8710
			1/128	1.4713e-6	2.1444

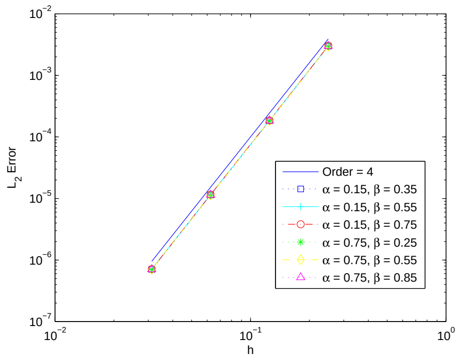

Figure 5.5: The spatial convergence orders for Example 5.2.

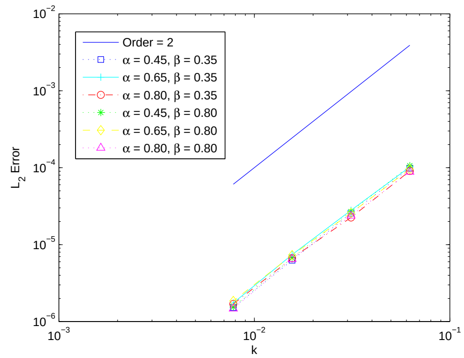

Figure 5.6: The temporal convergence orders for Example 5.2.

Example 5.3. In the example, when $p = 10^{-2}$, we consider (1.1)–(1.3), and the nonhomogeneous term f:

$$f(x,t) = \left(\frac{\Gamma(4+\beta)}{\Gamma(4-\alpha+\beta)} t^{-\alpha} + \frac{\pi^2 \Gamma(4+\beta)}{\Gamma(4+2\beta)} t^{\beta} + 4\pi^2 + \pi^4 + 10^{-2} \right) t^{3+\beta} \sin \pi x,$$

and the initial condition $u^0(x) = 0$ are chosen such that the analytical solution is

$$u(x,t) = t^{3+\beta} \sin \pi x.$$

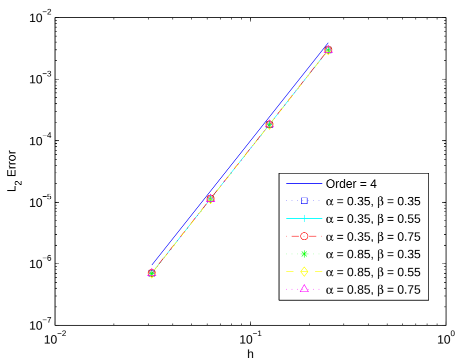

Figure 5.7: The spatial convergence orders for Example 5.3.

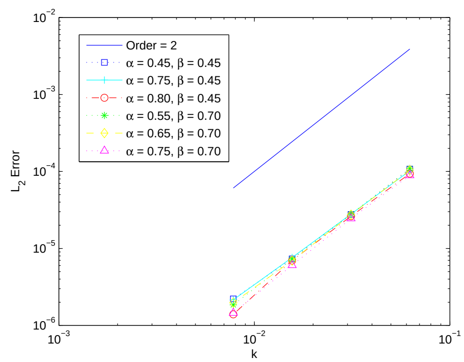

Figure 5.8: The temporal convergence orders for Example 5.3.

In Table 5.7, the L_2-norm errors and the corresponding space convergence orders with fixed $\alpha = 0.35,\ 0.85$ are presented, respectively. It can be seen that the space convergence orders reach order 4. In Table 5.8, the L_2-norm errors and the corresponding time convergence orders with fixed $\beta = 0.45,\ 0.70$ are presented, respectively. The convergence orders in time approximate order 2. These are in accordance with the theoretical analysis.

Table 5.7: The L_2 errors and convergence orders in space with fixed α for Example 5.3.

α	β	k	h	$E_2(h,k)$	Rate_2^x
0.35	0.35	1/2048	1/4	2.9802e-3	
			1/8	1.8273e-4	4.0277
			1/16	1.1361e-5	4.0075
			1/32	7.0302e-7	4.0144
	0.55	1/2048	1/4	2.9987e-3	
			1/8	1.8386e-4	4.0277
			1/16	1.1431e-5	4.0075
			1/32	7.0705e-7	4.0150
	0.75	1/2048	1/4	3.0140e-3	
			1/8	1.8479e-4	4.0277
			1/16	1.1490e-5	4.0075
			1/32	7.1096e-7	4.0144
0.85	0.35	1/2048	1/4	2.9512e-3	
			1/8	1.8095e-4	4.0276
			1/16	1.1250e-5	4.0076
			1/32	6.9496e-7	4.0168
	0.55	1/2048	1/4	2.9666e-3	
			1/8	1.8189e-4	4.0276
			1/16	1.1309e-5	4.0075
			1/32	6.9928e-7	4.0155
	0.75	1/2048	1/4	2.9781e-3	
			1/8	1.8260e-4	4.0276
			1/16	1.1354e-5	4.0074
			1/32	7.0348e-7	4.0126

Then we plot the figure of space convergence orders in Figure 5.7 when fixed $\alpha = 0.35$, 0.85 and $k = 1/2048$. In addition, Figure 5.7 presents the fourth-order convergence in the spatial direction. Also, we plot the figure of temporal convergence orders in Figure 5.8 when fixed $\beta = 0.45$, 0.70 and $h = 1/512$. We can see intuitively the second-order convergence in the temporal direction.

Table 5.8: The L_2 errors and convergence orders in time with fixed β for Example 5.3.

β	α	h	k	$E_2(h,k)$	Rate_2^t
0.45	0.45	1/512	1/16	1.0771e-4	
			1/32	2.7717e-5	1.9583
			1/64	7.3274e-6	1.9194
			1/128	2.2097e-6	1.7294
	0.75	1/512	1/16	1.0232e-4	
			1/32	2.7945e-5	1.8724
			1/64	7.6099e-6	1.8766
			1/128	2.1538e-6	1.8210
	0.80	1/512	1/16	9.3799e-5	
			1/32	2.6287e-5	1.8352
			1/64	6.9726e-6	1.9146
			1/128	1.3899e-6	2.3267
0.70	0.55	1/512	1/16	1.0948e-4	
			1/32	2.7947e-5	1.9700
			1/64	7.2725e-6	1.9422
			1/128	1.8453e-6	1.9788
	0.65	1/512	1/16	1.0402e-4	
			1/32	2.7442e-5	1.9224
			1/64	6.9482e-6	1.9817
			1/128	1.9755e-6	1.8144
	0.75	1/512	1/16	8.9152e-5	
			1/32	2.4529e-5	1.8618
			1/64	6.0514e-6	2.0191
			1/128	1.4491e-6	2.0621

6. Conclusion

This paper proposes and analyzes a compact finite difference scheme for the fourth-order subdiffusion equation with the Riemann–Liouville fractional integral. By using the discrete energy method, the Cholesky decomposition and the reduced-order method, the stability and convergence are proved. Numerical results confirm the theoretical analysis. The compact difference scheme is stable and convergent with the convergence order $\mathcal{O}(k^2+h^4)$.

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