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On Defects of Entire Curves of Finite Lower Order

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Abstract. In this paper we consider the relationship between the number of separated maximum points of an entire curve and the Baernstein's T^*-function. The results of Edrei, Goldberg, Krytov, Ostrovskii, Teichmüller are generalized. We also give example showing that the obtained estimate is sharp.

1. Introduction

We shall use standard notations of value distribution theory of meromorphic functions: $m(r,a,f)$ for the proximity function, $N(r,a,f)$ for the function counting a-points, $T(r,f)$ for Nevanlinna's characteristic, $\delta(a,f)$ for Nevanlinna's defect and λ, ρ for the lower order and order, respectively [13, 16].

Let $f(z)$ be a meromorphic function in $\mathbb{C}$, $\mathcal{L}(r,a,f) = \max_{|z|=r} \log^+ \frac{1}{|f(z)-a|}$ $(a \in \mathbb{C})$ and $\mathcal{L}(r,\infty,f) = \max_{|z|=r} \log^+ |f(z)|$. The quantity

$$\beta(a,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{T(r,f)}$$

is called Petrenko's deviation of a meromorphic function $f(z)$ at $a \in \overline{\mathbb{C}}$.

It is clear that $\delta(a,f) \le \beta(a,f)$ for $a \in \overline{\mathbb{C}}$. In 1969 Petrenko [29] obtained a sharp upper estimate of the magnitude of deviation of meromorphic functions of finite lower order.

Theorem 1.1. [29] *If $f(z)$ is a meromorphic function of finite lower order λ, then for $a \in \overline{\mathbb{C}}$,*

$$\beta(a,f) \le B(\lambda) := \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} & \text{if } \lambda \le 1/2, \\ \pi\lambda & \text{if } \lambda > 1/2. \end{cases}$$

In the case of $\lambda \le 1/2$, Theorem 1.1 was obtained by Goldberg and Ostrovskii in 1961 [12]. It should be mentioned here that the conjecture that $\beta(\infty,f) \le \pi\rho$ for entire

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functions of order ρ with $1/2 < \rho < \infty$ was stated in 1932 by Paley [28] and proved in 1969 by Govorov [14].

The sharp upper estimate of the sum of deviations was given by Marchenko and Shcherba in 1990 as a solution of Petrenko's problem given in his monograph [30]. They proved that the inequality $\sum_{a\in\overline{\mathbb{C}}}\beta(a,f)\leq 2B(\lambda)$ holds [26] for a meromorphic function $f(z)$ of finite lower order λ.

The theory of distribution of p-dimensional entire curves was developed in the years 1930–1950 by Cartan [4], H. Weyl, J. Weyl [34,35] and Ahlfors [1].

Let $\mathbb{C}^p$ be the p-dimensional complex space. For $\overrightarrow{a}=(a_1,a_2,\dots,a_p)$, $\overrightarrow{b}=(b_1,b_2,\dots,b_p)\in\mathbb{C}^p$ define a dot product $(\overrightarrow{a},\overrightarrow{b})=\sum_{k=1}^{p}a_k\bar{b}_k$ and a vector norm $\|\overrightarrow{a}\|=\sqrt{(\overrightarrow{a},\overrightarrow{a})}$.

A vector $\overrightarrow{G}(z)=(g_1(z),g_2(z),\dots,g_p(z))$, where $\{g_k(z)\}_{k=1}^{p}$ are entire functions, without common zeros, is called a *p-dimensional entire curve*. Thus $\overrightarrow{G}(z)$ is a holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}^p$.

We denote by $n(t,\overrightarrow{a},\overrightarrow{G})$ the number of zeros of the product $(\overrightarrow{G}(z),\overrightarrow{a})$ in the disc $\overline{K(0,t)}=\{z:|z|\leq t\}$, counted according to multiplicity. Each zero of the function $(\overrightarrow{G}(z),\overrightarrow{a})$ is called an $\overrightarrow{a}$-point of the entire curve $\overrightarrow{G}(z)$.

The $\overrightarrow{a}$-points counting function is defined as

$$N(r,\overrightarrow{a},\overrightarrow{G})=\int_0^r[n(t,\overrightarrow{a},\overrightarrow{G})-n(0,\overrightarrow{a},\overrightarrow{G})]\frac{dt}{t}+n(0,\overrightarrow{a},\overrightarrow{G})\log r.$$

The *proximity function* $m(r,\overrightarrow{a},\overrightarrow{G})$ is defined by

$$m(r,\overrightarrow{a},\overrightarrow{G})=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\overrightarrow{G}(re^{i\theta})\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(re^{i\theta}),\overrightarrow{a})|}\,d\theta.$$

The function $T(r,\overrightarrow{G})=\frac{1}{2\pi}\int_0^{2\pi}\log\|\overrightarrow{G}(re^{i\theta})\|\,d\theta$ is called the *characteristic* of the entire curve $\overrightarrow{G}(z)$.

The numbers

$$\rho=\limsup_{r\to\infty}\frac{\log T(r,\overrightarrow{G})}{\log r}\quad\text{and}\quad\lambda=\liminf_{r\to\infty}\frac{\log T(r,\overrightarrow{G})}{\log r}$$

are called respectively the *order* and the *lower order* of $\overrightarrow{G}(z)$ and the quantity

$$\delta(\overrightarrow{a},\overrightarrow{G})=\liminf_{r\to\infty}\frac{m(r,\overrightarrow{a},\overrightarrow{G})}{T(r,\overrightarrow{G})}$$

is called the *defect* of entire curve $\overrightarrow{G}(z)$ at the vector $\overrightarrow{a}$.

In 1933 Cartan [4] proved an analog of the first and second Nevanlinna fundamental theorems for entire curves (see also [31,32]).

