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# Analysis of Local Discontinuous Galerkin Method for the Variable-order Subdiffusion Equation with the Caputo–Hadamard Derivative

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Abstract. In this article, we investigate a high-order numerical method for the variable-order (VO) subdiffusion equation with the Caputo–Hadamard derivative. The temporal variable and the spatial variable are discretized by the finite difference method and the local discontinuous Galerkin method, respectively. Furthermore, for all variable-order $\alpha(t) \in (0,1)$, the stability and the optimal error estimates are proved for the presented scheme. Finally, several numerical tests are given to demonstrate optimal rates of convergence and show the efficiency of the method.

## 1. Introduction

Fractional calculus is a generalization from integer order to arbitrary order [18]. It has inspired many researchers to explore both theoretical and practical aspects of the subject due to its wide application in mathematical modeling of physics, engineering and biological phenomena [6, 16, 17, 24].

In recent decades, fractional partial differential equations (FPDEs) with Hadamard derivative and Caputo–Hadamard derivative have attracted great interest from some researchers. One of the important reasons is that it has been found that in some physical problems such as fracture analysis [2], mean square displacement of the particles [7], Lomnitz logarithmic creep law of materials [9] and so on, can be described by Hadamard derivative or Caputo–Hadamard FPDEs. The Hadamard derivative differs from the Caputo and Riemann–Liouville derivatives in the sense that the kernel of the integral involves a logarithmic function which has arbitrary exponents. The Riemann–Liouville and Hadamard fractional derivatives also have their own drawbacks, in that the constants are generally not equal to zero. Actually, some important works have been done on Hadamard fractional derivatives, which can help us better understand Hadamard fractional derivative [4, 14].

The Caputo–Hadamard derivative is a Caputo-type modification of the Hadamard derivative, and is more appropriate when it is applied for physically interpretable initial

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conditions. Some researchers have paid their attention to partial differential equations with Caputo–Hadamard derivative [1,8,10,11,28]. However, the investigations of numerical methods for FPDEs with Caputo–Hadamard derivative of variable order are rarely reported. Almeida [3] presented three types of variable order Caputo–Hadamard derivatives and studied the relations among them. Li and his coauthors [15] presented a local discontinuous Galerkin (LDG) method for Caputo–Hadamard FPDEs. Zheng [31] presented a logarithmic transformation which could transform the Caputo–Hadamard fractional models to the Caputo types.

Fractional order models which change with spatial and temporal variables were found in many physical problems, such as signature verification, algebraic structure, viscoelastic materials, and noise reduction [12,20,30,32]. Variable order fractional derivative is a good tool to eliminate the nonphysical singularity of the exact solutions for constant-order FPDEs and model multiphysics phenomena, and has many advantages in characterizing memory property of systems [21–23]. As far as we know, it is often more difficult to find the analytical solution for Caputo–Hadamard problems of variable-order than for Caputo–Hadamard models of constant order. Therefore, it is necessary to further develop the numerical methods of variable order Caputo–Hadamard problems, including those that have been used for constant-order problems. Notably, the research about numerical methods for variable-order Caputo–Hadamard FPDEs is very limited now. Inspired by this, we will consider the high order LDG method to solve the variable-order FPDEs with Caputo–Hadamard derivative in this paper.

Discontinuous Galerkin method has some properties and advantages. Like finite element methods, it easily handles complicated geometry and boundary conditions. The mass matrix is local to the cell due to the discontinuous basis. Furthermore, it communicates only with the immediate neighbors and disregards the order of the scheme and allows handling nodes in the mesh [27]. Due to its efficiency and flexibility regarding grid and shape functions, the discontinuous Galerkin method is a particularly effective tool to solve partial differential equations and has been applied to some FPDEs [13,15,19,25,29].

In this paper, we will present a LDG method for the following variable-order subdiffusion problem with the Caputo–Hadamard derivative

$$\begin{cases} u_t + {}_{a}^{CH}\partial_t^{\alpha(t)} u - u_{xx} = f(x,t), & (x,t) \in (0,b) \times (a,T],\ a>0, \\ u(x,a) = u_a(x), & x \in (0,b), \\ u(0,t) = u(b,t) = 0, & t \in (a,T], \end{cases} \tag{1.1}$$

where $0 < \alpha(t) < 1$, and $f$, $u_a$ are known smooth functions. The compactly supported or periodic boundary condition is considered.

The Caputo–Hadamard derivative of variable order $0<\alpha(t)<1$ is defined as

$$ {}_a^{CH}\partial_t^{\alpha(t)} z(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_a^t \log\left(\frac{t}{\xi}\right)^{-\alpha(t)} \frac{\partial z(\xi)}{\partial \xi}\, d\xi. $$

The rest of the paper is organized as follows. In Section 2, some useful symbols and auxiliary results are introduced. In Section 3, we give the fully discrete LDG method of the model (1.1), and discuss its stability and optimal error estimation in detail. Numerical experiments are carried out to verify the theoretical properties in Section 4.

