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Exponential Tikhonov Regularization Method for an Inverse Source Problem in a Sub-diffusion Equation

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Abstract. In this paper, it is considered an inverse space-dependent source problem of time-space fractional diffusion equation from the noisy final data in a bounded domain. Such a problem is mildly ill-posed. A new regularization method called the exponential Tikhonov method with a parameter γ is utilized to solve the problem, and its convergence rates are analyzed under an a-priori and an a-posteriori regularization parameter choice rule. A novel result indicates that the optimal convergence rate can be obtained and it is independent of the regularity information of the unknown source term when γ is less than or equal to zero. However, when γ is greater than zero, the optimal convergence rate depends on the value of γ related to the regularity of the unknown source but it does not have convergence saturation limit and can theoretically approach 1, which is superior to Tikhonov's regularization framework using the usual Sobolev space norm as a penalty term in a minimized functional. Numerical examples show that the proposed regularization method is effective and stable, and both parameter choice rules work well.

1. Introduction

Fractional-order diffusion equations have been widely studied and applied in many fields due to their superior properties in numerical calculations and practical applications, such as physics, finance, chemistry, biology and biochemistry, and so on [4, 5, 15, 16, 28].

However, in many practical applications, the source term, initial conditions, partial boundary conditions, diffusion coefficients or the order of fractional derivative may be unknown and need to be reconstructed based on the additional measurement data, which creates inverse problems for various fractional diffusion equations. Here, we consider an inverse problem of time-space fractional diffusion equation for reconstructing an unknown source.

Let T be a fixed constant and Ω be a bounded domain in $\mathbb{R}^d$ ($d \geq 1$) with sufficiently smooth boundary $\partial\Omega$, and $L^2(\Omega)$ be the square integrable function space with the

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scalar product $(\cdot,\cdot)$ defined by $(f,h) = \int_\Omega f(x)h(x)\,\mathrm{d}x$. Consider an initial-boundary value problem of time-space fractional diffusion equation with a homogeneous Dirichlet boundary condition as follows:

$$\begin{cases} \partial_{0+}^{\alpha} u(x,t) = -(-\Delta)^{\beta/2} u(x,t) + f(x)p(t), & (x,t) \in \Omega \times (0,T], \\ u(x,0) = 0, & x \in \overline{\Omega}, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \end{cases} \tag{1.1}$$

where u is an unknown function denoting the solute concentration and the model (1.1) is usually used to describe the anomalous diffusion of underground pollutants [16]. The fractional derivative $\partial_{0+}^{\alpha} u$ denotes the Caputo fractional left-sided derivative of order $\alpha \in (0,1)$ with respect to t defined by

$$\partial_{0+}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \quad 0 < t \le T,$$

where Γ is the Gamma function. The fractional Laplacian operator $(-\Delta)^{\beta/2}$ of order β $(1 < \beta \le 2)$ is defined by using the spectral decomposition of the Laplace operator. The definition is summarized in Definition 2.1. One can also see [24–26].

The inverse problem considered in this paper is to reconstruct the unknown source $f(x)$ from system (1.1) and the following terminal data

$$u(x,T) = g^{\delta}(x), \tag{1.2}$$

where $g^{\delta}(x) \in L^2(\Omega)$ is the measurement data of the exact value $g(x)$ with an error level

$$\|g^{\delta}(x) - g(x)\| \le \delta. \tag{1.3}$$

Here, δ is the noise level, and $\|\cdot\|$ is the L^2-norm induced by the scalar product.

The inverse source problem mentioned above is an ill-posed problem, one can see the detained statements in Section 2. Therefore some regularization techniques must be adopted to deal with this ill-posedness. For the time-space fractional diffusion equations (1.1), Huang et al. solved the Cauchy problem in [7] and Tatar et al. in [24–26] solved some inverse space-dependent source problems and identified the orders of time and space fractional derivatives. Li et al. [13] investigated an inverse time-dependent source problem and proved the unique existence of the solution to the forward problem as well as the uniqueness of the inverse problem. Also combining the boundary element method and the Tikhonov regularization reconstructed time-dependent source terms from additional point measurement data. Huang et al. studied the boundary controller design and stability analysis of the stabilization in [8]. Jia et al. used the total variational (TV) regularization methods to approximate the backward problem in [9]. Erdal et al. [11]

employed the quasi-reversible regularization method to solve the inverse source problem and provided corresponding convergence estimates. Tuan et al. [29] considered an inverse space-dependent source problem by using the Fourier truncation method and provided convergence estimates but no numerical example is given. Tomovski [27] proposed a numerical scheme to the generalized time-space fractional diffusion equation with composite fractional time derivatives for the first time.

For other inverse source problems in fractional diffusion equations, we refer to Sun, Yan and Liao [23], Chang, Sun and Wang [3], Sun and Chang [21], Ma, Prakash and Deiveegan [14], Xiong and Xue [32], Ruan, Zhang and Xiong [20] and the references therein. In [23], a simultaneously inverse problem for determining a space-dependent source and an order of the fractional derivative in time is considered. The uniqueness of the inverse problem is obtained by analytic continuation and Laplace transform, and the modified non-stationary iterative Tikhonov regularization method is used to obtain an approximation of the source function and fractional order. In [3], the authors proposed the modified quasi-boundary regularization method and the Landweber iterative regularization method for solving the inverse source problem, and provided corresponding convergence results and numerical examples. In [21], the Galerkin spectral method is used to a multi-term time-fractional diffusion equation, and the numerical stability and convergence of this method are studied. Also it is concluded that the spectral method itself can be used as a regularization method for some inverse problems.

The authors mainly focus on some regularization methods for solving inverse problems of time-space or time fractional diffusion equations and give convergence estimates under an a-priori and an a-posteriori choice rules of regularization parameters in the above literature. However, the optimal convergence rate always depends on the a-prior bound information of the unknown source term in most of the above papers. But the exact information about the regularity of the unknown source term is difficult to know in practical applications. In this article, we use a new regularization methods, which called the exponential Tikhonov regularization method with the exponential parameter γ, which was first proposed by Wang et al. in [31]. The main contributions of the paper can be organized as follows:

(1) The convergence estimates of the inverse problem are obtained under an a-priori and an a-posteriori choice rules of regularization parameters.

(2) The optimal rate of convergence can be obtained without knowing the exact regularity for the source function when $\gamma \leq 0$.

(3) A numerical scheme for implementing the proposed method is presented.

The paper is organized as follows. In Section 2, we propose ill-posedness and conditional stability of the inverse source problem and some preliminary results. In Section 3, we construct the exponential Tihkonov regularization method for solving the inverse source problem of time-space fractional diffusion equation, and provide convergence estimates under the a-prior and a-posterior regularization parameter choice rules. The numerical implementation for the direct problem is given in Section 4. Numerical results are investigated in Section 5. We end the paper with a brief concluding remark in Section 6.

2. Ill-posedness and conditional stability of the inverse problem

Throughout this article, we use the following definitions and lemmas.

Definition 2.1. [13] Suppose that $(\overline{\lambda_k}, \varphi_k)$ are the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω with the Dirichlet boundary condition on $\partial\Omega$:

$$\begin{cases} -\Delta\varphi_k = \overline{\lambda_k}\varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{in } \partial\Omega. \end{cases}$$

Let

$$\mathcal{H}_0^\beta(\Omega) := \left\{ u = \sum_{n=1}^{\infty} a_n\varphi_n : \|u\|^2_{\mathcal{H}_0^\beta(\Omega)} = \sum_{n=1}^{\infty} a_n^2 \overline{\lambda_n}^{\beta} < \infty \right\}.$$

Then if $u \in \mathcal{H}_0^\beta(\Omega)$, we define the operator $(-\Delta)^{\beta/2}$ by

$$(-\Delta)^{\beta/2} u = \sum_{n=1}^{\infty} a_n \overline{\lambda_n}^{\beta/2} \varphi_n,$$

which maps $\mathcal{H}_0^\beta(\Omega)$ onto $L^2(\Omega)$ with the following equivalence

$$\|u\|_{\mathcal{H}_0^\beta}(\Omega) = \|(-\Delta)^{\beta/2} u\|_{L^2(\Omega)}.$$

Note that if α tends to 1 and β tends to 2, the fractional derivative $\partial_{0+}^\alpha u$ tends to the first-order derivative u_t and the fractional Laplacian operator $(-\Delta)^{\beta/2}$ tends to the Laplacian operator $-\Delta$, and thus model (1.1) reproduces the standard diffusion equation.

Definition 2.2. [18] The Mittag–Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.3. [18] *The Laplace transform equality*

$$\int_0^\infty \exp(-pt) t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\pm\lambda t^\alpha)\, \mathrm{d}t = \frac{k! p^{\alpha-\beta}}{(p^\alpha \mp \lambda)^{k+1}}, \quad \Re(p) > |\lambda|^{1/\alpha}$$

holds, where $E_{\alpha,\beta}^{(k)}(y) = \frac{\mathrm{d}^k}{\mathrm{d}y^k} E_{\alpha,\beta}(y)$.

Lemma 2.4. [18] *Let* $\lambda > 0$, *then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0,\ 0 < \alpha < 1.$$

Lemma 2.5. [19] *For* $0 < \alpha < 1$, $\eta > 0$, *we have* $0 < E_{\alpha,1}(-\eta) < 1$ *and* $0 < E_{\alpha,\alpha}(-\eta) < \frac{1}{\Gamma(\alpha)}$. *Moreover,* $E_{\alpha,1}(-\eta)$ *is completely monotonic, that is*

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\eta^n}E_{\alpha,1}(-\eta) \geq 0, \quad \forall\, n \in \mathbb{N}\cup\{0\}.$$

Lemma 2.6. *If* $p(t) \in C[0,T]$ *satisfies* $p(t) \geq p_0 > 0$, $t \in [0,T]$, *we have*

$$\frac{C}{\lambda_n}p_0 \leq Q_n(T) \leq \frac{1}{\lambda_n}\|p\|_{C[0,T]}.$$

Proof. From Lemmas 2.4 and 2.5, it follows that

$$\begin{aligned} Q_n(T) &= \int_0^T p(\tau)(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)\,\mathrm{d}\tau \\ &\geq p_0 \int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)\,\mathrm{d}\tau \\ &= \frac{p_0}{\lambda_n}\big(1 - E_{\alpha,1}(-\lambda_n T^\alpha)\big) =: \frac{C}{\lambda_n}p_0, \end{aligned}$$

and

$$\begin{aligned} Q_n(T) &= \int_0^T p(\tau)(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)\,\mathrm{d}\tau \\ &\leq \|p\|_{C[0,T]} \int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)\,\mathrm{d}\tau \\ &= \frac{\|p\|_{C[0,T]}}{\lambda_n}\big(1 - E_{\alpha,1}(-\lambda_n T^\alpha)\big) \leq \frac{1}{\lambda_n}\|p\|_{C[0,T]}. \end{aligned}$$

□

Next, we discuss the ill-posedness and conditional stability of the inverse source problem (1.1) and (1.2).