Let $\overrightarrow{G}(z)$ be a p-dimensional entire curve and $\overrightarrow{a}$ be a p-dimensional complex vector such that $(\overrightarrow{G}(z), \overrightarrow{a}) \not\equiv 0$. We put [31]

$$\mathcal{L}(r, \overrightarrow{a}, \overrightarrow{G}) = \max_{|z|=r} \log \frac{\|\overrightarrow{G}(z)\| \cdot \|\overrightarrow{a}\|}{|(\overrightarrow{G}(z), \overrightarrow{a})|}.$$

The quantity

$$\beta(\overrightarrow{a}, \overrightarrow{G}) = \liminf_{r\to\infty} \frac{\mathcal{L}(r, \overrightarrow{a}, \overrightarrow{G})}{T(r, \overrightarrow{G})}$$

is called the *magnitude of deviation* of entire curve $\overrightarrow{G}(z)$ at vector $\overrightarrow{a}$. It is clear that $\delta(\overrightarrow{a}, \overrightarrow{G}) \le \beta(\overrightarrow{a}, \overrightarrow{G})$ for each $\overrightarrow{a} \in \mathbb{C}^p$. Petrenko obtained sharp estimate of deviation of entire curve of finite lower order.

Theorem 1.2. [31] *If a p-dimensional entire curve $\overrightarrow{G}(z)$ is of finite lower order λ, then for any $\overrightarrow{a} \in \mathbb{C}^p$ we have*

$$\beta(\overrightarrow{a}, \overrightarrow{G}) \le \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} & \text{for } \lambda \le 1/2, \\ \pi\lambda & \text{for } \lambda > 1/2. \end{cases}$$

Let $\nu(r)$ be the number of maximum modulus points of an entire function $f(z)$ on the circle $\{z : |z| = r\}$. In 1964 Erdős posed the following questions (see [17, Problem 2.16]):

Can we have a function $f(z) \not\equiv cz^n$ such that
(a) $\limsup_{r\to\infty} \nu(r) = \infty$; (b) $\liminf_{r\to\infty} \nu(r) = \infty$?

In 1968 Herzog and Piranian [18] found a positive solution of the Erdős problem (a). They gave a suitable example of an entire function of infinite lower order. In the case of entire functions of finite lower order the question (a) is still open (see also [24]).

In 1977 Clunie stated the same question as formulated in the Erdős's problem (b) (see [2, Problem 2.49]):

Is it true that $\liminf_{r\to\infty} \nu(r) < \infty$ for all transcendental entire functions f?

In [2] it was not mentioned that this question had been posed by Erdős first. Thus in [22] this problem was presented as Clunie's problem. In 2002 Piranian informed one of the authors by letter that this problem belonged originally to Erdős and was stated in 1964.

In 2024 Glücksam and Pardo-Simón found a positive solution of the Erdős's problem (b) [11].

In 1995 Marchenko introduced the term separated maximum modulus points of meromorphic functions [22] (see also [23, 25]). Let $f(z)$ be a meromorphic function in $\mathbb{C}$. For any $r \in (0, \infty)$ we denote by $p(r, \infty, f)$ the number of component intervals of the set

$$\{\theta : |f(re^{i\theta})| > 1\}$$

possessing at least one maximum modulus point of the function $f(z)$ on the circle $\{z : |z| = r\}$. We set

$$p(\infty, f) = \liminf_{r\to\infty} p(r, \infty, f).$$

In [22] was obtained a sharp estimate of $\beta(\infty, f)$ involving $p(\infty, f)$ for meromorphic functions of finite lower order.

Theorem 1.3. [22] *For a meromorphic function $f(z)$ of finite lower order λ we have*

$$\beta(\infty, f) \le \begin{cases} \frac{\pi\lambda}{p(\infty,f)} & \text{if } \frac{\lambda}{p(\infty,f)} \ge \frac{1}{2}, \\ \frac{\pi\lambda}{\sin\pi\lambda} & \text{if } p(\infty, f) = 1 \text{ and } \lambda < \frac{1}{2}, \\ \frac{\pi\lambda}{p(\infty,f)} \sin\frac{\pi\lambda}{p(\infty,f)} & \text{if } p(\infty, f) > 1 \text{ and } \frac{\lambda}{p(\infty,f)} < \frac{1}{2}. \end{cases}$$

Corollary 1.4. *If $f(z)$ is a meromorphic function of finite lower order λ, then*

$$\beta(\infty, f) \le \begin{cases} \frac{\pi\lambda}{\sin\pi\lambda} & \text{if } \lambda \le 1/2, \\ \pi\lambda & \text{if } \lambda > 1/2. \end{cases}$$

Petrenko's theorem (Theorem 1.1) follows from Corollary 1.4.

Corollary 1.5. *For a meromorphic function $f(z)$ of finite lower order λ and $\beta(\infty, f) > 0$, we have*

$$p(\infty, f) \le \max\left\{\left[\frac{\pi\lambda}{\beta(\infty, f)}\right], 1\right\} < \infty,$$

where $[x]$ means the integral part of the number x.

Corollary 1.6. *For an entire function $f(z)$ of finite lower order λ, we have*

$$p(\infty, f) \le \max\{[\pi\lambda], 1\} < \infty.$$

In 2004 Ciechanowicz and Marchenko [5] (see also [6, 7]) introduced the following generalization of the notion of separated maximum modulus points of a meromorphic function. Let $f(z)$ be a meromorphic function in $\mathbb{C}$ and $\phi(r)$ be a positive nondecreasing convex function of $\log r$ for $r > 0$, such that $\phi(r) = o(T(r, f))$ $(r \to \infty)$. Let denote by $\widehat{p}_\phi(r, \infty, f)$ the number of the component intervals of the set

$$\{\theta : \log|f(re^{i\theta})| > \phi(r)\}$$

possessing at least one maximum modulus point of the function $f(z)$ on the circle $\{z : |z| = r\}$. Let

$$\widehat{p}_\phi(\infty, f) = \liminf_{r\to\infty} \widehat{p}_\phi(r, \infty, f), \quad \widehat{p}(\infty, f) = \sup_\phi \widehat{p}_\phi(\infty, f).$$

If $\delta(\infty, f) > 0$ or $\beta(\infty, f) > 0$ and $f(z) \not\equiv \text{const}$, then $\widehat{p}(\infty, f) \ge p(\infty, f) \ge 1$. For entire functions we have $\delta(\infty, f) = 1$ and $\beta(\infty, f) \ge 1$. Thus for an entire function $f(z) \not\equiv \text{const}$ we have $\widehat{p}(\infty, f) \ge p(\infty, f) \ge 1$.