## 2. Notations and auxiliary results

Denote by $0=x_{1/2}<x_{3/2}<\cdots<x_{N+1/2}=b$ a partition of $\Omega=[0,b]$. Let $I_j=[x_{j-1/2},x_{j+1/2}]$, $j=1,\ldots,N$, and define $h_j=x_{j+1/2}-x_{j-1/2}$, $1\le j\le N$, $h=\max_{1\le j\le N} h_j$.

Denote $u^+_{j+1/2}=\lim_{t\to 0^+} u(x_{j+1/2}+t)$ and $u^-_{j+1/2}=\lim_{t\to 0^+} u(x_{j+1/2}-t)$. Let $u_h^n$ be the numerical solution which belongs to $V_h^k$:

$$ V_h^k=\{\omega:\omega\in P^k(I_j), x\in I_j, j=1,2,\ldots,N\}. $$

For error estimates, the following two special projections $\mathcal{Q}$ and $\mathcal{Q}^{\pm}$ are useful. For a given function $\mu(x)$, $x\in[0,b]$,

$$ \begin{aligned} &\int_{I_j}(\mathcal{Q}\mu(x)-\mu(x))\omega(x)\,dx=0, \quad \forall\,\omega\in P^k(I_j), \\ &\int_{I_j}(\mathcal{Q}^+\mu(x)-\mu(x))\omega(x)\,dx=0, \quad \forall\,\omega\in P^{k-1}(I_j), \\ &\mathcal{Q}^+\mu(x^+_{j-1/2})=\mu(x_{j-1/2}), \end{aligned} \tag{2.1} $$

and

$$ \begin{aligned} &\int_{I_j}(\mathcal{Q}^-\mu(x)-\mu(x))\omega(x)\,dx=0, \quad \forall\,\omega\in P^{k-1}(I_j), \\ &\mathcal{Q}^-\mu(x^-_{j+1/2})=\mu(x_{j+1/2}). \end{aligned} \tag{2.2} $$

According to [5, 26, 29], the following approximation result for $\mathcal{Q}$ and $\mathcal{Q}^{\pm}$ holds:

$$ \|\varrho\|+h\|\varrho\|_\infty+h^{1/2}\|\varrho\|_{\tau_h}\le Ch^{\min(k+1,\nu+1)}\|\mu\|_{\nu+1}, \tag{2.3} $$

where $\varrho=\mathcal{Q}\mu-\mu$ or $\varrho=\mathcal{Q}^{\pm}\mu-\mu$, and

$$ \|\varrho\|_{\tau_h}=\left(\frac{1}{2}\sum_{1\le j\le N}\left((\varrho^+_{j+1/2})^2+(\varrho^-_{j+1/2})^2\right)\right)^{1/2}. $$

In this paper, the positive number $C$ may have different values in different cases. The common symbols in Sobolev space are used. The scalar inner product on $L^2(\chi)$ can be expressed by $(\cdot,\cdot)_\chi$, and the related norm defined by $\|\cdot\|_\chi$. If $\chi=\Omega$, $\chi$ is dropped.

## 3. Fully discrete LDG scheme

Let $t_n = a + \frac{n}{M}(T-a)$, $\tau = t_n - t_{n-1}$. Temporal fractional derivatives $u_t$ and ${}_{a}^{CH}\partial_t^{\alpha(t)}u$ at $t_n$ are approximated by

$$u_t(x,t_n) = \frac{u(x,t_n) - u(x,t_{n-1})}{\tau} + \phi_1^n(x),$$

$$\begin{aligned}
&{}_{a}^{CH}\partial_t^{\alpha(t)}u(x,t_n)\\
&= \frac{1}{\Gamma(1-\alpha(t_n))}\int_a^{t_n} \log\left(\frac{t_n}{\xi}\right)^{-\alpha(t_n)} \frac{\partial u(x,\xi)}{\partial \xi}\, d\xi\\
&= \frac{1}{\Gamma(1-\alpha(t_n))}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \log\left(\frac{t_n}{\xi}\right)^{-\alpha(t_n)} \frac{\partial u(x,\xi)}{\partial \xi}\, d\xi\\
&= \frac{1}{\Gamma(1-\alpha(t_n))}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left(\frac{u(x,t_{k+1}) - u(x,t_k)}{\log\left(\frac{t_{k+1}}{t_k}\right)\xi} - \frac{u(x,t_{k+1}) - u(x,t_k)}{\log\left(\frac{t_{k+1}}{t_k}\right)\xi} + \frac{\partial u(x,\xi)}{\partial \xi}\right)\\
&\qquad \times \log\left(\frac{t_n}{\xi}\right)^{-\alpha(t_n)} d\xi\\
&= \frac{1}{\Gamma(2-\alpha(t_n))}\left(w_1^n u(x,t_n) + \sum_{k=1}^{n-1}(w_{k+1}^n - w_k^n)u(x,t_{n-k}) - w_n^n u(x,t_0)\right) + \phi_2^n(x),
\end{aligned}$$

where

$$w_k^n = \frac{\log\left(\frac{t_n}{t_{n-k}}\right)^{1-\alpha(t_n)} - \log\left(\frac{t_n}{t_{n-k+1}}\right)^{1-\alpha(t_n)}}{\log\left(\frac{t_{n-k+1}}{t_{n-k}}\right)}, \quad k = 1, 2, \ldots, n,$$

and the truncation error is $\phi^n(x) = \phi_1^n(x) + \phi_2^n(x)$. Using an analogue technique to Lemma 3.2 in [10], we could have the following result

$$\|\phi^n(x)\| \le C\tau,$$

where the constant $C > 0$ depends on the exact solution $u$ and the final time $T$.