Let the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition be $\overline{\lambda_n}$ and the corresponding eigenfunctions be $\varphi_n(x) \in H^2(\Omega)\cap H_0^1(\Omega)$. Then we have $-\Delta\varphi_n = \overline{\lambda_n}\varphi_n$ and $\varphi_n|_{\partial\Omega} = 0$. Counting according to the multiplicities, we can set: $0 < \overline{\lambda_1} \leq \overline{\lambda_2} \leq \cdots \leq \overline{\lambda_n} \leq \cdots$ and $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$. Then, for any given source $f(x) \in L^2(\Omega)$, by the Laplace transform (see Lemma 2.3) and the separation of variables we obtain the solution to system (1.1) as follows:

$$u(x,t) = \sum_{n=1}^\infty f_n Q_n(t)\varphi_n(x),$$

where $f_n = (f, \varphi_n)$, $\lambda_n = \overline{\lambda_n}^{\beta/2}$ and

$$Q_n(t) = \int_0^t p(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\,\mathrm{d}\tau. \tag{2.1}$$

Here we also give a well-posedness result for the direct problem (1.1), which is needed for subsequent analysis.

Lemma 2.7. [24] *Let $f(x) \in L^2(\Omega)$, $p(t) \in C^1[0,T]$. Then there exists a unique weak solution of the problem* (1.1) *such that $u \in L^2(0,T;\mathcal{H}_0^\beta(\Omega))$ and $\partial_{0+}^\alpha u \in L^2((0,T)\times\Omega)$. Moreover, there exists a constant C_0 such that the following inequality holds:*

$$\|u\|_{L^2(0,T;\mathcal{H}_0^\beta(\Omega))} + \|\partial_{0+}^\alpha u\|_{L^2((0,T)\times\Omega)} \le C_0 \|f\|_{L^2(\Omega)}.$$

Remark 2.8. In the above lemma, the original paper [24] only considered the well-posedness of the forward problem in the case of one-dimensional space. However, this result is also true for higher dimensional cases, because we do not need to estimate the eigenvalues and eigenfunctions of the elliptic operator in the proof process.

From the terminal condition $g(x) = u(x,T)$, we get

$$g(x) = \sum_{n=1}^\infty f_n Q_n(T) \varphi_n(x), \tag{2.2}$$

and

$$g_n = f_n Q_n(T), \tag{2.3}$$

where $g_n = (g, \varphi_n)$ and $Q_n(T) = \int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)\,\mathrm{d}\tau$. Since $0 < E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha) < 1$, we see that $f_n = 0$ if and only if $g_n = 0$ and $p \not\equiv 0$. This yields the uniqueness of the inverse source problem. From equation (2.3), we can easily get $f_n = \frac{g_n}{Q_n(T)}$. It follows that

$$f(x) = \sum_{n=1}^\infty \frac{g_n}{Q_n(T)} \varphi_n(x). \tag{2.4}$$

From Lemma 2.6, we know the denominator of (2.4) tends to zero when n is sufficiently large. Therefore, the above inverse problem is ill-posed and we need to add some regularization strategies. We also find that the inverse source problem for fractional diffusion equation is mildly ill-posed because the multiplier $1/Q_n(T)$ just grows linearly to λ_n, i.e., $1/Q_n(T) \sim \lambda_n$ as $n \to \infty$, which is very mild compared to the exponential growth $e^{\lambda_n T}$ for the case $\alpha = 1$.

If the exact solution $f(x)$ satisfies some a-priori bound conditions, the conditional stability of the inverse source problem can be obtained. To this end, we introduce an exponent operator of $(-\Delta)^{\beta/2}$ defined by

$$\exp(((-\Delta)^{\beta/2})^{\gamma}) = I + \frac{1}{1!}((-\Delta)^{\beta/2})^{\gamma} + \frac{1}{2!}((-\Delta)^{\beta/2})^{2\gamma} + \frac{1}{3!}((-\Delta)^{\beta/2})^{3\gamma} + \cdots + \frac{1}{n!}((-\Delta)^{\beta/2})^{n\gamma} + \cdots,$$

where I is a unit operator and $((-\Delta)^{\beta/2})^{k\gamma}\varphi_n = \lambda_n^{k\gamma}\varphi_n$, $k = 0, 1, 2, \ldots$. For $\gamma \in \mathbb{R}$ we define

$$D\left(\exp\left(\frac{((-\Delta)^{\beta/2})^{\gamma}}{2}\right)\right) = \left\{\psi \in L^2(\Omega) \,\Big|\, \sum_{n=1}^{\infty} \exp(\lambda_n^{\gamma})|(\psi, \varphi_n)|^2 < \infty\right\}, \tag{2.5}$$

and for $q > 0$ we define

$$D\big(((-\Delta)^{\beta/2})^{q}\big) = \left\{\psi \in L^2(\Omega) \,\Big|\, \sum_{n=1}^{\infty} \lambda_n^{2q}|(\psi, \varphi_n)|^2 < \infty\right\},$$

where $(\,\cdot\,,\cdot\,)$ is the inner product in $L^2(\Omega)$. Obviously, $D(\exp(((-\Delta)^{\beta/2})^{\gamma}/2))$ and $D(((-\Delta)^{\beta/2})^{q})$ are Hilbert spaces equipped with the following norms

$$\|\psi\|_{\gamma,\mathrm{Exp}} := \left(\sum_{n=1}^{\infty} \exp(\lambda_n^{\gamma})|(\psi, \varphi_n)|^2\right)^{1/2} \tag{2.6}$$

and

$$\|\psi\|_{D(((-\Delta)^{\beta/2})^{q})} := \left(\sum_{n=1}^{\infty} \lambda_n^{2q}|(\psi, \varphi_n)|^2\right)^{1/2},$$

respectively.

Theorem 2.9. *Assume that $p(t) \in C[0,T]$ satisfies $p(t) \geq p_0 > 0$, $t \in [0,T]$. Let $f(x) \in D\big(((-\Delta)^{\beta/2})^{q/2}\big)$ with $q > 0$, we have*

$$\|f\| \leq \frac{1}{\big(p_0(1 - E_{\alpha,1}(-\lambda_1 T^{\alpha}))\big)^{\frac{q}{q+2}}} M_1^{\frac{2}{q+2}} \|g\|^{\frac{q}{q+2}},$$

where M_1 is a positive constant such that $\|f\|_{D(((-\Delta)^{\beta/2})^{q/2})} \leq M_1$.

Moreover, let $f(x) \in D\big(\exp\big(\frac{((-\Delta)^{\beta/2})^{\gamma}}{2}\big)\big)$ with $\gamma \geq 0$, then

$$\|f\| \leq \frac{1}{\big(p_0(1 - E_{\alpha,1}(-\lambda_1 T^{\alpha}))\big)^{\frac{\gamma}{\gamma+2}}} M_2^{\frac{2}{\gamma+2}} \|g\|^{\frac{\gamma}{\gamma+2}},$$

where M_2 is a positive constant such that $\|f\|_{\gamma,\mathrm{Exp}} \leq M_2$.

Proof. Using Hölder's inequality, we can obtain

$$\begin{aligned}
\|f\|^2 &= \sum_{n=1}^{\infty} f_n^2 = \sum_{n=1}^{\infty} \left(\frac{g_n}{\int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau} \right)^2 \\
&= \sum_{n=1}^{\infty} \frac{g_n^{\frac{4}{q+2}}}{\left(\int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau \right)^2} g_n^{\frac{2q}{q+2}} \\
&\le \left(\sum_{n=1}^{\infty} \frac{g_n^2}{\left(\int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau \right)^{q+2}} \right)^{\frac{2}{q+2}} \left(\sum_{n=1}^{\infty} g_n^2 \right)^{\frac{q}{q+2}} \\
&= \left(\sum_{n=1}^{\infty} \frac{1}{\left(\int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau \right)^{q}} f_n^2 \right)^{\frac{2}{q+2}} \left(\sum_{n=1}^{\infty} g_n^2 \right)^{\frac{q}{q+2}} \\
&\le \frac{1}{\left(p_0 (1 - E_{\alpha,1}(-\lambda_1 T^\alpha)) \right)^{\frac{2q}{q+2}}} \left(\sum_{n=1}^{\infty} \lambda_n^q f_n^2 \right)^{\frac{2}{q+2}} \left(\sum_{n=1}^{\infty} g_n^2 \right)^{\frac{q}{q+2}} \\
&\le \frac{1}{\left(p_0 (1 - E_{\alpha,1}(-\lambda_1 T^\alpha)) \right)^{\frac{2q}{q+2}}} M_1^{\frac{4}{q+2}} \|g\|^{\frac{2q}{q+2}},
\end{aligned}$$

and

$$\begin{aligned}
&\|f\|^2 \\
&= \sum_{n=1}^{\infty} f_n^2 = \sum_{n=1}^{\infty} \left(\frac{g_n}{\int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau} \right)^2 \\
&= \sum_{n=1}^{\infty} \frac{g_n^{\frac{4}{\gamma+2}}}{\left(\int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau \right)^2} g_n^{\frac{2\gamma}{\gamma+2}} \\
&\le \left(\sum_{n=1}^{\infty} \frac{g_n^2}{\left(\int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau \right)^{\gamma+2}} \right)^{\frac{2}{\gamma+2}} \left(\sum_{n=1}^{\infty} g_n^2 \right)^{\frac{\gamma}{\gamma+2}} \\
&= \left(\sum_{n=1}^{\infty} \frac{\lambda_n^\gamma}{\lambda_n^\gamma \left(\int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau \right)^{\gamma}} f_n^2 \right)^{\frac{2}{\gamma+2}} \left(\sum_{n=1}^{\infty} g_n^2 \right)^{\frac{\gamma}{\gamma+2}} \\
&\le \left(\sum_{n=1}^{\infty} \frac{1}{\left(\int_0^T p(\tau)\lambda_n (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\,\mathrm{d}\tau \right)^{\gamma}} \exp(\lambda_n^\gamma) f_n^2 \right)^{\frac{2}{\gamma+2}} \left(\sum_{n=1}^{\infty} g_n^2 \right)^{\frac{\gamma}{\gamma+2}} \\
&\le \frac{1}{\left(p_0 (1 - E_{\alpha,1}(-\lambda_1 T^\alpha)) \right)^{\frac{2\gamma}{\gamma+2}}} \left(\sum_{n=1}^{\infty} \exp(\lambda_n^\gamma) f_n^2 \right)^{\frac{2}{\gamma+2}} \left(\sum_{n=1}^{\infty} g_n^2 \right)^{\frac{\gamma}{\gamma+2}} \\
&\le \frac{1}{\left(p_0 (1 - E_{\alpha,1}(-\lambda_1 T^\alpha)) \right)^{\frac{2\gamma}{\gamma+2}}} M_2^{\frac{4}{\gamma+2}} \|g\|^{\frac{2\gamma}{\gamma+2}}.
\end{aligned}$$

The proof is completed. □

3. Exponential Tikhonov regularization and convergence analysis

In this section, we propose an exponential Tikhonov regularization method to solve this ill-posed problem for obtaining stable approximations of $f(x)$, and present the corresponding convergence estimates under an a-priori and a-posterior regularization parameter choice rules.