Theorem 1.7. [5] *For a meromorphic function $f(z)$ of finite lower order λ, we have*

$$\beta(\infty, f) \leq \begin{cases} \frac{\pi\lambda}{\widehat{p}(\infty,f)} & \text{if } \frac{\lambda}{\widehat{p}(\infty,f)} \geq \frac{1}{2}, \\ \frac{\pi\lambda}{\sin \pi\lambda} & \text{if } \widehat{p}(\infty, f) = 1 \text{ and } \lambda < \frac{1}{2}, \\ \frac{\pi\lambda}{\widehat{p}(\infty,f)} \sin \frac{\pi\lambda}{\widehat{p}(\infty,f)} & \text{if } \widehat{p}(\infty, f) > 1 \text{ and } \frac{\lambda}{\widehat{p}(\infty,f)} < \frac{1}{2}. \end{cases}$$

Corollary 1.8. *For a meromorphic function $f(z)$ of finite lower order λ, we have*

$$\widehat{p}(\infty, f) \leq \max\left\{ \left[\frac{\pi\lambda}{\beta(\infty, f)}\right], 1 \right\}.$$

Corollary 1.9. *For an entire function $f(z)$ of finite lower order λ, we have*

$$\widehat{p}(\infty, f) \leq \max\{[\pi\lambda], 1\} < \infty.$$

In 2019 we introduced the term of separated maximum points of entire curves [19] (see also [36] and [20]). Let $\overrightarrow{G}(z) = (g_1(z), \dots, g_p(z))$ be an entire curve. For each p-dimensional complex vector $\overrightarrow{a}$ and the function $\phi(r)$, which is a positive, non-decreasing, convex function of $\log r$ for $r > 0$, such that $\phi(r) = o(T(r, \overrightarrow{G}))$ let $\widehat{p}_\phi(r, \overrightarrow{a}, \overrightarrow{G})$ be the number of component intervals of the set

$$\left\{ \theta : \log \frac{\|\overrightarrow{G}(re^{i\theta})\| \cdot \|\overrightarrow{a}\|}{|(\overrightarrow{G}(re^{i\theta}), \overrightarrow{a})|} > \phi(r) \right\}$$

possessing at least one maximum point of the function $\log \frac{\|\overrightarrow{G}(re^{i\theta})\| \cdot \|\overrightarrow{a}\|}{|(\overrightarrow{G}(re^{i\theta}), \overrightarrow{a})|}$. Let $\widehat{p}_\phi(\overrightarrow{a}, \overrightarrow{G}) = \liminf_{r\to\infty} \widehat{p}_\phi(r, \overrightarrow{a}, \overrightarrow{G})$,

$$\widehat{p}(\overrightarrow{a}, \overrightarrow{G}) = \sup_\phi \widehat{p}_\phi(\overrightarrow{a}, \overrightarrow{G}).$$

Theorem 1.10. [19, 36] *For a p-dimensional entire curve $\overrightarrow{G}(z)$ of finite lower order λ and $\overrightarrow{a} \in \mathbb{C}^p$, we have*

$$\beta(\overrightarrow{a}, \overrightarrow{G}) \leq \begin{cases} \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})} & \text{if } \frac{\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})} \geq \frac{1}{2}, \\ \frac{\pi\lambda}{\sin \pi\lambda} & \text{if } p(\overrightarrow{a}, \overrightarrow{G}) = 1 \text{ and } \lambda < \frac{1}{2}, \\ \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})} \sin \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})} & \text{if } \widehat{p}(\overrightarrow{a}, \overrightarrow{G}) > 1 \text{ and } \frac{\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})} < \frac{1}{2}. \end{cases}$$

Corollary 1.11. *For an entire curve $\overrightarrow{G}(z)$ of finite lower order λ and $\overrightarrow{a} \in \mathbb{C}^p$, we have*

$$\widehat{p}(\overrightarrow{a}, \overrightarrow{G}) \leq \max\left\{ 1, \left[\frac{\pi\lambda}{\beta(\overrightarrow{a}, \overrightarrow{G})}\right] \right\}.$$

2. Main results

Theorem 2.1. *Let $\overrightarrow{G}(z)$ be a p-dimensional entire curve of finite lower order $\lambda < \frac{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}{2}$ and $\overrightarrow{a} \in \mathbb{C}^p$. Then*

$$\limsup_{r\to\infty} \frac{\log \mu(r, \overrightarrow{a}, \overrightarrow{G})}{T(r, \overrightarrow{G})} \geq \frac{\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}}{\sin \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}} \left(\delta(\overrightarrow{a}, \overrightarrow{G}) - 1 + \cos \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a}, \overrightarrow{G})} \right),$$

where $\mu(r, \overrightarrow{a}, \overrightarrow{G}) = \min_{|z|=r} \frac{\|\overrightarrow{G}(z)\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(z), \overrightarrow{a})|}$.

Corollary 2.2. *Let $\overrightarrow{G}(z)$ be a p-dimensional entire curve of lower order $\lambda < 1/2$ and $\overrightarrow{a} \in \mathbb{C}^p$. Then*

$$\limsup_{r\to\infty} \frac{\log \mu(r, \overrightarrow{a}, \overrightarrow{G})}{T(r, \overrightarrow{G})} \geq \frac{\pi\lambda}{\sin \pi\lambda} \big(\delta(\overrightarrow{a}, \overrightarrow{G}) - 1 + \cos \pi\lambda\big),$$

where $\mu(r, \overrightarrow{a}, \overrightarrow{G}) = \min_{|z|=r} \frac{\|\overrightarrow{G}(z)\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(z), \overrightarrow{a})|}$.

The statement of Corollary 2.2 was obtained by Krytov [21]. In the case of meromorphic functions the result of Corollary 2.2 was obtained by Goldberg and Ostrovskii [13,27].

Corollary 2.3. *Suppose that $\overrightarrow{G}(z)$ is a p-dimensional entire curve of finite lower order $\lambda < \frac{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}{2}$ and $\delta(\overrightarrow{a}, \overrightarrow{G}) > 1 - \cos \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}$. Then there exists a sequence of circles $\{z : |z| = r_k\}$, $r_k \to \infty$, on which $\frac{\|\overrightarrow{G}(z)\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(z), \overrightarrow{a})|}$ tends to ∞ uniformly with respect to* $\arg z$.