Moreover, after some calculations, we could determine that $w_k^n$ has the following property

$$0 < w_n^n < w_{n-1}^n < \cdots < w_2^n < w_1^n. \tag{3.1}$$

Now we define the fully-discrete LDG method for the model (1.1). First, we rewrite (1.1) into the following first order system

$$p = u_x, \quad u_t + {}_{a}^{CH}\partial_t^{\alpha(t)}u - p_x = f(x,t).$$

Suppose $u_h^n, p_h^n \in V_h^k$ are the approximations of $u(\,\cdot\,,t_n)$, $p(\,\cdot\,,t_n)$, respectively, $f^n(x) =$

$f(x,t_n)$. Find $u_h^n, p_h^n \in V_h^k$, such that for all test functions $v, w \in V_h^k$,

$$
\begin{aligned}
&\left(\frac{1}{\tau}+\frac{w_1^n}{\Gamma(2-\alpha(t_n))}\right)\int_\Omega u_h^n v\,dx+\left(\int_\Omega p_h^n v_x\,dx-\sum_{j=1}^{N}\left((\widehat{p_h^n}v^-)_{j+1/2}-(\widehat{p_h^n}v^+)_{j-1/2}\right)\right)\\
&=\frac{1}{\Gamma(2-\alpha(t_n))}\left(\sum_{k=1}^{n-1}(w_k^n-w_{k+1}^n)\int_\Omega u_h^{n-k}v\,dx+w_n^n\int_\Omega u_h^0 v\,dx\right)\\
&\quad+\frac{1}{\tau}\int_\Omega u_h^{n-1}v\,dx+\int_\Omega f^n v\,dx,\\
&\int_\Omega p_h^n w\,dx+\int_\Omega u_h^n w_x\,dx-\sum_{j=1}^{N}\left((\widehat{u_h^n}w^-)_{j+1/2}-(\widehat{u_h^n}w^+)_{j-1/2}\right)=0.
\end{aligned}
\tag{3.2}
$$

The hat terms in (3.2) represent the numerical fluxes. The following numerical fluxes could be considered in order to guarantee stability:

$$
\widehat{u_h^n}=(u_h^n)^-,\quad \widehat{p_h^n}=(p_h^n)^+. \tag{3.3}
$$

We introduce the following notations which could be more convenient when discussing the theoretic properties of the scheme:

$$
\begin{aligned}
\Theta_\Omega(u_h^n,p_h^n;w,v)=&\int_\Omega u_h^n w_x\,dx-\sum_{j=1}^{N}\left(((u_h^n)^-w^-)_{j+1/2}-((u_h^n)^-w^+)_{j-1/2}\right)\\
&+\int_\Omega p_h^n v_x\,dx-\sum_{j=1}^{N}\left(((p_h^n)^+v^-)_{j+1/2}-((p_h^n)^+v^+)_{j-1/2}\right).
\end{aligned}
$$

### 3.1. Stability analysis

Without loss of generality, we consider the case $f(x,t)=0$ in discussing stability and convergence.

**Theorem 3.1.** *For compactly supported or periodic boundary conditions, the scheme* (3.2) *is unconditionally stable and satisfies*

$$
\|u_h^n\|\le\|u_h^0\|,\quad n=1,2,\dots,M.
$$

*Proof.* Based on the choice of fluxes (3.3), and taking the test functions $v=u_h^n$, $w=p_h^n$ in scheme (3.2), we can obtain

$$
\begin{aligned}
&\left(\frac{1}{\tau}+\frac{w_1^n}{\Gamma(2-\alpha(t_n))}\right)\|u_h^n\|^2+\|p_h^n\|^2+\Theta_\Omega(u_h^n,p_h^n;p_h^n,u_h^n)\\
&=\frac{1}{\Gamma(2-\alpha(t_n))}\left(\sum_{k=1}^{n-1}(w_k^n-w_{k+1}^n)\int_\Omega u_h^{n-k}u_h^n\,dx+w_n^n\int_\Omega u_h^0u_h^n\,dx\right)+\frac{1}{\tau}\int_\Omega u_h^{n-1}u_h^n\,dx.
\end{aligned}
\tag{3.4}
$$