Define a linear forward operator $K\colon L^2(\Omega) \to L^2(\Omega)$ that acts on $f(x)$ as

$$(Kf)(x) := \sum_{n=1}^{\infty} f_n \int_0^T p(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^\alpha)\, \mathrm{d}\tau\, \varphi_n(x).$$

Thus, (2.2) is rewritten as

$$Kf = g. \tag{3.1}$$

Obviously, $K\colon L^2(\Omega) \to L^2(\Omega)$ is a linear self-conjugate compact operator (see Lemma 2.7), and operator equation (3.1) is ill-posed. In order to stably reconstruct the source $f(x)$ from the noisy data $g^\delta(x)$ of $g(x)$, we minimize the following Tikhonov regularization functional with an exponential penalty:

$$J(f) = \|Kf - g^\delta\|^2 + \mu \|f\|^2_{\gamma,\mathrm{Exp}}, \tag{3.2}$$

where $\mu \in \mathbb{R}^+$ is a regularization parameter and the norm $\|\cdot\|_{\gamma,\mathrm{Exp}}$ is defined by (2.6). Obviously, the exponential Tikhonov functional (3.2) has a unique minimizer in $L^2(\Omega)$ according to standard linear inverse problem regularization theory [12].

Let $f^\delta_{\mu,\gamma}$ be the unique minimizer of exponential Tikhonov functional (3.2). By (2.5) and (2.6), we obtain

$$\|f\|^2_{\gamma,\mathrm{Exp}} = \left\| \exp\left(\frac{((-\Delta)^{\beta/2})^\gamma}{2} \right) f \right\|^2 = \left(\exp\left(\frac{((-\Delta)^{\beta/2})^\gamma}{2} \right) f, \exp\left(\frac{((-\Delta)^{\beta/2})^\gamma}{2} \right) f \right).$$

By the variational theory, then $f^\delta_{\mu,\gamma}$ satisfies the normal equation

$$K^*Kf^\delta_{\mu,\gamma} + \mu \exp(((-\Delta)^{\beta/2})^\gamma) f^\delta_{\mu,\gamma} = K^* g^\delta.$$

Through singular value decomposition [1] for the compact self-adjoint operator, we obtain

$$f^\delta_{\mu,\gamma}(x) = \sum_{n=1}^{\infty} \frac{Q_n(T)}{Q_n^2(T) + \mu \exp(\lambda_n^\gamma)}\, g_n^\delta \varphi_n(x), \tag{3.3}$$

where $Q_n(T)$ is defined by (2.1), $g_n^\delta = (g^\delta, \varphi_n)$. The function $f^\delta_{\mu,\gamma}(x)$ expressed in (3.3) is called a regularization solution of the inverse source problem with respect to μ. Similarly, the regularization solution for the exact value $g(x)$ is denoted by

$$f_{\mu,\gamma}(x) = \sum_{n=1}^{\infty} \frac{Q_n(T)}{Q_n^2(T) + \mu \exp(\lambda_n^\gamma)}\, g_n \varphi_n(x).$$

Remark 3.1. In the case of $\gamma = -\infty$ in the sense of limitation, the exponential Tikhonov regularization method degenerates to the standard Tikhonov regularization, and the corresponding regularization solution is

$$f_\mu(x) = \sum_{n=1}^{\infty} \frac{Q_n(T)}{Q_n^2(T) + \mu} g_n \varphi_n(x).$$

Theorem 3.2. *Let $p(t) \in C[0,T]$ satisfy $p(t) \geq p_0 > 0$, $t \in [0,T]$. Let $f_{\mu,\gamma}^\delta(x)$ be the regularization solution with respect to the noise data $g^\delta(x)$ and the noise assumption $\|g^\delta(x) - g(x)\| \leq \delta$ be held.*

(i) *In the case of $\gamma \leq 0$: Suppose that there exists a positive constant M_1 such that $\|f\|_{D(((-\Delta)^{\beta/2})^q)} \leq M_1$, the choice $\mu = \left(\frac{\delta}{M_1}\right)^{\frac{2}{q+1}}$ for $q \in (0,2]$ or $\mu = \left(\frac{\delta}{M_1}\right)^{\frac{2}{3}}$ for $q > 2$ yields the convergence estimate*

$$\|f_{\mu,\gamma}^\delta(x) - f(x)\| \leq \begin{cases} (C_1 + C_2) M_1^{\frac{1}{3}} \delta^{\frac{2}{3}}, & q \geq 2, \\ (C_1 + C_2) M_1^{\frac{1}{q+1}} \delta^{\frac{q}{q+1}}, & 0 < q < 2, \end{cases}$$

where C_1, C_2 are two positive constants depending on λ_1, $\underline{C}$, p_0 and q.

(ii) *In the case of $\gamma > 0$: Suppose that there exists a positive constant M_2 such that $\|f\|_{\gamma,\mathrm{Exp}} \leq M_2$, the choice $\mu = \left(\frac{\delta}{M_2}\right)^{\frac{8+4\gamma}{6+\gamma}}$ yields the convergence estimate*

$$\|f_{\mu,\gamma}^\delta(x) - f(x)\| \leq (C_3 + C_4) M_2^{\frac{4}{6+\gamma}} \delta^{\frac{2+\gamma}{6+\gamma}},$$

where C_3, C_4 are two positive constants depending on λ_1, $\underline{C}$, p_0 and γ.

Proof. From Lemma 2.6, it follows that

$$\frac{\underline{C}}{\lambda_n} p_0 \leq Q_n(T) \leq \frac{1}{\lambda_n} \|p\|_{C[0,T]}.$$

By the triangular inequality, one has

$$\|f_{\mu,\gamma}^\delta(x) - f(x)\| \leq \|f_{\mu,\gamma}^\delta(x) - f_{\mu,\gamma}(x)\| + \|f_{\mu,\gamma}(x) - f(x)\|. \tag{3.4}$$

(i) For $\gamma \leq 0$, the first term on the right-hand side of inequality (3.4) is

$$\begin{aligned} \|f_{\mu,\gamma}^\delta(x) - f_{\mu,\gamma}(x)\|^2 &= \left\| \sum_{n=1}^{\infty} \frac{Q_n(T)}{Q_n^2(T) + \mu \exp(\lambda_n^\gamma)} (g_n^\delta - g_n) \varphi_n(x) \right\|^2 \\ &\leq \sum_{n=1}^{\infty} \left| \frac{\frac{1}{\lambda_n} \|p\|_{C[0,T]}}{\left(\frac{\underline{C}}{\lambda_n} p_0\right)^2 + \mu \exp(0)} \right|^2 |g_n^\delta - g_n|^2 \\ &\leq \sup_n \left| \frac{\lambda_n \|p\|_{C[0,T]}}{\underline{C}^2 p_0^2 + \lambda_n^2 \mu} \right|^2 \sum_{n=1}^{\infty} |g_n^\delta - g_n|^2 \leq \left(\frac{\|p\|_{C[0,T]}}{2\underline{C} p_0} \right)^2 \frac{\delta^2}{\mu}. \end{aligned}$$

It follows that

$$\|f^\delta_{\mu,\gamma}(x) - f_{\mu,\gamma}(x)\| \le C_1 \frac{\delta}{\sqrt{\mu}}, \tag{3.5}$$

where $C_1 = \frac{\|p\|_{C[0,T]}}{2\underline{C}p_0}$. The second term on the right-hand side of inequality (3.4) is

$$\begin{aligned}
&\|f_{\mu,\gamma}(x) - f(x)\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \left(\frac{Q_n(T)}{Q_n^2(T) + \mu \exp(\lambda_n^\gamma)} - \frac{1}{Q_n(T)} \right) g_n \varphi_n(x) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \frac{-\mu \exp(\lambda_n^\gamma)}{Q_n^2(T) + \mu \exp(\lambda_n^\gamma)} f_n \varphi_n(x) \right\|^2 \le \sum_{n=1}^{\infty} \left(\frac{\mu \exp(\lambda_n^\gamma)}{\left(\frac{\underline{C}}{\lambda_n} p_0\right)^2 + \mu \exp(\lambda_n^\gamma)} \right)^2 |f_n|^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{\mu \lambda_n^2 \exp(\lambda_n^\gamma)}{\underline{C}^2 p_0^2 + \mu \lambda_n^2 \exp(\lambda_n^\gamma)} \right)^2 |f_n|^2 \le \sum_{n=1}^{\infty} \left(\frac{\mu \lambda_n^{2-q} \exp(1)}{\underline{C}^2 p_0^2 + \mu \lambda_n^2} \right)^2 \lambda_n^{2q} |f_n|^2 \\
&\le \sup_n \left(\frac{\mu \lambda_n^{2-q} \exp(1)}{\underline{C}^2 p_0^2 + \mu \lambda_n^2} \right)^2 \sum_{n=1}^{\infty} \lambda_n^{2q} |f_n|^2.
\end{aligned} \tag{3.6}$$