Corollary 2.4. *Suppose that $\overrightarrow{G}(z)$ is a p-dimensional entire curve of lower order $\lambda < 1/2$ and $\delta(\overrightarrow{a}, \overrightarrow{G}) > 1 - \cos \pi\lambda$. Then there is a sequence $r_n \to \infty$, such that $\frac{\|\overrightarrow{G}(r_n e^{i\theta})\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(r_n e^{i\theta}), \overrightarrow{a})|}$ tends uniformly to ∞ for $\theta \in [0, 2\pi]$.*

The statement of Corollary 2.4 was obtained by Krytov [21]. In the case of meromorphic functions the result of Corollary 2.4 was obtained earlier by Goldberg and Ostrovskii [13, 27] and Edrei [8].

It is necessary to admit that in 1939 Teichmüller [33] proved that for the meromorphic function $f(z)$ of order $\rho < 1/2$ such that $\delta(\infty, f) > 1 - \cos \pi\rho$ it holds for all $\theta \in [0, 2\pi]$ that

$$\limsup_{r\to\infty} |f(re^{i\theta})| = \infty.$$

Therefore Teichmüller get the result of Corollary 2.4 in the case of meromorphic functions such that $\delta(\infty, f) > \frac{1-\cos\pi\rho}{1-\epsilon\cos\pi\rho}$ $(0 < \epsilon < 1)$.

3. Auxiliary results

Let $\overrightarrow{G}(z)$ be a p-dimensional entire curve, $\overrightarrow{a} \in \mathbb{C}^p$ and let $\phi(r)$ be a positive, nondecreasing, convex function of $\log r$ such that $\phi(r) = o(T(r, \overrightarrow{G}))$. We consider the function given by

$$u_\phi(z) = \max\left\{\log \frac{\|\overrightarrow{G}(z)\| \cdot \|\overrightarrow{a}\|}{|(\overrightarrow{G}(z), \overrightarrow{a})|}, \phi(|z|)\right\}.$$

In [19] we proved the following lemma.

Lemma 3.1. [19] *The function $u_\phi(z)$ is a δ-subharmonic function in $\mathbb{C}$, i.e.,*

$$u_\phi(z) = u_1(z) - u_2(z),$$

where $u_1(z)$, $u_2(z)$ are subharmonic functions in $\mathbb{C}$ and

$$\frac{1}{2\pi}\int_0^{2\pi} u_2(re^{i\theta})\, d\theta = N(r, \overrightarrow{a}, \overrightarrow{G}).$$

Let [3, 19]

$$m^*(r, \theta, u_\phi) = \sup_{|E|=2\theta} \frac{1}{2\pi}\int_E u_\phi(re^{i\varphi})\, d\varphi, \quad T^*(r, \theta, u_\phi) = m^*(r, \theta, u_\phi) + N(r, \overrightarrow{a}, \overrightarrow{G}),$$

where $r \in (0, \infty)$, $\theta \in [0, \pi]$, E is a measurable set and $|E|$ is the Lebesgue measure of E. Now for each $t \in (0, +\infty)$, consider the set

$$F_t = \{re^{i\varphi} : u_\phi(re^{i\varphi}) > t\},$$

and let

$$\widetilde{u}_\phi(re^{i\varphi}) = \sup\{t : re^{i\varphi} \in F_t^*\},$$

where F_t^* is the symmetric rearrangement of the set F_t [15].

The function $\widetilde{u}_\phi(re^{i\varphi})$ is non-negative and non-increasing in the interval $[0, \pi]$, even with respect to ϕ and for each fixed $r > 0$ equimeasurable with $u_\phi(re^{i\varphi})$. Moreover, it satisfies the equalities

$$\widetilde{u}_\phi(r) = \max\left\{\log \max_{|z|=r} \frac{\|\overrightarrow{G}(z)\| \cdot \|\overrightarrow{a}\|}{|(\overrightarrow{G}(z), \overrightarrow{a})|}, \phi(r)\right\},$$

$$\widetilde{u}_\phi(re^{i\pi}) = \max\left\{\log \min_{|z|=r} \frac{\|\overrightarrow{G}(z)\| \cdot \|\overrightarrow{a}\|}{|(\overrightarrow{G}(z), \overrightarrow{a})|}, \phi(r)\right\},$$

$$m^*(r, \theta, u_\phi) = \frac{1}{\pi}\int_0^\theta \widetilde{u}_\phi(re^{i\varphi})\, d\varphi.$$

From Baernstein's theorem [3], the function $T^*(r,\theta,u_\phi)$ is subharmonic in $D=\{re^{i\theta}: 0<r<\infty, 0<\theta<\pi\}$, continuous in $D\cup(-\infty,0)\cup(0,\infty)$ and logarithmically convex in $r>0$ for each fixed $\theta\in[0,\pi]$. Moreover,

$$\begin{aligned} T^*(r,0,u_\phi) &= N(r,\overrightarrow{a},\overrightarrow{G}),\\ T^*(r,\pi,u_\phi) &= T(r,\overrightarrow{G})+o(T(r,\overrightarrow{G})) \quad (r\to\infty),\\ \frac{\partial}{\partial\theta}T^*(r,\theta,u_\phi) &= \frac{\widetilde{u}_\phi(re^{i\theta})}{\pi} \quad \text{for } 0\le\theta\le\pi. \end{aligned}$$

Let $\alpha(r)$ be a real-valued function of a real variable r and define

$$L\alpha(r)=\liminf_{h\to 0}\frac{\alpha(re^h)+\alpha(re^{-h})-2\alpha(r)}{h^2}.$$

When $\alpha(r)$ is twice differentiable in r, then $L\alpha(r)=r\frac{d}{dr}\big(r\frac{d}{dr}\alpha(r)\big)$.

In [19] we proved the following lemma.