In each cell $I_j = [x_{j-1/2}, x_{j+1/2}]$, we have

$$
\begin{aligned}
&\Theta_{I_j}(u_h^n, p_h^n; p_h^n, u_h^n) \\
&= \int_{I_j} u_h^n (p_h^n)_x \, dx - ((u_h^n)^-(p_h^n)^-)_{j+1/2} + ((u_h^n)^-(p_h^n)^+)_{j-1/2} \\
&\quad + \int_{I_j} p_h^n (u_h^n)_x \, dx - ((p_h^n)^+(u_h^n)^-)_{j+1/2} + ((p_h^n)^+(u_h^n)^+)_{j-1/2} \\
&= ((u_h^n)^-(p_h^n)^-)_{j+1/2} - ((u_h^n)^+(p_h^n)^+)_{j-1/2} - ((u_h^n)^-(p_h^n)^-)_{j+1/2} \\
&\quad + ((u_h^n)^-(p_h^n)^+)_{j-1/2} - ((p_h^n)^+(u_h^n)^-)_{j+1/2} + ((p_h^n)^+(u_h^n)^+)_{j-1/2}.
\end{aligned} \tag{3.5}
$$

Summing (3.5) from 1 to $N$ over $j$, and after some calculations, we could easily get

$$
\Theta_\Omega(u_h^n, p_h^n; p_h^n, u_h^n) = 0. \tag{3.6}
$$

Considering (3.1), (3.4), (3.6) and the Cauchy–Schwarz inequality, we establish the following inequality

$$
\begin{aligned}
&\left(\frac{1}{\tau} + \frac{w_1^n}{\Gamma(2-\alpha(t_n))}\right) \|u_h^n\| \\
&\le \frac{1}{\Gamma(2-\alpha(t_n))} \left(\sum_{k=1}^{n-1} (w_k^n - w_{k+1}^n) \|u_h^{n-k}\| + w_n^n \|u_h^0\|\right) + \frac{1}{\tau}\|u_h^{n-1}\|.
\end{aligned} \tag{3.7}
$$

Next mathematical induction would be used to prove Theorem 3.1. Setting $n = 1$ in (3.7), we know

$$
\left(\frac{1}{\tau} + \frac{w_1^1}{\Gamma(2-\alpha(t_1))}\right) \|u_h^1\| \le \frac{1}{\Gamma(2-\alpha(t_1))\tau} w_1^1 \|u_h^0\| + \frac{1}{\tau}\|u_h^0\|,
$$

that is,

$$
\|u_h^1\| \le \|u_h^0\|.
$$

Now assume that the following inequality

$$
\|u_h^i\| \le \|u_h^0\|, \quad i = 1, 2, 3, \ldots, n-1 \tag{3.8}
$$

holds, we need to get

$$
\|u_h^n\| \le \|u_h^0\|.
$$

Based on (3.8), we can have the following inequality

$$
\begin{aligned}
&\left(\frac{1}{\tau} + \frac{w_1^n}{\Gamma(2-\alpha(t_n))}\right) \|u_h^n\| \\
&\le \frac{1}{\Gamma(2-\alpha(t_n))} \left(\sum_{k=1}^{n-1} (w_k^n - w_{k+1}^n) \|u_h^{n-k}\| + w_n^n \|u_h^0\|\right) + \frac{1}{\tau}\|u_h^{n-1}\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(2-\alpha(t_n))}\left(\sum_{i=1}^{n-1}(w_k^n - w_{k+1}^n) + w_n^n\right)\|u_h^0\| + \frac{1}{\tau}\|u_h^0\| \\
&= \frac{1}{\Gamma(2-\alpha(t_n))} w_1^n \|u_h^0\| + \frac{1}{\tau}\|u_h^0\|.
\end{aligned}
$$

Obviously, we can get

$$
\|u_h^n\| \le \|u_h^0\|.
$$

Theorem 3.1 is proved. □

## 3.2. Convergence

**Theorem 3.2.** *Let $u \in H^{k+1}(\Omega) \cap H^2(\Omega)$ be the exact solution of the model (1.1), and $u_h^n$ be the numerical solution of the scheme (3.2), then the following error estimates*

$$
\|u(x,t_n) - u_h^n\| \le C(\tau + h^{k+1})
$$

*holds, where $C$ is a positive constant and depends on $u$ and $T$.*

*Proof.*

$$
\begin{aligned}
&e_u^n = u(x,t_n) - u_h^n = \xi_u^n - \eta_u^n, \quad \xi_u^n = \mathcal{Q}^- e_u^n, \quad \eta_u^n = \mathcal{Q}^- u(x,t_n) - u(x,t_n), \\
&e_p^n = p(x,t_n) - p_h^n = \xi_p^n - \eta_p^n, \quad \xi_p^n = \mathcal{Q}^+ e_p^n, \quad \eta_p^n = \mathcal{Q}^+ p(x,t_n) - p(x,t_n).
\end{aligned} \tag{3.9}
$$

