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Existence and Uniqueness for the Cauchy Problem of Semilinear Heat Equations on Stratified Lie Groups in the Critical Sobolev Space

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Abstract. The aim of this paper is to give existence and uniqueness results for solutions to the Cauchy problem of semilinear heat equations on stratified Lie groups $\mathbb{G}$ with the small initial data belonging to the critical Sobolev space. We consider a power type nonlinearity that behaves like $|u|^\alpha$ or $|u|^{\alpha-1}u$ ($\alpha > 1$).

1. Introduction and main result

On the Euclidean space $\mathbb{R}^n$, there are many results for the Cauchy problem of the semilinear heat equations

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = F(u(t,x)), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\mathbb{R}^+$ denotes the interval $(0,\infty)$, Δ is a laplacian on $\mathbb{R}^n$ and the unknown function u is real valued. One assumes that the nonlinear functions $F\colon \mathbb{R} \to \mathbb{R}$ satisfy that $F(0) = 0$ and that there exists $\alpha > 1$ such that

$$|F(u) - F(v)| \le C(|u|^{\alpha-1} + |v|^{\alpha-1})|u - v|. \tag{1.2}$$

In particular, F is locally Lipchitz continuous on $\mathbb{R}$. In [20], Weissler showed that for any $u_0 \in L^{p_c}(\mathbb{R}^d)$, there exist $T > 0$ and a unique local mild solution $u(t,x)$ to (1.1) in $C([0,T]; L^{p_c}(\mathbb{R}^n))$, where $p_c = n(\alpha-1)/2$ is the critical Lebesgue index. Moreover in [21], Weissler also showed the existence of global mild solutions to (1.1) for small initial data $u_0 \in L^{p_c}(\mathbb{R}^n)$ (see also [7]). Being a mild solution to (1.1) means that it solves the integral equation

$$u(t,x) = e^{t\Delta} u_0(x) + \int_0^t e^{(t-\tau)\Delta} F(u(\tau,x))\, d\tau, \tag{1.3}$$

where $e^{t\Delta} u_0(x) = (u_0 * E_t)(x)$ and $E_t(x)$ is the heat kernel on $\mathbb{R}^n$. Furthermore in [13], Ribaud showed that for any small initial data u_0 in the critical Sobolev space $L^p_{s_c}(\mathbb{R}^n)$,

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where $\max\{1, p_c/\alpha\} < p \le p_c$ and $s_c = n/p - 2/(\alpha - 1)$, there exists a unique global solution to (1.3) in $C([0,\infty); L^p_{s_c}(\mathbb{R}^n))$.

On the other hand, on stratified Lie groups $\mathbb{G}$ with the homogeneous dimension N (as a typical example, the Heisenberg group $\mathbb{H}^d$ with $N = 2d + 2$), in the last decade, one considers the Cauchy problem for the following semilinear heat equations (for instance, we refer to [2, 8, 10, 12, 16] and references therein. We also refer to [3, 11, 14])

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) = F(u(t,x)), & (t,x) \in \mathbb{R}^+ \times \mathbb{G}, \\ u(0,x) = u_0(x), & x \in \mathbb{G}, \end{cases} \tag{1.4}$$

where $\mathcal{L}$ is a sublaplacian on $\mathbb{G}$ and the unknown function u is real valued. Particularly we consider that the nonlinear functions $F\colon \mathbb{R} \to \mathbb{R}$ satisfy the condition (1.2). In [8], we showed the existence and uniqueness of time local mild solution to (1.4) for the initial data u_0 belonging to $L^p_s(\mathbb{G})$ for $s > s_c$, where $s_c = N/p - 2/(\alpha - 1)$ is the scaling critical index in the Sobolev space on startified Lie groups, and the mild solution to (1.4) is the solution of the integral equation

$$u(t,x) = e^{-t\mathcal{L}}u_0(x) + \int_0^t e^{-(t-\tau)\mathcal{L}}F(u(\tau,x))\,d\tau, \tag{1.5}$$

where $e^{-t\mathcal{L}}u_0(x) = (u_0 * h_t)(x)$ and $h_t(x)$ is the heat kernel on $\mathbb{G}$.

Remark 1.1. If we assume that $1 < p < \infty$, $s_c < 0$ and $s = 0$, for any initial data $u_0 \in L^p_s(\mathbb{G})$, there exists $T_0 > 0$ and a unique solution $u(t,x)$ to (1.5) with F satisfying (1.3) in $C([0,T_0); L^p(\mathbb{G}))$ (for details, see Theorem 1.2 in [8]). Note that if $s_c < 0$, $p_c < p$. We use this fact later.

