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Increasing Stability in an Inverse Boundary Value Problem—Bayesian Viewpoint

Pu-Zhao Kow and Jenn-Nan Wang*

Abstract. Motivated by the recent work of Abraham and Nickl on the statistical Calderón problem [2], we revisit the increasing stability phenomenon in the inverse boundary value problem for the stationary wave equation with a potential using the Bayesian approach. In this paper, rather than the Dirichlet-to-Neumann map, we consider another type of boundary measurements called the impedance-to-Neumann map. Its graph forms a subset of Cauchy data. We show the consistency of the posterior mean with a contraction rate demonstrating the phenomenon of increasing stability.

1. Introduction

In this work, we study the inverse boundary value problem for the stationary wave equation with frequency κ in $\mathbb{R}^d$ ($d \geq 3$), which is modeled by the Helmholtz equation with a potential. Let D be a bounded Lipschitz domain in $\mathbb{R}^d$ satisfying

$$x \cdot \nu \geq c_0 > 0 \quad \text{for all } x \in \partial D, \tag{1.1}$$

where ν is the unit normal derivative on ∂D. Furthermore, assume $D \subset B_R$ for some $R > 0$. We consider the following impedance boundary-value problem for Helmholtz equation with a potential

$$\begin{cases} (\Delta + \kappa^2 + q(x))u = 0 & \text{in } D, \\ \partial_\nu u - \mathbf{i}\kappa u = g & \text{on } \partial D \end{cases} \tag{1.2}$$

with $g \in L^2(\partial D)$ and $\kappa > 0$ is the frequency (or wave number). Throughout the paper, we consider $\kappa \geq 1$. The potential function q is real-valued and satisfies

$$\|q\|_{L^\infty(D)} \leq \min\left\{M, \frac{\kappa^2}{16MR^2}, \frac{\kappa^2}{4d-6}\right\} \tag{1.3}$$

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*Corresponding author.

for some $M > 0$. It is easy to see that one can choose $\kappa_0 = \kappa_0(D, M) > 0$ such that

$$\text{for each } \kappa \geq \kappa_0, \text{ (1.3) can be guaranteed by } \|q\|_{L^\infty(D)} \leq M. \tag{1.4}$$

For the well-posedness of the boundary value problem (1.2), we show in Theorem B.5 in Appendix B that there exists a unique solution $u \in H^1(D)$ to (1.2) satisfying

$$\|\nabla u\|_{L^2(D)}^2 + \kappa^2 \|u\|_{L^2(D)}^2 + \|\nabla u\|_{L^2(\partial D)}^2 + \kappa^2 \|u\|_{L^2(\partial D)}^2 \leq C \|g\|_{L^2(\partial D)}^2 \tag{1.5}$$

for some positive constant $C = C(D, c_0)$, see also Remark B.4 for its optimality. We remark that the main tool used in the proof is the Rellich identities (see Lemma B.2). Accordingly, we can define the following bounded linear operator

$$\mathcal{M}_{q,\kappa^2} \colon L^2(\partial D) \to L^2(\partial D), \quad \mathcal{M}_{q,\kappa^2}[g] := \partial_\nu u\big|_{\partial D}, \tag{1.6}$$

which is called the *impedance-to-Neumann map*.

1.1. Deterministic inverse problem

We prove the following stability estimate in the determination of the potential by the measurement $\mathcal{M}_{q,\kappa^2}$ in the deterministic case.

Theorem 1.1. (see also Theorem 2.7) *Let $m \geq 0$ and $s > m + \frac{d}{2}$ be integers. Assume that $M > 0$ and D is a bounded $C^{m,1}$-domain in $\mathbb{R}^d$ satisfying* (1.1)*. Let $q_1, q_2 \in H^{2s}(D)$ be* real-valued *functions satisfying* (1.3), $\operatorname{supp}(q_1 - q_2) \Subset D$ *and* $\sup_{j=1,2} \|q_j\|_{H^{2s}(D)} \leq M$*. Then there exists a constant $C = C(D, s, m, M, \operatorname{supp}(q_1 - q_2)) > 0$ such that*

$$\|q_1 - q_2\|_{H_{\overline{D}}^{-s}} := \|\chi_D(q_1 - q_2)\|_{H^{-s}(\mathbb{R}^d)} \leq C\kappa^{m+3}\mathcal{E} + C\left(\kappa + \log\frac{1}{\mathcal{E}}\right)^{-(s-\frac{d}{2})} \tag{1.7}$$

for all $\kappa \geq 1$ provided $\mathcal{E} := \|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{H^m(\partial D) \to L^2(\partial D)} < 1/e$.

Remark 1.2. The regularity assumption on ∂D is to guarantee that the boundary Sobolev space $H^m(\partial D)$ is well-defined. One also can refer e.g., the monographs [29, 31] for more details about the Sobolev space $H_{\overline{D}}^{-s}$. By slightly modifying the ideas, one can also obtain an analogue result for the impedance-to-Dirichlet map $g \mapsto u|_{\partial D}$, where u is the unique solution of (1.2) satisfying (1.5). The stability estimate in (1.7) consists of two terms. The logarithmic term reflects the ill-posedness of this inverse boundary value problem and may be shown to be optimal by carrying out Mandache's method [30]. However, this logarithmic term decreases as the frequency k increases, and the estimate becomes a Hölder type. The transition from a logarithmic estimate to a Hölder estimate as $\kappa \to \infty$ justifies the phenomenon of increasing stability rigorously.

Before going further, we would like to discuss some related works in the deterministic setting. Assuming that κ^2 is not a Dirichlet eigenvalue of $-\Delta - q(x)$ on D, the Dirichlet boundary value problem of $(\Delta + \kappa^2 + q(x))u = 0$ in D with any suitable Dirichlet data $u|_{\partial D} = f$ is well-posed. Consequently, the Dirichlet-to-Neumann (DN) map

$$\Lambda_{q,\kappa^2} \colon f \mapsto \partial_\nu u\big|_{\partial D}$$

is well-defined. The typical inverse problem is to determine $q(x)$ from Λ_{q,κ^2}. For general κ^2, one may replace the measurement Λ_{q,κ^2} by the set of Cauchy data

$$\mathcal{C}_{q,\kappa^2} := \left\{ \left(u|_{\partial D}, \partial_\nu u\big|_{\partial D}\right) : \ u \in H^1(D) \text{ satisfies } (\Delta + \kappa^2 + q)u = 0 \text{ in } D \right\}$$

endowed with the Hausdorff distance

$$\begin{aligned} \operatorname{dist}(\mathcal{C}_{q_1,\kappa^2}, \mathcal{C}_{q_2,\kappa^2}) = \max \Bigg\{ &\max_{(f,g)\in\mathcal{C}_{q_1}} \max_{(\widetilde{f},\widetilde{g})\in\mathcal{C}_{q_2}} \frac{\|(f,g) - (\widetilde{f},\widetilde{g})\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{1/2}}}, \\ &\max_{(f,g)\in\mathcal{C}_{q_2}} \max_{(\widetilde{f},\widetilde{g})\in\mathcal{C}_{q_1}} \frac{\|(f,g) - (\widetilde{f},\widetilde{g})\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{1/2}}} \Bigg\}. \end{aligned} \tag{1.8}$$

where

$$\|(f,g)\|_{H^{1/2}\oplus H^{1/2}} = \left(\|f\|^2_{H^{1/2}(\partial D)} + \|g\|^2_{H^{-1/2}(\partial D)} \right)^{1/2}.$$

The global injectivity of $q \mapsto \Lambda_{q,\kappa^2}$ or $q \mapsto \mathcal{C}_{q,\kappa^2}$ has been established under different smoothness assumptions on q, see [6–8, 11, 42, 49]. Logarithmic type stability estimates for this inverse problem could be found in [3, 6, 40, 41, 43]. The optimality of the logarithmic stability estimates (in terms of exponential instability) were proved in [20, 21, 30]. Taking the frequency κ into consideration, the increasing stability estimates at the high frequency were derived in [22–25, 44, 45]. On the other hand, following Mandache's approach [30], one can show that the increasing stability estimates are optimal [21, 27]. In this paper, we prove the stability estimate of the inverse boundary value problem (see Theorem 1.1) in terms of an alternative measurement (the impedance-to-Neumann map (1.6)), which has the following two advantages:

- can be easily quantified in terms of the operator norm (compare with the Cauchy data set $\mathcal{C}_{q,\kappa^2}$ with Hausdorff distance (1.8)); and
- there is no eigenvalue issue in this formulation (compare with the DN map Λ_{q,κ^2}).

1.2. Statistical models

From now on, we additionally assume that D has smooth boundary ∂D. In a recent paper [2], Abraham and Nickl study the Calderón problem on determination of the conductivity parameter by the corresponding DN map, based on statistical noise models. Their paper gives rigorous statistical guarantees for the performance of the Bayesian approach to such statistical Calderón problem, a typical nonlinear inverse problem. Their results also provide us some interpretations of Alessandrini's stability estimate [3] and Manache's exponential instability [30] from the viewpoint of the Bayesian de-noise methodology. In this work, we would like to extend Abraham and Nickl's results to the stationary wave equation with a potential (1.2), especially, to verify the increasing stability in the perspective of statistical Bayesian methodology in the non-linear settings. The study of inverse problems in the Bayesian inversion framework has recently attracted much attention since Stuart's seminal article [48] (see also [10]). In addition, the monographs [17, 35] provide mathematical foundations of statistical inverse problems in great detail. On the other hand, some computational aspects of the inversion theory can be found in [26]. For further results on the Bayesian inverse problems in the non-linear settings, we refer the reader to other interesting papers [1, 14, 18, 32–34, 36–39, 51].

Before stating the main results of this paper, we would like to briefly describe three noise models mentioned in [2]. Let us define the map

$$\widetilde{\mathcal{M}}_{q,\kappa^2} := \mathcal{M}_{q,\kappa^2} - \mathcal{M}_{0,\kappa^2},$$

where $\mathcal{M}_{0,\kappa^2}$ is the impedance-to-Neumann map (1.6) corresponding to $q = 0$. Let $\mathbb{1}_{I_p}$ be the indicator of I_p, where $\{I_p\}_{p\leq P}$ is a collection of disjoint measurable subsets of ∂D. Denote $\psi_j = c_j \mathbb{1}_{I_j}$, where c_j is the normalization constant so that $\|\psi_j\|_{L^2(\partial D)} = 1$. We modify the electrode model [2, (1.2)] by considering the following model:

$$\widetilde{Y}_{j\ell} = \langle \widetilde{\mathcal{M}}_{q,\kappa^2}[\psi_j], \psi_\ell \rangle_{L^2(\partial D)} + \varepsilon \widetilde{g}_{j\ell}, \quad \widetilde{g}_{j\ell} \overset{\text{iid}}{\sim} \mathbb{C}\mathcal{N}(0,1), \quad j,\ell \leq P. \tag{1.9}$$

Hereafter, $\mathbb{C}\mathcal{N}(0,1)$ denotes the complex normal defined by $\zeta \sim \mathbb{C}\mathcal{N}(0,1)$ if and only if $\zeta = \Re\zeta + \mathbf{i}\Im\zeta$, where $\Re\zeta$, $\Im\zeta$ are iid standard normals, denoted by $\mathcal{N}(0,1)$. For simplicity, we assume that the noise level $\varepsilon > 0$ is uniform for all $j, \ell \leq P$.

We now introduce another measurement model based on the Laplace–Beltrami operator on ∂D. Let $\{\phi_j\} = \{\phi_j\}_{j=1}^{\infty}$ be the set of real-valued eigenfunctions of the Laplace–Beltrami operator on ∂D, which forms an orthonormal basis of $L^2(\partial D)$. Scaling $\{\phi_j\}$ appropriately, $\{\phi_j^{(p)}\}$ also forms an orthonormal basis of $H^p(\partial D)$ with $p \in \mathbb{R}$, where $H^p(\partial D)$ is the $L^2(\partial D)$-based Sobolev space defined on ∂D, provided that ∂D is sufficiently smooth, with the convention $H^0(\partial D) = L^2(\partial D)$. The data in the spectral noise model is given by

$$\widetilde{Y}_{j\ell} = \langle \widetilde{\mathcal{M}}_{q,\kappa^2}[\phi_j^{(p)}], \phi_j^{(0)} \rangle_{L^2(\partial D)} + \varepsilon \widetilde{g}_{j\ell}, \quad \widetilde{g}_{j\ell} \overset{\text{iid}}{\sim} \mathbb{C}\mathcal{N}(0,1), \quad j,\ell \leq P \tag{1.10}$$

with $p \in \mathbb{R}$. According to [2], the parameter p is chosen by the experimenter and it reflects how the signal-to-error ratio varies with the frequency j of $\phi_j^{(p)}$, as p increases, the signal at high frequencies (p is large) decreases compared to the signal at low frequencies.

We want to make further remarks about (1.10). Note that we can identify the Hilbert space $H^p(\partial D)$ over the complex field with the Hilbert space $H^p_{\mathbb{R}}(\partial D)$ over the real field $\mathbb{R}$. Now the model (1.10) can be written as

$$\begin{aligned}\Re\widetilde{Y}_{j\ell} &= \langle \Re\widetilde{\mathcal{M}}_{q,\kappa^2}[\phi_j^{(p)}], \phi_\ell^{(0)}\rangle_{L^2(\partial D)} + \varepsilon\Re\widetilde{g}_{j\ell},\\ \Im\widetilde{Y}_{j\ell} &= \langle \Im\widetilde{\mathcal{M}}_{q,\kappa^2}[\phi_j^{(p)}], \phi_\ell^{(0)}\rangle_{L^2(\partial D)} + \varepsilon\Im\widetilde{g}_{j\ell}.\end{aligned}$$

Note that $\widetilde{\mathcal{M}}_{q,\kappa^2}$ is a complex linear map on $H^p(\partial D)$. It is not difficult to see that $\Re\widetilde{\mathcal{M}}_{q,\kappa^2}$ is a *real* linear map on $H^p_{\mathbb{R}}(\partial D)$ and

$$\Re\widetilde{\mathcal{M}}_{q,\kappa^2}[\mathbf{i}\phi_j^{(p)}] = \Re\big(\mathbf{i}\widetilde{\mathcal{M}}_{q,\kappa^2}[\phi_j^{(p)}]\big) = -\Im\widetilde{\mathcal{M}}_{q,\kappa^2}[\phi_j^{(p)}].$$

By writing $g_{j\ell} = \Re\widetilde{g}_{j\ell}$, $g'_{j\ell} = -\Im\widetilde{g}_{j\ell}$ and $Y_{j\ell} = (\Re\widetilde{Y}_{j\ell}, -\Im\widetilde{Y}_{j\ell})^\intercal$, we see that the model (1.10) is equivalent to

$$Y_{j\ell} = \begin{cases}\langle \Re\widetilde{\mathcal{M}}_{q,\kappa^2}[\phi_j^{(p)}], \phi_\ell^{(0)}\rangle_{L^2(\partial D)} + \varepsilon g_{j\ell},\\ \langle \Re\widetilde{\mathcal{M}}_{q,\kappa^2}[\mathbf{i}\phi_j^{(p)}], \phi_\ell^{(0)}\rangle_{L^2(\partial D)} + \varepsilon g'_{j\ell}\end{cases}$$

for $g_{j\ell}, g'_{j\ell} \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$. In other words, $\widetilde{\mathcal{M}}_{q,\kappa^2}$ acting on $H^r(\partial D)$ is completely determined by $\Re\widetilde{\mathcal{M}}_{q,\kappa^2}$ acting on $H^p_{\mathbb{R}}(\partial D)$ and vice versa.

The third model studied here is a continuous model, which can be formally considered as the limit model of the discrete one (1.2) as $j,\ell \to \infty$. To be precise, we consider a Gaussian white noise model on a space of Hilbert–Schmidt operators (a separable Hilbert space). Each real linear operator $T\colon H^p_{\mathbb{R}}(\partial D) \to L^2(\partial D)$ can be represented as follows: for any real $f \in H^p(\partial D)$,

$$\begin{aligned}T(f) &:= \sum_{j,\ell=1}^{\infty} t_{j\ell}\langle f, \phi_j^{(p)}\rangle_{H^p(\partial D)}\phi_\ell^{(0)} = \sum_{j,\ell=1}^{\infty} t_{j\ell} b_{j\ell}^{(p)}(f),\\ T(\mathbf{i}f) &:= -\sum_{j,\ell=1}^{\infty} t'_{j\ell}\langle f, \phi_j^{(p)}\rangle_{H^p(\partial D)}\phi_\ell^{(0)} = \sum_{j,\ell=1}^{\infty} t'_{j\ell} \widetilde{b}_{j\ell}^{(p)}(\mathbf{i}f),\end{aligned} \tag{1.11}$$

where

$$b_{j\ell}^{(p)}(f) = \phi_j^{(p)} \otimes \phi_\ell^{(0)}(f) = \langle f, \phi_j^{(p)}\rangle_{H^p(\partial D)}\phi_\ell^{(0)},$$

$$\widetilde{b}_{j\ell}^{(p)}(\mathbf{i}f) = \mathbf{i}\phi_j^{(p)} \otimes \phi_\ell^{(0)}(\mathbf{i}f) = -\langle f, \phi_j^{(p)}\rangle_{H^p(\partial D)}\phi_\ell^{(0)} \quad \text{and} \quad \sum_{j,\ell=1}^{\infty}\big(t_{j,\ell}^2 + (t'_{j\ell})^2\big) < \infty.$$

Denote $\mathbb{H}_p$ the space of all real-valued Hilbert–Schmidt operators $H^p_{\mathbb{R}}(\partial D) \to L^2(\partial D)$ and

$\mathbb{H}_p$ itself a Hilbert space with the inner product

$$\begin{aligned}\langle S,T\rangle_{\mathbb{H}_p} &= \sum_{j,\ell=1}^{\infty}(s_{j\ell}t_{j\ell}+s'_{j\ell}t'_{j\ell}) \\ &= \sum_{j,\ell=1}^{\infty}\Big(\langle S\phi_j^{(p)},\phi_\ell^{(0)}\rangle_{L^2(\partial D)}\langle T\phi_j^{(p)},\phi_\ell^{(0)}\rangle_{L^2(\partial D)} \\ &\qquad + \langle S(\mathbf{i}\phi_j^{(p)}),\phi_\ell^{(0)}\rangle_{L^2(\partial D)}\langle T(\mathbf{i}\phi_j^{(p)}),\phi_\ell^{(0)}\rangle_{L^2(\partial D)}\Big),\end{aligned}$$

where $s_{j\ell}$, $s'_{j\ell}$, $t_{j\ell}$, $t'_{j\ell}$ are defined as in (1.11).

The continuous model with the Gaussian white noise defined on the space of Hilbert–Schmidt operators $\mathbb{H}_p$ is given by

$$Y = \Re\widetilde{\mathcal{M}}_{q,\kappa^2} + \varepsilon\mathbb{W} \tag{1.12a}$$

which is realized as a Gaussian process indexed by $\mathbb{H}_p$, namely,

$$\langle Y,T\rangle_{\mathbb{H}_p} = \langle\Re\widetilde{\mathcal{M}}_{q,\kappa^2},T\rangle_{\mathbb{H}_p} + \varepsilon\langle\mathbb{W},T\rangle_{\mathbb{H}_p} \quad \text{for all } T\in\mathbb{H}_p, \tag{1.12b}$$

where

$$\langle\mathbb{W},T\rangle_{\mathbb{H}_p} = \sum_{j,\ell=1}^{\infty} g_{j\ell}\langle T\phi_j^{(p)},\phi_\ell^{(0)}\rangle_{L^2(\partial D)} + \sum_{j,\ell=1}^{\infty} g'_{j\ell}\langle T(\mathbf{i}\phi_j^{(p)}),\phi_\ell^{(0)}\rangle_{L^2(\partial D)}$$

for $g_{j,\ell}, g'_{j,\ell} \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$. In other words, the process $\mathbb{W}$ is an *isonormal Gaussian process* indexed by the Hilbert space $\mathbb{H}_p$, see e.g., the monographs [17, 35]. Note that

$$\mathbb{E}[\mathbb{W}(T)\mathbb{W}(S)] = \langle T,S\rangle_{\mathbb{H}_p} \quad \text{for all } T,S\in\mathbb{H}_p.$$

Let $\mathbb{P}_\varepsilon^{q,\kappa^2}$ (also depends on p) denote the probability law of Y in (1.12a) and $\mathbb{E}_\varepsilon^{q,\kappa^2}$ be the corresponding expectation. One sees that $\mathbb{P}_\varepsilon^{0,\kappa^2}$ is the probability law of $\varepsilon\mathbb{W}$. We mainly focus on the model (1.12a), see Theorems 1.3 and 1.7 below. By following the ideas in [2, Appendix D], one can also obtain similar results for the model (1.9) and (1.10). The work [2] establishes the "equivalence" of three models described above for Calderón's problem. Likewise, the same proofs work for the measurement $\mathcal{M}_{q,\kappa^2}$ here.