Using the inequality (2.3), we can obtain approximate results for $\eta_u^n$ and $\eta_p^n$. Next, we estimate $\xi_u^n$ and $\xi_p^n$. By taking the fluxes (3.3), the following error equation holds:

$$
\begin{aligned}
&\left(\frac{1}{\tau} + \frac{w_1^n}{\Gamma(2-\alpha(t_n))}\right)\int_\Omega e_u^n v\,dx + \int_\Omega e_p^n w\,dx + \Theta_\Omega(e_u^n, e_p^n; w, v) + \int_\Omega \phi^n(x) v\,dx \\
&-\frac{1}{\tau}\int_\Omega e_u^{n-1} v\,dx + \frac{1}{\Gamma(2-\alpha(t_n))}\left(\sum_{k=1}^{n-1}(w_{k+1}^n - w_k^n)\int_\Omega e_u^{n-k} v\,dx - w_n^n \int_\Omega e_u^0 v\,dx\right) = 0.
\end{aligned} \tag{3.10}
$$

Utilizing (3.9), with $v = \xi_u^n$ and $w = \xi_p^n$, the equation (3.10) is rewritten as

$$
\begin{aligned}
&\left(\frac{1}{\tau} + \frac{w_1^n}{\Gamma(2-\alpha(t_n))}\right)\|\xi_u^n\|^2 + \|\xi_p^n\|^2 + \Theta_\Omega(\xi_u^n, \xi_p^n; \xi_p^n, \xi_u^n) \\
&= \frac{1}{\Gamma(2-\alpha(t_n))}\left(\sum_{k=1}^{n-1}(w_k^n - w_{k+1}^n)\int_\Omega \xi_u^{n-k}\xi_u^n\,dx + w_n^n\int_\Omega \xi_u^0 \xi_u^n\,dx\right) \\
&\quad - \int_\Omega \phi^n(x)\xi_u^n\,dx + \left(\frac{1}{\tau} + \frac{w_1^n}{\Gamma(2-\alpha(t_n))}\right)\int_\Omega \eta_u^n \xi_u^n\,dx + \Theta_\Omega(\eta_u^n, \eta_p^n; \xi_p^n, \xi_u^n) \\
&\quad + \int_\Omega \eta_p^n \xi_p^n\,dx + \frac{1}{\tau}\int_\Omega \xi_u^{n-1}\xi_u^n\,dx - \frac{1}{\tau}\int_\Omega \eta_u^{n-1}\xi_u^n\,dx \\
&\quad - \frac{1}{\Gamma(2-\alpha(t_n))}\left(\sum_{k=1}^{n-1}(w_k^n - w_{k+1}^n)\int_\Omega \eta_u^{n-k}\xi_u^n\,dx + w_n^n\int_\Omega \eta_u^0\xi_u^n\,dx\right).
\end{aligned} \tag{3.11}
$$

With the properties (2.1) and (2.2), we can obtain

$$\Theta_\Omega(\eta_u^n, \eta_p^n; \xi_p^n, \xi_u^n) = 0.$$

Seeing (3.11), we could have the following equality

$$\begin{aligned}
&\left(\frac{1}{\tau} + \frac{w_1^n}{\Gamma(2-\alpha(t_n))}\right)\|\xi_u^n\|^2 + \|\xi_p^n\|^2 \\
&= \frac{1}{\Gamma(2-\alpha(t_n))}\left(\sum_{k=1}^{n-1}(w_k^n - w_{k+1}^n)\int_\Omega \xi_u^{n-k}\xi_u^n\,dx + w_n^n\int_\Omega \xi_u^0\xi_u^n\,dx\right) \\
&\quad + \frac{1}{\tau}\int_\Omega \xi_u^{n-1}\xi_u^n\,dx - \int_\Omega \phi^n(x)\xi_u^n\,dx + \int_\Omega \eta_p^n\xi_p^n\,dx + \mathfrak{m}_1 + \mathfrak{m}_2,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\mathfrak{m}_1 &= \frac{1}{\Gamma(2-\alpha(t_n))}\left(w_1^n\int_\Omega \eta_u^n\xi_u^n\,dx - \sum_{k=1}^{n-1}(w_k^n - w_{k+1}^n)\int_\Omega \eta_u^{n-k}\xi_u^n\,dx - w_n^n\int_\Omega \eta_u^0\xi_u^n\,dx\right), \\
\mathfrak{m}_2 &= \int_\Omega \left(\frac{\eta_u^n - \eta_u^{n-1}}{\tau}\right)\xi_u^n\,dx.
\end{aligned}$$

Next we begin to estimate the terms $\mathfrak{m}_i$, $i = 1, 2$. With the help of

$$\left\|\frac{\eta_u^j - \eta_u^{j-1}}{\tau}\right\| \le \left\|\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\frac{\partial}{\partial t}(\mathcal{Q}^- u(x,t) - u(x,t))\,dt\right\| \le Ch^{k+1}\|u_t\|_{L^\infty(H^2(\Omega))},$$

we can get

$$\mathfrak{m}_1 \le Ch^{k+1}\|\xi_u^n\| \quad \text{and} \quad \mathfrak{m}_2 = \int_\Omega \left(\frac{\eta_u^n - \eta_u^{n-1}}{\tau}\right)\xi_u^n\,dx \le Ch^{k+1}\|u_t\|_{L^\infty(H^2(\Omega))}\|\xi_u^n\|.$$