Therefore, it is a natural and interesting problem that of the existence and uniqueness of mild solution to (1.4) in the critical space $L^p_{s_c}(\mathbb{G})$. The aim of this paper is to give the following existence and uniqueness result for global in time solutions to (1.5) for a small data u_0 belonging to the critical space $L^p_{s_c}(\mathbb{G})$. The significance of this result is that we extend the existence and uniqueness results obtained by Ribaud [13] on the Euclidean space to $\mathbb{G}$. We note that the embedding $L^p_{s_c}(\mathbb{G}) \subset L^{p_c}(\mathbb{G})$ holds by the Sobolev embedding theorem (see Proposition 3.5 below).

Theorem 1.2. *Let* $p_c = N(\alpha - 1)/2$ *and*

$$\max\left\{1, \frac{p_c}{\alpha}\right\} < p \le p_c.$$

Assume (1.2) *holds. There exists* $\varepsilon > 0$ *such that the following holds: For any initial data* $u_0 \in L^p_{s_c}(\mathbb{G})$ *with* $\|u_0\|_{L^{p_c}} \le \varepsilon$*, then there exists a unique global solution* $u(t,x)$ *to* (1.5) *in*

$C([0,\infty); L^p_{s_c}(\mathbb{G}))$. *Furthermore for any* $q > \alpha$ *satisfying* $0 < b(q) < 1/\alpha$, *the solution* u *satisfies* $u \in C((0,\infty); L^q(\mathbb{G}))$ *and*

$$\sup_{t>0} \|t^{b(q)} u(t)\|_q \le 2C \|u_0\|_{p_c}$$

for some $C > 0$, *where* $b(q) = N(1/p_c - 1/q)/2$.

Remark 1.3. The assumption for p in Theorem 1.2 means $0 \le s_c \le 2$. If $s_c = 0$, then $p = p_c$.

To prove Theorem 1.2, we first give the existence and uniqueness result (see Lemma 4.1 below) for the solution to (1.5) for a small data u_0 belonging to the critical Lebesgue space $L^{p_c}(\mathbb{G})$, where $p_c = N(\alpha-1)/2$ according to [7,20]. The method mainly used is the standard method, that is, the Banach fixed point theorem, as in Euclidean space.

The content of the paper is as follows. In Section 2, we introduce the definition and properties of stratified Lie groups $\mathbb{G}$. We also introduce the properties of the heat kernel on $\mathbb{G}$ and $L^\alpha - L^\beta$ estimate (see Proposition 2.4). In Section 3, we recall the Sobolev spaces L^p_s on $\mathbb{G}$. Especially, we introduce Sobolev embedding theorem (see Proposition 3.5) and $\dot{L}^p_{s+\theta} - \dot{L}^p_s$ estimate (see Proposition 3.6). Finally, in Section 4, we prove Theorem 1.2. Throughout this paper, the letters C will be used to denote positive constants, which are independent of the main variables involved and whose values may vary at every occurrence.

2. Stratified Lie groups

2.1. Definition and properties

We recall the definition of stratified Lie groups (see [1,4–6,15,18] and references therein). In particular, we adopt the definition introduced in [1]. A Lie group $(\mathbb{R}^d, \cdot)$ is called a stratified Lie group (a homogeneous Carnot group) and is denoted by $\mathbb{G} = (\mathbb{R}^d, \cdot, \delta_\lambda)$ if it satisfies the following two conditions:

(i) For $d = d_1 + \cdots + d_r$, $d_1, \ldots, d_r \in \mathbb{N}$ and $r \ge 1$, the decomposition $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_r}$ holds and for any $\lambda > 0$, the dilation $\delta_\lambda \colon \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \ldots, x^{(r)}) := (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}),$$

where $x^{(k)} \in \mathbb{R}^{d_k}$ for $k = 1, \ldots, r$, is an automorphism of the group $(\mathbb{R}^d, \cdot)$.

(ii) Let d_1 be as in (i) and $X_1, \ldots, X_{d_1}$ be the left invariant vector fields on $(\mathbb{R}^d, \cdot)$ such that $X_k(0) = \frac{\partial}{\partial x_k}\big|_{x=0}$ for $k = 1, \ldots, d_1$. Then Hörmander condition

$$\operatorname{rank}(\operatorname{Lie}\{X_1, \ldots, X_{d_1}\}(x)) = d$$

holds for any $x \in \mathbb{R}^d$, that is, the iterated commutators of $X_1, \ldots, X_{d_1}$ span the Lie algebra $\mathfrak{g}$ of $(\mathbb{R}^d, \cdot)$.

As a remark, the definition of stratified Lie groups implies that the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ admits a stratification, i.e., a direct sum decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

such that $[V_1, V_{i-1}] = V_i$ if $2 \le i \le r$ and $[V_1, V_r] = \{0\}$. It is known that the composition law $\cdot$ of stratified Lie groups takes the following form (for instance, see [1]).