1.3. Statistical inverse problem

Let $D_0 \Subset D$ be an open domain[1], $\alpha > 0$ and $M > 0$. We define

$$\begin{aligned}\mathcal{V}_{D_0} &= \big\{q\in C^0(\overline{D},\mathbb{R}) : q(x)=0 \text{ in } D\setminus\overline{D_0}\big\}, \\ \mathcal{V}_{D_0}^{\alpha}(M) &= \big\{q\in\mathcal{V}_{D_0} : \|q\|_{H^\alpha(D)}\le M\big\}.\end{aligned}$$

We will prove a contraction result for the continuous model (1.12a) in the next theorem.

[1]This means that $\overline{D_0}\subset D$.

Theorem 1.3. *Let $M > 0$ and $0 < c_0 < 1$ be real parameters. Let D be a bounded smooth domain in $\mathbb{R}^d$ satisfying* (1.1) *and $D_0 \Subset D$. Let $p \geq 2d-1$ and let α, β be integers satisfying $\alpha > \beta \geq 7d/2$. Then there exist positive constants $C = C(D, c_0, p, \beta, M, D_0)$ and $\kappa_0 = \kappa_0(\alpha, D, M)$, and for each $\kappa \geq \kappa_0$ there exists a measurable function $q_{\varepsilon,\kappa^2} = q_{\varepsilon,\kappa^2}(Y)$ of the observations $Y \sim \mathbb{P}_\varepsilon^{q,\kappa^2}$ such that*

$$\sup_{\kappa \geq \kappa_0,\, q \in \mathcal{V}_{D_0}^{\alpha}(M)} \mathbb{P}_\varepsilon^{q,\kappa^2}\big(\|q_{\varepsilon,\kappa^2} - q\|_\infty > C\xi_\kappa(\varepsilon)\big) \to 0 \quad \text{as } \varepsilon \to 0,$$

where the factor $\xi_\kappa(\varepsilon)$ is explicitly defined by

$$\xi_\kappa(\varepsilon) := \kappa^{d/2+1} \varepsilon^{\frac{\alpha}{2(\alpha+d)(p-d+1)}} + \left(\kappa + \log\frac{1}{\varepsilon}\right)^{-d/2} \tag{1.13}$$

for all sufficiently small $\varepsilon > 0$.

It is important to point out that the contraction rate $\xi_k(\varepsilon)$ consists of two parts: a logarithmic rate and a Hölder rate. The logarithmic rate decreases as κ increases. In other words, the rate becomes Hölder-type dominated at high frequencies. Theorem 1.3 reflects the phenomenon of increasing stability in the determination of the potential by the impedance-to-Neumann map as explained in Theorem 1.1.

In accordance with (1.13), even though the statistical rate is dominated by the term $\kappa^{d/2+1}\varepsilon^{\frac{\alpha}{2(\alpha+d)(p-d+1)}}$ at large κ, the constant $\kappa^{d/2+1}$ there suggests the decline in experimental quality if κ is "too large". Since the limit in Theorem 1.3 is uniform with respect to κ, we could choose κ as a function of the noise level ε of the statistical model described in (1.12a) and (1.12b), for example, we can take

$$\kappa(\varepsilon) = \frac{1}{\varepsilon^\theta} \quad \text{with} \quad \left(\frac{d}{2} + 1\right)\theta \leq \frac{\alpha}{2(\alpha+d)(p-d+1)}, \tag{1.14}$$

which gives

$$\begin{aligned} \xi(\varepsilon) = \xi_{\kappa(\varepsilon)}(\varepsilon) &= \varepsilon^{\frac{\alpha}{2(\alpha+d)(p-d+1)} - \left(\frac{d}{2}+1\right)\theta} + \left(\frac{1}{\varepsilon^\theta} + \log\frac{1}{\varepsilon}\right)^{-d/2} \\ &\leq \varepsilon^{\frac{\alpha}{2(\alpha+d)(p-d+1)} - \left(\frac{d}{2}+1\right)\theta} + \varepsilon^{d\theta/2} \leq 2\varepsilon^{\theta_0}, \end{aligned}$$

where $\theta_0 := \min\big\{\frac{\alpha}{2(\alpha+d)(p-d+1)} - \big(\frac{d}{2}+1\big)\theta, \frac{d\theta}{2}\big\}$. Note that $\theta_0 \in (0,1)$. We now obtain a corollary from Theorem 1.3.

Corollary 1.4. *Assume that the assumptions of Theorem 1.3 hold. Let κ be chosen as in* (1.14), *and write $\mathbb{P}_\varepsilon^q = \mathbb{P}_\varepsilon^{q,\kappa^2}$ and $q_\varepsilon = q_{\varepsilon,\kappa^2}$. Then there exist positive constants $C = C(D, c_0, p, \beta, M, D_0)$ such that*

$$\sup_{q \in \mathcal{V}_{D_0}^{\alpha}(M)} \mathbb{P}_\varepsilon^q\big(\|q_\varepsilon - q\|_\infty > C\varepsilon^{\theta_0}\big) \to 0 \quad \text{as } \varepsilon \to 0, \tag{1.15}$$

where $\theta_0 = \min\big\{\frac{\alpha}{2(\alpha+d)(p-d+1)} - \big(\frac{d}{2}+1\big)\theta, \frac{d\theta}{2}\big\}$.

A Hölder contraction rate in (1.15) indicates that this inverse boundary value problem is "mildly" ill-posed by aligning the wave number κ wisely with the noise level ε. The Hölder contraction rate corresponds to the Hölder stability estimate in the inverse boundary problem for the wave equation [4].

The construction of the estimator q_{ε,κ^2} in Theorem 1.3 follows from the Bayesian approach to inverse problems explained in great details in [10,17,35,48]. Roughly speaking, this estimator is constructed by the posterior mean arising from some given Gaussian process prior. To this end, we would like to discuss the existence of a posterior distribution in the Gaussian white noise model. As above, $\mathbb{W}$ is a centered Gaussian white noise indexed by $T \in \mathbb{H}_p$, which we also denote as $(\mathbb{W}(T) : T \in \mathbb{H}_p)$, with covariance $\mathbb{E}(\mathbb{W}(T)\mathbb{W}(S)) = \langle T, S\rangle_{\mathbb{H}_p}$. Since the covariance operator of $\mathbb{W}$ is not of trace class, $\mathbb{W}$ cannot be realized as a random element in $\mathbb{H}_p$. To overcome this inconvenience, we can expand the space of $\mathbb{H}_p$ to a weighed Hilbert space as described in [34, Section 7.4, (110)]. Similar to [2, (13)], by the Cameron–Martin theorem, one can show that the law $\mathbb{P}_\varepsilon^{q,\kappa^2}$ is dominated by the law $\mathbb{P}_\varepsilon^{0,\kappa^2}$ (i.e., the law $\varepsilon\mathbb{W}$) with the log-likehood function

$$\ell(q) \equiv \log \mathsf{p}_\varepsilon^{q,\kappa^2}(Y) := \log \frac{\mathrm{d}\mathbb{P}_\varepsilon^{q,\kappa^2}}{\mathrm{d}\mathbb{P}_\varepsilon^{0,\kappa^2}}(Y) = \frac{1}{\varepsilon^2}\langle \Re\widetilde{\mathcal{M}}_{q,\kappa^2}, Y\rangle_{\mathbb{H}_p} - \frac{1}{2\varepsilon^2}\|\Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p}^2 \tag{1.16}$$

for $q \in \mathcal{V}_{D'}$ with $D' \Subset D$, see also [34, Section 7.4, (110)]. The derivation of (1.16) requires the Borel measurability of the mapping $q \mapsto \Re\widetilde{\mathcal{M}}_{q,\kappa^2}$ from the (Polish) space $\mathcal{V}_{D'}$ equipped with the $\|\cdot\|_\infty$-topology into the Hilbert space $\mathbb{H}_p$, which can be guaranteed by Lemma 2.6 below.

Assume that Π is a prior probability distribution on $(\mathcal{V}_{D'}, \mathcal{B}_{\mathcal{V}_{D'}})$, where $\mathcal{B}_{\mathcal{V}_{D'}}$ is the Borel σ-field of the (Polish) space $\mathcal{V}_{D'}$. The Bayes theorem implies

$$\Pi(B|Y) = \frac{\int_B \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)\,\mathrm{d}\Pi(q)}{\int_{\mathcal{V}_{D'}} \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)\,\mathrm{d}\Pi(q)} \quad \text{for all } B \in \mathcal{B}_{\mathcal{V}_{D'}}, \tag{1.17}$$

see e.g., [34, Section 7.4, (111)]. In what follows, we denote $\mathbb{E}^\Pi(\cdot)$ the expectation operator with respect to the prior and $\mathbb{E}^\Pi(\,\cdot\,|Y)$ the expectation operator with respect to the posterior.

Inspire by the prior construction introduced in [2], here we consider the priors that are given by appropriate scalings of a Gaussian process prior. For this end, a base prior Π' satisfying the following assumption is chosen (we consider priors which are slightly smooter than [2, Assumption 1] in view of the stability estimate proved in Theorem 2.7 below, see also [35, Condition 2.2.1]):

Assumption 1.5. *Let* Π' *be a centered Gaussian Borel probability measure on the Banach space* $C^0(\overline{D})$*, and let* α*,* β *be integers satisfying* $\alpha > \beta \geq 7d/2$*. Assume that*

$\Pi'(H^\beta(D)) = 1$ *and the reproducing kernel Hilbert space (RKHS)* $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ *of* Π' *is continuously embedded into the Sobolev space* $H^\alpha(D)$.

Example 1.6. As explained in [2], the restrictions of Gaussian processes with covariance given by Whittle–Matérn kernels satisfy Assumption 1.5 for any α, β satisfying $2 + \frac{d}{2} < \beta < \alpha - \frac{d}{2}$ and $\mathcal{H} = H^\alpha(D)$.

Now let $\zeta\colon D \to [0,1]$ be a smooth cutoff function satisfying that $\zeta = 1$ on D_0 and $\mathrm{supp}(\zeta) \subset D_1$ where $D_0 \Subset D_1 \Subset D$. The induced prior on q is given by

$$q = \varepsilon^{\frac{d}{\alpha+d}} \zeta\theta' \quad \text{where } \theta' \sim \Pi'. \tag{1.18}$$

The key parameter ε can be interpreted as a penalized parameter. The law on q is denoted by Π_ε. We also assume that the "true" potential q_0 lies inside the induced priors on q, i.e.,

$$q_0 = \varepsilon^{\frac{d}{\alpha+d}} \zeta\theta_0 \quad \text{for some } \theta_0 \in H^\alpha(D) \text{ with } \mathrm{supp}(\theta_0) \subset D_0.$$

We now state a key contraction result for the posterior distribution. Theorem 1.3 then follows from this contraction result. Let $\Pi(\,\cdot\,|Y)$ be the posterior distribution of q conditioned on the observations Y in the model (1.12a).

Theorem 1.7. *Let* $M > 0$, $0 < c_0 < 1$ *and* $p > 2d - 2$ *be given parameters. Assume that* D *is a bounded smooth domain in* $\mathbb{R}^d$ *satisfying* (1.1) *and* $D_0 \Subset D_1 \Subset D$. *Let* α, β *be integers satisfying* $\alpha > \beta \geq 7d/2$. *The base prior* Π' *satisfies Assumption* 1.5 *and the rescaled prior* Π_ε *is given in* (1.18). *Assume that the "ground truth"* q_0 *belongs to the set*

$$\mathcal{Q} := \mathcal{V}_{D_0} \cap \big\{ q \in \mathcal{H} : \|q\|_{\mathcal{H}} \leq M \big\},$$

where $\mathcal{H}$ *is the RKHS given in Assumption* 1.5. *Then there exists a positive constant* $C = C(D, c_0, p, \alpha, \beta, M, D_1, D_0)$ *such that*

$$\Pi_\varepsilon\big(\|q - q_0\|_{L^\infty(D)} > C\xi_\kappa(\varepsilon) | Y \big) \xrightarrow{\mathbb{P}_\varepsilon^{q_0,\kappa^2}} 0 \quad \text{as } \varepsilon \to 0, \tag{1.19}$$

where $\xi_\kappa(\varepsilon)$ *is given in* (1.13). *In addition, there exists a constant* $\kappa_0 = \kappa_0(\alpha, D, M)$ *such that for each* $K > C$, *it holds that*

$$\sup_{\kappa \geq \kappa_0,\, q_0 \in \mathcal{Q}} \mathbb{P}_\varepsilon^{q_0,\kappa^2} \big(\|\mathbb{E}^{\Pi_\varepsilon}(q|Y) - q_0\|_{L^\infty(D)} > K\xi_\kappa(\varepsilon) \big) \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.20}$$

By setting $q_{\varepsilon,\kappa^2} = \mathbb{E}^{\Pi_\varepsilon}(q|Y)$, it is clear that Theorem 1.3 is an easy consequence of (1.20). Indeed, since $\alpha > 7d/2$, we can choose an integer $\beta \geq 7d/2$, and, therefore, Assumption 1.5 holds.

1.4. Organization of the paper

We postpone the proof of Theorem 1.1 to Appendix A. In order to explain the ideas clearly, we split the proof of Theorem 1.7 in Sections 2–4. In order to make the paper self-contained, we also provide a proof of the well-posedness for the impedance boundary-value problem (1.2) in Appendix B.

2. Stability estimate in terms of Hilbert—Schmidt norm

In order to prove Theorem 1.7, we need to measure $\Re\mathcal{M}_{q_1,\kappa^2}-\Re\mathcal{M}_{q_2,\kappa^2}$ (as well as $\mathcal{M}_{q_1,\kappa^2}-\mathcal{M}_{q_2,\kappa^2}$, see Lemma 2.5 below) in terms of Hilbert–Schmidt norm rather than the operator norm used in Theorem 1.1. In order to do so, we will consider the *low rank approximation* by projecting $\Re\widetilde{\mathcal{M}}_{q,\kappa^2}$ onto a finite-dimensional subspace by employing the idea in [2]. Our focus here is to keep track of the dependence of the key parameter κ. Recall that the collection $\{(b_{j\ell}^{(r)},\widetilde{b}_{j\ell}^{(r)})\}_{j,\ell\in\mathbb{N}}$ forms an orthonormal basis of the space of Hilbert–Schmidt operators mapping from $H_{\mathbb{R}}^r(\partial D)$ into $L^2(\partial D)$, where

$$\begin{aligned} b_{j\ell}^{(r)}(f) &= \phi_j^{(r)}\otimes\phi_\ell^{(0)}(f) = \langle f,\phi_j^{(r)}\rangle_{H^r(\partial D)}\phi_\ell^{(0)}, \\ \widetilde{b}_{j\ell}^{(r)}(\mathbf{i}f) &= \mathbf{i}\phi_j^{(r)}\otimes\phi_\ell^{(0)}(\mathbf{i}f) = -\langle f,\phi_j^{(r)}\rangle_{H^r(\partial D)}\phi_\ell^{(0)} \end{aligned}$$

for all real-valued $f\in H^r(\partial D)$. For separable Hilbert spaces A and B, let $\mathcal{L}(A,B)$ be the space of bounded linear operators mapping from A into B endowed with the operator norm $\|\cdot\|_{\mathcal{L}(A,B)}$, and let $\mathcal{L}_2(A,B)$ be the space of Hilbert–Schmidt operators mapping from A into B equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{L}_2(A,B)}$. Moreover, similar as above, define the orthonormal basis $\{(b_{j\ell}^{(p,r)},\widetilde{b}_{j\ell}^{(p,r)})\}_{j,\ell\in\mathbb{N}}$ of $\mathcal{L}_2(H_{\mathbb{R}}^p,H^r)=\mathcal{L}_2(H_{\mathbb{R}}^p(\partial D),H^r(\partial D))$ by

$$\begin{aligned} b_{j\ell}^{(p,r)}(f) &= \phi_j^{(p)}\otimes\phi_\ell^{(r)}(f) = \langle f,\phi_j^{(p)}\rangle_{H^r(\partial D)}\phi_\ell^{(r)}, \\ \widetilde{b}_{j\ell}^{(p,r)}(\mathbf{i}f) &= \mathbf{i}\phi_j^{(p)}\otimes\phi_\ell^{(r)}(\mathbf{i}f) = -\langle f,\phi_j^{(p)}\rangle_{H^r(\partial D)}\phi_\ell^{(r)} \end{aligned}$$

for all real-valued $f\in H^r(\partial D)$. Using this convention, one sees that $\mathbb{H}_q=\mathcal{L}_2(H_{\mathbb{R}}^q,H^0)$. We first recall two lemmas controlling Hilbert–Schmidt norms for different domains and codomains in terms of each other, and in terms of operator norms.

Lemma 2.1. [2, Lemma 17] *For $p,r,s,t\in\mathbb{R}$, let*

$$T\in\operatorname{span}\big\{(b_{j\ell}^{(p,r)},\widetilde{b}_{j\ell}^{(p,r)}):1\le j\le J,1\le\ell\le K\big\}.$$

Then there is a constant C, depending on D and the differences $r-p$ and $s-t$, such that

$$\|T\|_{\mathcal{L}_2(H_{\mathbb{R}}^r,H^s)}\le C\big(1+J^{\frac{1}{d-1}}\big)^{(p-r)_+}\big(1+K^{\frac{1}{d-1}}\big)^{(s-t)_+}\|T\|_{\mathcal{L}_2(H_{\mathbb{R}}^p,H^t)},$$

where $x_+=\max\{x,0\}$ for $x\in\mathbb{R}$.

Lemma 2.2. [2, Lemma 18] *For $p, r, s, t \in \mathbb{R}$ with $p \le r$ and $s \le t$, let $T \in \mathcal{L}(H_{\mathbb{R}}^{p-(d-1)}, H^t)$. Then $T \in \mathcal{L}_2(H_{\mathbb{R}}^r, H^s)$ and there exists a constant C, depending on D and $r-p$, $s-t$, such that*

$$\|T\|_{\mathcal{L}_2(H_{\mathbb{R}}^r, H^s)} \le C\|T\|_{\mathcal{L}(H_{\mathbb{R}}^{p-(d-1)}, H^t)} \tag{2.1}$$

and the following low-rank approximation *holds:*

$$\|T - \pi_{JK}T\|_{\mathcal{L}_2(H_{\mathbb{R}}^r, H^s)} \le C\|T\|_{\mathcal{L}(H_{\mathbb{R}}^{p-(d-1)}, H^t)} \max\left\{(1 + J^{\frac{1}{d-1}})^{p-r}(1 + K^{\frac{1}{d-1}})^{s-t}\right\}, \tag{2.2}$$

where the projection map π_{JK} is given by

$$\begin{aligned} \pi_{JK}T &= \left(\sum_{j\le J, \ell \le K} \langle T, b_{j\ell}^{(r)} \rangle_{\mathbb{H}_r} b_{j\ell}^{(r)}, \sum_{j\le J, \ell\le K} \langle T, \widetilde{b}_{j\ell}^{(r)} \rangle_{\mathbb{H}_r} \widetilde{b}_{j\ell}^{(r)} \right) \\ &= \left(\sum_{j\le J, \ell\le K} \langle T\phi_j^{(r)}, \phi_\ell^{(0)} \rangle_{L^2(\partial D)} b_{j\ell}^{(r)}, \sum_{j\le J, \ell\le K} \langle T(\mathbf{i}\phi_j^{(r)}), \phi_\ell^{(0)} \rangle_{L^2(\partial D)} \widetilde{b}_{j\ell}^{(r)} \right). \end{aligned}$$

We now show the following lemma.

Lemma 2.3. *Assume that D is a bounded smooth domain in $\mathbb{R}^d$ satisfying* (1.1)*. Let q_1, q_2 be* real-valued *functions satisfying* (1.3)*. Suppose further that $\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} < 1$. Then for parameters satisfying $d-1 < m \le p$, there exists a positive constant $C_0 = C_0(D, c_0, p, m)$, which is independent of κ, such that*

$$\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathcal{L}(H_{\mathbb{R}}^m, L^2)} \le C_0\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p}^{\frac{d-1-m}{d-1-p}}. \tag{2.3}$$

Note that it suffices to take $C_0(D, c_0, p, m) > 1$.