Notice that $\|\xi_u^0\| = 0$. Considering the Cauchy–Schwarz inequality in (3.12), we can obtain

$$\begin{aligned}
\left(1 + \frac{w_1^n\tau}{\Gamma(2-\alpha(t_n))}\right)\|\xi_u^n\| &\le \frac{\tau}{\Gamma(2-\alpha(t_n))}\sum_{k=1}^{n-1}(w_k^n - w_{k+1}^n)\|\xi_u^{n-k}\| + \|\xi_u^{n-1}\| \\
&\quad + C\tau(h^{k+1} + \tau).
\end{aligned} \tag{3.13}$$

Mathematical induction will be used. First, let $n = 1$ in (3.13), we can obtain

$$\|\xi_u^1\| \le C\tau(h^{k+1} + \tau).$$

Then assume that

$$\|\xi_u^j\| \le Cj\tau(h^{k+1} + \tau), \quad j = 1, 2, \ldots, n-1. \tag{3.14}$$

According to (3.1), (3.13) and (3.14), we have

$$\begin{aligned}&\left(1+\frac{w_1^n\tau}{\Gamma(2-\alpha(t_n))}\right)\|\xi_u^n\|\\ &\le \frac{\tau}{\Gamma(2-\alpha(t_n))}\sum_{k=1}^{n-1}(w_k^n-w_{k+1}^n)\|\xi_u^{n-k}\|+\|\xi_u^{n-1}\|+C\tau(h^{k+1}+\tau)\\ &\le \left((n-1)\left(\frac{w_1^n\tau}{\Gamma(2-\alpha(t_n))}+1\right)+1\right)C\tau(h^{k+1}+\tau).\end{aligned}$$

Dividing both sides of the inequality by $\left(1+\frac{w_1^n\tau}{\Gamma(2-\alpha(t_n))}\right)$, we can easily get

$$\|\xi_u^n\|\le Cn\tau(h^{k+1}+\tau).$$

Based on the triangle inequality and the approximation result (2.3), the proof of Theorem 3.2 is completed. □

## 4. Numerical experiment

In this section, we first give some details of how the numerical procedure is implemented. Then we will present some numerical examples to verify the theoretical results and show the efficiency of the scheme (3.2) for variable-order subdiffusion equation with Caputo–Hadamard derivative.

### 4.1. Algorithm implementation

We perform our computations using MATLAB 2018b and the following basis functions for $x\in I_j$:

$$\varphi_1^j=1,\quad \varphi_2^j=\frac{x-\left(\frac{x_{j-1/2}+x_{j+1/2}}{2}\right)}{h_j},\quad \varphi_3^j=\left(\frac{x-\left(\frac{x_{j-1/2}+x_{j+1/2}}{2}\right)}{h_j}\right)^2.$$

Here we briefly show how to implement the procedure for piecewise $P^2$ polynomials, others can be implemented by a similar procedure. By taking the test functions $v=\Phi_k^j$, $w=\Phi_k^j$ in (3.2), we can implement the procedure. The main steps are as follows.

To begin with, we consider the case $n=0$,

$$\begin{bmatrix}A&&\\&\ddots&\\&&A\end{bmatrix}_{2(k+1)N\times 2(k+1)N}\begin{bmatrix}\overrightarrow{U^0}\\ \overrightarrow{P^0}\end{bmatrix}_{2(k+1)N\times 1}=\begin{bmatrix}\overrightarrow{R^0}\\ \overrightarrow{0}\end{bmatrix}_{2(k+1)N\times 1} \tag{4.1}$$

where

$$A = h\begin{bmatrix} 1 & 0 & 1/12 \\ 0 & 1/12 & 0 \\ 1/12 & 0 & 1/80 \end{bmatrix},$$

$\overrightarrow{R^0}$ can be obtained from the initial condition, so we can easily solve the solution vector $\big[\overrightarrow{U^1},\ \overrightarrow{P^1}\big]^{\mathrm{T}}$. Then for case $n \geq 1$, we have

$$\begin{bmatrix}
A_n & & & & & B_1 & C_1 & & & \\
& \ddots & & & & & \ddots & \ddots & & \\
& & \ddots & & & & & \ddots & \ddots & \\
& & & A_n & & & & & B_1 & C_1 \\
& & & & A_n & C_1 & & & & B_1 \\
B_2 & & & & C_2 & A & & & & \\
C_2 & B_2 & & & & & A & & & \\
& \ddots & \ddots & & & & & \ddots & & \\
& & \ddots & \ddots & & & & & \ddots & \\
& & & C_2 & B_2 & & & & & A
\end{bmatrix}_{6N\times 6N}
\begin{bmatrix} \overrightarrow{U^n} \\ \overrightarrow{P^n} \end{bmatrix}_{6N\times 1}
= \begin{bmatrix} \overrightarrow{R^n} \\ \overrightarrow{0} \end{bmatrix}_{6N\times 1}, \tag{4.2}$$

where

$$\overrightarrow{U^n} = (u_1^{n1}, u_2^{n1}, u_3^{n1}, \ldots, u_1^{nN}, u_2^{nN}, u_3^{nN}), \quad \overrightarrow{P^n} = (p_1^{n1}, p_2^{n1}, p_3^{n1}, \ldots, p_1^{nN}, p_2^{nN}, p_3^{nN}).$$

Here we give the general form of all the global matrices, while $B_1$, $B_2$, $C_1$ and $C_2$ in (4.2) need further calculation. It should be noted that since we have chosen a uniform grid, we find that the elements of the internal part of the global matrix are all the same through calculation. In fact, when calculating the $t_n$ layer, the global matrix on the left is the same except for $A_n = \big(\frac{1}{\tau} + \frac{w_1^n}{\Gamma(2-\alpha(t_n))}\big)A$.