Obviously, for the first case of $q \ge 2$ we can estimate

$$\sup_n \frac{\mu \lambda_n^{2-q} \exp(1)}{\underline{C}^2 p_0^2 + \mu \lambda_n^2} \le \frac{\mu \lambda_n^{2-q} \exp(1)}{\underline{C}^2 p_0^2} = \frac{\exp(1)}{\lambda_n^{q-2} \underline{C}^2 p_0^2} \mu \le \frac{\exp(1)}{\lambda_1^{q-2} \underline{C}^2 p_0^2} \mu;$$

and for the second case $q < 2$, we have

$$\sup_n \frac{\mu \lambda_n^{2-q} \exp(1)}{\underline{C}^2 p_0^2 + \mu \lambda_n^2} \le \sup_{t>0} \frac{t^{2-q} \exp(1)}{\underline{C}^2 p_0^2 + \mu t^2} \mu \le \underline{C}^{-q} p_0^{-q} \exp(1) \frac{2-q}{2} \left(\frac{2-q}{q} \right)^{-\frac{q}{2}} \mu^{\frac{q}{2}}.$$

Combining the two above cases with the aid of (3.6), for $0 < \mu < 1$ one has

$$\|f_{\mu,\gamma}(x) - f(x)\| \le \begin{cases} C_2 M_1 \mu, & q \ge 2, \\ C_2 M_1 \mu^{\frac{q}{2}}, & 0 < q < 2, \end{cases}$$

where

$$C_2 = \begin{cases} \frac{\exp(1)}{\lambda_1^{q-2} \underline{C}^2 p_0^2}, & q \ge 2, \\ \underline{C}^{-q} p_0^{-q} \exp(1) \frac{2-q}{2} \left(\frac{2-q}{q}\right)^{-\frac{q}{2}}, & 0 < q < 2. \end{cases}$$

Then, we have

$$\begin{aligned}
\|f^\delta_{\mu,\gamma}(x) - f(x)\| &\le \|f^\delta_{\mu,\gamma}(x) - f_{\mu,\gamma}(x)\| + \|f_{\mu,\gamma}(x) - f(x)\| \\
&\le \begin{cases} C_1 \frac{\delta}{\sqrt{\mu}} + C_2 M_1 \mu, & q \ge 2, \\ C_1 \frac{\delta}{\sqrt{\mu}} + C_2 M_1 \mu^{\frac{q}{2}}, & 0 < q < 2, \end{cases}
\end{aligned}$$

which implies that

$$\|f_{\mu,\gamma}^{\delta}(x)-f(x)\| \le \begin{cases}(C_1+C_2)M_1^{\frac{1}{3}}\delta^{\frac{2}{3}}, & q\ge 2,\\ (C_1+C_2)M_1^{\frac{1}{q+1}}\delta^{\frac{q}{q+1}}, & 0<q<2,\end{cases}$$

under the choice of $\mu=\big(\frac{\delta}{M_1}\big)^{\frac{2}{3}}$ for $q\ge 2$ or $\mu=\big(\frac{\delta}{M_1}\big)^{\frac{2}{q+1}}$ for $0<q<2$.

(ii) For $\gamma>0$, the first term on the right-hand side of inequality (3.4) become

$$\begin{aligned}&\|f_{\mu,\gamma}^{\delta}(x)-f_{\mu,\gamma}(x)\|^2\\ &=\left\|\sum_{n=1}^{\infty}\frac{Q_n(T)}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}(g_n^{\delta}-g_n)\varphi_n(x)\right\|^2\le\sum_{n=1}^{\infty}\left|\frac{\frac{1}{\lambda_n}\|p\|_{C[0,T]}}{\big(\frac{\underline{C}}{\lambda_n}p_0\big)^2+\mu\exp(\lambda_n^{\gamma})}\right|^2|g_n^{\delta}-g_n|^2\\ &\le\sup_n\left|\frac{\lambda_n\|p\|_{C[0,T]}}{\underline{C}^2p_0^2+\mu\lambda_n^{\gamma+2}}\right|^2\sum_{n=1}^{\infty}|g_n^{\delta}-g_n|^2\le\left|\frac{(1+\gamma)^{\frac{1+\gamma}{2+\gamma}}\|p\|_{C[0,T]}}{(2+\gamma)\underline{C}^{\frac{2(1+\gamma)}{2+\gamma}}p_0^{\frac{2(1+\gamma)}{2+\gamma}}}\mu^{-\frac{1}{2+\gamma}}\right|^2\sum_{n=1}^{\infty}|g_n^{\delta}-g_n|^2.\end{aligned}$$

Thus there exist $C_3=\frac{(1+\gamma)^{\frac{1+\gamma}{2+\gamma}}\|p\|_{C[0,T]}}{(2+\gamma)\underline{C}^{\frac{2(1+\gamma)}{2+\gamma}}p_0^{\frac{2(1+\gamma)}{2+\gamma}}}$ such that

$$\|f_{\mu,\gamma}^{\delta}(x)-f_{\mu,\gamma}(x)\|\le C_3\frac{\delta}{\mu^{\frac{1}{2+\gamma}}}.$$

The second term on the right-hand side of inequality (3.4) is

$$\begin{aligned}&\|f_{\mu,\gamma}(x)-f(x)\|^2\\ &=\left\|\sum_{n=1}^{\infty}\left(\frac{Q_n(T)}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}-\frac{1}{Q_n(T)}\right)g_n\varphi_n(x)\right\|^2=\left\|\sum_{n=1}^{\infty}\frac{-\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}f_n\varphi_n(x)\right\|^2\\ &=\sum_{n=1}^{\infty}\frac{\mu^2\exp(\lambda_n^{\gamma})}{(Q_n^2(T)+\mu\exp(\lambda_n^{\gamma}))^2}\exp(\lambda_n^{\gamma})|f_n|^2\le\left(\sup_n\frac{\mu\exp\big(\frac{\lambda_n^{\gamma}}{2}\big)}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right)^2\sum_{n=1}^{\infty}\exp(\lambda_n^{\gamma})|f_n|^2.\end{aligned}$$

Obviously, there exists an integer $N_0>0$ such that $\lambda_n^2\le\exp(\lambda_n^{\gamma})$ for $n\ge N_0$. Thus, for the case of $n\ge N_0$ we can derive that

$$\begin{aligned}\frac{\mu\exp\big(\frac{\lambda_n^{\gamma}}{2}\big)}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}&\le\frac{\mu\exp\big(\frac{\lambda_n^{\gamma}}{2}\big)}{\big(\frac{\underline{C}}{\lambda_n}p_0\big)^2+\mu\exp(\lambda_n^{\gamma})}\le\frac{\mu\exp\big(\frac{\lambda_n^{\gamma}}{2}+\lambda_n^{\gamma}\big)}{\underline{C}^2p_0^2+\mu\exp(2\lambda_n^{\gamma})}\\ &\le\sup_{t>0}\frac{\mu t^3}{\underline{C}^2p_0^2+\mu t^4}\le\frac{3^{\frac{3}{4}}}{4\sqrt{\underline{C}p_0}}\mu^{\frac{1}{4}}.\end{aligned}$$

For the case of $n\le N_0$ we have

$$\frac{\mu\exp\big(\frac{\lambda_n^{\gamma}}{2}\big)}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\le\frac{\mu\exp\big(\frac{\lambda_{N_0}^{\gamma}}{2}\big)}{\big(\frac{\underline{C}}{\lambda_{N_0}}p_0\big)^2+\mu\exp(\lambda_n^{\gamma})}\le\frac{\mu\exp\big(\frac{\lambda_{N_0}^{\gamma}}{2}\big)}{\frac{\underline{C}^2}{\lambda_{N_0}^2}p_0^2}\le\frac{\lambda_{N_0}^2\exp\big(\frac{\lambda_{N_0}^{\gamma}}{2}\big)}{\underline{C}^2p_0^2}\mu.$$

Combining the above two cases yields that

$$\|f_{\mu,\gamma}(x) - f(x)\| \le C_4 M_2 \mu^{\frac{1}{4}}$$

for $0 < \mu < 1$, where $C_4 = \max\left\{\frac{3^{\frac{3}{4}}}{4\sqrt{\underline{C}p_0}}, \frac{\lambda_{N_0}^2 \exp\left(\frac{\lambda_{N_0}^{\gamma}}{2}\right)}{\underline{C}^2 p_0^2}\right\}$. Therefore, we have

$$\|f_{\mu,\gamma}^{\delta}(x) - f(x)\| \le \|f_{\mu,\gamma}^{\delta}(x) - f_{\mu,\gamma}(x)\| + \|f_{\mu,\gamma}(x) - f(x)\| = C_3 \frac{\delta}{\mu^{\frac{1}{2+\gamma}}} + C_4 M_2 \mu^{\frac{1}{4}}.$$

Then choosing the regularization parameter $\mu = \left(\frac{\delta}{M_2}\right)^{\frac{8+4\gamma}{6+\gamma}}$ yields

$$\|f_{\mu,\gamma}^{\delta}(x) - f(x)\| \le (C_3 + C_4) M_2^{\frac{4}{6+\gamma}} \delta^{\frac{2+\gamma}{6+\gamma}}.$$

The proof is completed. □

Theorem 3.2 gives an a-priori strategy for choosing regularization parameters, in which the regularity of the exact solution $f(x)$ should be known exactly before choosing. However, the regularity of $f(x)$ is not known in many practical applications. Therefore, it is necessary to study posterior strategies for selecting regularization parameters. Here, we adopt the Morozov discrepancy principle [30] to choose regularization parameters, and give the convergence estimates of the corresponding regularization solutions. According to the Morozov principle, the regularization parameter μ is chosen to be the solution to the discrepancy equation

$$\|K f_{\mu,\gamma}^{\delta} - g^{\delta}\| = \rho\delta, \tag{3.7}$$

where $\rho > 1$ is a given constant. The solvability of the discrepancy equation (3.7) is guaranteed by the following lemma for $0 < \rho\delta < \|g^{\delta}\|$.