Lemma 3.2. [19] *Let $\overrightarrow{G}(z)$ be a p-dimensional entire curve and $\overrightarrow{a}$ be a p-dimensional complex vector. For almost all $\theta\in[0,\pi]$ and for all $r>0$ such that the function $\frac{\|\overrightarrow{G}(z)\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(z),\overrightarrow{a})|}$ has neither zeros nor poles in $\{z:|z|=r\}$, we have*

$$LT^*(r,\theta,u_\phi)\ge -\frac{\widehat{p}_\phi^2(r,\overrightarrow{a},\overrightarrow{G})}{\pi}\frac{\partial\widetilde{u}_\phi(r,\theta)}{\partial\theta}.$$

Lemma 3.3. [22] *Let the function $f(x)$ be non-decreasing on the interval $[a,b]$ and let $\varphi(x)$ be a non-negative function having a bounded derivative of the interval $[a,b]$. Then*

$$\int_a^b f'(x)\varphi(x)\,dx\le f(b)\varphi(b)-f(a)\varphi(a)-\int_a^b\varphi'(x)f(x)\,dx.$$

We will remind the definition of the Pólya peaks for a monotonic functions [30]. Let $T(r)$ be an increasing and continuous for $r\ge r_0$ function of finite lower order λ. The sequence $\{r_k\}$ is called a sequence of Pólya peaks of the function $T(r)$ if there are sequences $\{a_k\}$, $\{A_k\}$ and $\{\epsilon_k\}$ of non-negative numbers such that

$$\lim_{k\to\infty}a_k=\lim_{k\to\infty}\epsilon_k=0,\quad \lim_{k\to\infty}A_k=\lim_{k\to\infty}a_kr_k=\infty,$$

and for all $r\in[a_kr_k,A_kr_k]$ and for $k>k_0$ we have

$$T(r)\ge(1-\epsilon_k)\left(\frac{r}{r_k}\right)^\lambda T(r_k).$$

Lemma 3.4. [30, p. 40] *Let S_k and R_k be two sequences such that*

$$\lim_{k\to\infty}S_k=\lim_{k\to\infty}R_k=\lim_{k\to\infty}\frac{R_k}{S_k}=\infty,$$

and for each k the numbers $2S_k$ and $2R_k$ are Pólya peaks of the function $T(r)$. Then for each positive number ϵ there exists $k_0(\epsilon)$ such that for each $k > k_0$ we have

$$\frac{T(2S_k)}{S_k^\lambda} + \frac{T(2R_k)}{R_k^\lambda} < \epsilon \int_{2S_k}^{R_k} \frac{T(r)}{r^{\lambda+1}}\, dr.$$

In our later considerations instead of the function $T(r)$ we will be using the Nevallina's characteristic of a p-dimensional entire curve $\overrightarrow{G}(z)$ of finite lower order λ. From Lemma 3.4 we have

$$\frac{T(2S_k, \overrightarrow{G})}{S_k^\lambda} + \frac{T(2R_k, \overrightarrow{G})}{R_k^\lambda} < \epsilon \int_{2S_k}^{R_k} \frac{T(r, \overrightarrow{G})}{r^{\lambda+1}}\, dr \quad (k \to \infty). \tag{3.1}$$

4. Proof of Theorem 2.1

If $\widehat{p}(\overrightarrow{a}, \overrightarrow{G}) = +\infty$ then by Theorem 1.3 we have $\beta(\overrightarrow{a}, \overrightarrow{G}) = 0$. Thus $\delta(\overrightarrow{a}, \overrightarrow{G}) = 0$, so the right side of inequality in the statement of Theorem 2.1 is equal to zero and left side is non-negative.

Let now $\widehat{p}(\overrightarrow{a}, \overrightarrow{G}) < \infty$. If $\delta(\overrightarrow{a}, \overrightarrow{G}) \le 1 - \cos \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a}, \overrightarrow{G})}$ then Theorem 2.1 is obviously. Let $\delta(\overrightarrow{a}, \overrightarrow{G}) > 1 - \cos \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a}, \overrightarrow{G})}$. Then $\delta(\overrightarrow{a}, \overrightarrow{G}) > 0$ and for every $\phi(r)$ we have $\widehat{p}_\phi(\overrightarrow{a}, \overrightarrow{G}) \ge 1$. We shall first consider the case $\lambda > 0$. We put [9, 10, 22]

$$\sigma(r) = \int_0^\pi T^*(r, \theta, u_\phi) \sin \frac{\lambda\theta}{\widehat{p}_\phi(\overrightarrow{a}, \overrightarrow{G})}\, d\theta,$$

where $T^*(r, \theta, u_\phi) = T^*(re^{i\theta}, u_\phi)$.

Since $T^*(re^{i\theta}, u_\phi)$ is a convex function of $\log r$, it follows that for all $r > 0$ and $h > 0$ we have

$$T^*(re^h, \theta, u_\phi) + T^*(re^{-h}, \theta, u_\phi) - 2T^*(r, \theta, u_\phi) \ge 0.$$

Thus by Fatou's lemma for all $r > 0$ we have

$$L\sigma(r) \ge \int_0^\pi LT^*(r, \theta, u_\phi) \sin \frac{\lambda\theta}{\widehat{p}_\phi(\overrightarrow{a}, \overrightarrow{G})}\, d\theta \ge 0. \tag{4.1}$$

It follows from this inequality that $\sigma(r)$ is a convex function of $\log r$, and so $r\sigma'_-(r)$ is an increasing function on $(0, \infty)$, where $\sigma'_-(r)$ is the left-hand derivative of $\sigma(r)$ at the point r. Therefore, for almost all $r > 0$,

$$L\sigma(r) = r\frac{d}{dr}\big(r\sigma'_-(r)\big).$$

It follows from (4.1) and Lemma 3.2 that for almost all $r > 0$,

$$r\frac{d}{dr}\big(r\sigma'_-(r)\big) \ge -\int_0^\pi \frac{\widehat{p}_\phi^2(r, \overrightarrow{a}, \overrightarrow{G})}{\pi} \frac{\partial \widetilde{u}_\phi(r, \theta)}{\partial\theta} \sin \frac{\lambda\theta}{\widehat{p}_\phi(r, \overrightarrow{a}, \overrightarrow{G})}\, d\theta. \tag{4.2}$$