According to the result of (4.1), $\overrightarrow{R^1}$ can be calculated, and subsequently, the vector $\big[\overrightarrow{U^2}, \overrightarrow{P^2}\big]^{\mathrm{T}}$ can also be solved in (4.2). Similar to the above steps, $\overrightarrow{R^n}$ can be calculated based on $\big[\overrightarrow{U^j}, \overrightarrow{P^j}\big]^{\mathrm{T}}$, $j = 0, 1, \ldots, n-1$. Finally, we can find $\big[\overrightarrow{U^n}, \overrightarrow{P^n}\big]^{\mathrm{T}}$.

## 4.2. Numerical test

Consider the model problem (1.1)

$$\begin{cases} u_t + {}_1^{CH}\partial_t^{\alpha(t)} u - u_{xx} = f(x,t), & (x,t) \in (0,1) \times (1,2], \\ u(x,1) = 0, & x \in (0,1), \\ u(0,t) = u(1,t) = 0, & t \in (1,2]. \end{cases}$$

The exact solution is $u(x,t) = ((\log t)^{\alpha(t)} + \log(t)^2)\sin(2\pi x)$. The proposed scheme is implemented for different values $\alpha(t)$ by taking piecewise $P^k$ $(k = 0, 1, 2)$ polynomials as basis functions.

Table 4.1: Errors versus $N$, order for different $\alpha(t)$ with $M = 500$, $T = 2$.

| $\alpha$ | $P^k$ | $N$ | $L^\infty$-error | order | $L^2$-error | order |
|---|---|---|---|---|---|---|
| $\alpha(t) = \frac{5+e^t}{19}$ | $P^0$ | 5 | 7.9068E-01 | - | 3.3618E-01 | - |
| | | 10 | 3.9760E-01 | 0.9918 | 1.6395E-01 | 1.0396 |
| | | 15 | 2.6533E-01 | 0.9976 | 1.0880E-01 | 1.0113 |
| | | 20 | 1.9907E-01 | 0.9988 | 8.1470E-02 | 1.0055 |
| | $P^1$ | 5 | 3.1678E-01 | - | 8.5416E-02 | - |
| | | 10 | 8.1973E-02 | 1.9503 | 2.1497E-02 | 1.9904 |
| | | 15 | 3.6826E-02 | 1.9735 | 9.5648E-03 | 1.9972 |
| | | 20 | 2.0643E-02 | 2.0121 | 5.3823E-03 | 1.9986 |
| | $P^2$ | 5 | 4.0219E-02 | - | 8.4732E-03 | - |
| | | 10 | 5.0349E-03 | 2.9979 | 1.0789E-03 | 2.9733 |
| | | 15 | 1.5379E-03 | 2.9250 | 3.2182E-04 | 2.9835 |
| | | 20 | 6.4867E-04 | 3.0007 | 1.3819E-04 | 2.9387 |
| $\alpha(t) = \frac{\sin(t)+\cos(t)}{36}$ | $P^0$ | 5 | 9.2217E-01 | - | 3.9204E-01 | - |
| | | 10 | 4.6292E-01 | 0.9943 | 1.9088E-01 | 1.0384 |
| | | 15 | 3.0883E-01 | 0.9983 | 1.2663E-01 | 1.0120 |
| | | 20 | 2.3167E-01 | 0.9992 | 9.4815E-02 | 1.0059 |
| | $P^1$ | 5 | 3.6884E-01 | - | 9.9453E-02 | - |
| | | 10 | 9.5335E-02 | 1.9519 | 2.5018E-02 | 1.9910 |
| | | 15 | 4.2782E-02 | 1.9762 | 1.1131E-02 | 1.9973 |
| | | 20 | 2.3948E-02 | 2.0169 | 6.2640E-03 | 1.9985 |
| | $P^2$ | 5 | 4.6779E-02 | - | 9.8647E-03 | - |
| | | 10 | 5.8596E-03 | 2.9970 | 1.2581E-03 | 2.9711 |
| | | 15 | 1.7554E-03 | 2.9449 | 3.8248E-04 | 2.9365 |
| | | 20 | 7.5485E-04 | 2.9730 | 1.7855E-04 | 2.6480 |

First, we consider the spatial order of convergence of the scheme (3.2). By choosing a fixed temporal step $\tau = \frac{1}{500}$, the spatial accuracy is verified. Numerical results for different variable-order fractional parameter $\alpha(t) = \frac{5+e^t}{19}$, $\frac{\sin(t)+\cos(t)}{36}$ are shown in Table 4.1. From Table 4.1, we can easily find that the convergence rate in spatial direction is close to $k+1$ for piecewise $P^k$ elements.