Lemma 3.3. *Let $g^{\delta} \in L^2(\Omega)$ and $\theta(\mu) = \|K f_{\mu,\gamma}^{\delta} - g^{\delta}\|^2$. Then the following results hold:*

(1) *$\theta(\mu)$ is a continuous function;*

(2) *$\lim_{\mu\to 0}\theta(\mu) = 0$, $\lim_{\mu\to\infty}\theta(\mu) = \|g^{\delta}\|^2$;*

(3) *$\theta(\mu)$ is a strictly increasing function for $\mu \in (0,\infty)$.*

Proof. (1) By simple calculations, we have

$$\begin{aligned}\theta(\mu) = \|K f_{\mu,\gamma}^{\delta} - g^{\delta}\|^2 &= \left\|\sum_{n=1}^{\infty}\left(\frac{Q_n^2(T)}{Q_n^2(T) + \mu\exp(\lambda_n^{\gamma})} - 1\right) g_n^{\delta}\varphi_n(x)\right\|^2 \\ &= \sum_{n=1}^{\infty}\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T) + \mu\exp(\lambda_n^{\gamma})}\right|^2 |g_n^{\delta}|^2 \le \sum_{n=1}^{\infty}|g_n^{\delta}|^2 = \|g^{\delta}\|^2 < +\infty,\end{aligned}$$

which implies that $\theta(\mu)$ is continuous on $[0,+\infty)$.

(2) Owing to continuity of the $\theta(\mu)$ on $[0,+\infty)$, we have

$$\lim_{\mu\to 0}\theta(\mu)=\lim_{\mu\to 0}\sum_{n=1}^{\infty}\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right|^2|g_n^{\delta}|^2=\sum_{n=1}^{\infty}\lim_{\mu\to 0}\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right|^2|g_n^{\delta}|^2=0,$$

and

$$\begin{aligned}\left|\theta(\mu)-\|g^{\delta}\|^2\right| &= \left|\sum_{n=1}^{\infty}\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right|^2|g_n^{\delta}|^2-\sum_{n=1}^{\infty}|g_n^{\delta}|^2\right|\\ &= \left|\sum_{n=1}^{\infty}\left(\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right|^2-1\right)|g_n^{\delta}|^2\right|\\ &\le \sum_{n=1}^{\infty}\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}+1\right|\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}-1\right||g_n^{\delta}|^2\\ &\le 2\sum_{n=1}^{\infty}\left|\frac{Q_n^2(T)}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right||g_n^{\delta}|^2\\ &\le 2\sum_{n=1}^{\infty}|g_n^{\delta}|^2=2\|g^{\delta}\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\lim_{\mu\to\infty}\left|\theta(\mu)-\|g^{\delta}\|^2\right| &= \lim_{\mu\to\infty}\left|\sum_{n=1}^{\infty}\left(\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right|^2-1\right)|g_n^{\delta}|^2\right|\\ &= \sum_{n=1}^{\infty}\left|\lim_{\mu\to\infty}\left(\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right|^2-1\right)|g_n^{\delta}|^2\right|\\ &= 0,\end{aligned}$$

thus we have

$$\lim_{\mu\to\infty}\theta(\mu)=\|g^{\delta}\|^2.$$

(3) Because $\theta(\mu)$ uniformly converges with respect to μ, it can be differentiated term by term. Thus, for any sufficiently small $\varepsilon_0>0$, by simple computation we know

$$\begin{aligned}\theta'(\mu) &= 2\sum_{n=1}^{\infty}\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right|\frac{\exp(\lambda_n^{\gamma})(Q_n^2(T)+\mu\exp(\lambda_n^{\gamma}))-\mu\exp(\lambda_n^{\gamma})\exp(\lambda_n^{\gamma})}{(Q_n^2(T)+\mu\exp(\lambda_n^{\gamma}))^2}|g_n^{\delta}|^2\\ &= 2\sum_{n=1}^{\infty}\left|\frac{\mu\exp(\lambda_n^{\gamma})}{Q_n^2(T)+\mu\exp(\lambda_n^{\gamma})}\right|\frac{\exp(\lambda_n^{\gamma})Q_n^2(T)}{(Q_n^2(T)+\mu\exp(\lambda_n^{\gamma}))^2}|g_n^{\delta}|^2\\ &\le \frac{1}{2}\sum_{n=1}^{\infty}\frac{\exp(\lambda_n^{\gamma})Q_n^2(T)}{Q_n^2(T)\mu\exp(\lambda_n^{\gamma})}|g_n^{\delta}|^2\\ &\le \frac{1}{2\mu}|g_n^{\delta}|^2<+\infty\end{aligned}$$

for all $\mu \in [\varepsilon_0, +\infty)$.

Obviously, we find that $\theta'(\mu) > 0$ is true by differentiating term by term. Therefore, the function $\theta(\mu)$ is strictly increasing in $(0, \infty)$. This proof is completed. □

Theorem 3.4. *Let $p(t) \in C[0,T]$ satisfy $p(t) \geq p_0 > 0$ for all $t \in [0,T]$. Assume that the noise data $g^\delta(x)$ satisfies* (1.3)*, and $0 < \rho\delta < \|g^\delta\|$ for $\rho > 1$. Let $f^\delta_{\mu,\gamma}(x)$ be the exponential regularization solution in which the regularization parameter μ is selected by the Morozov's discrepancy principle* (3.7)*.*

(i) *In the case of $\gamma \leq 0$: Suppose that there exists $M_1 > 0$ such that $\|f\|_{D(((-\Delta)^{\beta/2})^q)} \leq M_1$, then the following convergence estimate holds:*

$$\|f^\delta_{\mu,\gamma}(x) - f(x)\| \leq \begin{cases} \left(C_1 \left(\frac{C_6 M_1}{\rho-1}\right)^{\frac{1}{1+1}} + \frac{M_1^{\frac{1}{q+1}}(1+\rho)^{\frac{q}{q+1}}}{\left(p_0(1-E_{\alpha,1}(-\lambda_1 T^\alpha))\right)^{\frac{q}{q+1}}} \right) \delta^{1/2}, & q \geq 1, \\ \left(C_1 \left(\frac{C_6 M_1}{\rho-1}\right)^{\frac{1}{q+1}} + \frac{M_1^{\frac{1}{q+1}}(1+\rho)^{\frac{q}{q+1}}}{\left(p_0(1-E_{\alpha,1}(-\lambda_1 T^\alpha))\right)^{\frac{q}{q+1}}} \right) \delta^{\frac{q}{q+1}}, & q \in (0,1). \end{cases}$$

(ii) *In the case of $\gamma > 0$: Suppose that there exists $M_2 > 0$ such that $\|f\|_{\gamma,\mathrm{Exp}} \leq M_2$, then the following convergence estimate holds:*

$$\|f^\delta_{\mu,\gamma}(x) - f(x)\| \leq \frac{1}{\left(p_0(1 - E_{\alpha,1}(-\lambda_1 T^\alpha))\right)^{\frac{\gamma}{\gamma+2}}} (2M_2)^{\frac{2}{\gamma+2}} (1+\rho)^{\frac{\gamma}{\gamma+2}} \delta^{\frac{\gamma}{\gamma+2}}.$$

Proof. (i) The proof for $\gamma \leq 0$. The discrepancy principle (3.7) for choosing the regularization parameter μ yields that

$$\begin{aligned} \rho\delta = \|Kf^\delta_{\mu,\gamma} - g^\delta\| &= \left\| \sum_{n=1}^\infty \left(\frac{Q_n^2(T)}{Q_n^2(T) + \mu\exp(\lambda_n^\gamma)} - 1 \right) g_n^\delta \varphi_n(x) \right\| \\ &= \left\| \sum_{n=1}^\infty \frac{\mu\exp(\lambda_n^\gamma)}{Q_n^2(T) + \mu\exp(\lambda_n^\gamma)} g_n^\delta \varphi_n(x) \right\| \\ &\leq \left\| \sum_{n=1}^\infty \frac{\mu\exp(\lambda_n^\gamma)}{Q_n^2(T) + \mu\exp(\lambda_n^\gamma)} (g_n^\delta - g_n)\varphi_n(x) \right\| + \left\| \sum_{n=1}^\infty \frac{\mu\exp(\lambda_n^\gamma)}{Q_n^2(T) + \mu\exp(\lambda_n^\gamma)} g_n \varphi_n(x) \right\| \\ &\leq \delta + \left\| \sum_{n=1}^\infty \frac{\mu\exp(\lambda_n^\gamma)}{Q_n^2(T) + \mu\exp(\lambda_n^\gamma)} g_n \varphi_n(x) \right\|. \end{aligned}$$

By the result of $\frac{C}{\lambda_n} p_0 \leq Q_n(T) \leq \frac{1}{\lambda_n}\|p\|_{C[0,T]}$, it follows that

$$\begin{aligned} &(\rho-1)^2\delta^2 \\ &\leq \left\| \sum_{n=1}^\infty \frac{\mu\exp(\lambda_n^\gamma)}{Q_n^2(T) + \mu\exp(\lambda_n^\gamma)} g_n \varphi_n(x) \right\|^2 = \sum_{n=1}^\infty \left(\frac{\mu\exp(\lambda_n^\gamma) Q_n(T) \lambda_n^{-q}}{Q_n^2(T) + \mu\exp(\lambda_n^\gamma)} \right)^2 \lambda_n^{2q} |f_n|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty}\left(\frac{\mu\exp(\lambda_n^\gamma)\|p\|_{C[0,T]}\lambda_n^{-1-q}}{\left(\frac{\underline{C}}{\lambda_n}p_0\right)^2+\mu\exp(\lambda_n^\gamma)}\right)^2\lambda_n^{2q}|f_n|^2 \le \sum_{n=1}^{\infty}\left(\frac{\mu\exp(1)\|p\|_{C[0,T]}\lambda_n^{1-q}}{\underline{C}^2p_0^2+\mu\lambda_n^2}\right)^2\lambda_n^{2q}|f_n|^2\\
&\le \left(\sup_n \frac{\mu\exp(1)\|p\|_{C[0,T]}\lambda_n^{1-q}}{\underline{C}^2p_0^2+\mu\lambda_n^2}\right)^2\sum_{n=1}^{\infty}\lambda_n^{2q}|f_n|^2.
\end{aligned}
$$