By definition $\widehat{p}_\phi(r,\overrightarrow{a},\overrightarrow{G})$ takes only the integral values. Thus for $r > r_0$ we have $\widehat{p}_\phi(\overrightarrow{a},\overrightarrow{G}) \le \widehat{p}_\phi(r,\overrightarrow{a},\overrightarrow{G})$. From this and (4.2) it follows that for almost all $r > r_0$,

$$r\frac{d}{dr}\big(r\sigma'_-(r)\big) \ge -\int_0^\pi \frac{\widehat{p}^2_\phi(\overrightarrow{a},\overrightarrow{G})}{\pi}\frac{\partial\widetilde{u}_\phi(r,\theta)}{\partial\theta}\sin\frac{\lambda\theta}{\widehat{p}_\phi(\overrightarrow{a},\overrightarrow{G})}\,d\theta. \tag{4.3}$$

If there are neither zeros nor poles of $\frac{\|\overrightarrow{G}(z)\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(z),\overrightarrow{a})|}$ on the circle $\{z : |z| = r\}$ for $r > 0$, the function $u_\phi(r,\theta) = \max\big(\log\frac{\|\overrightarrow{G}(re^{i\theta})\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(re^{i\theta}),\overrightarrow{a})|},\phi(r)\big)$ fulfills the Lipschitz condition in $\theta \in [0,2\pi]$. Therefore $\widetilde{u}_\phi(r,\theta)$ also fulfills the Lipschitz condition on $[0,\pi]$ [15]. This implies that the function $\widetilde{u}_\phi(r,\theta)$ is absolutely continuous on $[0,\pi]$. Integrating twice by parts the right side of (4.3), we have for almost all $r > r_0$,

$$\begin{aligned} r\frac{d}{dr}\big(r\sigma'_-(r)\big) &\ge -\frac{\widehat{p}^2_\phi(\overrightarrow{a},\overrightarrow{G})}{\pi}\widetilde{u}_\phi(r,\pi)\sin\frac{\lambda\pi}{\widehat{p}_\phi(\overrightarrow{a},\overrightarrow{G})} \\ &\quad + \lambda\widehat{p}_\phi(\overrightarrow{a},\overrightarrow{G})T^*(r,\pi,u_\phi)\cos\frac{\lambda\pi}{\widehat{p}_\phi(\overrightarrow{a},\overrightarrow{G})} \\ &\quad - \lambda\widehat{p}_\phi(\overrightarrow{a},\overrightarrow{G})N(r,\overrightarrow{a},\overrightarrow{G}) + \lambda^2\sigma(r) \\ &:= h(r) + \lambda^2\sigma(r). \end{aligned} \tag{4.4}$$

Dividing both sides of (4.4) by $r^{\lambda+1}$ and integrating by parts over the interval $[2S_k, R_k]$, where S_k, R_k are the sequences described in (3.1) we have

$$\int_{2S_k}^{R_k}\frac{h(r)}{r^{\lambda+1}}\,dr + \lambda^2\int_{2S_k}^{R_k}\frac{\sigma(r)}{r^{\lambda+1}}\,dr \le \int_{2S_k}^{R_k}\frac{1}{r^\lambda}\frac{d}{dr}\big(r\sigma'_-(r)\big)\,dr = I. \tag{4.5}$$

Invoking Lemma 3.3 we get

$$I \le \frac{\sigma'_-(r)}{r^{\lambda+1}}\Big|_{2S_k}^{R_k} + \lambda\int_{2S_k}^{R_k}\frac{\sigma'_-(r)}{r^\lambda}\,dr. \tag{4.6}$$

The function $\sigma(r)$ is a convex function of $\log r$ on the interval $(0,+\infty)$, i.e., $g(t) = \sigma(e^t)$ is convex on $(-\infty,\infty)$. Thus the function $g(t)$ satisfies a Lipschitz condition on each interval $[a,b] \subset (0,+\infty)$, so is also absolutely continuous on each interval. Then the function $\sigma(r) = g(\log r)$ is also absolutely continuous on the interval $[a,b] \subset (0,+\infty)$. Integrating by parts the integral in the inequality (4.6), we have

$$\int_{2S_k}^{R_k}\frac{\sigma'_-(r)}{r^\lambda}\,dr = \int_{2S_k}^{R_k}\frac{\sigma'(r)}{r^\lambda}\,dr = \frac{\sigma(R_k)}{R_k^\lambda} - \frac{\sigma(2S_k)}{(2S_k)^\lambda} + \lambda\int_{2S_k}^{R_k}\frac{\sigma(r)}{r^{\lambda+1}}\,dr. \tag{4.7}$$

By (4.5), (4.6) and (4.7) we have

$$\int_{2S_k}^{R_k}\frac{h(r)}{r^{\lambda+1}}\,dr \le \left(\frac{\sigma'_-(r)}{r^{\lambda-1}} + \lambda\frac{\sigma(r)}{r^\lambda}\right)\Big|_{2S_k}^{R_k}. \tag{4.8}$$

By the definition of $\sigma(r)$ we get

$$0 \le \sigma(R) \le \pi(1+o(1))T(R,\overrightarrow{G}) < 2\pi T(R,\overrightarrow{G}) \quad (R\to\infty). \tag{4.9}$$

The function $r\sigma'_{-}(r)$ is non-decreasing on $(0,\infty)$, hence

$$\begin{aligned}\sigma(2R) \ge \sigma(2R)-\sigma(R) &= \int_R^{2R} \sigma'(r)\,dr = \int_R^{2R} \frac{r\sigma'_{-}(r)}{r}\,dr\\ &\ge R\sigma'_{-}(R)\int_R^{2R}\frac{dr}{r} = R\sigma'_{-}(R)\log 2.\end{aligned}$$

Consequently, we have

$$R\sigma'_{-}(R) \le \frac{1}{\log 2}\sigma(2R) \le \frac{2\pi}{\log 2}T(2R,\overrightarrow{G}) \quad (R\to\infty). \tag{4.10}$$