Table 4.2: Errors versus $N$, order for different $\alpha(t)$ with $M = 5000$, $T = 2$.

| $\alpha$ | $P^k$ | $N$ | $L^\infty$-error | order | $L^2$-error | order |
|---|---|---|---|---|---|---|
| $\alpha(t) = \frac{\sqrt{t}-\sin(t)}{48}$ | $P^0$ | 5 | 9.2298E-01 | - | 3.9239E-01 | - |
| | | 10 | 4.6331E-01 | 0.9943 | 1.9104E-01 | 1.0384 |
| | | 15 | 3.0908E-01 | 0.9983 | 1.2674E-01 | 1.0120 |
| | | 20 | 2.3186E-01 | 0.9992 | 9.4893E-02 | 1.0059 |
| | $P^1$ | 5 | 3.6920E-01 | - | 9.9535E-02 | - |
| | | 10 | 9.9548E-02 | 1.9514 | 2.5038E-02 | 1.9911 |
| | | 15 | 4.2865E-02 | 1.9746 | 1.1140E-02 | 1.9974 |
| | | 20 | 2.4016E-02 | 2.0139 | 6.2686E-03 | 1.9987 |
| | $P^2$ | 5 | 4.6845E-02 | - | 9.8724E-03 | - |
| | | 10 | 5.8644E-03 | 2.9978 | 1.2573E-03 | 2.9731 |
| | | 15 | 1.7857E-03 | 2.9326 | 3.7683E-04 | 2.9717 |
| | | 20 | 7.5547E-04 | 2.9903 | 1.6556E-04 | 2.8589 |
| $\alpha(t) = \frac{6t-5}{50t}$ | $P^0$ | 5 | 9.0948E-01 | - | 3.8665E-01 | - |
| | | 10 | 4.5659E-01 | 0.9941 | 1.8827E-01 | 1.0382 |
| | | 15 | 3.0461E-01 | 0.9983 | 1.2491E-01 | 1.0119 |
| | | 20 | 2.2851E-01 | 0.9992 | 9.3521E-02 | 1.0059 |
| | $P^1$ | 5 | 3.6398E-01 | - | 9.8087E-02 | - |
| | | 10 | 9.4222E-02 | 1.9497 | 2.4675E-02 | 1.9910 |
| | | 15 | 4.2395E-02 | 1.9696 | 1.0978E-02 | 1.9974 |
| | | 20 | 2.3819E-02 | 2.0041 | 6.1778E-03 | 1.9986 |
| | $P^2$ | 5 | 4.6256E-02 | - | 9.7289E-03 | - |
| | | 10 | 5.7797E-03 | 3.0006 | 1.2393E-03 | 2.9727 |
| | | 15 | 1.7913E-03 | 2.8890 | 3.7244E-04 | 2.9651 |
| | | 20 | 7.6214E-04 | 2.9705 | 1.6561E-04 | 2.8172 |

Next we consider the case $\alpha(t) = \frac{\sqrt{t}-\sin(t)}{48}$ and $\frac{6t-5}{50t}$. Spatial meshes $h = \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}$

and a small time step $\tau = \frac{1}{5000}$ are considered in Table 4.2. We can see that the errors in $L^2$ and $L^\infty$-norm reach optimal order of accuracy for piecewise $P^k$ polynomials, which is consistent with the theoretical results. Finally, we test the temporal convergence rate of the scheme (3.2) for $\alpha(t) = \frac{1+2t}{8}$ by piecewise $P^0$ polynomials. Taking the spatial mesh size $h = \tau$, and the temporal meshes $\tau = \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$, respectively. One can observe that the temporal convergence rate is first order in Figure 4.1, which also supports the theoretical results.

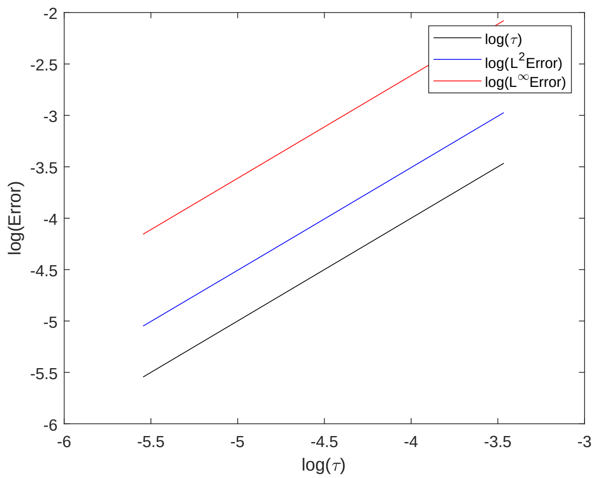


Figure 4.1: $L^2$ errors and $L^\infty$ errors versus $\tau$, order for $\alpha(t) = \frac{1+2t}{8}$, $M = N$, $k = 0$.

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