It is easy to derive that

$$
\sup_n \frac{\mu\exp(1)\|p\|_{C[0,T]}\lambda_n^{1-q}}{\underline{C}^2p_0^2+\mu\lambda_n^2} \le \frac{\exp(1)\|p\|_{C[0,T]}\lambda_1^{1-q}}{\underline{C}^2p_0^2}\mu
$$

for $q \ge 1$, and

$$
\begin{aligned}
\sup_n \frac{\mu\exp(1)\|p\|_{C[0,T]}\lambda_n^{1-q}}{\underline{C}^2p_0^2+\mu\lambda_n^2} &\le \frac{\exp(1)\|p\|_{C[0,T]}(1-q)^{\frac{1-q}{2}}}{2(1+q)^{-\frac{1+q}{2}}\underline{C}^{1+q}p_0^{1+q}}\mu^{\frac{1+q}{2}}\\
&\le \frac{\exp(1)\|p\|_{C[0,T]}(1-q)^{\frac{1-q}{2}}}{(2-q)\underline{C}^{1+q}p_0^{1+q}}\mu^{\frac{1+q}{2}}
\end{aligned}
$$

for $0 < q < 1$. Thus, by direct computations we have

$$
\frac{1}{\sqrt{\mu}} \le \begin{cases} \left(\frac{C_6M_1}{\rho-1}\right)^{\frac{1}{1+1}}\delta^{-\frac{1}{2}}, & q \ge 1,\\ \left(\frac{C_6M_1}{\rho-1}\right)^{\frac{1}{q+1}}\delta^{-\frac{1}{q+1}}, & 0<q<1,\end{cases}
$$

where $C_6 = \max\left\{\frac{\exp(1)\|p\|_{C[0,T]}\lambda_1^{1-q}}{\underline{C}^2p_0^2}, \frac{\exp(1)\|p\|_{C[0,T]}(1-q)^{\frac{1-q}{2}}}{(2-q)\underline{C}^{1+q}p_0^{1+q}}\right\}$. By the use of inequality (3.5), we obtain

$$
\|f_{\mu,\gamma}^\delta(x) - f_{\mu,\gamma}(x)\| \le \begin{cases} C_1\left(\frac{C_6M_1}{\rho-1}\right)^{\frac{1}{1+1}}\delta^{1/2}, & q\ge 1,\\ C_1\left(\frac{C_6M_1}{\rho-1}\right)^{\frac{1}{q+1}}\delta^{\frac{q}{q+1}}, & 0<q<1.\end{cases} \tag{3.8}
$$

On the other hand, we have

$$
\begin{aligned}
&\|K(f_{\mu,\gamma}-f)\|\\
&= \left\|\sum_{n=1}^{\infty}Q_n(T)\left(\frac{Q_n(T)}{Q_n^2(T)+\mu\exp(\lambda_n^\gamma)} - \frac{1}{Q_n(T)}\right)g_n\varphi_n(x)\right\|\\
&= \left\|\sum_{n=1}^{\infty}\frac{-\mu\exp(\lambda_n^\gamma)}{Q_n^2(T)+\mu\exp(\lambda_n^\gamma)}g_n\varphi_n(x)\right\|\\
&\le \left\|\sum_{n=1}^{\infty}\frac{-\mu\exp(\lambda_n^\gamma)}{Q_n^2(T)+\mu\exp(\lambda_n^\gamma)}(g_n-g_n^\delta)\varphi_n(x)\right\| + \left\|\sum_{n=1}^{\infty}\frac{-\mu\exp(\lambda_n^\gamma)}{Q_n^2(T)+\mu\exp(\lambda_n^\gamma)}g_n^\delta\varphi_n(x)\right\|.
\end{aligned}
$$

Notice that

$$\left\| \sum_{n=1}^{\infty} \frac{-\mu \exp(\lambda_n^{\gamma})}{Q_n^2(T) + \mu \exp(\lambda_n^{\gamma})} (g_n - g_n^{\delta}) \varphi_n(x) \right\| \leq \delta,$$

$$\|K f_{\mu,\gamma}^{\delta} - g^{\delta}\| = \left\| \sum_{n=1}^{\infty} \frac{-\mu \exp(\lambda_n^{\gamma})}{Q_n^2(T) + \mu \exp(\lambda_n^{\gamma})} g_n^{\delta} \varphi_n(x) \right\|,$$

we have

$$\|K(f_{\mu,\gamma} - f)\| \leq \delta + \rho\delta = (1+\rho)\delta. \tag{3.9}$$

From inequality (3.9) and the conditional stability estimate in Theorem 2.9, we obtain

$$\begin{aligned} \|f_{\mu,\gamma} - f\| &\leq \frac{1}{\big(p_0(1 - E_{\alpha,1}(-\lambda_1 T^{\alpha}))\big)^{\frac{q}{q+1}}} M_1^{\frac{1}{q+1}} \|K(f_{\mu,\gamma} - f)\|^{\frac{q}{q+1}} \\ &\leq \frac{1}{\big(p_0(1 - E_{\alpha,1}(-\lambda_1 T^{\alpha}))\big)^{\frac{q}{q+1}}} M_1^{\frac{1}{q+1}} (1+\rho)^{\frac{q}{q+1}} \delta^{\frac{q}{q+1}}. \end{aligned} \tag{3.10}$$

By combining (3.8) with (3.10), the assertion of the theorem is proved for the case of $\gamma \leq 0$.

(ii) The proof for $\gamma > 0$. Since $f_{\mu,\gamma}^{\delta}(x)$ is the exponential regularization solution, i.e., the minimum of regularized functional (3.2), one has

$$\|K f_{\mu,\gamma}^{\delta} - g^{\delta}\|^2 + \mu \|f_{\mu,\gamma}^{\delta}\|_{\gamma,\mathrm{Exp}}^2 \leq \|K f - g^{\delta}\|^2 + \mu \|f\|_{\gamma,\mathrm{Exp}}^2 = \|g - g^{\delta}\|^2 + \mu \|f\|_{\gamma,\mathrm{Exp}}^2.$$

The discrepancy principle (3.7) for choosing the regularization parameter μ directly yields that

$$\rho^2 \delta^2 + \mu \|f_{\mu,\gamma}^{\delta}\|_{\gamma,\mathrm{Exp}}^2 \leq \|g - g^{\delta}\|^2 + \mu \|f\|_{\gamma,\mathrm{Exp}}^2,$$

and

$$\|f_{\mu,\gamma}^{\delta}\|_{\gamma,\mathrm{Exp}}^2 \leq \|f\|_{\gamma,\mathrm{Exp}}^2 + \frac{1-\rho^2}{\mu} \delta^2 \leq \|f\|_{\gamma,\mathrm{Exp}}^2 \leq M_2^2.$$

Thereby, one has

$$\|f_{\mu,\gamma}^{\delta} - f\|_{\gamma,\mathrm{Exp}} \leq \|f_{\mu,\gamma}^{\delta}\|_{\gamma,\mathrm{Exp}} + \|f\|_{\gamma,\mathrm{Exp}} \leq 2M_2.$$

On the other hand, we have

$$\|K f_{\mu,\gamma}^{\delta} - g\| \leq \|K f_{\mu,\gamma}^{\delta} - g^{\delta}\| + \|g^{\delta} - g\| \leq \rho\delta + \delta = (1+\rho)\delta.$$

From the latter conditional stability in Theorem 2.9, we obtain

$$\begin{aligned} \|f_{\mu,\gamma}^{\delta} - f\| &\leq \frac{1}{\big(p_0(1 - E_{\alpha,1}(-\lambda_1 T^{\alpha}))\big)^{\frac{\gamma}{\gamma+2}}} M_1^{\frac{1}{q+1}} \|K(f_{\mu,\gamma} - f)\|^{\frac{q}{q+1}} \\ &\leq \frac{1}{\big(p_0(1 - E_{\alpha,1}(-\lambda_1 T^{\alpha}))\big)^{\frac{\gamma}{\gamma+2}}} (2M_2)^{\frac{2}{\gamma+2}} (1+\rho)^{\frac{\gamma}{\gamma+2}} \delta^{\frac{\gamma}{\gamma+2}}. \end{aligned}$$

The proof is completed. □

Remark 3.5. The results of Theorems 3.2 and 3.4 show that its optimal convergence order is independent of the q value when $\gamma \leq 0$, which means that the exponential regularization method can be implemented and its optimal convergence rate can be achieved without the exact value of the q (the exact information on the regularity of $f(x)$). In this sense, the proposed exponential regularization method is superior to the regularization methods using the usual Sobolev's space norm as a penalty term in the minimizing functional [14, 32]. However, if we want to achieve the optimal convergence rate in the case of $\gamma > 0$, we need to know the exact information on the regularity of $f(x)$, i.e., the value of γ.

4. Numerical implementations

In this section, we present numerical methods for implementing the exponential Tikhonov regularisation method. To obtain the measurements $g(x)$, we use the finite difference method to solve direct problem (1.1). For the inverse problem, we use the series expression (3.3), where the Mittag–Leffler function is computed by the one that comes with `MATLAB`.

Here we assume $\Omega = (0, 2)$. Take the grid size for time and space variable in the finite difference algorithm are $\triangle t = \frac{T}{N}$ and $\triangle x = \frac{2}{M}$, respectively. The grid points on the time interval $[0, T]$ are labeled $t_n = n\triangle t$, $n = 0, 1, \ldots, N$, and the grid points in the space interval $[0, 2]$ are $x_i = i\triangle x$, $i = 0, 1, \ldots, M$. Moreover, we denote $f_i = f(x_i)$, $p_j = p(t_j)$, and the approximate values of function u at the grid points are denoted $u_i^n \approx u(x_i, t_n)$.

The time-fractional derivative is approximated by

$$\partial_{0+}^{\alpha} u(x_i, t_j) \approx \frac{(\triangle t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} (u_i^{j-k} - u_i^{j-k-1})((k+1)^{1-\alpha} - k^{1-\alpha}) \tag{4.1}$$

for $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$. This scheme was used in [17, 35].

In the following, we use a numerical scheme for the Riesz derivative to approximate the fractional Laplace operator. From the numerical point of view, Yang et al. [33] pointed out that the one-dimensional fractional Laplacian operator with homogeneous Dirichlet boundary condition defined by the spectral decomposition is equivalent to the Riesz fractional derivative, which is also confirmed in literature [34]. To discretize the fractional derivatives in space, we use the α-order fractional difference method from [2], i.e.,

$$\frac{\partial^{\beta} u}{\partial |x|^{\beta}}(x_i, t_j) \approx -(\triangle x)^{-\beta} \sum_{m=i-M}^{i} s_m^{(\beta)} u(x_{i-m}, t_j), \tag{4.2}$$

where

$$s_m^{(\beta)} = \frac{(-1)^m \Gamma(\beta+1)}{\Gamma(\frac{\beta}{2} - m + 1)\Gamma(\frac{\beta}{2} + m + 1)}, \quad m = \pm 1, \pm 2, \ldots.$$

Using the fact that u_i^0 $(0 \le i \le M)$, $u_0^j = u_M^j$ $(0 \le j \le N)$, the above formulas (4.1) and (4.2) directly lead to the high order finite difference numerical scheme for (1.1), i.e.,

$$
(4.3) \qquad \begin{cases} u_i^j + \frac{\eta}{(\triangle x)^\beta} \sum_{m=0}^{M} s_{i-m}^{(\beta)} u_m^j \\ \quad = \sum_{k=0}^{j-2} (b_k - b_{k+1}) u_i^{j-k-1} + \eta f_i p_j, & 1 \le j \le N,\ 1 \le i \le M-1, \\ u_i^0 = 0, & 1 \le i \le M-1, \\ u_0^j = u_M^j, & 0 \le j \le N, \end{cases}
$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$, $\eta = \Gamma(2-\alpha) \cdot (\triangle t)^\alpha$.