Remark 2.4. Note that by (2.1), for each $m + d - 1 \le p$, we have

$$\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} \le C\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathcal{L}(H_{\mathbb{R}}^m, L^2)} \tag{2.4}$$

for some $C = C(D, m, p)$.

Proof of Lemma 2.3. It is not difficult to check that the operator norm of a linear operator between separable Hilbert spaces is bounded by its Hilbert norm: For each $m \in \mathbb{R}$, one has

$$\begin{aligned} &\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathcal{L}(H_{\mathbb{R}}^m, L^2)} \\ &\le \|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_m} \\ &\le \|\Re\mathcal{M}_{q_1,\kappa^2} - \pi_{JJ}\Re\mathcal{M}_{q_1,\kappa^2}\|_{\mathbb{H}_m} + \|\Re\mathcal{M}_{q_2,\kappa^2} - \pi_{JJ}\Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_m} \\ &\quad + \|\pi_{JJ}\Re\mathcal{M}_{q_1,\kappa^2} - \pi_{JJ}\Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_m}. \end{aligned}$$

Using (2.2)[2] in Lemma 2.2, together with the fact $\|\Re\mathcal{M}_{q_j,\kappa^2}\|_{\mathcal{L}(L^2_{\mathbb{R}},L^2)} \le \|\mathcal{M}_{q_j,\kappa^2}\|_{\mathcal{L}(L^2,L^2)} \le C(D,c_0)$ (which is a consequence of the energy estimate (1.5)), for each $j=1,2$, we know that there exists a positive constant $C = C(D,c_0,m)$ such that

$$\|\Re\mathcal{M}_{q_j,\kappa^2} - \pi_{JJ}\Re\mathcal{M}_{q_j,\kappa^2}\|_{\mathbb{H}_m} \le C(1+J^{\frac{1}{d-1}})^{d-1-m} \quad \text{for all } m \ge d-1. \tag{2.5}$$

On the other hand, by Lemma 2.1[3], we see that there exists a positive constant $C = C(D,p,m)$ such that

$$\begin{aligned} &\|\pi_{JJ}\Re\mathcal{M}_{q_1,\kappa^2} - \pi_{JJ}\Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_m} \\ &\le C(1+J^{\frac{1}{d-1}})^{p-m}\|\pi_{JJ}\Re\mathcal{M}_{q_1,\kappa^2} - \pi_{JJ}\Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} \\ &\le C(1+J^{\frac{1}{d-1}})^{p-m}\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} \quad \text{for all } p \ge m. \end{aligned}$$

From three estimates above, it follows easily that there exists a positive constant $C = C(D,c_0,p,m)$ such that

$$\begin{aligned} &\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathcal{L}(H^m_{\mathbb{R}},L^2)} \\ &\le C\big((1+J^{\frac{1}{d-1}})^{d-1-m} + (1+J^{\frac{1}{d-1}})^{p-m}\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p}\big) \\ &\le C\big(J^{\frac{d-1-m}{d-1}} + J^{\frac{p-m}{d-1}}\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p}\big) \end{aligned}$$

for all $d-1 \le m \le p$ and integers $J \ge 1$. We now restrict the parameters $d-1 < m \le p$. Since $\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} < 1$, choosing

$$J = \left\lfloor \|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p}^{\frac{d-1}{d-1-p}} \right\rfloor$$

in the estimate above, (2.3) follows immediately. □

We next show that measurements $\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}$ and $\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}$ are equivalent.

Lemma 2.5. *Suppose that all assumptions in Lemma 2.3 are satisfied. Then for each $d-1 < m \le p-d+1$, we have*

$$\|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} = \|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p}$$

and

$$\begin{aligned} \|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_1,\kappa^2}\|_{\mathcal{L}(H^m_{\mathbb{R}},L^2)} &\le \|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_1,\kappa^2}\|_{\mathcal{L}(H^m,L^2)} \\ &\le 2\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_1,\kappa^2}\|_{\mathcal{L}(H^m_{\mathbb{R}},L^2)}. \end{aligned}$$

Here $\|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p}$ is defined in terms of the basis $\{b^{(p)}_{j\ell}\}$.

[2] We choose $s=t=0$, $r=m \ge d-1$, $p=d-1$, $K=J$ and $T=\Re\mathcal{M}_{q_j,\kappa^2}$ (for $j=0,1$).

[3] We choose $s=t=0$, $r=m$, $K=J$ and $T=\pi_{JJ}\Re\mathcal{M}_{q_1,\kappa^2} - \pi_{JJ}\Re\mathcal{M}_{q_1,\kappa^2}$.

Proof. The first estimate is obvious since

$$(\Im\mathcal{M}_{q_1,\kappa^2} - \Im\mathcal{M}_{q_2,\kappa^2})(\phi_j^{(p)}) = (\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2})(\mathbf{i}\phi_j^{(p)}) \quad \text{for all } j.$$

The first inequality in the second estimate is clear. On the other hand, we can derive

$$\begin{aligned}
&\|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{\mathcal{L}(H^m,L^2)} \\
&= \sup_{f\neq 0} \frac{\|(\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2})(f)\|_{L^2(\partial D)}}{\|f\|_{H^m(\partial D)}} \\
&\leq \sup_{f\neq 0} \left(\frac{\|(\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2})(f)\|_{L^2(\partial D)}}{\|f\|_{H^m(\partial D)}} + \frac{\|(\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2})(\mathbf{i}f)\|_{L^2(\partial D)}}{\|\mathbf{i}f\|_{H^m(\partial D)}} \right) \\
&\leq 2\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_1,\kappa^2}\|_{\mathcal{L}(H^m_{\mathbb{R}},L^2)},
\end{aligned}$$

which implies the second inequality of the second estimate. □

We now prove the continuity of the mapping $q \mapsto \Re\mathcal{M}_{q,\kappa^2}$ in terms of Hilbert–Schmidt norm.

Lemma 2.6. *For each $p > 2d-2$, we have*

$$\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} \leq C\|q_1 - q_2\|_{L^\infty(D)}$$

for some positive constant $C = C(D, c_0, p)$, which is independent of κ.

Proof. For $p > 2d-2$, we can choose m satisfying $d-1 < m \leq p-d+1$. For each $j = 1, 2$, let u_j be the solution of (1.2) with $q = q_j$, then

$$\begin{cases} (\Delta + \kappa^2 + q_1)(u_1 - u_2) = (q_2 - q_1)u_2 & \text{in } D, \\ \partial_\nu(u_1 - u_2) - \mathbf{i}\kappa(u_1 - u_2) = 0 & \text{on } \partial D. \end{cases}$$

By Theorem B.3, we can obtain

$$\begin{aligned}
\|\nabla(u_1 - u_2)\|_{L^2(\partial D)} &\leq C\|(q_1 - q_2)u_2\|_{L^2(D)} \leq C\|u_2\|_{L^2(D)}\|q_1 - q_2\|_{L^\infty(D)} \\
&\leq C\kappa^{-1}\|g\|_{L^2(\partial D)}\|q_1 - q_2\|_{L^\infty(D)} \leq C\kappa^{-1}\|g\|_{H^m(\partial D)}\|q_1 - q_2\|_{L^\infty(D)},
\end{aligned}$$

which implies

$$\|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{\mathcal{L}(H^m,L^2)} \leq C\|q_1 - q_2\|_{L^\infty(D)}$$

for some positive constant $C = C(D, c_0)$. Our lemma then follows from (2.4) and Lemma 2.5. □

We end this section by proving a result analogue to Theorem 1.1, but in terms of Hilbert–Schmidt norms.

Theorem 2.7. *Let D be a bounded smooth domain in $\mathbb{R}^d$ satisfying* (1.1)*. Assume that q_1, q_2 are* real-valued *functions satisfying* (1.3)*,* $\operatorname{supp}(q_1 - q_2) \subset D$ *and* $\sup_{j=1,2} \|q_j\|_{H^\beta(D)} \leq M$ *for some integer $\beta \geq 7d/2$, and fix $p \geq 2d-1$. Then there exist a positive constant $C = C(D, c_0, p, \beta, M, \operatorname{supp}(q_1 - q_2))$, independent of κ, such that*

$$\|q_1 - q_2\|_{L^\infty(D)} \leq C\widetilde{\xi}_\kappa\big(\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p}\big),$$

where $\widetilde{\xi}_\kappa$ is given by

$$\widetilde{\xi}_\kappa(\zeta) = \begin{cases} 0 & \text{if } \zeta = 0, \\ \kappa^{d/2+1}\zeta^{\frac{1}{2(p-d+1)}} + \big(\kappa + \frac{1}{p-d+1}\log\frac{1}{\zeta}\big)^{-d/2} & \text{if } 0 < \zeta < \frac{1}{(2C_0e)^{p-d+1}}, \\ 1 & \text{otherwise.} \end{cases} \tag{2.6}$$

Here C_0 is the constant obtained in Lemma 2.3*.*

Proof. We write $\beta = 3s - d$ and s is an integer satisfying $s \geq \frac{3d}{2}$, and so $\|q_j\|_{H^{2s}(D)} \leq \|q_j\|_{H^\beta(D)} \leq M$. By the Sobolev embedding and Theorem 1.1 (with $m = d$), one sees that there exists a positive constant $C = C(D, \beta, M, \operatorname{supp}(q_1 - q_2))$ such that

$$\begin{aligned} &\|q_1 - q_2\|_{L^\infty(D)} \\ &\leq C\|(q_1 - q_2)\chi_D\|_{H^{s-d}(\mathbb{R}^d)} \leq C\|(q_1 - q_2)\chi_D\|_{H^{-s}(\mathbb{R}^d)}^{1/2}\|q_1 - q_2\|_{H^{3s-d}(D)}^{1/2} \\ &\leq C\kappa^{d/2+3/2}\mathcal{E}^{1/2} + C\left(\kappa + \log\frac{1}{\mathcal{E}}\right)^{-\frac{1}{2}(s-\frac{d}{2})} \leq C\kappa^{d/2+3/2}\mathcal{E}^{1/2} + C\left(\kappa + \log\frac{1}{\mathcal{E}}\right)^{-d/2} \end{aligned} \tag{2.7}$$

provided $\mathcal{E} = \|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{\mathcal{L}(H^d,L^2)} < 1/e$. Combining Lemmas 2.3 and 2.5 (with $m = d$) yields that

$$\mathcal{E} \leq \|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathcal{L}(H^d_{\mathbb{R}},L^2)} \leq 2C_0\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p}^{\frac{1}{p-d+1}}$$

with $C_0 = C_0(D, c_0, p)$ given in Lemma 2.3, provided $\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} < 1$. Therefore the condition $\mathcal{E} < 1/e$ can be guaranteed as long as

$$\zeta := \|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} < \frac{1}{(2C_0e)^{p-d+1}}. \tag{2.8}$$

Hence, whenever (2.8) holds, we obtain from (2.7) there exists a positive constant $C = C(D, c_0, p, \beta, M, \operatorname{supp}(q_1 - q_2))$ such that

$$\|q_1 - q_2\|_{L^\infty(D)} \leq C\kappa^{d/2+1}\zeta^{\frac{1}{2(p-d+1)}} + C\left(\kappa + \frac{1}{p-d+1}\log\frac{1}{\zeta}\right)^{-d/2}.$$

On the other hand, if $\zeta \geq 1/(2C_0e)^{p-d+1}$, we simply consider the trivial bound $\|q_1 - q_2\|_{L^\infty(D)} \leq 2M$ and the proof is completed. □

3. Tests and priors' properties

To prove Theorem 1.7, motivated by [15], we would like to prove the existence of certain test functions, i.e., $\{0,1\}$-valued measurable functions, by showing the existence of appropriate estimators having good concentration properties. Recall that $\mathbb{P}_\varepsilon^{q,\kappa^2}$ is the probability law of Y arising from (1.12a) and $\mathbb{E}_\varepsilon^{q,\kappa^2}$ is the corresponding expectation. Using (2.5), we can prove the following lemma by following the argument in [2, Lemma 8].

Lemma 3.1. *Assume that D satisfies* (1.1)*. Let $q_0 \in L^\infty(D)$ be the "ground truth" with $\|q_0\|_{L^\infty(D)} \le M_0$ for some $M_0 > 0$. Let $M_1 > 0$ and denote $\kappa_0 = \kappa_0(D, \max\{M_0, M_1\})$ the positive constant given in* (1.4)*. Fix any wave number $\kappa \ge \kappa_0$ and real parameters $0 < \delta < 1$ as well as $p \ge (d-1)/\delta$. Let $\eta_\varepsilon > 0$ satisfy*

$$\eta_\varepsilon \varepsilon^{-(1-\delta)} \to \infty \quad \text{as } \varepsilon \to 0.$$

For any $\tau > 0$, we write $C_\tau = \sqrt{2(1+2\tau+2\tau^2)}$. Then there exist a test $\psi = \psi(Y)$ with $Y \sim \mathbb{P}_\varepsilon^{q_0,\kappa^2}$ such that for all sufficiently small $\varepsilon > 0$, one has

$$\mathbb{E}_\varepsilon^{q_0,\kappa^2} \psi \le 2\exp\big(-\tau(\eta_\varepsilon/\varepsilon)^2\big) \tag{3.1a}$$

and for each $q \in L^\infty(D)$ with $\|q\|_{L^\infty(D)} \le M_1$ and $\|\Re\widetilde{\mathcal{M}}_{q,\kappa^2} - \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}\|_{\mathbb{H}_p} \ge 2C_\tau\eta_\varepsilon$, we have

$$\mathbb{E}_\varepsilon^{q,\kappa^2}[1-\psi] \le 2\exp\big(-\tau(\eta_\varepsilon/\varepsilon)^2\big). \tag{3.1b}$$

Proof. For any measurable set $\mathcal{A}$, we denote $\mathbb{1}\mathcal{A}$ the characteristic function of $\mathcal{A}$. We define the random element $\widehat{\mathcal{M}}$ by

$$\widehat{\mathcal{M}} = \left(\sum_{j,\ell \le J_\varepsilon} \widehat{\mathcal{M}}_{j\ell} b_{j\ell}^{(p)}, \sum_{j,\ell \le J_\varepsilon} \widehat{\mathcal{M}}'_{j\ell} \widetilde{b}_{j\ell}^{(p)} \right)$$

where $J_\varepsilon = \lfloor \eta_\varepsilon/\varepsilon \rfloor$ and

$$\begin{aligned} \widehat{\mathcal{M}}_{j\ell} &= \langle Y, (b_{j\ell}^{(p)}, 0)\rangle_{\mathbb{H}_p} = \langle \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}(\phi_j^{(p)}), \phi_\ell^{(0)}\rangle_{L^2(\partial D)} + \varepsilon g_{j\ell}, \\ \widehat{\mathcal{M}}'_{j\ell} &= \langle Y, (0, \widetilde{b}_{j\ell}^{(p)})\rangle_{\mathbb{H}_p} = \langle \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}(\mathbf{i}\phi_j^{(p)}), \phi_\ell^{(0)}\rangle_{L^2(\partial D)} + \varepsilon g'_{j\ell} \end{aligned}$$

where from (1.12b) we see that $g_{j\ell} = \langle \mathbb{W}, (b_{j\ell}^{(p)}, 0)\rangle_{\mathbb{H}_p} \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$ and $g'_{j\ell} = \langle \mathbb{W}, (0, \widetilde{b}_{j\ell}^{(p)})\rangle_{\mathbb{H}_p} \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$.

We want to show that $\widehat{\mathcal{M}}$ is a legitimate estimator of $\widetilde{\mathcal{M}}_{q,\kappa^2}$. Let $C > 0$ be a positive constant to be chosen later. It is easy to see that

$$\begin{aligned} \mathbb{P}_\varepsilon^{q,\kappa^2}\big(\|\widehat{\mathcal{M}} - \Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} > C\eta_\varepsilon\big) &\le \mathbb{1}\left\{ \|\Re\widetilde{\mathcal{M}}_{q,\kappa^2} - \pi_{J_\varepsilon J_\varepsilon}\Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} > \frac{1}{2}C\eta_\varepsilon \right\} \\ &\quad + \mathbb{P}_\varepsilon^{q,\kappa^2}\left(\|\widehat{\mathcal{M}} - \pi_{J_\varepsilon J_\varepsilon}\Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} > \frac{1}{2}C\eta_\varepsilon \right), \end{aligned} \tag{3.2}$$

which is known as a *bias-variance trade-off* inequality.

We first estimate the bias term. Similar to (2.5) (with $m = p$), we obtain that

$$\|\Re\widetilde{\mathcal{M}}_{q,\kappa^2} - \pi_{J_\varepsilon J_\varepsilon}\Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} \leq C_1 J_\varepsilon^{\frac{d-1-p}{d-1}}$$

for some positive constant $C_1 = C_1(D, c_0, p)$. Hence we can estimate the bias term as

$$\begin{aligned}
&\mathbb{1}\left\{\|\Re\widetilde{\mathcal{M}}_{q,\kappa^2} - \pi_{J_\varepsilon J_\varepsilon}\Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} > \frac{1}{2}C\eta_\varepsilon\right\} \\
&\leq \mathbb{1}\left\{C_1 J_\varepsilon^{1-\frac{p}{d-1}} > \frac{1}{2}C\eta_\varepsilon\right\} \leq \mathbb{1}\left\{C_1\left(\frac{\eta_\varepsilon}{2\varepsilon}\right)^{1-\frac{p}{d-1}} > \frac{1}{2}C\eta_\varepsilon\right\} \\
&= \mathbb{1}\left\{C_1 2^{\frac{p}{d-1}}\varepsilon^{\frac{p}{d-1}-1}\varepsilon^{-(1-\delta)\frac{p}{d-1}} > C(\eta_\varepsilon\varepsilon^{-(1-\delta)})^{\frac{p}{d-1}}\right\} \\
&= \mathbb{1}\left\{C_1 2^{\frac{p}{d-1}}\varepsilon^{\frac{p\delta}{d-1}-1} > C(\eta_\varepsilon\varepsilon^{-(1-\delta)})^{\frac{p}{d-1}}\right\}.
\end{aligned}$$

Since $p\delta \geq d-1$ and $\eta_\varepsilon\varepsilon^{-(1-\delta)} \to \infty$, we conclude that

$$\mathbb{1}\left\{\|\Re\widetilde{\mathcal{M}}_{q,\kappa^2} - \pi_{J_\varepsilon J_\varepsilon}\Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} > \frac{1}{2}C\eta_\varepsilon\right\} \leq \mathbb{1}\left\{C_1 J_\varepsilon^{1-\frac{p}{d-1}} > \frac{1}{2}C\eta_\varepsilon\right\} = 0$$

for all sufficiently small $\varepsilon > 0$.