Moreover, in view of the monotonicity of $R\sigma'_{-}(R)$ we have for $R\ge 1$,

$$R\sigma'_{-}(R) \ge \sigma'_{-}(1) = C. \tag{4.11}$$

By (4.8), (4.9), (4.10) and (4.11) we have

$$\int_{2S_k}^{R_k} \frac{h(r)}{r^{\lambda+1}}\,dr \le 2\pi\left(\frac{1}{\log 2}+\lambda\right)\frac{T(2R_k,\overrightarrow{G})}{R_k^{\lambda}} - \frac{C}{(2S_k)^{\lambda}} \quad (k\to\infty).$$

It follows from (3.1) that for $k > k_0(\epsilon)$,

$$\int_{2S_k}^{R_k} \frac{h(r)}{r^{\lambda+1}}\,dr < \epsilon\int_{2S_k}^{R_k}\frac{T(r,\overrightarrow{G})}{r^{\lambda+1}}\,dr.$$

Therefore there exists a sequence $r_k \in [2S_k, R_k]$ such that $h(r_k) < \varepsilon T(r_k,\overrightarrow{G})$. Since $S_k\to\infty$ it follows that $r_k\to\infty$ as $k\to\infty$.

Recalling the definition of $h(r)$ we have for $k > k_0$,

$$\begin{aligned}&\frac{\widehat{p}_{\phi}^{2}(\overrightarrow{a},\overrightarrow{G})}{\pi}\Bigg(\frac{\pi\lambda}{\widehat{p}_{\phi}(\overrightarrow{a},\overrightarrow{G})}T^*(r_k,\pi,u_\phi)\cos\frac{\lambda\pi}{\widehat{p}_{\phi}(\overrightarrow{a},\overrightarrow{G})}\\ &\qquad -\frac{\pi\lambda}{\widehat{p}_{\phi}(\overrightarrow{a},\overrightarrow{G})}N(r_k,\overrightarrow{a},\overrightarrow{G}) - \widetilde{u}_\phi(r_k,\pi)\sin\frac{\lambda\pi}{\widehat{p}_{\phi}(\overrightarrow{a},\overrightarrow{G})}\Bigg)\\ &< \epsilon T(r_k,\overrightarrow{G}).\end{aligned} \tag{4.12}$$

The quantity $\widehat{p}_{\phi}(\overrightarrow{a},\overrightarrow{G})$ is an entire non-negative number. Since $\widehat{p}(\overrightarrow{a},\overrightarrow{G}) = \sup_\phi \widehat{p}_{\phi}(\overrightarrow{a},\overrightarrow{G})$ there is the function $\phi(r)$, such that $\widehat{p}_{\phi}(\overrightarrow{a},\overrightarrow{G}) = \widehat{p}(\overrightarrow{a},\overrightarrow{G})$. If we apply the inequality (4.12) to the function ϕ, then we have

$$\begin{aligned}&\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}T^*(r_k,\pi,u_\phi)\cos\frac{\lambda\pi}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}\\ &\quad -\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}N(r_k,\overrightarrow{a},\overrightarrow{G}) - \widetilde{u}_\phi(r_k,\pi)\sin\frac{\lambda\pi}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}\\ &< \epsilon T(r_k,\overrightarrow{G}) \quad (k\to\infty).\end{aligned} \tag{4.13}$$

Since

$$\begin{aligned}
T^*(r,\pi,u_\phi) &= \frac{1}{\pi}\int_0^\pi \widetilde{u}_\phi(r,\theta)\,d\theta + N(r,\overrightarrow{a},\overrightarrow{G})\\
&= \frac{1}{2\pi}\int_0^{2\pi} u_\phi(r,\theta)\,d\theta + N(r,\overrightarrow{a},\overrightarrow{G})\\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\frac{\|\overrightarrow{G}(re^{i\theta})\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(re^{i\theta}),\overrightarrow{a})|}\,d\theta + o(T(r,\overrightarrow{G})) + N(r,\overrightarrow{a},\overrightarrow{G})\\
&= m(r,\overrightarrow{a},\overrightarrow{G}) + N(r,\overrightarrow{a},\overrightarrow{G}) + o(T(r,\overrightarrow{G}))\\
&= T(r,\overrightarrow{G}) + o(T(r,\overrightarrow{G})),
\end{aligned}$$

then by (4.13) we have

$$\begin{aligned}
&\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}T(r_k,\overrightarrow{G})\cos\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})} - \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}N(r_k,\overrightarrow{a},\overrightarrow{G}) - \widetilde{u}_\phi(r_k,\pi)\sin\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}\\
&\quad < \epsilon T(r_k,\overrightarrow{G}) \quad (k\to\infty).
\end{aligned}$$

Since $\delta(\overrightarrow{a},\overrightarrow{G}) = 1 - \limsup_{r\to\infty}\frac{N(r,\overrightarrow{a},\overrightarrow{G})}{T(r,\overrightarrow{G})}$, then

$$N(r,\overrightarrow{a},\overrightarrow{G}) < (1-\delta(\overrightarrow{a},\overrightarrow{G})+\epsilon)T(r,\overrightarrow{G}) \quad (r\to\infty).$$

Hence

$$\begin{aligned}
\widetilde{u}_\phi(r,\pi) &= \max\left(\min_{|z|=r}\log\frac{\|\overrightarrow{G}(z)\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(z),\overrightarrow{a})|},\phi(r)\right)\\
&\le \min_{|z|=r}\log\frac{\|\overrightarrow{G}(z)\|\cdot\|\overrightarrow{a}\|}{|(\overrightarrow{G}(z),\overrightarrow{a})|} + \phi(r)\\
&= \log\mu(r,\overrightarrow{a},\overrightarrow{G}) + o(T(r,\overrightarrow{G})) \quad (r\to\infty).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}T(r_k,\overrightarrow{G})\cos\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})} - \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}(1-\delta(\overrightarrow{a},\overrightarrow{G})+\epsilon)T(r_k,\overrightarrow{G})\\
&\quad - \log\mu(r_k,\overrightarrow{a},\overrightarrow{G})\sin\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}\\
&< \epsilon T(r_k,\overrightarrow{G}) \quad (k\to\infty).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sin\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}\limsup_{r\to\infty}\frac{\log\mu(r,\overrightarrow{a},\overrightarrow{G})}{T(r,\overrightarrow{G})}\\
&\ge \frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})}\left(\delta(\overrightarrow{a},\overrightarrow{G}) - 1 + \cos\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a},\overrightarrow{G})} - \epsilon\right) - \epsilon.
\end{aligned}$$

Taking $\epsilon\to 0^+$ we get the statement of Theorem 2.1 for $\lambda > 0$. The proof for $\lambda = 0$ can be obtained similarly [22].