Then (4.3) can be rewritten in the form of a matrix equation, i.e.,

$$
\mathbf{A}\mathbf{U}^j = \sum_{k=1}^{j-1} (b_{j-k-1} - b_{j-k}) \mathbf{U}^k + \eta p_j \mathbf{f}, \quad \mathbf{U}^0 = \mathbf{0},
$$

where $\mathbf{U}^k = (u_1^k, u_2^k, \ldots, u_{M-1}^k)^T$, $\mathbf{f} = (f_1, f_2, \ldots, f_{M-1})^T$ and

$$
\mathbf{A} = \begin{pmatrix} 1 + \frac{\eta}{(\triangle x)^\beta} s_0^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_{-1}^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_{-2}^{(\beta)} & \cdots & \frac{\eta}{(\triangle x)^\beta} s_{-M+3}^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_{-M+2}^{(\beta)} \\ \frac{\eta}{(\triangle x)^\beta} s_1^{(\beta)} & 1 + \frac{\eta}{(\triangle x)^\beta} s_0^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_{-1}^{(\beta)} & \cdots & \frac{\eta}{(\triangle x)^\beta} s_{-M+4}^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_{-M+3}^{(\beta)} \\ \frac{\eta}{(\triangle x)^\beta} s_2^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_1^{(\beta)} & 1 + \frac{\eta}{(\triangle x)^\beta} s_0^{(\beta)} & \cdots & \frac{\eta}{(\triangle x)^\beta} s_{-M+5}^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_{-M+4}^{(\beta)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\eta}{(\triangle x)^\beta} s_{M-2}^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_{M-3}^{(\beta)} & \frac{\eta}{(\triangle x)^\beta} s_{M-4}^{(\beta)} & \cdots & \frac{\eta}{(\triangle x)^\beta} s_1^{(\beta)} & 1 + \frac{\eta}{(\triangle x)^\beta} s_0^{(\beta)} \end{pmatrix}.
$$

We take $g = U^N$ as the exact final data.

5. Numerical experiment

In this section, we present some numerical experiments to show the proposed exponential regularization method and the corresponding convergence rate. The numerical examples are constructed in the following way.

We selected the exact solution $f(x)$ and obtained the exact data function $g(x)$ using the finite difference method stated in Section 4. Then we added a random perturbation to each function giving vectors g^δ. Finally we obtained the regularization solution using (3.3).

In our experiments, we always set $T = 1.0$. The noisy data g^δ is given in the following

$$
g^\delta = g + \sigma g \cdot (2 \cdot \mathrm{rand}(\mathrm{size}(g)) - 1).
$$

The corresponding noise level is calculated by $\delta = \|g^\delta - g\|$.

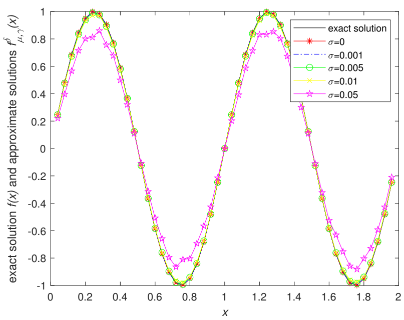
To show the accuracy of numerical solution, we compute the approximate L^2 error denoted by

$$e(f,\sigma) = \|f(x) - f^{\delta}_{\mu,\gamma}(x)\|,$$

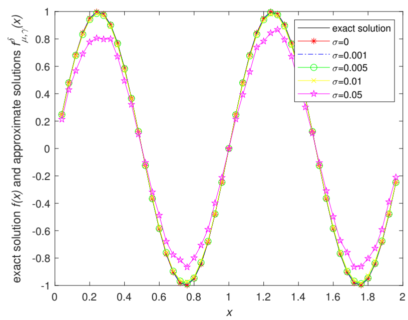
and the approximate relative error in L^2 norm denote by

$$e_r(f,\sigma) = \|f(x) - f^{\delta}_{\mu,\gamma}(x)\|/\|f(x)\|.$$

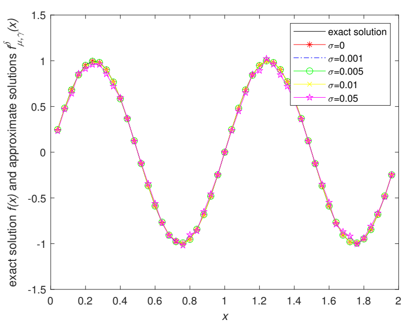
Example 5.1. Let $d = 1$, $\Omega = (0, 2)$, $\alpha = 0.7$, $\beta = 1.3$. Take a source function $f(x) = \sin(2\pi x)$ and $p(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + 4\pi^2 t^2$.



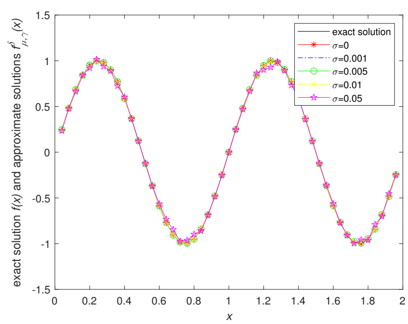
(a) $\gamma = -1$, $q = 0.5$



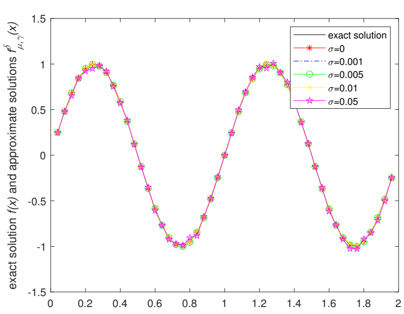
(b) $\gamma = -5$, $q = 0.5$



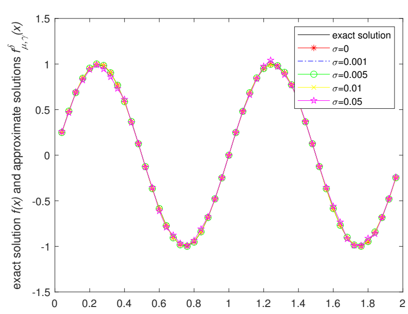
(c) $\gamma = -1$, $q = 1$



(d) $\gamma = -5$, $q = 1$



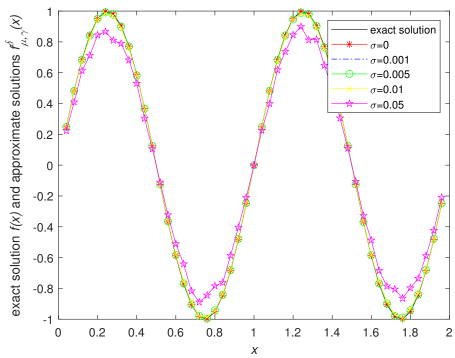
(e) $\gamma = -1$, $q = 2.5$



(f) $\gamma = -5$, $q = 2.5$

Figure 5.1: The exact and regularized source terms given by the a priori parameter choice rule for Example 5.1.

The numerical results by using the a priori parameter choice rule for various noise levels $\sigma = 0.001,\ 0.005,\ 0.01,\ 0.05$ in the case of $\gamma = -1, -5$ are showed in Figure 5.1 in which we use $\mu = 0.01\big(\frac{\delta}{M_1}\big)^{\frac{2}{3}}$ for $q \geq 2$ and $\mu = 0.01\big(\frac{\delta}{M_1}\big)^{\frac{2}{q+1}}$ for $q \in (0, 2]$. We can see that the numerical results are in good agreement with the exact shape. In Figure 5.2 we use $\mu = (1e-17)\delta^{\frac{8+4\gamma}{6+\gamma}}$ for $\gamma = 1/2$.



(a) $\gamma = 1/2$

Figure 5.2: The exact and regularized source terms given by the a priori parameter choice rule for Example 5.1.

Here, we take the same random numbers for different noise level σ to conduct numerical simulations. Numerical results are showed in Table 5.1. We can see that numerical results for the prior choice of regularization parameters are consistent with theoretical results of Theorem 3.2 for $\gamma \leq 0$, and the convergence rate is independent of the value of γ for $\gamma \leq 0$.

Table 5.1: The relative error $e_r(f, \sigma)$ for different values of σ.

σ	$\gamma=-1$			$\gamma=-5$		
	$q=0.5$	$q=1$	$q=2.5$	$q=0.5$	$q=1$	$q=2.5$
0.001	0.0014	7.8525e-04	7.2547e-04	0.0015	7.6346e-04	6.5724e-04
0.005	0.0096	0.0039	0.0030	0.098	0.0034	0.0029
0.01	0.0198	0.0069	0.0061	0.0190	0.0067	0.0064
0.05	0.1487	0.0401	0.0293	0.1529	0.0362	0.0279

In the posterior, the crucial work is the choice of regularization parameter. As suggested in [6, 10], we choose

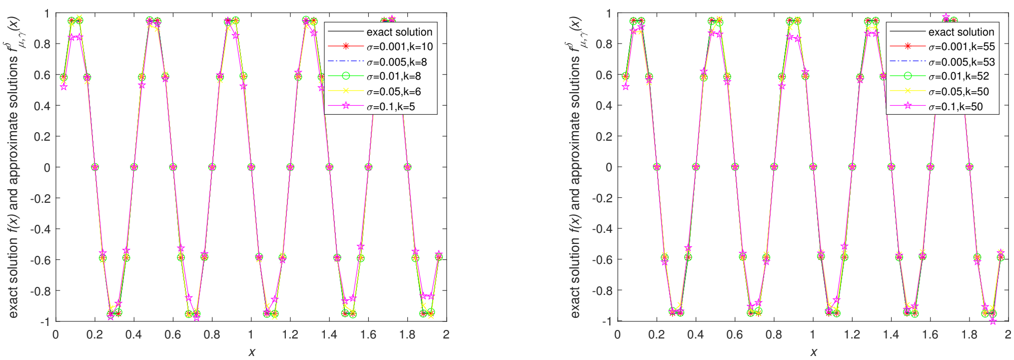
$$\mu_k = \mu_0 r^k, \quad 0 < r < 1,\ k = 0, 1, \ldots. \tag{5.1}$$

Meanwhile, the key work is to find an appropriate stopping rule in the iteration algorithm. In this paper, we suggest the well-known Morozov's discrepancy principle, i.e., we choose k^* satisfying inequality

$$E_{k^*} < \rho\delta < E_{k-1}, \quad k \le k^*, \tag{5.2}$$

where $\rho > 1$ is a constant.