Next, we estimate the variance term. Applying Parseval's identity yields

$$\begin{aligned}
\|\widehat{\mathcal{M}} - \pi_{J_\varepsilon J_\varepsilon}\Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p}^2 &= \sum_{j,\ell\leq J_\varepsilon}|\widehat{\mathcal{M}}_{j\ell} - \langle\Re\widetilde{\mathcal{M}}_{q,\kappa^2}(\phi_j^{(p)}), \phi_\ell^{(0)}\rangle_{L^2(\partial D)}|^2 \\
&\quad + \sum_{j,\ell\leq J_\varepsilon}|\widehat{\mathcal{M}}'_{j\ell} - \langle\Re\widetilde{\mathcal{M}}_{q,\kappa^2}(\mathbf{i}\phi_j^{(p)}), \phi_\ell^{(0)}\rangle_{L^2(\partial D)}|^2 \\
&= \varepsilon^2\sum_{j,\ell\leq J_\varepsilon}\big(|g_{j\ell}|^2 + |g'_{j\ell}|^2\big).
\end{aligned}$$

Using the tail inequality in [2, (36)] or [17, Theorem 3.1.9], we have

$$\begin{aligned}
&\Pr\left(\sum_{j,\ell\leq J_\varepsilon}\big(|g_{j\ell}|^2 + |g'_{j\ell}|^2\big) \geq 2\big(J_\varepsilon^2 + 2J_\varepsilon\sqrt{x} + 2x\big)\right) \\
&\leq \Pr\left(\sum_{j,\ell\leq J_\varepsilon}|g_{j\ell}|^2 \geq J_\varepsilon^2 + 2J_\varepsilon\sqrt{x} + 2x\right) + \Pr\left(\sum_{j,\ell\leq J_\varepsilon}|g'_{j\ell}|^2 \geq J_\varepsilon^2 + 2J_\varepsilon\sqrt{x} + 2x\right) \\
&\leq 2e^{-x}.
\end{aligned}$$

We now choose $x = \tau(\eta_\varepsilon/\varepsilon)^2$ in the above inequality and see that

$$\begin{aligned}
2e^{-\tau(\eta_\varepsilon/\varepsilon)^2} &\geq \Pr\left(\sum_{j,\ell\leq J_\varepsilon}\big(|g_{j\ell}|^2 + |g'_{j\ell}|^2\big) \geq 2(1 + 2\tau + 2\tau^2)\left(\frac{\eta_\varepsilon}{\varepsilon}\right)^2\right) \\
&= \Pr\big(\|\widehat{\mathcal{M}} - \pi_{J_\varepsilon J_\varepsilon}\Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} \geq \sqrt{2(1 + 2\tau + 2\tau^2)}\eta_\varepsilon\big).
\end{aligned}$$

It follows from above that $\widehat{\mathcal{M}}$ is indeed a valid estimator. This can be seen by choosing $C = C_\tau = \sqrt{2(1+2\tau+2\tau^2)}$ in (3.2) we reach

$$\mathbb{P}_\varepsilon^{q,\kappa^2}\big(\|\widehat{\mathcal{M}} - \Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} > C_\tau \eta_\varepsilon\big) \le 2e^{-\tau(\eta_\varepsilon/\varepsilon)^2} \tag{3.3}$$

which is valid for all $\kappa \ge \kappa_0$ and for all $q \in L^\infty(D)$ with $\|q\|_\infty \le \max\{M_0, M_1\}$.

Finally, we want to verify that $\psi_\varepsilon(Y) := \mathbb{1}\big\{\|\widehat{\mathcal{M}} - \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}\|_{\mathbb{H}_p} > C_\tau\eta_\varepsilon\big\}$ satisfies (3.1a) and (3.1b). One can choose $q = q_0$ in (3.3) and see that

$$\mathbb{E}_\varepsilon^{q_0,\kappa^2}\psi = \mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(\|\widehat{\mathcal{M}} - \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}\|_{\mathbb{H}_p} > C_\tau\eta_\varepsilon\big) \le 2e^{-\tau(\eta_\varepsilon/\varepsilon)^2},$$

which verifies (3.1a). On the other hand, for each $q \in L^\infty(D)$ with $\|q\|_{L^\infty(D)} \le M_1$ and $\|\Re\widetilde{\mathcal{M}}_{q,\kappa^2} - \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}\|_{\mathbb{H}_p} \ge 2C_\tau\eta_\varepsilon$, we have

$$\begin{aligned}
\mathbb{E}_\varepsilon^{q,\kappa^2}[1-\psi] &= \mathbb{P}_\varepsilon^{q,\kappa^2}\big(\|\widehat{\mathcal{M}} - \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}\|_{\mathbb{H}_p} \le C_\tau\eta_\varepsilon\big)\\
&\le \mathbb{P}_\varepsilon^{q,\kappa^2}\big(\|\Re\widetilde{\mathcal{M}}_{q,\kappa^2} - \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}\|_{\mathbb{H}_p} - \|\widehat{\mathcal{M}} - \Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p} \le C_\tau\eta_\varepsilon\big)\\
&= \mathbb{P}_\varepsilon^{q,\kappa^2}\big(\|\Re\widetilde{\mathcal{M}}_{q,\kappa^2} - \Re\widetilde{\mathcal{M}}_{q_0,\kappa^2}\|_{\mathbb{H}_p} - C_\tau\eta_\varepsilon \le \|\widehat{\mathcal{M}} - \Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p}\big)\\
&\le \mathbb{P}_\varepsilon^{q,\kappa^2}\big(C_\tau\eta_\varepsilon \le \|\widehat{\mathcal{M}} - \Re\widetilde{\mathcal{M}}_{q,\kappa^2}\|_{\mathbb{H}_p}\big).
\end{aligned}$$

Finally, combining the above inequality with (3.3) yields (3.1b). □

Let $\mathbb{K}(\mathsf{p},\mathsf{q}) = \mathbb{E}_{\mathsf{p}}\log\frac{\mathsf{p}}{\mathsf{q}} \equiv \mathbb{E}_{X\sim\mathsf{p}}\log\frac{\mathsf{p}}{\mathsf{q}}(X)$ be the Kullback–Leibler divergence between distributions with densities p and q. Let $\mathsf{p}_\varepsilon^{q,\kappa^2}$ be the probability density given in (1.16). We also denote Var_q the variance operator associated to the probability measure $\mathbb{P}_\varepsilon^{q,\kappa^2}$. Following the same argument as in [2, Lemma 9], one can easily derive

$$\mathbb{K}(\mathsf{p}_\varepsilon^{q_0,\kappa^2}, \mathsf{p}_\varepsilon^{q_1,\kappa^2}) = \frac{1}{2}\varepsilon^{-2}\|\Re\mathcal{M}_{q_0,\kappa^2} - \Re\mathcal{M}_{q_1,\kappa^2}\|_{\mathbb{H}_p}^2 \tag{3.4a}$$

and

$$\mathrm{Var}_{q_0}\left(\log\frac{\mathsf{p}_\varepsilon^{q_0,\kappa^2}}{\mathsf{p}_\varepsilon^{q_1,\kappa^2}}\right) = \varepsilon^{-2}\|\Re\mathcal{M}_{q_0,\kappa^2} - \Re\mathcal{M}_{q_1,\kappa^2}\|_{\mathbb{H}_p}^2 \tag{3.4b}$$

for all $q_0, q_1 \in L^\infty(D)$ with $\|q_0\|_{L^\infty(D)} \le M_0$, $\|q_1\|_{L^\infty(D)} \le M_1$ and $\kappa \ge \kappa_0(D, \max\{M_0, M_1\})$. We now define the Kullback–Leibler ball $B_{\mathrm{KL}}^\varepsilon(\eta)$ with radius η centered at q_0 by

$$B_{\mathrm{KL}}^\varepsilon(\eta) := \left\{q \in L^\infty(D) : \mathbb{K}(\mathsf{p}_\varepsilon^{q_0,\kappa^2}, \mathsf{p}_\varepsilon^{q,\kappa^2}) \le (\eta/\varepsilon)^2, \mathrm{Var}_{q_0}\left(\log\frac{\mathsf{p}_\varepsilon^{q_0,\kappa^2}}{\mathsf{p}_\varepsilon^{q,\kappa^2}}\right) \le (\eta/\varepsilon)^2\right\},$$

see also [18, (A4)]. From (3.4a), (3.4b) and Lemma 2.6, for each $p > 2d-2$ there exists a positive constant $c = c(D, c_0, p)$ such that

$$\begin{aligned}
\big\{q \in C^0(\overline{D}) : \|q - q_0\|_{L^\infty(D)} \le c\eta\big\} &\subset \big\{q \in L^\infty(D) : \|\Re\mathcal{M}_{q,\kappa^2} - \Re\mathcal{M}_{q_0,\kappa^2}\|_{\mathbb{H}_p} \le \eta\big\}\\
&\subset B_{\mathrm{KL}}^\varepsilon(\eta) \quad \text{for all } \eta > 0,
\end{aligned}$$

hence

$$\{q \in C^0(\overline{D}) : \|q - q_0\|_{L^\infty(D)} \leq c\eta^2\} \subset B_{\mathrm{KL}}^{\varepsilon}(\eta) \tag{3.5}$$

for all *sufficiently small* $\eta > 0$. With the preceding preparations, we now prove the following support result for the prior Π_ε along the lines of [2, Lemma 11], roughly indicating that the prior puts a sufficient amount of "mass" near the true parameter.

Lemma 3.2. *Let $M > 0$, $0 < c_0 < 1$ and $p > 2d - 2$ be real parameters. Let D be a bounded smooth domain in $\mathbb{R}^d$ satisfying* (1.1) *and let $D_0 \Subset D$. Assume that α, β are integers satisfying $\alpha > \beta \geq 7d/2$, the base prior Π' satisfies Assumption* 1.5 *and Π_ε is the prior arising from* (1.18). *Suppose that $q_0 \in \mathcal{Q}$, where $\mathcal{Q}$ is the set given in Theorem* 1.7. *Let $\eta_\varepsilon = \varepsilon^{1-\frac{d}{\alpha+d}}$, then there exist positive constants $\kappa_0 = \kappa_0(\alpha, D, M)$ and $\gamma = \gamma(\alpha, D, c_0, p, M)$, which is independent of D_0, q_0, such that*

$$\Pi_\varepsilon\big(B_{\mathrm{KL}}^{\varepsilon}(\eta_\varepsilon)\big) \geq e^{-\gamma(\eta_\varepsilon/\varepsilon)^2}$$

for all $\kappa \geq \kappa_0$ and for all sufficiently small $\varepsilon > 0$.

Proof. Note that Π_ε's RKHS is $\mathcal{H}_\varepsilon = \{\zeta\theta' : \theta' \in \mathcal{H}\}$ with norm $\|\cdot\|_{\mathcal{H}_\varepsilon}$ satisfying the bound

$$\|\zeta\theta'\|_{\mathcal{H}_\varepsilon} \leq \varepsilon^{-\frac{d}{\alpha+d}}\|\theta'\|_{\mathcal{H}} = \frac{\eta_\varepsilon}{\varepsilon}\|\theta'\|_{\mathcal{H}}.$$

Since $q_0 = \zeta q_0$ (recall that $\mathrm{supp}(q_0) \subset D_0$ and $\zeta \equiv 1$ on D_0), $q_0 \in \mathcal{H}$ and $\|q_0\|_{\mathcal{H}} \leq M$, by choosing $\theta' = q_0$ in the above equation, we see that

$$\|q_0\|_{\mathcal{H}_\varepsilon} \leq \frac{\eta_\varepsilon}{\varepsilon}\|q_0\|_{\mathcal{H}} \leq \frac{M\eta_\varepsilon}{\varepsilon}.$$

Hence from [17, Corollary 2.6.18], one has

$$\begin{aligned} \Pi_\varepsilon\big(\|q - q_0\|_{L^\infty(D)} \leq c\eta_\varepsilon^2\big) &\geq \exp\left(-\frac{1}{2}\|q_0\|_{\mathcal{H}_\varepsilon}^2\right)\Pi_\varepsilon\big(\|q\|_{L^\infty(D)} \leq c\eta_\varepsilon^2\big) \\ &\geq e^{-\frac{1}{2}M^2(\eta_\varepsilon/\varepsilon)^2}\Pi'\left(\|\theta'\|_{L^\infty(D)} \leq c\frac{\eta_\varepsilon^3}{\varepsilon}\right), \end{aligned} \tag{3.6}$$

where the last inequality follows from (1.18).

We denote $N(B_{\mathcal{H}}, \|\cdot\|_{L^\infty(D)}, \delta)$ the smallest number of $\|\cdot\|_{L^\infty(D)}$-balls of radius δ needed to cover the unit ball $B_{\mathcal{H}}$ in $\mathcal{H}$. Since $\mathcal{H}$ embeds continuously into $H^\alpha(I_d)$ for some sufficiently large cube I_d, then

$$\log N(B_{\mathcal{H}}, \|\cdot\|_{L^\infty(D)}, \delta) \leq K\delta^{-d/\alpha}$$

for some positive constant $K = K(\alpha, D)$, see [17, after Corollary 4.3.38]. It then follows from [28, Theorem 1.2] that

$$\Pi'\left(\|\theta'\|_{L^\infty(D)} \leq c\frac{\eta_\varepsilon^3}{\varepsilon}\right) \geq e^{-c'(\eta_\varepsilon^3/\varepsilon)^{-s}} \tag{3.7}$$

for some constant $c' = c'(c,K)$, where s is such that $\frac{d}{\alpha} = \frac{2s}{2+s}$, i.e., $s = \frac{2d}{2\alpha-d}$.

By Assumption 1.5 and Sobolev embedding, one has $\mathcal{H} \subset H^\alpha(D) \subset C^0(\overline{D})$, and hence there exists a positive constant $M_0 = M_0(\alpha, D, M)$ such that

$$\|q_0\|_{L^\infty(D)} \leq M_0,$$

therefore (3.5) is valid for all $\kappa \geq \kappa_0(D, M_0)$. We now combine (3.5), (3.6) and (3.7) to obtain that

$$\Pi_\varepsilon\big(B^\varepsilon_{\mathrm{KL}}(\eta_\varepsilon)\big) \geq e^{-\frac{1}{2}M^2(\eta_\varepsilon/\varepsilon)^2} e^{-c'(\eta_\varepsilon^3/\varepsilon)^{-2d/(2\alpha-d)}}$$

for all sufficiently small $\varepsilon > 0$, and together with the fact $(\eta_\varepsilon^3/\varepsilon)^{-2d/(2\alpha-d)} = (\eta_\varepsilon/\varepsilon)^2$, the lemma is proved. □

4. Posterior contraction

This section is devoted to the proof of Theorem 1.7. Following the ideas in [2], we first establish two results about posterior asymptotic. Recall that D_1 satisfies $D_0 \Subset D_1 \Subset D$. Choosing $\Pi = \Pi_\varepsilon$ in (1.17) yields

$$\Pi_\varepsilon(B|Y) = \frac{\int_B \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)/\mathsf{p}_\varepsilon^{q_0,\kappa^2}(Y)\,\mathrm{d}\Pi_\varepsilon(q)}{\int_{\mathcal{V}_{D_1}} \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)/\mathsf{p}_\varepsilon^{q_0,\kappa^2}(Y)\,\mathrm{d}\Pi_\varepsilon(q)} \quad \text{for all } B \in \mathcal{B}_{\mathcal{V}_{D_1}}. \tag{4.1}$$

We first estimate the size of the denominator in (4.1), which is similar to [2, Lemma 14]. The proof modifies the ideas in [17, Lemma 7.3.4].

Lemma 4.1. *Suppose all assumptions in Lemma 3.2 hold. Then*

$$\sup_{\kappa \geq \kappa_0,\, q_0 \in \mathcal{Q}} \mathbb{P}_\varepsilon^{q_0,\kappa^2}(L_{q_0}^{\complement}) \to 0 \quad \textit{as } \varepsilon \to 0,$$

where $L_{q_0}^{\complement}$ is the complement of the event

$$L_{q_0} = \left\{ \int_{\mathcal{V}_{D_1}} (\mathsf{p}_\varepsilon^{q,\kappa^2}/\mathsf{p}_\varepsilon^{q_0,\kappa^2})\,\mathrm{d}\Pi_\varepsilon(q) \geq e^{-(\gamma+2)(\eta_\varepsilon/\varepsilon)^2} \right\}.$$

Proof. We follow the argument used in the proof of [1, Lemma 21]. From Jensen's inequality, we have

$$\int_{\mathcal{V}_{D_1}} (\mathsf{p}_\varepsilon^{q,\kappa^2}/\mathsf{p}_\varepsilon^{q_0,\kappa^2})\,\mathrm{d}\Pi_\varepsilon(q) \geq \Pi_\varepsilon\big(B^\varepsilon_{\mathrm{KL}}(\eta_\varepsilon)\big)\exp\left(\int_{B^\varepsilon_{\mathrm{KL}}(\eta_\varepsilon)} \log(\mathsf{p}_\varepsilon^{q,\kappa^2}/\mathsf{p}_\varepsilon^{q_0,\kappa^2})\,\mathrm{d}\widetilde{\Pi}_\varepsilon(q)\right),$$

where $\widetilde{\Pi}_\varepsilon = \Pi_\varepsilon/\Pi_\varepsilon\big(B^\varepsilon_{\mathrm{KL}}(\eta_\varepsilon)\big)$. Combining the above equation with Lemma 3.2 implies

$$\begin{aligned}
\mathbb{P}_\varepsilon^{q_0,\kappa^2}(L_{q_0}^{\complement}) &= \mathbb{P}_\varepsilon^{q_0,\kappa^2}\left(\int_{\mathcal{V}_{D_1}} (\mathsf{p}_\varepsilon^{q,\kappa^2}/\mathsf{p}_\varepsilon^{q_0,\kappa^2})\,\mathrm{d}\Pi_\varepsilon(q) < e^{-(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\right) \\
&\leq \mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(\Pi_\varepsilon\big(B^\varepsilon_{\mathrm{KL}}(\eta_\varepsilon)\big)e^X \leq e^{-(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\big) \\
&\leq \mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(e^{-\gamma(\eta_\varepsilon/\varepsilon)^2}e^X \leq e^{-(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\big) = \mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(X \leq -2(\eta_\varepsilon/\varepsilon)^2\big),
\end{aligned} \tag{4.2}$$

where

$$X := \int_{B^{\varepsilon}_{\mathrm{KL}}(\eta_\varepsilon)} \log(\mathsf{p}^{q,\kappa^2}_{\varepsilon}/\mathsf{p}^{q_0,\kappa^2}_{\varepsilon})\,\mathrm{d}\widetilde{\Pi}_{\varepsilon}(q) = -\int_{B^{\varepsilon}_{\mathrm{KL}}(\eta_\varepsilon)} \log(\mathsf{p}^{q_0,\kappa^2}_{\varepsilon}/\mathsf{p}^{q,\kappa^2}_{\varepsilon})\,\mathrm{d}\widetilde{\Pi}_{\varepsilon}(q).$$

Thus, applying Fubini's theorem and the definition of $B^{\varepsilon}_{\mathrm{KL}}(\eta_\varepsilon)$ gives

$$\mathbb{E}^{q_0,\kappa^2}_{\varepsilon} X \geq -\sup_{q\in B^{\varepsilon}_{\mathrm{KL}}(\eta_\varepsilon)} \mathbb{E}^{q_0,\kappa^2}_{\varepsilon} \log(\mathsf{p}^{q_0,\kappa^2}_{\varepsilon}/\mathsf{p}^{q,\kappa^2}_{\varepsilon}) \geq -(\eta_\varepsilon/\varepsilon)^2.$$

From (4.2) and Chebyshev's inequality, we now have

$$\begin{aligned} \mathbb{P}^{q_0,\kappa^2}_{\varepsilon}(L^{\complement}_{q_0}) &\leq \mathbb{P}^{q_0,\kappa^2}_{\varepsilon}\big(X - \mathbb{E}^{q_0,\kappa^2}_{\varepsilon} X \leq -(\eta_\varepsilon/\varepsilon)^2\big) \leq \mathbb{P}^{q_0,\kappa^2}_{\varepsilon}\big(|X - \mathbb{E}^{q_0,\kappa^2}_{\varepsilon} X| \geq (\eta_\varepsilon/\varepsilon)^2\big) \\ &\leq (\eta_\varepsilon/\varepsilon)^{-4}\,\mathrm{Var}^{q_0,\kappa^2}_{\varepsilon} X. \end{aligned} \tag{4.3}$$

Finally, by Jensen's inequality, Fubini's theorem and the definition of $B^{\varepsilon}_{\mathrm{KL}}(\eta_\varepsilon)$ again, we can estimate

$$\begin{aligned} \mathrm{Var}^{q_0,\kappa^2}_{\varepsilon} X &= \mathbb{E}^{q_0,\kappa^2}_{\varepsilon}\left(\int_{B^{\varepsilon}_{\mathrm{KL}}(\eta_\varepsilon)} \log(\mathsf{p}^{q,\kappa^2}_{\varepsilon}/\mathsf{p}^{q_0,\kappa^2}_{\varepsilon})\,\mathrm{d}\widetilde{\Pi}_{\varepsilon}(q) - \mathbb{E}^{q_0,\kappa^2}_{\varepsilon} X\right)^2 \\ &\leq \mathbb{E}^{q_0,\kappa^2}_{\varepsilon}\int_{B^{\varepsilon}_{\mathrm{KL}}(\eta_\varepsilon)} \big(\log(\mathsf{p}^{q,\kappa^2}_{\varepsilon}/\mathsf{p}^{q_0,\kappa^2}_{\varepsilon}) - \mathbb{E}^{q_0,\kappa^2}_{\varepsilon}\log(\mathsf{p}^{q,\kappa^2}_{\varepsilon}/\mathsf{p}^{q_0,\kappa^2}_{\varepsilon})\big)^2\,\mathrm{d}\widetilde{\Pi}_{\varepsilon}(q) \\ &= \int_{B^{\varepsilon}_{\mathrm{KL}}(\eta_\varepsilon)} \mathrm{Var}^{q_0,\kappa^2}_{\varepsilon}\log(\mathsf{p}^{q,\kappa^2}_{\varepsilon}/\mathsf{p}^{q_0,\kappa^2}_{\varepsilon})\,\mathrm{d}\widetilde{\Pi}_{\varepsilon}(q) \leq (\eta_\varepsilon/\varepsilon)^2, \end{aligned}$$

and then combine this with (4.3) to obtain

$$\mathbb{P}^{q_0,\kappa^2}_{\varepsilon}(L^{\complement}_{q_0}) \leq (\eta_\varepsilon/\varepsilon)^{-2}. \tag{4.4}$$

Observe that the right-hand side of (4.4) is independent of both κ and q_0. Thus, our lemma follows. □

We now prove the following two results (see Lemmas 4.2 and 4.4 below) using the method in [2, Lemmas 12 and 13] whose the ideas are taken from Bayesian nonparametric statistics [16, 50].