5. Example

For any $\lambda > 0$ and for any $n \in \mathbb{N}$ we consider the meromorphic function $F(z) = f_{\lambda/n}(z^n)$, where $f_\rho(z)$ is a meromorphic function given by Teichmüller [33] (see also [13, p. 282]). The function $f_\rho(z)$ is a meromorphic function of order $\rho : 0 < \rho < 1/2$, $\delta(\infty, f_\rho) = 1 - \cos \pi\rho$ and $|f_\rho(-r)| \le 2$ for $r \ge 0$.

Clearly $F(z)$ is a meromorphic function of lower order $\lambda : 0 < \lambda/n < 1/2$, $\widehat{p}(\infty, F) = n$, $\delta(\infty, F) = 1 - \cos \frac{\pi\lambda}{n}$ and $|F(\sqrt[n]{-1} \cdot r)| \le 2$ for $r \ge 0$.

Since the function $F(z)$ is a meromorphic function there are the entire functions $g_1(z)$ and $g_2(z)$ such that

$$F(z) = \frac{g_1(z)}{g_2(z)}.$$

Let $\overrightarrow{G}_F(z) = \{h_0(z), h_1(z), \dots, h_{p-1}(z)\}$, where

$$h_k(z) = C_{p-1}^k g_1^{p-1-k}(z) g_2^k(z), \quad k = 0, 1, \dots, p-1,$$

$\overrightarrow{a} = (0, 0, \dots, 0, 1)$. We have

$$\begin{aligned}
T(r, \overrightarrow{G}_F) &= \frac{1}{2\pi} \int_0^{2\pi} \log \|\overrightarrow{G}_F(re^{i\varphi})\| \, d\varphi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \left(|g_1(re^{i\varphi})|^2 + |g_2(re^{i\varphi})|^2\right)^{\frac{p-1}{2}} d\varphi \\
&= \frac{1}{2\pi} \frac{p-1}{2} \int_0^{2\pi} \log \left(|F(re^{i\varphi})|^2 + 1\right) d\varphi + \frac{p-1}{2\pi} \int_0^{2\pi} \log |g_2(re^{i\varphi})| \, d\varphi \\
&= \frac{p-1}{2\pi} \int_0^{2\pi} \log^+ |F(re^{i\varphi})| \, d\varphi + (p-1)N(r, 0, g_2) + O(1) \\
&= (p-1)m(r, \infty, F) + (p-1)N(r, \infty, F) + O(1) \\
&= (p-1)T(r, F) + O(1).
\end{aligned}$$

$$\begin{aligned}
N(r, \overrightarrow{a}, \overrightarrow{G}_F) &= \frac{1}{2\pi} \int_0^{2\pi} \log \left|(\overrightarrow{G}_F(re^{i\theta}), \overrightarrow{a})\right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |g_2^{p-1}(re^{i\theta})| \, d\theta \\
&= (p-1)N(r, 0, g_2) + O(1) = (p-1)N(r, \infty, F) + O(1).
\end{aligned}$$

Hence the lower order of $\overrightarrow{G}_F$ is λ ($\lambda = \rho$), $0 < \lambda < n/2$ and

$$\delta(\overrightarrow{a}, \overrightarrow{G}_F) = \delta(\infty, F) = 1 - \cos \frac{\pi\lambda}{n}.$$

For $z = \sqrt[n]{-1} \cdot r$ ($r \ge 0$), we have

$$\begin{aligned}
\log \frac{\|\overrightarrow{G}_F(z)\| \cdot \|\overrightarrow{a}\|}{|(\overrightarrow{G}_F(z), \overrightarrow{a})|} &= \log \frac{\|\overrightarrow{G}_F(z)\|}{|g_2^{p-1}(z)|} = \log \frac{|g_2(z)|^{p-1}(1 + |F(z)|^2)^{\frac{p-1}{2}}}{|g_2(z)|^{p-1}} \\
&= \frac{p-1}{2} \log(1 + |F(z)|^2) \le \frac{p-1}{2} |F(z)|^2 \le 2(p-1).
\end{aligned}$$

Therefore

$$\widehat{p}(\overrightarrow{a}, \overrightarrow{G}_F) = n.$$

Hence for any natural number $p \geq 2$, any natural number n and for any $\lambda > 0$ such that $\lambda < n/2$ there is an entire curve $\overrightarrow{G}_F \colon \mathbb{C} \to \mathbb{C}^p$ of lower order λ and vector $\overrightarrow{a} \in \mathbb{C}^p$ such that

$$\widehat{p}(\overrightarrow{a}, \overrightarrow{G}_F) = n, \quad \lambda < \frac{n}{2} = \frac{\widehat{p}(\overrightarrow{a}, \overrightarrow{G}_F)}{2}, \quad \delta(\overrightarrow{a}, \overrightarrow{G}_F) = 1 - \cos\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a}, \overrightarrow{G}_F)}$$

and for $z = \sqrt[n]{-1} \cdot r$ $(r \geq 0)$ we have

$$\log\frac{\|\overrightarrow{G}_F(z)\| \cdot \|\overrightarrow{a}\|}{|(\overrightarrow{G}_F(z), \overrightarrow{a})|} \leq 2(p-1).$$

The example of the entire curve $\overrightarrow{G}_F(z)$ proves that the condition $\delta(\overrightarrow{a}, \overrightarrow{G}) > 1 - \cos\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a}, \overrightarrow{G})}$ in Corollary 2.3 can not be replaced by $\delta(\overrightarrow{a}, \overrightarrow{G}) \geq 1 - \cos\frac{\pi\lambda}{\widehat{p}(\overrightarrow{a}, \overrightarrow{G})}$.

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