Example 5.2. Let $d = 1$, $\Omega = (0, 2)$, $\alpha = 0.7$, $\beta = 1.3$. Take a source function $f(x) = \sin(5\pi x)$ and $p(t) = \exp(-t)$. The noise levels $\sigma = 0.001$, 0.005, 0.01, 0.05, 0.1, the regularization parameter μ_k defined in (5.1) are chosen by $\mu_0 = 1$, $r = 0.4$.



(a) $\gamma = -1$ (b) $\gamma = 1/2$

Figure 5.3: The exact and regularized source terms given by the a posteriori parameter choice rule for Example 5.2.

The numerical inversions are showed in Figure 5.3, where the regularization parameters are provided by the solution to the inequality (5.2) with $\rho = 1.1$.

Remark 5.3. In the numerical implementation, we adopt a non-iterative algorithm to reconstruct the unknown source in one-dimensional case. Because the penalty term of exponential Tikhonov regularization is much more complex than that of usual Tikhonov regularization, it is still difficult to solve the minimizer to the variational problem directly by iterative regularization under the optimization framework. In addition, we usually need to compute the gradient of a forward operator by the iterative method, and we usually use numerical differential method to approximately substitute the gradient when it is hard to obtain the exact expression for the gradient. So we need to introduce a numerical differentiation step τ. However, a lot of numerical examples are sensitive to τ. In this computation, we can successfully avoid the problem of selection of the differential step. Because the forward operator is linear to f and the gradient can be replaced by a substitute where the gradient is not dependent to τ, see also our previous work [22]. This will be our work in the future.

6. Concluding remarks

This paper studies the inverse problem for recovering the space-dependent source of time-space fractional diffusion equation. The conditional stability of the inverse problem is analyzed by combining two priori bounded conditions. Then we use the exponential Tikhonov regularization method to solve the inverse problem, and the corresponding convergence rates is obtained under the a-priori and a-posteriori choice strategies of regularization parameters. In the case of $\gamma \leq 0$, the exponential Tikhonov regularization method is independent of the value of q and its optimal convergence rate can be achieved for any $f(x) \in D\big(((-\Delta)^{\beta/2})^q\big)$, $q > 0$. In other words, the proposed method is valid even if the exact source $f(x)$ belongs to $L^2(\Omega)$ in the sense of $q \to 0$. Hence, in the case $\gamma \leq 0$ the proposed method is applicable to more realistic sources. Although the exponential regularization method has higher convergence rate for $\gamma > 0$, it requires a stronger regularity of the exact source, i.e., $f(x) \in D\Big(\exp\big(\frac{((-\Delta)^{\beta/2})^\gamma}{2}\big)\Big)$, and the corresponding convergence rate dependents on the value of γ.

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References

[1] W. Audeh and F. Kittaneh, *Singular value inequalities for compact operators*, Linear Algebra Appl. **437** (2012), no. 10, 2516–2522.

[2] C. Çelik and M. Duman, *Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative*, J. Comput. Phys. **231** (2012), no. 4, 1743–1750.

[3] M. Chang, L. Sun and Y. Wang, *Two regularization methods for identifying the unknown source in a multiterm time-fractional diffusion equation*, Rocky Mountain J. Math. **53** (2023), no. 5, 1387–1414.

[4] A. V. Chechkin, R. Gorenflo and I. M. Sokolov, *Fractional diffusion in inhomogeneous media*, J. Phys. A **38** (2005), no. 42, L679–L684.

[5] B. Chopard, D. R. de Sousa, J. Lätt, L. Mountrakis, F. Dubois, C. Yourassowsky, P. V. Antwerpen, O. Eker, L. Vanhamme, D. Perez-Morga, G. Courbebaisse,

E. Lorenz, A. G. Hoekstra and K. Z. Boudjeltia, *A physical description of the adhesion and aggregation of platelets*, R. Soc. Open Sci. **4** (2017), no. 4, 170219, 10 pp.

[6] M. Hanke and C. W. Groetsch, *Nonstationary iterated Tikhonov regularization*, J. Optim. Theory Appl. **98** (1998), no. 1, 37–53.

[7] F. Huang and F. Liu, *The space-time fractional diffusion equation with Caputo derivatives*, J. Appl. Math. Comput. **19** (2005), no. 1-2, 179–190.

[8] J. Huang and H.-C. Zhou, *Boundary stabilization for time-space fractional diffusion equation*, Eur. J. Control **65** (2022), Paper No. 100639, 6 pp.

[9] J. Jia, J. Peng, J. Gao and Y. Li, *Backward problem for a time-space fractional diffusion equation*, Inverse Probl. Imaging **12** (2018), no. 3, 773–799.

[10] B. Kaltenbacher, A. Neubauer and O. Scherzer, *Iterative Regularization Methods for Nonlinear Ill-posed Problems*, Radon Series on Computational and Applied Mathematics **6**, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[11] E. Karapinar, D. Kumar, R. Sakthivel, N. H. Luc and N. H. Can, *Identifying the space source term problem for time-space-fractional diffusion equation*, Adv. Difference Equ. **2020**, Paper No. 557, 23 pp.

[12] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, Second edition, Applied Mathematical Sciences **120**, Springer, New York, 2011.

[13] Y. S. Li and T. Wei, *An inverse time-dependent source problem for a time-space fractional diffusion equation*, Appl. Math. Comput. **336** (2018), 257–271.

[14] Y.-K. Ma, P. Prakash and A. Deiveegan, *Generalized Tikhonov methods for an inverse source problem of the time-fractional diffusion equation*, Chaos Solitons Fractals **108** (2018), 39–48.

[15] M. M. Meerschaert and E. Scalas, *Coupled continuous time random walks in finance*, Phys. A **370** (2006), no. 1, 114–118.

[16] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 77 pp.

[17] D. A. Murio, *Implicit finite difference approximation for time fractional diffusion equations*, Comput. Math. Appl. **56** (2008), no. 4, 1138–1145.

[18] I. Podlubny, *Fractional Differential Equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering **198**, Academic Press, San Diego, CA, 1999.

[19] H. Pollard, *The completely monotonic character of the Mittag–Leffler function* $E_a(-x)$, Bull. Amer. Math. Soc. **54** (1948), 1115–1116.

[20] Z. Ruan, S. Zhang and S. Xiong, *Solving an inverse source problem for a time fractional diffusion equation by a modified quasi-boundary value method*, Evol. Equ. Control Theory **7** (2018), no. 4, 669–682.

[21] L. L. Sun and M. L. Chang, *Galerkin spectral method for a multi-term time-fractional diffusion equation and an application to inverse source problem*, Netw. Heterog. Media **18** (2023), no. 1, 212–243.

[22] L. L. Sun and X. B. Yan, *Inverse source problem for a multiterm time-fractional diffusion equation with nonhomogeneous boundary condition*, Adv. Math. Phys. **2020**, Art. ID 1825235, 14 pp.

[23] L. Sun, X. Yan and K. Liao, *Simultaneous inversion of a fractional order and a space source term in an anomalous diffusion model*, J. Inverse Ill-Posed Probl. **30** (2022), no. 6, 791–805.

[24] S. Tatar and S. Ulusoy, *An inverse source problem for a one-dimensional space-time fractional diffusion equation*, Appl. Anal. **94** (2015), no. 11, 2233–2244.

[25] S. Tatar, R. Tinaztepe and S. Ulusoy, *Determination of an unknown source term in a space-time fractional diffusion equation*, J. Fract. Calc. Appl. **6** (2015), no. 1, 83–90.

[26] S. Tatar, R. Tınaztepe and S. Ulusoy, *Simultaneous inversion for the exponents of the fractional time and space derivatives in the space-time fractional diffusion equation*, Appl. Anal. **95** (2016), no. 1, 1–23.

[27] Ž. Tomovski, T. Sandev, R. Metzler and J. Dubbeldam, *Generalized space-time fractional diffusion equation with composite fractional time derivative*, Phys. A **391** (2012), no. 8, 2527–2542.

[28] S. A. Trigger, *Anomalous transport in velocity space: from Fokker–Planck to the general equation*, J. Phys. A **43** (2010), no. 28, 285005, 15 pp.

[29] N. H. Tuan and L. D. Long, *Fourier truncation method for an inverse source problem for space-time fractional diffusion equation*, Electron. J. Differential Equations **2017**, Paper No. 122, 16 pp.

[30] Z. Wang and J. Liu, *New model function methods for determining regularization parameters in linear inverse problems*, Appl. Numer. Math. **59** (2009), no. 10, 2489–2506.

[31] Z. Wang, S. Qiu, S. Yu, B. Wu and W. Zhang, *Exponential Tikhonov regularization method for solving an inverse source problem of time fractional diffusion equation*, J. Comput. Math. **41** (2023), no. 2, 173–190.

[32] X. Xiong and X. Xue, *A fractional Tikhonov regularization method for identifying a space-dependent source in the time-fractional diffusion equation*, Appl. Math. Comput. **349** (2019), 292–303.

[33] Q. Yang, F. Liu and I. Turner, *Numerical methods for fractional partial differential equations with Riesz space fractional derivatives*, Appl. Math. Model. **34** (2010), no. 1, 200–218.

[34] Y.-X. Zhang, J. Jia and L. Yan, *Bayesian approach to a nonlinear inverse problem for a time-space fractional diffusion equation*, Inverse Problems **34** (2018), no. 12, 125002, 19 pp.

[35] P. Zhuang and F. Liu, *Implicit difference approximation for the time fractional diffusion equation*, J. Appl. Math. Comput. **22** (2006), no. 3, 87–99.

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