Lemma 4.2. *Suppose all assumptions in Lemma* 3.2 *hold. Then there exists a positive constant $M' > 0$, independent of κ, such that*

$$\sup_{\kappa\geq\kappa_0,\, q_0\in\mathcal{Q}} \mathbb{P}^{q_0,\kappa^2}_{\varepsilon}\big(\Pi_\varepsilon(\|q\|_{H^\beta(D)} > M'|Y) > e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big) \to 0 \quad \textit{as } \varepsilon \to 0.$$

Remark 4.3. By Sobolev embedding theorem and adjusting the constant M', Lemma 4.2 remains valid if $\|q\|_{H^\beta(D)}$ is replaced by $\|q\|_{L^\infty(D)}$.

Proof of Lemma 4.2. Let L_{q_0} be the event as in Lemma 4.1. In view of the posterior distribution (4.1), by Fubini's theorem and the identity $\mathbb{E}_\varepsilon^{q_0,\kappa^2}\big(\mathsf{p}_\varepsilon^{q,\kappa^2}/\mathsf{p}_\varepsilon^{q_0,\kappa^2}\big)(Y) = 1$, we have

$$
\begin{aligned}
\mathbb{E}_\varepsilon^{q_0,\kappa^2}\big(\mathbb{1}_{L_{q_0}}\Pi_\varepsilon(B|Y)\big) &= \mathbb{E}_\varepsilon^{q_0,\kappa^2}\left(\mathbb{1}_{L_{q_0}}\frac{\int_B \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)/\mathsf{p}_\varepsilon^{q_0,\kappa^2}(Y)\,\mathrm{d}\Pi_\varepsilon(q)}{\int_{\mathcal{V}_{D_1}} \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)/\mathsf{p}_\varepsilon^{q_0,\kappa^2}(Y)\,\mathrm{d}\Pi_\varepsilon(q)}\right)\\
&\le e^{(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\mathbb{E}_\varepsilon^{q_0,\kappa^2}\left(\int_B \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)/\mathsf{p}_\varepsilon^{q_0,\kappa^2}(Y)\,\mathrm{d}\Pi_\varepsilon(q)\right)\\
&\le e^{(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\int_B \mathbb{E}_\varepsilon^{q_0,\kappa^2}(\mathsf{p}_\varepsilon^{q,\kappa^2}/\mathsf{p}_\varepsilon^{q_0,\kappa^2})(Y)\,\mathrm{d}\Pi_\varepsilon(q)\\
&= e^{(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\Pi_\varepsilon(B)
\end{aligned} \tag{4.5}
$$

for all $B \in \mathcal{B}_{\mathcal{V}_{D_1}}$. Let M' be a positive parameter to be chosen later. By Markov's inequality (see e.g., [12]) and choosing $B = \big\{\|q\|_{H^\beta(D)} > M'\big\}$ in the above inequality, we see that

$$
\begin{aligned}
&\mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(\Pi_\varepsilon(\|q\|_{H^\beta(D)} > M'|Y) > e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big)\\
&\le \mathbb{P}_\varepsilon^{q_0,\kappa^2}(L_{q_0}^{\complement}) + \mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(\mathbb{1}_{L_{q_0}}\Pi_\varepsilon(\|q\|_{H^\beta(D)} > M'|Y) > e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big)\\
&\le \mathbb{P}_\varepsilon^{q_0,\kappa^2}(L_{q_0}^{\complement}) + e^{(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\mathbb{E}_\varepsilon^{q_0,\kappa^2}\big(\mathbb{1}_{L_{q_0}}\Pi_\varepsilon(\|q\|_{H^\beta(D)} > M'|Y)\big)\\
&\le \mathbb{P}_\varepsilon^{q_0,\kappa^2}(L_{q_0}^{\complement}) + e^{(2\gamma+6)(\eta_\varepsilon/\varepsilon)^2}\Pi_\varepsilon(\|q\|_{H^\beta(D)} > M').
\end{aligned} \tag{4.6}
$$

In conjunction with the facts that $\varepsilon^{-d/(\alpha+d)} = \eta_\varepsilon/\varepsilon$ and that $\|\zeta\theta'\|_{H^\beta(D)} \le C\|\zeta\|_{H^\beta(D)} \times \|\theta'\|_{H^\beta(D)}$ for some positive constant $C = C(D,\beta)$, one can deduce that

$$
\begin{aligned}
\Pi_\varepsilon\big(\|q\|_{H^\beta(D)} > M'\big) &= \Pi_\varepsilon\big(\|\varepsilon^{d/(\alpha+d)}\zeta\theta'\|_{H^\beta(D)} > M'\big)\\
&\le \Pi'\big(\|\theta'\|_{H^\beta(D)} > (\eta_\varepsilon/\varepsilon)\|\zeta\|_{H^\beta(D)}^{-1}C^{-1}M'\big).
\end{aligned} \tag{4.7}
$$

We now want to apply Fernique's theorem following the ideas in [35, Step 1 in Theorem 2.2.2]. In view of the separability $H^\beta(D)$, the Hahn–Banach theorem, and the hypothesis $\Pi'(H^\beta(D)) = 1$, we obtain that

$$
\Pr\left(\sup_{T\in\mathcal{T}}|T(\theta')| = \|\theta'\|_{H^\beta(D)} < \infty\right) = 1,
$$

where $\mathcal{T}$ is a countable family of $(H^\beta(D))'$. Fernique's theorem [17, Theorem 2.1.20] implies initially that $\mathbb{E}'\|\theta'\|_{H^\beta(D)} \le C'$ for some positive constant C' depending only on the base prior Π', and similar to [35, (2.21)] one has

$$
\begin{aligned}
&\Pi'\big(\|\theta'\|_{H^\beta(D)} > (\eta_\varepsilon/\varepsilon)\|\zeta\|_{H^\beta(D)}^{-1}C^{-1}M'\big)\\
&\le \Pi'\big(\|\theta'\|_{H^\beta(D)} - \mathbb{E}'\|\theta'\|_{H^\beta(D)} > (\eta_\varepsilon/\varepsilon)\|\zeta\|_{H^\beta(D)}^{-1}(2C)^{-1}M'\big)\\
&\le \exp\big(-c'(\eta_\varepsilon/\varepsilon)^2\|\zeta\|_{H^\beta(D)}^{-1}M'\big).
\end{aligned}
$$

Hence, given any $c > 0$, one can choose $M' > 0$ such that

$$\Pi'\big(\|\theta'\|_{H^\beta(D)} > (\eta_\varepsilon/\varepsilon)\|\zeta\|_{H^\beta(D)}^{-1}C^{-1}M'\big) \le e^{-c(\eta_\varepsilon/\varepsilon)^2}. \tag{4.8}$$

We combine (4.6), (4.7) and (4.8) to obtain

$$\mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(\Pi_\varepsilon(\|q\|_{H^\beta(D)} > M'|Y) > e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big) \le \mathbb{P}_\varepsilon^{q_0,\kappa^2}(L_{q_0}^{\complement}) + e^{(2\gamma+6-c)(\eta_\varepsilon/\varepsilon)^2}$$

Finally, choosing $c > 2\gamma+6$, our lemma immediately follows from Lemma 4.1. □

Lemma 4.4. *Under the assumptions of Lemma 3.2, there exists a positive constant $C_1 > 0$, which is independent of κ, such that*

$$\sup_{\kappa\ge\kappa_0,\, q_0\in\mathcal{Q}} \mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(\Pi_\varepsilon\big(\|\Re\mathcal{M}_{q,\kappa^2} - \Re\mathcal{M}_{q_0,\kappa^2}\|_{\mathbb{H}_p} > C_1\eta_\varepsilon|Y\big) > 2e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big) \to 0$$

as $\varepsilon \to 0$.

Proof. Let $M' > 0$ be the positive constant obtained in Lemma 4.2 and $C > 0$ be a positive constant to be determined later. Define the set

$$S = \big\{q \in \mathcal{B}_{\mathcal{V}_{D_1}} : \|\Re\mathcal{M}_{q,\kappa^2} - \Re\mathcal{M}_{q_0,\kappa^2}\|_{\mathbb{H}_p} > C\eta_\varepsilon, \|q\|_{L^\infty(D)} \le M'\big\},$$

then it is readily seen that

$$\begin{aligned}
&\Pi_\varepsilon\big(\|\Re\mathcal{M}_{q,\kappa^2} - \Re\mathcal{M}_{q_0,\kappa^2}\|_{\mathbb{H}_p} > C\eta_\varepsilon|Y\big)\\
&\le \Pi_\varepsilon(S|Y) + \Pi_\varepsilon\big(\|q\|_{L^\infty(D)} > M'|Y\big)\\
&= \Pi_\varepsilon(S|Y)\mathbb{1}_{L_{q_0}^{\complement}} + \Pi_\varepsilon(S|Y)\psi\mathbb{1}_{L_{q_0}} + \Pi_\varepsilon(S|Y)(1-\psi)\mathbb{1}_{L_{q_0}} + \Pi_\varepsilon\big(\|q\|_{L^\infty(D)} > M'|Y\big)\\
&\le \mathbb{1}_{L_{q_0}^{\complement}} + \psi + \Pi_\varepsilon(S|Y)(1-\psi)\mathbb{1}_{L_{q_0}} + \Pi_\varepsilon\big(\|q\|_{L^\infty(D)} > M'|Y\big)
\end{aligned}$$

where ψ is the test given in Lemma 3.1 and L_{q_0} is the event defined in Lemma 4.1. Accordingly, we can upper bound the probability of the event

$$B := \big\{\Pi_\varepsilon\big(\|\Re\mathcal{M}_{q,\kappa^2} - \Re\mathcal{M}_{q_0,\kappa^2}\|_{\mathbb{H}_p} > C\eta_\varepsilon|Y\big) > 2e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big\}$$

by

$$\mathbb{P}_\varepsilon^{q_0,\kappa^2}(L_{q_0}^{\complement}) + \mathbb{E}_\varepsilon^{q_0,\kappa^2}\psi + \mathbb{P}_\varepsilon^{q_0,\kappa^2}\big(\Pi_\varepsilon(S|Y)(1-\psi)\mathbb{1}_{L_{q_0}} > e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big). \tag{4.9}$$

Similar to (4.5), using the posterior distribution (4.1), the definition of L_{q_0}, Fubini's theorem and Lemma 3.1, for each $\tau > 0$, one can estimate

$$\begin{aligned}
&\mathbb{E}_\varepsilon^{q_0,\kappa^2}\big(\Pi_\varepsilon(S|Y)(1-\psi)\mathbb{1}_{L_{q_0}}\big)\\
&= \mathbb{E}_\varepsilon^{q_0,\kappa^2}\left(\mathbb{1}_{L_{q_0}}\frac{\int_S \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)/\mathsf{p}_\varepsilon^{q_0,\kappa^2}(Y)\,\mathrm{d}\Pi_\varepsilon(q)}{\int_{\mathcal{V}_{D_1}} \mathsf{p}_\varepsilon^{q,\kappa^2}(Y)/\mathsf{p}_\varepsilon^{q_0,\kappa^2}(Y)\,\mathrm{d}\Pi_\varepsilon(q)}\right)\\
&\le e^{(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\mathbb{E}_\varepsilon^{q_0,\kappa^2}\int_S (1-\psi)(Y)(\mathsf{p}_\varepsilon^{q,\kappa^2}/\mathsf{p}_\varepsilon^{q_0,\kappa^2})(Y)\,\mathrm{d}\Pi_\varepsilon(q)\\
&\le e^{(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\int_S \mathbb{E}_\varepsilon^{q,\kappa^2}((1-\psi)(Y))\,\mathrm{d}\Pi_\varepsilon(q) \le 2e^{(\gamma+2-\tau)(\eta_\varepsilon/\varepsilon)^2}.
\end{aligned} \tag{4.10}$$

Hence, by Markov's inequality and (4.10), we have that

$$\begin{aligned}\mathbb{P}_{\varepsilon}^{q_0,\kappa^2}\big(\Pi_\varepsilon(S|Y)(1-\psi)\mathbb{1}_{L_{q_0}} > e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big) &\le e^{(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\mathbb{E}_{\varepsilon}^{q_0,\kappa^2}\big(\Pi_\varepsilon(S|Y)(1-\psi)\mathbb{1}_{L_{q_0}}\big)\\ &\le e^{(2\gamma+6-\tau)(\eta_\varepsilon/\varepsilon)^2},\end{aligned}$$

and thus from (4.9) we reach

$$\mathbb{P}_{\varepsilon}^{q_0,\kappa^2}(B) \le \mathbb{P}_{\varepsilon}^{q_0,\kappa^2}(L_{q_0}^{\complement}) + \mathbb{E}_{\varepsilon}^{q_0,\kappa^2}\psi + e^{(2\gamma+6-\tau)(\eta_\varepsilon/\varepsilon)^2}.$$

We choose $\tau > 2\gamma+6$ and the corresponding C_τ given in Lemma 3.1 (and set $C_1 := C_\tau$), this lemma is proved in view of Lemmas 3.1 and 4.1. □

Now we are ready to prove the theorem of contraction result.

Proof of Theorem 1.7. Let M' and C_1 be positive constants defined in Lemmas 4.2 and 4.4, respectively. By Theorem 2.7, for each sufficiently small $\varepsilon > 0$, one observes the implication

$$\begin{gathered}\|q_j\|_{H^\beta(D)} \le M',\ \ j=1,2, \quad \text{and} \quad \|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} \le C_1\eta_\varepsilon\\ \text{together imply } \|q_1-q_2\|_{L^\infty(D)} \le C\widetilde{\xi}_\kappa(C_1\eta_\varepsilon),\end{gathered}$$

where $\widetilde{\xi}_\kappa$ is given by (2.6). It is not difficult to compute that

$$\widetilde{\xi}_\kappa(C_1\eta_\varepsilon) \le C_2\xi_\kappa(\varepsilon) \quad \text{with} \quad \xi_\kappa(\varepsilon) := \kappa^{d/2+1}\varepsilon^{\frac{\alpha}{2(\alpha+d)(p-d+1)}} + \left(\kappa + \log\frac{1}{\varepsilon}\right)^{-d/2}$$

for all sufficiently small $\varepsilon > 0$. Therefore, we reach

$$\begin{aligned}&\Pi_\varepsilon\big(\|q-q_0\|_{L^\infty(D)} > C_2\xi_\kappa(\varepsilon)|Y\big)\\ &\le \Pi_\varepsilon\big(\|q\|_{H^\beta(D)} > M'|Y\big) + \Pi_\varepsilon\big(\|\Re\mathcal{M}_{q_1,\kappa^2} - \Re\mathcal{M}_{q_2,\kappa^2}\|_{\mathbb{H}_p} > C_1\eta_\varepsilon|Y\big).\end{aligned} \tag{4.11}$$

Combining (4.11) with Lemmas 4.2 and 4.4 gives the contraction rate (1.19).

Next we would like to prove the consistency of the posterior mean $\mathbb{E}^{\Pi_\varepsilon}(q|Y)$. To begin, let γ be the constant given in Lemma 3.2. Recall the event L_{q_0} defined in Lemma 4.1. Define the event

$$\mathcal{A} := L_{q_0} \cap \big\{\Pi_\varepsilon\big(\|q-q_0\|_{L^\infty(D)} > C_2\xi_\kappa(\varepsilon)|Y\big) \le 3e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\big\},$$

and it is readily seen that, for each constant $K > 0$ (to be determined later) and for any sufficiently small ε,

$$\begin{aligned}&\mathbb{P}_{\varepsilon}^{q_0,\kappa^2}\big(\|\mathbb{E}^{\Pi_\varepsilon}(q|Y) - q_0\|_{L^\infty(D)} > K\xi_\kappa(\varepsilon)\big)\\ &\le \mathbb{P}_{\varepsilon}^{q_0,\kappa^2}(\mathcal{A}^{\complement}) + \mathbb{P}_{\varepsilon}^{q_0,\kappa^2}\big(\|\mathbb{E}^{\Pi_\varepsilon}(q-q_0|Y)\|_{L^\infty(D)}\mathbb{1}_{\mathcal{A}} > K\xi_\kappa(\varepsilon)\big).\end{aligned} \tag{4.12}$$

In view of (4.11), we can see that

$$
\begin{aligned}
\mathcal{A}^{\complement} &= L_{q_0}^{\complement} \cup \bigl\{\Pi_\varepsilon\bigl(\|q-q_0\|_{L^\infty(D)} > C_2\xi_\kappa(\varepsilon)|Y\bigr) > 3e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\bigr\} \\
&\subset L_{q_0}^{\complement} \cup \bigl\{\Pi_\varepsilon\bigl(\|\Re\mathcal{M}_{q,\kappa^2} - \Re\mathcal{M}_{q_0,\kappa^2}\|_{\mathbb{H}_p} > C_1\eta_\varepsilon\bigr) > 2e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\bigr\} \\
&\quad \cup \bigl\{\Pi_\varepsilon\bigl(\|q\|_{H^\beta(D)} > M'|Y\bigr) > e^{-(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}\bigr\},
\end{aligned}
$$

and then from Lemmas 4.1, 4.2 and 4.4, it follows that

$$
\sup_{\kappa\geq\kappa_0,\, q_0\in\mathcal{Q}} \mathbb{P}_\varepsilon^{q_0,\kappa^2}(\mathcal{A}^{\complement}) \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.13}
$$

On the other hand, by Jensen's and Cauchy–Schwartz's inequalities, one can estimate

$$
\begin{aligned}
&\|\mathbb{E}^{\Pi_\varepsilon}(q-q_0|Y)\|_{L^\infty(D)}\mathbb{1}_{\mathcal{A}} \\
&\leq C_2\xi_\kappa(\varepsilon) + \mathbb{E}^{\Pi_\varepsilon}\bigl(\|q-q_0\|_{L^\infty(D)}\mathbb{1}\bigl\{\|q-q_0\|_{L^\infty(D)} > C_2\xi_\kappa(\varepsilon)\bigr\}|Y\bigr)\mathbb{1}_{\mathcal{A}} \\
&\leq C_2\xi_\kappa(\varepsilon) + \bigl(\mathbb{E}^{\Pi_\varepsilon}(\|q-q_0\|_{L^\infty(D)}^2|Y)\bigr)^{1/2}\Pi_\varepsilon\bigl(\|q-q_0\|_{L^\infty(D)} > C_2\xi_\kappa(\varepsilon)|Y\bigr)^{1/2}\mathbb{1}_{\mathcal{A}}.
\end{aligned}
$$

Choosing $K > C_2$ implies

$$
\begin{aligned}
&\bigl\{\|\mathbb{E}^{\Pi_\varepsilon}(q-q_0|Y)\|_{L^\infty(D)}\mathbb{1}_{\mathcal{A}} \geq K\xi_\kappa(\varepsilon)\bigr\} \\
&\subset \bigl\{\bigl(\mathbb{E}^{\Pi_\varepsilon}(\|q-q_0\|_{L^\infty(D)}^2|Y)\bigr)^{1/2}\Pi_\varepsilon\bigl(\|q-q_0\|_{L^\infty(D)} > C_2\xi_\kappa(\varepsilon)|Y\bigr)^{1/2}\mathbb{1}_{\mathcal{A}} \geq (K-C_2)\xi_\kappa(\varepsilon)\bigr\}.
\end{aligned}
$$

From Markov's and Cauchy–Schwartz's inequalities, it yields

$$
\begin{aligned}
&\mathbb{P}_\varepsilon^{q_0,\kappa^2}\bigl(\|\mathbb{E}^{\Pi_\varepsilon}(q-q_0|Y)\|_{L^\infty(D)}\mathbb{1}_{\mathcal{A}} \geq K\xi_\kappa(\varepsilon)\bigr) \\
&\leq \frac{1}{(K-C_2)\xi_\kappa(\varepsilon)}\mathbb{E}_\varepsilon^{q_0,\kappa^2}\bigl(\bigl(\mathbb{E}^{\Pi_\varepsilon}(\|q-q_0\|_{L^\infty(D)}^2|Y)\bigr)^{1/2} \\
&\quad \times \Pi_\varepsilon\bigl(\|q-q_0\|_{L^\infty(D)} > C_2\xi_\kappa(\varepsilon)|Y\bigr)^{1/2}\mathbb{1}_{\mathcal{A}}\bigr) \\
&\leq \frac{1}{(K-C_2)\xi_\kappa(\varepsilon)}\mathbb{E}_\varepsilon^{q_0,\kappa^2}\bigl(\mathbb{E}^{\Pi_\varepsilon}(\|q-q_0\|_{L^\infty(D)}^2|Y)\mathbb{1}_{\mathcal{A}}\bigr)^{1/2} \\
&\quad \times \mathbb{E}_\varepsilon^{q_0,\kappa^2}\bigl(\Pi_\varepsilon(\|q-q_0\|_{L^\infty(D)} > C_2\xi_\kappa(\varepsilon)|Y)\mathbb{1}_{\mathcal{A}}\bigr)^{1/2} \\
&\leq \frac{\sqrt{3}e^{-\frac{1}{2}(\gamma+4)(\eta_\varepsilon/\varepsilon)^2}}{(K-C_2)\xi_\kappa(\varepsilon)}\mathbb{E}_\varepsilon^{q_0,\kappa^2}\bigl(\mathbb{E}^{\Pi_\varepsilon}(\|q-q_0\|_{L^\infty(D)}^2|Y)\mathbb{1}_{\mathcal{A}}\bigr)^{1/2}.
\end{aligned} \tag{4.14}
$$

By Fubini's theorem and the definition of $\mathcal{A}$, one can compute that

$$
\begin{aligned}
&\mathbb{E}_\varepsilon^{q_0,\kappa^2}\bigl(\mathbb{E}^{\Pi_\varepsilon}(\|q-q_0\|_{L^\infty(D)}^2|Y)\mathbb{1}_{\mathcal{A}}\bigr) \\
&\leq e^{(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\mathbb{E}_\varepsilon^{q_0,\kappa^2}\left(\int \|q-q_0\|_{L^\infty(D)}^2 \frac{\mathsf{p}_\varepsilon^{q,\kappa^2}}{\mathsf{p}_\varepsilon^{q_0,\kappa^2}}(Y)\,\mathrm{d}\Pi_\varepsilon(q)\right) \\
&\leq e^{(\gamma+2)(\eta_\varepsilon/\varepsilon)^2}\mathbb{E}^{\Pi_\varepsilon}\bigl(\|q-q_0\|_{L^\infty(D)}^2\bigr).
\end{aligned}
$$

Putting together the inequality above and (4.14) yields

$$\mathbb{P}_{\varepsilon}^{q_0,\kappa^2}\big(\|\mathbb{E}^{\Pi_\varepsilon}(q-q_0|Y)\|_{L^\infty(D)}\mathbb{1}_{\mathcal{A}} \geq K\xi_\kappa(\varepsilon)\big) \leq \frac{\sqrt{3}e^{-(\eta_\varepsilon/\varepsilon)^2}}{(K-C)\xi_\kappa(\varepsilon)}\mathbb{E}^{\Pi_\varepsilon}\big(\|q-q_0\|^2_{L^\infty(D)}\big). \tag{4.15}$$

Since

$$\mathbb{E}^{\Pi_\varepsilon}\big(\|q-q_0\|^2_{L^\infty(D)}\big) \leq 2\big(\|q_0\|^2_{L^\infty(D)} + \mathbb{E}^{\Pi_\varepsilon}\|q\|^2_{L^\infty(D)}\big),$$

$q = \varepsilon^{\alpha/(\alpha+d)}\zeta\theta'$, and $\mathbb{E}^{\Pi'}\|\theta'\|^2_{L^\infty(D)}$ is finite, by Sobolev embedding theorem, we see that $\mathbb{E}^{\Pi_\varepsilon}\big(\|q-q_0\|^2_{L^\infty(D)}\big)$ is uniformly bounded for $\kappa \geq \kappa_0$ and $q_0 \in \mathcal{Q}$. Finally, the limit (1.20) follows from (4.12), (4.13) and (4.15), as well as the fact $e^{-(\eta_\varepsilon/\varepsilon)^2}/\xi_\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. □

A. Stability estimate of the inverse problem

In this section, we are devoted to the proof of the stability estimate in Theorem 1.1. We first observe the symmetry property of the impedance-to-Dirichlet operator.

Lemma A.1. *Let $q \in L^\infty(D)$ be real-valued functions satisfying* (1.3)*. Then*

$$\int_{\partial D}\big(\mathcal{M}_{q,\kappa^2}[g_1]\big)g_2\,\mathrm{d}S = \int_{\partial D} g_1\mathcal{M}_{q,\kappa^2}[g_2]\,\mathrm{d}S \quad \textit{for all } g_1, g_2 \in L^2(\partial D).$$

Proof. Let $u_1, u_2 \in H^1(D)$ satisfy (1.2) and (1.5) with $g = g_1, g_2$, respectively. By direct computations, we obtain

$$\begin{aligned} 0 &= -\mathbf{i}\kappa\left(\int_{\partial D}(\partial_\nu u_1)u_2\,\mathrm{d}S - \int_{\partial D} u_1\partial_\nu u_2\,\mathrm{d}S\right) \\ &= \int_{\partial D}\partial_\nu u_1(\partial_\nu u_2 - \mathbf{i}\kappa u_2)\,\mathrm{d}S - \int_{\partial D}(\partial_\nu u_1 - \mathbf{i}\kappa u_1)\partial_\nu u_2\,\mathrm{d}S \\ &= \int_{\partial D}\big(\mathcal{M}_{q,\kappa^2}[g_1]\big)g_2\,\mathrm{d}S - \int_{\partial D} g_1\big(\mathcal{M}_{q,\kappa^2}[g_2]\big)\,\mathrm{d}S \end{aligned}$$

and hence the lemma. □

With the above symmetry property at hand, we are now able to prove the following crucial integral identity.

Lemma A.2. *Let $m \geq 0$ and $q_1, q_2 \in L^\infty(D)$ be real-valued functions satisfying* (1.3)*. Given any $g_1, g_2 \in L^2(\partial D)$, let $u_1, u_2 \in H^1(D)$ satisfy* (1.2) *and* (1.5) *corresponding to $q = q_j$ and $g = g_j$, $j = 1, 2$. Then*

$$\begin{aligned} &\left|\int_D (q_1-q_2)u_1u_2\,\mathrm{d}x\right| \\ &\leq \kappa^{-1}\|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{H^m(\partial D)\to L^2(\partial D)}\big(\|\partial_\nu u_1\|_{H^m(\partial D)} + \kappa\|u_1\|_{H^m(\partial D)}\big) \\ &\quad \times \big(\|\partial_\nu u_2\|_{L^2(\partial D)} + \kappa\|u_2\|_{L^2(\partial D)}\big). \end{aligned}$$

Proof. Straightforward computations show that

$$\begin{aligned}-\mathbf{i}\kappa\int_D (q_1-q_2)u_1u_2\,\mathrm{d}x &= -\mathbf{i}\kappa\left(\int_{\partial D}(\partial_\nu u_1)u_2\,\mathrm{d}S - \int_{\partial D} u_1\partial_\nu u_2\,\mathrm{d}S\right)\\ &= \int_{\partial D}\partial_\nu u_1(\partial_\nu u_2 - \mathbf{i}\kappa u_2)\,\mathrm{d}S - \int_{\partial D}(\partial_\nu u_1 - \mathbf{i}\kappa u_1)\partial_\nu u_2\,\mathrm{d}S\\ &= \int_{\partial D}\big(\mathcal{M}_{q_1,\kappa^2}[g_1]\big)g_2\,\mathrm{d}S - \int_{\partial D} g_1\mathcal{M}_{q_2,\kappa}[g_2]\,\mathrm{d}S.\end{aligned}$$

Combining the above equation with Lemma A.1, we have

$$-\mathbf{i}\kappa\int_D (q_1-q_2)u_1u_2\,\mathrm{d}x = \int_{\partial D}\big((\mathcal{M}_{q_1,\kappa^2}-\mathcal{M}_{q_2,\kappa^2})[g_1]\big)g_2\,\mathrm{d}S.$$

Application of Hölder's inequality gives

$$\kappa\left|\int_D (q_1-q_2)u_1u_2\,\mathrm{d}x\right| \le \|g_1\|_{H^m(\partial D)}\|g_2\|_{L^2(\partial D)}\|\mathcal{M}_{q_1,\kappa^2}-\mathcal{M}_{q_2,\kappa^2}\|_{H^m(\partial D)\to L^2(\partial D)}$$

and notice

$$\|g_j\|_{H^m(\partial D)} \le \|\partial_\nu u_j\|_{H^m(\partial D)} + \kappa\|u_j\|_{H^m(\partial D)},$$

the lemma is obvious. □

In order to make the paper self-contained, we recall the complex geometric optics (CGO) solutions described in [25, Lemma 2.1] or [24, Proposition 3.2], see also [19, 49].

Lemma A.3. *Let $d \ge 3$ and $\sigma > d/2$ be integers. Assume that $\zeta = \eta + \mathbf{i}\xi$ ($\eta, \xi \in \mathbb{R}^d$) satisfies*

$$|\eta|^2 = \kappa^2 + |\xi|^2 \quad \textit{and} \quad \eta\cdot\xi = 0 \quad (\Longleftrightarrow \zeta\cdot\zeta = \kappa^2).$$

Then there exist constants $C_* > 0$ and $C > 0$, independent of κ, such that if $|\xi| > C_*\|q\|_{H^\sigma(D)}$ then there exists a solution u to the equation $(\Delta + \kappa^2 + q(x))u = 0$ in D of the form

$$u(x) = e^{\mathbf{i}\zeta\cdot x}(1+\psi(x)), \quad \|\psi\|_{H^\sigma(D)} \le \frac{C}{|\xi|}\|q\|_{H^\sigma(D)}.$$

For our purpose, we will choose $\sigma = 2s$ for integer $s > d/2$. For later convenience, we denote $\mathcal{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$. By the trace theorem, e.g., [29, Theorem 9.4], we have

$$\begin{aligned}\|u_j\|_{H^m(\partial D)} &\le C\big(\|u_j\|_{L^2(D)} + \|\nabla^{\otimes(m+1)}u_j\|_{L^2(D)}\big),\\ \|\partial_\nu u_j\|_{L^2(\partial D)} &\le C\big(\|u_j\|_{L^2(D)} + \|\nabla^{\otimes(m+2)}u_j\|_{L^2(D)}\big)\end{aligned} \tag{A.1}$$

for some constant $C = C(D,m) > 0$, where $(\nabla^{\otimes\ell})_{i_1\cdots i_\ell} = \partial_{i_1}\cdots\partial_{i_\ell}$. Thus, we can substitute the CGO solutions into the identity in Lemma A.2. We now able to prove the following lemma.

Lemma A.4. *Suppose that all assumptions in Theorem* 1.1 *hold. Let C_* be the constant given in Lemma* A.3. *Then there exists a constant $C = C(s, m, D, M, \mathrm{supp}(q_1 - q_2)) > 0$ such that*

$$\begin{aligned}
&\left|((q_1 - q_2)\chi_D)\widehat{\ }(r\omega)\right| \\
&\equiv \left|\int_{\mathbb{R}^d} \chi_D(q_1 - q_2)e^{-\mathbf{i}r\omega\cdot x}\mathrm{d}x\right| \\
&\le C\kappa^{m+3}e^{Ca}\|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{H^m(\partial D)\to L^2(\partial D)} \\
&\quad + \frac{C}{a}\left(\int_{\mathbb{R}^d}\langle \mathsf{y}\rangle^{-2s}\left(\int_{\mathbb{R}^d}\langle -\mathsf{x} + r\omega - \mathsf{y}\rangle^{-2s}|((q_1 - q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2\,\mathrm{d}\mathsf{x}\right)\mathrm{d}\mathsf{y}\right)^{1/2}
\end{aligned}$$

for all $r \ge 0$, $\omega \in \mathcal{S}^{d-1}$, $a > C_*M$ with $\kappa^2 + a^2 > \frac{r^2}{4}$ and $\kappa \ge 1$, where $\langle \mathsf{y}\rangle = (1 + |\mathsf{y}|^2)^{1/2}$. Hereafter, x and y denote the phase variables in the Fourier transform.

Proof. Fix any $\omega \in \mathcal{S}^{d-1}$, since $d \ge 3$, one can choose $\omega^\perp, \widetilde{\omega}^\perp \in \mathcal{S}^{d-1}$ satisfying $\omega\cdot\omega^\perp = \omega\cdot\widetilde{\omega}^\perp = \omega^\perp\cdot\widetilde{\omega}^\perp = 0$. Like in [25, Lemma 3.1], we set

$$\xi_1 = a\omega^\perp, \quad \eta_1 = -\frac{r}{2}\omega + \left(\kappa^2 + a^2 - \frac{r^2}{4}\right)^{1/2}\widetilde{\omega}^\perp, \quad \xi_2 = -\xi_1, \quad \eta_2 = -r\omega - \eta_1,$$

and thus for each $j = 1, 2$ we see that

$$\xi_j\cdot\eta_j = 0, \quad |\eta_j|^2 = \kappa^2 + |\xi_j|^2, \quad |\xi_j| = a \ge C_*M \ge C_*\|q_j\|_{H^{2s}(D)}.$$

For each $j = 1, 2$, consider the CGO solutions with $q = q_j$ described in Lemma A.3:

$$u_j(x) = e^{\mathbf{i}\zeta_j\cdot x}(1 + \psi_j(x)), \quad \|\psi_j\|_{H^{2s}(D)} \le \frac{C}{|\xi_j|}\|q_j\|_{H^{2s}(D)} \le \frac{CM}{a} < \frac{CM}{C_*M} = \frac{C}{C_*}.$$

We now plug those u_j into the inequality in Lemma A.2 to obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\chi_D(q_1 - q_2)(1 + \psi_1)(1 + \psi_2)e^{-\mathbf{i}r\omega\cdot x}\,\mathrm{d}x\right| \\
&= \left|\int_D (q_1 - q_2)u_1u_2\,\mathrm{d}x\right| \\
&\le \kappa^{-1}\|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{H^m(\partial D)\to L^2(\partial D)}\big(\|\partial_\nu u_1\|_{H^m(\partial D)} + \kappa\|u_1\|_{H^m(\partial D)}\big) \\
&\quad \times \big(\|\partial_\nu u_2\|_{L^2(\partial D)} + \kappa\|u_2\|_{L^2(\partial D)}\big),
\end{aligned}$$

which implies

$$\begin{aligned}
&\left|\mathscr{F}[\chi_D(q_1 - q_2)](r\omega)\right| \\
&\equiv \left|\int_{\mathbb{R}^d}\chi_D(q_1 - q_2)e^{-\mathbf{i}r\omega\cdot x}\,\mathrm{d}x\right| \\
&\le \kappa^{-1}\|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{H^m(\partial D)\to L^2(\partial D)}\big(\|\partial_\nu u_1\|_{H^m(\partial D)} + \kappa\|u_1\|_{H^m(\partial D)}\big) \\
&\quad \times \big(\|\partial_\nu u_2\|_{L^2(\partial D)} + \kappa\|u_2\|_{L^2(\partial D)}\big) \\
&\quad + \left|\int_D (q_1 - q_2)e^{-\mathbf{i}r\omega\cdot x}(\psi_1 + \psi_2 + \psi_1\psi_2)\,\mathrm{d}x\right|.
\end{aligned} \tag{A.2}$$

We pick $R_0 = R_0(D) > 0$ such that $D \subset B_{R_0}(0)$. For each $j = 1, 2$, since

$$\|\psi_j\|_{L^\infty(D)} \le C\|\psi_j\|_{H^{2s}(D)} \le C \quad (\text{since } s > d/2),$$

we have $|u_j(x)| \le Ce^{|\xi_j|R_0} = Ce^{aR_0}$ for all $x \in D$, which gives

$$\|u_j\|_{L^2(D)} \le Ce^{aR_0}.$$

We can estimate

$$\begin{aligned}\|\nabla u_j\|_{L^2(D)} &= \|\mathbf{i}u_j\zeta_j + e^{\mathbf{i}\langle\zeta_j,\cdot\rangle}\nabla\psi_j\|_{L^2(D)} \le |\zeta_j|\|u_j\|_{L^2(D)} + e^{|\xi_j|R_0}\|\nabla\psi_j\|_{L^2(D)} \\ &\le C(a+\kappa)e^{aR_0} + Ce^{aR_0} \le C\kappa e^{Ca},\end{aligned}$$

and, inductively,

$$\|\nabla^{\otimes\ell}u_j\|_{L^2(D)} \le C\kappa^\ell e^{Ca} \quad \text{for all } \ell \in \mathbb{N}.$$

Therefore, by (A.1), we have

$$\|\partial_\nu u_j\|_{H^\ell(\partial D)} + \kappa\|u_j\|_{H^\ell(\partial D)} \le C\kappa^{\ell+2}e^{Ca} \quad \text{for all } \ell \in \mathbb{N}\cup\{0\}. \tag{A.3}$$

We now choose $\chi \in C_c^\infty(D)$ with $0 \le \chi \le 1$ in $\mathbb{R}^d$ satisfying $\chi = 1$ near $\operatorname{supp}(q_1 - q_2)$, and aim to estimate

$$\left|\int_D (q_1 - q_2)e^{-\mathbf{i}r\omega\cdot x}(\psi_1 + \psi_2 + \psi_1\psi_2)\,\mathrm{d}x\right| \equiv \left|\int_{\mathbb{R}^d}((q_1 - q_2)\chi_D)(x)e^{-\mathbf{i}r\omega\cdot x}\Psi(x)\chi(x)\,\mathrm{d}x\right|$$

with $\Psi(x) = \psi_1(x) + \psi_2(x) + \psi_1(x)\psi_2(x)$ by modifying some ideas in [24, Lemma 3.4]. It is not difficult to see that

$$\|\Psi\|_{H^{2s}(D)} \le \frac{C}{a}.$$

Since D is an extension domain, one can find an extension $\Psi_{\rm ext} \in H^{2s}(\mathbb{R}^d)$ with $\Psi_{\rm ext}|_D = \Psi$. Using the Parseval's identity and the convolution identity for Fourier transform, one has

$$\begin{aligned}&\int_{\mathbb{R}^d}((q_1 - q_2)\chi_D)(x)e^{-\mathbf{i}r\omega\cdot x}\Psi_{\rm ext}(x)\chi(x)\,\mathrm{d}x \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d}((q_1 - q_2)\chi_D)\widehat{\ }(\mathsf{x})(e^{-\mathbf{i}\langle r\omega,\cdot\rangle}\Psi_{\rm ext}\chi)\widehat{\ }(-\mathsf{x})\,\mathrm{d}\mathsf{x} \\ &= (2\pi)^{-2d}\int_{\mathbb{R}^d}((q_1 - q_2)\chi_D)\widehat{\ }(\mathsf{x})((e^{-\mathbf{i}\langle r\omega,\cdot\rangle}\chi)\widehat{\ } * (\Psi_{\rm ext})\widehat{\ })(-\mathsf{x})\,\mathrm{d}\mathsf{x}.\end{aligned}$$

By Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}((q_1 - q_2)\chi_D)\widehat{\ }(\mathsf{x})((e^{-\mathbf{i}\langle r\omega,\cdot\rangle}\chi)\widehat{\ } * (\Psi_{\rm ext})\widehat{\ })(-\mathsf{x})\,\mathrm{d}\mathsf{x}\right| \\ &\le \int_{\mathbb{R}^d}|((q_1 - q_2)\chi_D)\widehat{\ }(\mathsf{x})|\left|((e^{-\mathbf{i}\langle r\omega,\cdot\rangle}\chi)\widehat{\ } * (\Psi_{\rm ext})\widehat{\ })(-\mathsf{x})\right|\,\mathrm{d}\mathsf{x}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})| \left| \int_{\mathbb{R}^d} (e^{-\mathbf{i}\langle r\omega,\cdot\rangle}\chi)\widehat{\ }(-\mathsf{x}-\mathsf{y})(\Psi_{\mathrm{ext}})\widehat{\ }(\mathsf{y})\,\mathrm{d}\mathsf{y}\right| \mathrm{d}\mathsf{x} \\
&\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})||(e^{-\mathbf{i}\langle r\omega,\cdot\rangle}\chi)\widehat{\ }(-\mathsf{x}-\mathsf{y})|\,\mathrm{d}\mathsf{x}\right) |(\Psi_{\mathrm{ext}})\widehat{\ }(\mathsf{y})|\,\mathrm{d}\mathsf{y} \\
&\le \left(\int_{\mathbb{R}^d} \langle \mathsf{y}\rangle^{-4s} \left(\int_{\mathbb{R}^d} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})||(e^{-\mathbf{i}\langle r\omega,\cdot\rangle}\chi)\widehat{\ }(-\mathsf{x}-\mathsf{y})|\,\mathrm{d}\mathsf{x}\right)^2 \mathrm{d}\mathsf{y}\right)^{1/2} \|\Psi_{\mathrm{ext}}\|_{H^{2s}(\mathbb{R}^d)} \\
&= \left(\int_{\mathbb{R}^d} \langle \mathsf{y}\rangle^{-4s} \left(|((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})||\widehat{\chi}(-\mathsf{x}+r\omega-\mathsf{y})|\,\mathrm{d}\mathsf{x}\right)^2 \mathrm{d}\mathsf{y}\right)^{1/2} \|\Psi_{\mathrm{ext}}\|_{H^{2s}(\mathbb{R}^d)} \\
&\le \left(\int_{\mathbb{R}^d} \langle \mathsf{y}\rangle^{-4s} \left(\int_{\mathbb{R}^d} \langle -\mathsf{x}+r\omega-\mathsf{y}\rangle^{-2s} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2\,\mathrm{d}\mathsf{x}\right) \mathrm{d}\mathsf{y}\right)^{1/2} \\
&\quad \times \|\chi\|_{H^s(\mathbb{R}^d)} \|\Psi_{\mathrm{ext}}\|_{H^{2s}(\mathbb{R}^d)}.
\end{aligned}
$$

It is easy to see that $\|\chi\|_{H^s(\mathbb{R}^d)} \le C(s,D)$, and thus

$$
\begin{aligned}
&\left| \int_D (q_1-q_2) e^{-\mathbf{i} r\omega\cdot x}(\psi_1+\psi_2+\psi_1\psi_2)\,\mathrm{d}x \right| \\
&\le C \left(\int_{\mathbb{R}^d} \langle \mathsf{y}\rangle^{-4s} \left(\int_{\mathbb{R}^d} \langle -\mathsf{x}+r\omega-\mathsf{y}\rangle^{-2s} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2\,\mathrm{d}\mathsf{x}\right) \mathrm{d}\mathsf{y}\right)^{1/2} \|\Psi_{\mathrm{ext}}\|_{H^{2s}(\mathbb{R}^d)}.
\end{aligned}
$$

Note that the above inequality does not depend on which extension Ψ_{ext} of Ψ is chosen. In view of the equivalence (see e.g., [31, Chapter 3])

$$
\inf_{\substack{\Psi_{\mathrm{ext}}\in H^{2s}(\mathbb{R}^d) \\ \Psi_{\mathrm{ext}}|_D=\Psi}} \|\Psi_{\mathrm{ext}}\|_{H^{2s}(\mathbb{R}^d)} \cong \|\Psi\|_{H^{2s}(D)} \le \frac{C}{a},
$$

we obtain

$$
\begin{aligned}
&\left| \int_D (q_1-q_2) e^{-\mathbf{i} r\omega\cdot x}(\psi_1+\psi_2+\psi_1\psi_2)\,\mathrm{d}x \right| \\
&\le \frac{C}{a} \left(\int_{\mathbb{R}^d} \langle \mathsf{y}\rangle^{-4s} \left(\int_{\mathbb{R}^d} \langle -\mathsf{x}+r\omega-\mathsf{y}\rangle^{-2s} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2\,\mathrm{d}\mathsf{x}\right) \mathrm{d}\mathsf{y}\right)^{1/2}.
\end{aligned} \tag{A.4}
$$

The lemma is proved by combining (A.2), (A.3) and (A.4). □

Similar to [25, Lemma 3.2] or [24, Lemma 3.5], we can easily prove the following corollary by choosing a suitable parameter a.

Corollary A.5. *Suppose that all assumptions in Theorem* 1.1 *hold. Let* C_* *be the constant given in Lemma* A.3 *and* $R_* > C_* M$*. Denote* $\mathcal{E} := \|\mathcal{M}_{q_1,\kappa^2} - \mathcal{M}_{q_2,\kappa^2}\|_{H^m(\partial D)\to L^2(\partial D)}$*. Then there exists a constant* $C = C(s,m,D,M,\operatorname{supp}(q_1-q_2)) > 0$ *such that the following statement holds for all* $r \ge 0$*,* $\omega \in \mathcal{S}^{d-1}$ *and* $\kappa \ge 1$*: If* $0 \le r \le \kappa + R_*$ *then*

$$
\begin{aligned}
&\left|((q_1-q_2)\chi_D)\widehat{\ }(r\omega)\right| \\
&\le C\kappa^{m+3} e^{CR_*}\mathcal{E} + \frac{C}{R_*} \left(\int_{\mathbb{R}^d} \langle \mathsf{y}\rangle^{-4s} \left(\int_{\mathbb{R}^d} \langle -\mathsf{x}+r\omega-\mathsf{y}\rangle^{-2s} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2\,\mathrm{d}\mathsf{x}\right) \mathrm{d}\mathsf{y}\right)^{1/2};
\end{aligned}
$$

otherwise if $r > k + R_*$ then

$$\begin{aligned}&\big|((q_1 - q_2)\chi_D)\widehat{\ }(r\omega)\big| \\ &\le C\kappa^{m+3}e^{Cr}\mathcal{E} + \frac{C}{r}\left(\int_{\mathbb{R}^d}\langle \mathsf{y}\rangle^{-4s}\left(\int_{\mathbb{R}^d}\langle -\mathsf{x} + r\omega - \mathsf{y}\rangle^{-2s}|((q_1 - q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2\,\mathrm{d}\mathsf{x}\right)\mathrm{d}\mathsf{y}\right)^{1/2}.\end{aligned}$$

Proof. If $0 \le r \le k + R_*$, we take $a = R_*$; otherwise, we set $a = r$. □

We now estimate the H^{-s}-norm of $q_1 - q_2$ following the argument in [24, Lemma 3.6].

Lemma A.6. *As in Corollary* A.5*, there exists a constant C, depending on s, m, D, M,* $\mathrm{supp}(q_1 - q_2)$*, such that*

$$\begin{aligned}\|(q_1 - q_2)\chi_D\|_{H^{-s}(\mathbb{R}^d)} &\le C\kappa^{m+3}(e^{CR_*} + \chi(T)e^{CT})\mathcal{E} \\ &\quad + \frac{C}{R_*}\|(q_1 - q_2)\chi_D\|_{H^{-s}(\mathbb{R}^d)} + CT^{-(s-\frac{d}{2})}\end{aligned}$$

for all $T \ge \kappa + R_*$, where $0 \le \chi(T) \le 1$ is a continuous function with $\chi(\kappa + R_*) = 0$.

Proof. Using the polar coordinates $\mathsf{x} = r\omega$, we write

$$\|(q_1 - q_2)\chi_D\|^2_{H^{-s}(\mathbb{R}^d)} = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}I_1 &:= \int_0^{\kappa+R_*}\int_{\mathcal{S}^{d-1}}|((q_1 - q_2)\chi_D)\widehat{\ }(r\omega)|^2(1 + r^2)^{-s}r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}r, \\ I_2 &:= \int_{\kappa+R_*}^{T}\int_{\mathcal{S}^{d-1}}|((q_1 - q_2)\chi_D)\widehat{\ }(r\omega)|^2(1 + r^2)^{-s}r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}r, \\ I_3 &:= \int_T^{\infty}\int_{\mathcal{S}^{d-1}}|((q_1 - q_2)\chi_D)\widehat{\ }(r\omega)|^2(1 + r^2)^{-s}r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}r.\end{aligned}$$

It is not difficult to estimate I_3. Indeed, since $\mathrm{supp}(q_1 - q_2) \subset D$, Hölder's inequality implies $|((q_1 - q_2)\chi_D)\widehat{\ }(r\omega)| \le C\|q_1 - q_2\|_{L^2(D)} \le C$, and

$$I_3 \le C\int_T^{\infty}\int_{\mathcal{S}^{d-1}}(1 + r^2)^{-s}r^{d-1}\,\mathrm{d}\omega\,\mathrm{d}r \le CT^{-(2s-d)}.$$

On the other hand, the following inequality can be proved as in [24, (3.18)]:

$$\begin{aligned}&\int_{\mathbb{R}^d}\langle \mathsf{z}\rangle^{-2s}\int_{\mathbb{R}^d}\langle \mathsf{y}\rangle^{-4s}\int_{\mathbb{R}^d}\langle -\mathsf{x} + \mathsf{z} - \mathsf{y}\rangle^{-2s}|((q_1 - q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2\,\mathrm{d}\mathsf{x}\,\mathrm{d}\mathsf{y}\,\mathrm{d}\mathsf{z} \\ &\le C\|(q_1 - q_2)\chi_D\|^2_{H^{-s}(D)}.\end{aligned} \tag{A.5}$$

Recalling that

$$\int_0^{\infty}(1 + r^2)^{-s}r^{d-1}\,\mathrm{d}r < \infty,$$

and using Corollary A.5 and (A.5), we can derive

$$\begin{aligned} I_1 &\le C\kappa^{2(m+3)} e^{CR_*}\mathcal{E}^2 \\ &\quad + \frac{C}{R_*^2}\int_{|\mathsf{z}|<\kappa+R_*} \langle \mathsf{z}\rangle^{-2s} \int_{\mathbb{R}^d} \langle \mathsf{y}\rangle^{-4s} \int_{\mathbb{R}^d} \langle -\mathsf{x}+\mathsf{z}-\mathsf{y}\rangle^{-2s} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2 \,\mathrm{d}\mathsf{x}\,\mathrm{d}\mathsf{y}\,\mathrm{d}\mathsf{z} \\ &\le C\kappa^{2(m+3)} e^{CR_*}\mathcal{E}^2 + \frac{C}{R_*^2}\|(q_1-q_2)\chi_D\|^2_{H^{-s}(D)} \end{aligned}$$

and

$$\begin{aligned} I_2 &\le C\kappa^{2(m+3)} e^{CT}\mathcal{E}^2 \\ &\quad + \frac{C}{R_*^2}\int_{\kappa+R_*<|\mathsf{z}|<T} \langle \mathsf{z}\rangle^{-2s} \int_{\mathbb{R}^d} \langle \mathsf{y}\rangle^{-4s} \int_{\mathbb{R}^d} \langle -\mathsf{x}+\mathsf{z}-\mathsf{y}\rangle^{-2s} |((q_1-q_2)\chi_D)\widehat{\ }(\mathsf{x})|^2 \,\mathrm{d}\mathsf{x}\,\mathrm{d}\mathsf{y}\,\mathrm{d}\mathsf{z} \\ &\le C\kappa^{2(m+3)} e^{CT}\mathcal{E}^2 + \frac{C}{R_*^2}\|(q_1-q_2)\chi_D\|^2_{H^{-s}(D)}. \end{aligned}$$

By the definition of I_2, we can define $I_2 = 0$ if $T = \kappa + R_*$. Finally, the proof of the lemma is completed by combining all the above inequalities. □

With Lemma A.4 at hand, we are now ready to prove Theorem 1.1 using similar arguments as in [24, Theorem 2.1].

Proof of Theorem 1.1. One can fix a sufficiently large

$$R_* = R_*(s, m, D, M, \operatorname{supp}(q_1 - q_2))$$

in Lemma A.6 to obtain

$$\|(q_1-q_2)\chi_D\|_{H^{-s}(\mathbb{R}^d)} \le C\kappa^{m+3}(e^{CR_*} + \chi(T)e^{CT})\mathcal{E} + CT^{-(s-\frac{d}{2})}. \tag{A.6}$$

We now restrict $\mathcal{E} < 1/e$ so that $\log\frac{1}{\mathcal{E}} > 1$. We consider the following two cases:

$$\text{(i) } \kappa + R_* \le p\log\frac{1}{\mathcal{E}}, \quad \text{(ii) } \kappa + R_* > p\log\frac{1}{\mathcal{E}},$$

where $p > 0$ will be determined later.

Case (i). For $\kappa + R_* \le p\log\frac{1}{\mathcal{E}}$, we choose $T = p\log\frac{1}{\mathcal{E}}$. Then it is easy to see that

$$\kappa + \log\frac{1}{\mathcal{E}} \le \kappa + R_* + \log\frac{1}{\mathcal{E}} \le (1+p)\log\frac{1}{\mathcal{E}} = \frac{1+p}{p}T,$$

and, since $s > d/2$, the following inequality

$$T^{-(s-\frac{d}{2})} \le C_1\left(\kappa + \log\frac{1}{\mathcal{E}}\right)^{-(s-\frac{d}{2})}$$

holds for all $C_1 \geq \left(\frac{1+p}{p}\right)^{s-\frac{d}{2}}$. We want to choose C_1 and p so that

$$\kappa^{m+3}\mathcal{E}^{1-Cp} = \kappa^{m+3}e^{CT}\mathcal{E} \leq C_1\left(\kappa + \log\frac{1}{\mathcal{E}}\right)^{-(s-\frac{d}{2})},$$

equivalently,

$$(m+3)\log\kappa + (Cp-1)\log\frac{1}{\mathcal{E}} + \left(s - \frac{d}{2}\right)\log\left(\kappa + \log\frac{1}{\mathcal{E}}\right) \leq \log C_1. \tag{A.7}$$

Note that $\kappa \leq \kappa + R_* \leq p\log\frac{1}{\mathcal{E}}$ and hence

$$\begin{aligned}
&(\text{LHS of (A.7)}) \\
&\leq (m+3)\log\left(p\log\frac{1}{\mathcal{E}}\right) + (Cp-1)\log\frac{1}{\mathcal{E}} + \left(s-\frac{d}{2}\right)\log\left((1+p)\log\frac{1}{\mathcal{E}}\right) \\
&\leq \left(m+3+s-\frac{d}{2}\right)\log(1+p) + (Cp-1)\log\frac{1}{\mathcal{E}} + \left(m+3s-\frac{d}{2}\right)\log\log\frac{1}{\mathcal{E}}.
\end{aligned}$$

We now set $p = \frac{1}{2C}$ to obtain

$$(\text{LHS of (A.7)}) \leq \left(m+3+s-\frac{d}{2}\right)\log\left(\frac{2C+1}{2C}\right) - \frac{1}{2}\log\frac{1}{\mathcal{E}} + \left(m+3s-\frac{d}{2}\right)\log\log\frac{1}{\mathcal{E}},$$

and (A.7) holds if

$$\log C_1 \geq \sup_{0<\epsilon<1/e}\left(m+3+s-\frac{d}{2}\right)\log\left(\frac{2C+1}{2C}\right) - \frac{1}{2}\log\frac{1}{\epsilon} + \left(m+3s-\frac{d}{2}\right)\log\log\frac{1}{\epsilon}.$$

Finally, from (A.6) with $T = p\log\frac{1}{\mathcal{E}} = \frac{1}{2C}\log\frac{1}{\mathcal{E}}$, it follows

$$\|(q_1-q_2)\chi_D\|_{H^{-s}(\mathbb{R}^d)} \leq C\left(\kappa + \log\frac{1}{\mathcal{E}}\right)^{-(s-\frac{d}{2})}.$$

Case (ii). When $\kappa + R_* > p\log\frac{1}{\mathcal{E}} = \frac{1}{2C}\log\frac{1}{\mathcal{E}}$, choosing $T = \kappa + R_*$ and using the fact $\chi(\kappa + R_*) = 0$, we have

$$\begin{aligned}
&\|(q_1-q_2)\chi_D\|_{H^{-s}(\mathbb{R}^d)} \\
&\leq C\kappa^{m+3}e^{CR_*}\mathcal{E} + C(\kappa+R_*)^{-(s-\frac{d}{2})} \leq C\kappa^{m+3}\mathcal{E} + C\left(\kappa + \frac{1}{2}R_*\right)^{-(s-\frac{d}{2})} \\
&\leq C\kappa^{m+3}\mathcal{E} + C\left(\frac{1}{2}\kappa + \frac{1}{4C}\log\frac{1}{\mathcal{E}}\right)^{-(s-\frac{d}{2})} \leq C\kappa^{m+3}\mathcal{E} + C\left(\kappa + \log\frac{1}{\mathcal{E}}\right)^{-(s-\frac{d}{2})},
\end{aligned}$$

where we recall that $R_* = R_*(s, m, D, M, \operatorname{supp}(q_1 - q_2))$. □

B. Well-posedness of the impedance-boundary value problem

Let $\kappa > 0$, $q \in L^\infty(D)$, $F \in L^2(D)$ and $g \in L^2(\partial D)$. A function $u \in H^1(D)$ is called a (weak) solution of

$$\begin{cases} (\Delta + \kappa^2 + q(x))u = -F & \text{in } D, \\ \partial_\nu u - \mathbf{i}\kappa u = g & \text{on } \partial D \end{cases} \tag{B.1}$$

if

$$a(u,v) = (F,v)_{L^2(D)} + \langle g, v\rangle_{\partial D} \quad \text{for all } v \in H^1(D), \tag{B.2}$$

where

$$a(w,v) := (\nabla w, \nabla v)_{L^2(D)} - ((\kappa^2 + q)w, v)_{L^2(D)} - \mathbf{i}\kappa\langle w, v\rangle_{\partial D}$$

and $\langle\,\cdot\,,\cdot\,\rangle_{\partial D}$ is the duality pair on ∂D.

We first prove the following lemma similar to [13, Lemma 2.2].

Lemma B.1. *Let $u \in H^1(D)$ be a solution of* (B.1). *For each $\delta_1, \delta_2 > 0$, there hold*

$$\begin{aligned} \|\nabla u\|^2_{L^2(D)} \le{}& (\kappa^2 + \|q\|_{L^\infty(D)} + \delta_1)\|u\|^2_{L^2(D)} \\ &+ \left(\frac{\delta_1}{2\kappa^2} + \frac{1}{2\delta_1}\right)\left(\|F\|^2_{L^2(D)} + \|g\|^2_{L^2(\partial D)}\right) \end{aligned} \tag{B.3a}$$

and

$$\|u\|^2_{L^2(\partial D)} \le \frac{\delta_2}{\kappa}\|u\|^2_{L^2(D)} + \frac{1}{\delta_2\kappa}\|F\|^2_{L^2(D)} + \frac{1}{\kappa^2}\|g\|^2_{L^2(\partial D)}. \tag{B.3b}$$

Proof. We choose $v = u$ in (B.2), and take the real and imaginary parts, we get

$$\|\nabla u\|^2_{L^2(D)} - \int_D (\kappa^2 + q(x))|u(x)|^2\,\mathrm{d}x = \Re\big((F,u)_{L^2(D)} + \langle g,u\rangle_{\partial D}\big), \tag{B.4a}$$

$$-\mathbf{i}\kappa\|u\|^2_{L^2(\partial D)} = \Im\big((F,u)_{L^2(D)} + \langle g,u\rangle_{\partial D}\big). \tag{B.4b}$$

It is straightforward to derive (B.3b) from (B.4b). Similarly, from (B.4a), we have

$$\begin{aligned} \|\nabla u\|^2_{L^2(D)} \le{}& \left(\kappa^2 + \|q\|_{L^\infty(D)} + \frac{\delta_1}{2}\right)\|u\|^2_{L^2(D)} + \frac{1}{2\delta_1}\|F\|^2_{L^2(D)} \\ &+ \frac{\delta_1}{2}\|u\|^2_{L^2(\partial D)} + \frac{1}{2\delta_1}\|g\|^2_{L^2(\partial D)}. \end{aligned} \tag{B.5}$$

Substituting $\delta_2 = \kappa$ into (B.3b) and combining the resulting equation with (B.5) easily imply (B.3a). □

We also need the following lemma, which can be proved using the same argument as in [13, Lemma 2.3]. So we omit the details here.

Lemma B.2 (Rellich)**.** *For each $u \in H^2(D)$, the following identities hold*

$$\Re(u, x\cdot\nabla u)_{L^2(D)} = -\frac{d}{2}\|u\|_{L^2(D)}^2 + \frac{1}{2}\langle x\cdot\nu, |u|^2\rangle_{\partial D},$$
$$\Re(\nabla u, \nabla(x\cdot\nabla u))_{L^2(D)} = \frac{2-d}{2}\|\nabla u\|_{L^2(D)}^2 + \frac{1}{2}\langle x\cdot\nu, |\nabla u|^2\rangle_{\partial D}.$$

We now prove the following wave-number explicit estimate for the solution of the boundary value problem (B.1) similar to [13, Theorem 2.4].

Theorem B.3. *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$, which is star-shaped with respect to a ball, that is,*

$$x\cdot\nu \geq c_0 > 0 \quad \text{for all } x \in \partial D. \tag{B.6}$$

Let $M > 0$ and the potential function q satisfies $\|q\|_{L^\infty(D)} \leq \min\left\{M, \frac{\kappa^2}{16MR^2}, \frac{\kappa^2}{4d-6}\right\}$, where $R > 0$ be any number such that $D \subset B_R$. Then there exists a positive constant $C = C(D, c_0)$ such that

$$\begin{aligned} &\|\nabla u\|_{L^2(D)}^2 + \kappa^2\|u\|_{L^2(D)}^2 + \|\nabla u\|_{L^2(\partial D)}^2 + \kappa^2\|u\|_{L^2(\partial D)}^2 \\ &\leq C(1+\kappa^{-2})\big(\|F\|_{L^2(D)}^2 + \|g\|_{L^2(\partial D)}^2\big), \end{aligned} \tag{B.7}$$

which holds true for all $\kappa > 0$ and for all solution $u \in H^1(D)$ of (B.1).

Remark B.4. The estimate (B.7) is almost optimal for large $\kappa > 1$ in the following perspectives.

- In [5, Lemma 5.5], the authors showed that, if D is a ball then there exist a $g \in L^2(\partial D)$ and a solution $u \in H^1(D)$ of (B.1) with $q \equiv 0$ and $F \equiv 0$ such that $\kappa\|u\|_{L^2(D)} \gtrsim \|g\|_{L^2(\partial D)}$.
- In [47, Lemma 4.12], the author proved that given any bounded Lipschitz domain D, there exist $F \in L^2(D)$ and a solution $u \in H^1(D)$ of (B.1) with $q \equiv 0$ and $g \equiv 0$ such that $\kappa\|u\|_{L^2(D)} \gtrsim \|F\|_{L^2(D)}$.

We also refer to, e.g., [9, 46] and the references therein for related results about this topic.

Proof of Theorem B.3. Using mollifiers, it suffices to show the theorem for $u \in H^2(D)$ (also see the proof of [13, Theorem 2.4]). Choosing $v = x\cdot\nabla u$ in the real part of (B.2) and using Lemma B.2, we have

$$\begin{aligned} \Re\big((F,v)_{L^2(D)} + \langle g, v\rangle_{\partial D}\big) &= \frac{2-d}{2}\|\nabla u\|_{L^2(D)}^2 + \frac{1}{2}\langle x\cdot\nu, |\nabla u|^2\rangle_{\partial D} + \frac{d\kappa^2}{2}\|u\|_{L^2(D)}^2 \\ &\quad - \frac{\kappa^2}{2}\langle x\cdot\nu, |u|^2\rangle_{\partial D} - \Re(qu,v)_{L^2(D)} + \kappa\Im\langle u, v\rangle_{\partial D}, \end{aligned}$$

and hence from (B.6)

$$\begin{aligned}\frac{d\kappa^2}{2}\|u\|^2_{L^2(D)} &= \frac{d-2}{2}\|\nabla u\|^2_{L^2(D)} + \Re(qu,v)_{L^2(D)} - \kappa\Im\langle u,v\rangle_{\partial D}\\ &\quad - \frac{1}{2}\langle x\cdot\nu, |\nabla u|^2\rangle_{\partial D} + \frac{\kappa^2}{2}\langle x\cdot\nu, |u|^2\rangle_{\partial D} + \Re\big((F,v)_{L^2(D)} + \langle g,v\rangle_{\partial D}\big)\\ &\le \frac{d-2}{2}\|\nabla u\|^2_{L^2(D)} + R\|q\|_{L^\infty(D)}\left(\frac{1}{2\delta_1}\|u\|^2_{L^2(D)} + \frac{\delta_1}{2}\|\nabla u\|^2_{L^2(D)}\right)\\ &\quad + \frac{R}{2\delta_2}\|F\|^2_{L^2(D)} + \frac{R\delta_2}{2}\|\nabla u\|^2_{L^2(D)} + \frac{R}{2\delta_3}\|g\|^2_{L^2(\partial D)} + \frac{R\delta_3}{2}\|\nabla u\|^2_{L^2(\partial D)}\\ &\quad + \frac{\kappa R}{\delta_4}\|u\|^2_{L^2(\partial D)} + \kappa R\delta_4\|\nabla u\|^2_{L^2(\partial D)} - \frac{c_0}{2}\|\nabla u\|^2_{L^2(\partial D)} + \frac{\kappa^2 R}{2}\|u\|^2_{L^2(\partial D)}.\end{aligned}$$

Setting $\delta_3 = \frac{c_0}{4R}$ and $\delta_4 = \frac{c_0}{8\kappa R}$ in the above equation yields

$$\begin{aligned}\frac{d\kappa^2}{2}\|u\|^2_{L^2(D)} &\le \left(\frac{d-2}{2} + \frac{R\delta_2}{2} + \frac{R\|q\|_{L^\infty(D)}\delta_1}{2}\right)\|\nabla u\|^2_{L^2(D)}\\ &\quad + \frac{R\|q\|_{L^\infty(D)}}{2\delta_1}\|u\|^2_{L^2(D)} + \left(\frac{8\kappa^2R^2}{c_0} + \frac{\kappa^2 R}{2}\right)\|u\|^2_{L^2(\partial D)}\\ &\quad - \frac{c_0}{4}\|\nabla u\|^2_{L^2(\partial D)} + \frac{R}{2\delta_2}\|F\|^2_{L^2(D)} + \frac{2R^2}{c_0}\|g\|^2_{L^2(\partial D)}.\end{aligned} \tag{B.8}$$

Combining (B.3a), (B.3b) and (B.8), we obtain

$$\begin{aligned}&d\kappa^2\|u\|^2_{L^2(D)} + \frac{c_0}{2}\|\nabla u\|^2_{L^2(\partial D)}\\ &\le \left(\big(d-2+R\delta_2+R\|q\|_{L^\infty(D)}\delta_1\big)\big(\kappa^2 + \|q\|_{L^\infty(D)} + \delta_5\big) + \frac{R\|q\|_{L^\infty(D)}}{\delta_1}\right)\|u\|^2_{L^2(D)}\\ &\quad + \big(d-2+R\delta_2+R\|q\|_{L^\infty(D)}\delta_1\big)\left(\frac{\delta_5}{2\kappa^2} + \frac{1}{2\delta_5}\right)\big(\|F\|^2_{L^2(D)} + \|g\|^2_{L^2(\partial D)}\big)\\ &\quad + \kappa^2\left(\frac{16R^2}{c_0} + R\right)\left(\frac{\delta_6}{\kappa}\|u\|^2_{L^2(D)} + \frac{1}{\delta_6\kappa}\|F\|^2_{L^2(D)} + \frac{1}{\kappa^2}\|g\|^2_{L^2(\partial D)}\right)\\ &\quad + \frac{R}{\delta_2}\|F\|^2_{L^2(D)} + \frac{2R^2}{c_0}\|g\|^2_{L^2(\partial D)}.\end{aligned} \tag{B.9}$$

Since $\|q\|_{L^\infty(D)} \le M$, we choose $\delta_1 = \frac{1}{4MR}$, $\delta_2 = \frac{1}{4R}$ and compute that

$$\begin{aligned}c_1 &:= d\kappa^2 - \big(d-2+R\delta_2+R\|q\|_{L^\infty(D)}\delta_1\big)\big(\kappa^2 + \|q\|_{L^\infty(D)} + \delta_5\big) - \frac{R\|q\|_{L^\infty(D)}}{\delta_1}\\ &\quad - \kappa^2\left(\frac{16R^2}{c_0} + R\right)\frac{\delta_6}{\kappa}\\ &= 2\kappa^2 - \left(\frac{1}{4} + \frac{\|q\|_{L^\infty(D)}}{4M}\right)\kappa^2 - \left(\frac{4d-7}{4} + \frac{\|q\|_{L^\infty(D)}}{4M}\right)\big(\|q\|_{L^\infty(D)} + \delta_5\big)\\ &\quad - 4MR^2\|q\|_{L^\infty(D)} - \kappa^2\left(\frac{16R^2}{c_0} + R\right)\frac{\delta_6}{\kappa}\\ &\ge \frac{3}{2}\kappa^2 - \frac{2d-3}{2}\big(\|q\|_{L^\infty(D)} + \delta_5\big) - 4MR^2\|q\|_{L^\infty(D)} - \kappa^2\left(\frac{16R^2}{c_0} + R\right)\frac{\delta_6}{\kappa}.\end{aligned}$$

Next, from $4MR^2\|q\|_{L^\infty(D)} \leq \kappa^2/4$ and $\frac{2d-3}{2}\|q\|_{L^\infty(D)} \leq \kappa^2/4$ (see (1.3)), it follows that

$$c_1 \geq \kappa^2 - \frac{2d-3}{2}\delta_5 - \kappa^2\left(\frac{16R^2}{c_0} + R\right)\frac{\delta_6}{\kappa}.$$

Now choosing $\delta_5 = \frac{\kappa^2}{4d-6}$ and $\delta_6 = \frac{\kappa}{4}\big(\frac{16R^2}{c_0} + R\big)^{-1}$ implies $c_1 \geq \frac{\kappa^2}{2}$. Thus, by (B.9), we have

$$\begin{aligned}
\frac{\kappa^2}{2}\|u\|_{L^2(D)}^2 + \frac{c_0}{2}\|\nabla u\|_{L^2(\partial D)}^2 &\leq c_1\|u\|_{L^2(D)}^2 + \frac{c_0}{2}\|\nabla u\|_{L^2(\partial D)}^2 \\
&\leq \left(\frac{1}{8} + \frac{(2d-3)^2}{2\kappa^2} + 4\left(\frac{16R^2}{c_0} + R\right)^2 + 16R^2\right)\|F\|_{L^2(D)}^2 \\
&\quad + \left(\frac{1}{8} + \frac{(2d-3)^2}{2\kappa^2} + \frac{16R^2}{c_0} + R + \frac{2R^2}{c_0}\right)\|g\|_{L^2(\partial D)}^2.
\end{aligned}$$

Combining the above inequality with Lemma B.1 (with $\delta_1 = \kappa^2$ and $\delta_2 = \kappa$) immediately yields (B.7). □

By the Fredholm alternative principle as in [13, Theorem 2.5], we finally conclude that

Theorem B.5. *Suppose that all assumptions in Theorem* B.3 *hold. Then there exists a unique solution* $u \in H^1(D)$ *to* (B.1) *and the estimate* (B.7) *is satisfied.*

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References

[1] K. Abraham, *Nonparametric Bayesian posterior contraction rates for scalar diffusions with high-frequency data*, Bernoulli **25** (2019), no. 4A, 2696–2728.

[2] K. Abraham and R. Nickl, *On statistical Calderón problems*, Math. Stat. Learn. **2** (2019), no. 2, 165–216.

[3] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27** (1988), no. 1-3, 153–172.

[4] G. Alessandrini and J. Sylvester, *Stability for a multidimensional inverse spectral theorem*, Comm. Partial Differential Equations **15** (1990), no. 5, 711–736.

[5] D. Baskin, E. A. Spence and J. Wunsch, *Sharp high-frequency estimates for the Helmholtz equation and applications to boundary integral equations*, SIAM J. Math. Anal. **48** (2016), no. 1, 229–267.

[6] E. Blåsten, O. Y. Imanuvilov and M. Yamamoto, *Stability and uniqueness for a two-dimensional inverse boundary value problem for less regular potentials*, Inverse Probl. Imaging **9** (2015), no. 3, 709–723.

[7] E. Blåsten, L. Tzou and J.-N. Wang, *Uniqueness for the inverse boundary value problem with singular potentials in 2D*, Math. Z. **295** (2020), no. 3-4, 1521–1535.

[8] A. L. Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case*, J. Inverse Ill-Posed Probl. **16** (2008), no. 1, 19–33.

[9] S. N. Chandler-Wilde, I. G. Graham, S. Langdon and E. A. Spence, *Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering*, Acta Numer. **21** (2012), 89–305.

[10] M. Dashti and A. M. Stuart, *The Bayesian approach to inverse problems*, in: *Handbook of Uncertainty Quantification*, 311–428, Springer, Cham, 2017.

[11] D. Dos Santos Ferreira, C. E. Kenig and M. Salo, *Determining an unbounded potential from Cauchy data in admissible geometries*, Comm. Partial Differential Equations **38** (2013), no. 1, 50–68.

[12] R. Durrett, *Probability—Theory and examples*, Fifth edition, Cambridge Series in Statistical and Probabiilstic Mathematics **49**, Cambridge University Press, Cambridge, 2019.

[13] X. Feng, J. Lin and C. Lorton, *An efficient numerical method for acoustic wave scattering in random media*, SIAM/ASA J. Uncertain. Quantif. **3** (2015), no. 1, 790–822.

[14] T. Furuya, P.-Z. Kow and J.-N. Wang, *Consistency of the Bayes method for the inverse scattering problem*, Inverse Problems **40** (2024), no. 5, Paper No. 055001, 24 pp.

[15] S. Ghosal, J. K. Ghosh and A. W. Van Der Vaart, *Convergence rates of posterior distributions*, Ann. Statist. **28** (2000), no. 2, 500–531.

[16] S. Ghosal and A. Van Der Vaart, *Fundamentals of Nonparametric Bayesian Inference*, Cambridge Series in Statistical and Probabilistic Mathematics **44**, Cambridge University Press, Cambridge, 2017.

[17] E. Giné and R. Nickl, *Mathematical Foundations of Infinite-dimensional Statistical Models*, Cambridge Series in Statistical and Probabilistic Mathematics **40**, Cambridge University Press, New York, 2016.

[18] M. Giordano and R. Nickl, *Consistency of Bayesian inference with Gaussian process priors in an elliptic inverse problem*, Inverse problems **36** (2020), no. 8, 085001, 35 pp.

[19] P. Hähner, *A periodic Faddeev-type solution operator*, J. Differential Equations **128** (1996), no. 1, 300–308.

[20] M. I. Isaev, *Exponential instability in the inverse scattering problem on an energy interval*, Funktsional Anal. i Prilozhen. **47** (2013), no. 3, 28–36; translation in Funct. Anal. Appl. **47** (2013), no. 3, 187–194.

[21] ———, *Instability in the Gel'fand inverse problem at high energies*, Appl. Anal. **92** (2013), no. 11, 2262–2274.

[22] M. I. Isaev and R. G. Novikov, *Energy and regularity dependent stability estimates for the Gel'fand inverse problem in multidimensions*, J. Inverse Ill-Posed Probl. **20** (2012), no. 3, 313–325.

[23] V. Isakov, *Increasing stability for the Schrödinger potential from the Dirichlet-to-Neumann map*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), no. 3, 631–640.

[24] V. Isakov, R.-Y. Lai and J.-N. Wang, *Increasing stability for the conductivity and attenuation coefficients*, SIAM J. Math. Anal. **48** (2016), no. 1, 569–594.

[25] V. Isakov, S. Nagayasu, G. Uhlmann and J.-N. Wang, *Increasing stability of the inverse boundary value problem for the Schrödinger equation*, in: *Inverse Problems and Applications*, 131–141, Contemp. Math. **615**, Amer. Math. Soc., Providence, RI, 2014.

[26] J. Kaipio and E. Somersalo, *Statistical and Computational Inverse Problems*, Applied Mathematical Sciences **160**, Springer-Verlag, New York, 2005.

[27] P.-Z. Kow, G. Uhlmann and J.-N. Wang, *Optimality of increasing stability for an inverse boundary value problem*, SIAM J. Math. Anal. **53** (2021), no. 6, 7062–7080.

[28] W. V. Li and W. Linde, *Approximation, metric entropy and small ball estimates for Gaussian measures*, Ann. Probab. **27** (1999), no. 3, 1556–1578.

[29] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications: I*, Die Grundlehren der mathematischen Wissenschaften **181**, Springer-Verlag, New York, 1972.

[30] N. Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems **17** (2001), no. 5, 1435–1444.

[31] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[32] F. Monard, R. Nickl and G. P. Paternain, *Efficient nonparametric Bayesian inference for X-ray transforms*, Ann. Statist. **47** (2019), no. 2, 1113–1147.

[33] ———, *Consistent inversion of noisy non-Abelian X-ray transforms*, Comm. Pure Appl. Math. **74** (2021), no. 5, 1045–1099.

[34] R. Nickl, *Bernstein–von Mises theorems for statistical inverse problems I: Schrödinger equation*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 8, 2697–2750.

[35] ———, *Bayesian Non-linear Statistical Inverse Problems*, Zurich Lectures in Advanced Mathematics, EMS Press, Berlin, 2023.

[36] R. Nickl and G. P. Paternain, *On some information-theoretic aspects of non-linear statistical inverse problems*, in: *ICM—International Congress of Mathematicians: Vol. 7, Sections 15–20*, 5516–5538, EMS Press, Berlin, 2023.

[37] R. Nickl and J. Söhl, *Nonparametric Bayesian posterior contraction rates for discretely observed scalar diffusions*, Ann. Statist. **45** (2017), no. 4, 1664–1693.

[38] ———, *Bernstein-von Mises theorems for statistical inverse problems II: Compound Poisson processes*, Electron. J. Stat. **13** (2019), no. 2, 3513–3571.

[39] R. Nickl and S. Wang, *On polynomial-time computation of high-dimensional posterior measures by Langevin-type algorithms*, J. Eur. Math. Soc. (JEMS) **26** (2024), no. 3, 1031–1112.

[40] R. G. Novikov, *New global stability estimates for the Gel'fand–Calderon inverse problem*, Inverse Problems **27** (2011), no. 1, 015001, 21 pp.

[41] R. G. Novikov and M. Santacesaria, *A global stability estimate for the Gel'fand–Calderón inverse problem in two dimensions*, J. Inverse Ill-Posed Probl. **18** (2010), no. 7, 765–785.

[42] ———, *Monochromatic reconstruction algorithms for two-dimensional multi-channel inverse problems*, Int. Math. Res. Not. IMRN **2013**, no. 6, 1205–1229.

[43] M. Santacesaria, *New global stability estimates for the Calderón problem in two dimensions*, J. Inst. Math. Jussieu **12** (2013), no. 3, 553–569.

[44] ———, *Stability estimates for an inverse problem for the Schrödinger equation at negative energy in two dimensions*, Appl. Anal. **92** (2013), no. 8, 1666–1681.

[45] ———, *A Hölder-logarithmic stability estimate for an inverse problem in two dimensions*, J. Inverse Ill-Posed Probl. **23** (2015), no. 1, 51–73.

[46] S. Sauter and C. Torres, *Stability estimate for the Helmholtz equation with rapidly jumping coefficients*, Z. Angew. Math. Phys. **69** (2018), no. 6, Paper No. 139, 30 pp.

[47] E. A. Spence, *Wavenumber-explicit bounds in time-harmonic acoustic scattering*, SIAM J. Math. Anal. **46** (2014), no. 4, 2987–3024.

[48] A. M. Stuart, *Inverse problems: a Bayesian perspective*, Acta Numer. **19** (2010), 451–559.

[49] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. (2) **125** (1987), no. 1, 153–169.

[50] A. W. Van Der Vaart and J. H. Van Zanten, *Rates of contraction of posterior distributions based on Gaussian process priors*, Ann. Statist. **36** (2008), no. 3, 1435–1463.

[51] S. J. Vollmer, *Posterior consistency for Bayesian inverse problems through stability and regression results*, Inverse Problems **29** (2013), no. 12, 125011, 32 pp.

Pu-Zhao Kow
Department of Mathematical Sciences, National Chengchi University, Taipei 116, Taiwan
E-mail address: `pzkow@g.nccu.edu.tw`

Jenn-Nan Wang
Institute of Applied Mathematical Sciences, National Taiwan University, Taipei 106, Taiwan
E-mail address: `jnwang@math.ntu.edu.tw`