Proposition 2.1. [1] *Let $\mathbb{G} = (\mathbb{R}^d, \cdot, \delta_\lambda)$ be stratified Lie groups. The composition law $\mathbb{G} \times \mathbb{G} \ni (x, y) \mapsto x \cdot y \in \mathbb{G}$ has polynomial component functions. Moreover we have for $x, y \in \mathbb{R}^d$,*

$$(x \cdot y)_1 = x_1 + y_1, \quad (x \cdot y)_j = x_j + y_j + Q_j(x, y), \quad 2 \le j \le d,$$

and the following facts hold:

(i) *$Q_j(x, y)$ only depends on $x_1, \dots, x_{j-1}$ and $y_1, \dots, y_{j-1}$,*

(ii) *$Q_j(x, y)$ is a sum of mixed monomials in x, y,*

(iii) *there exists $\sigma_j > 0$ such that $Q_j(\delta_\lambda x, \delta_\lambda y) = \lambda^{\sigma_j} Q_j(x, y)$.*

More precisely, $Q_j(x, y)$ only depends on the x_k's and y_k's with $\sigma_k < \sigma_j$.

The number r is called the step of $\mathbb{G}$, and the number

$$N = \sum_{k=1}^{r} k d_k$$

is called the homogeneous dimension of $\mathbb{G}$. Let $X_1, \dots, X_d$ be the Jacobian basis of $\mathfrak{g}$, i.e.,

$$X_j(0) = \frac{\partial}{\partial x_j}\Big|_{x=0} \quad \text{for } j = 1, \dots, d$$

(for details, see [1]). We can also denote the Jacobian basis $\{X_1, \dots, X_d\}$ by

$$X_1^{(1)}, \dots, X_{d_1}^{(1)}, \dots, X_1^{(r)}, \dots, X_{d_r}^{(r)}.$$

Then $X_k^{(i)}$, $1 \le i \le r$, takes the form

$$X_k^{(i)} = \frac{\partial}{\partial x_k^{(i)}} + \sum_{l=i+1}^{r} \sum_{m=1}^{d_l} a_{k,m}^{(i,l)}(x^{(1)}, \dots, x^{(l-i)}) \frac{\partial}{\partial x_m^{(l)}}, \quad 1 \le k \le d_i,$$

where $a_{k,m}^{(i,l)}$ is a δ_λ-homogeneous polynomial function of degree $l - i$ (see [1,15,18] for more details), namely, $a_{k,m}^{(i,j)}(\delta_\lambda(x)) = \lambda^{l-i} a_{k,m}^{(i,j)}(x)$. Especially, for $i = 1$, we have

$$X_k^{(1)} = \frac{\partial}{\partial x_k^{(1)}} + \sum_{l=2}^{r} \sum_{m=1}^{d_l} a_{k,m}^{(1,l)}(x^{(1)}, \dots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \quad 1 \le k \le d_1.$$

We denote by X_k for $1 \le k \le d_1$, the left invariant vector fields $X_k^{(1)}$ in the first stratum of $\mathbb{G}$. The left invariant vector fields $X_1, \dots, X_{d_1}$ are called the (Jacobian) generators of $\mathbb{G}$. A sublaplacian on $\mathbb{G}$ is denoted by $\mathcal{L} = -\sum_{i=1}^{d_1} X_i^2$. The operator $\mathcal{L}$ is not elliptic but hypoelliptic. $\mathcal{L}$ is δ_λ homogeneous of degree two, that is, for any $\lambda > 0$, we have

$$\mathcal{L}(u(\delta_\lambda(x))) = \lambda^2 (\mathcal{L}u)(\delta_\lambda(x))$$

for any $x \in \mathbb{G}$. Since the system $\mathbf{X} = \{X_1, \dots, X_{d_1}\}$ satisfies the Hörmander condition, the Carnot–Carathéodory distance $\rho_{\mathbf{X}}(x, x')$ is well-defined (see [1, 18] for details). We denote by $\rho(x)$ the distance from the origin, i.e., $\rho(x) = \rho_{\mathbf{X}}(\boldsymbol{e}, x)$, where $\boldsymbol{e}$ is a neutral element of $\mathbb{G}$. The homogeneous of degree of ρ is one, that is,

$$\rho(\delta_\lambda(x)) = \lambda \rho(x), \quad x \in \mathbb{G}$$

for any $\lambda > 0$ (see [1, Proposition 5.2.6]). The following estimate also holds:

$$\rho(x'^{-1} \cdot x) \le \rho(x) + \rho(x').$$

Define

$$|x|_{\mathbb{G}} = \left(\sum_{j=1}^{r} |x^{(j)}|^{\frac{2r!}{j}} \right)^{\frac{1}{2r!}}$$

for $x = (x^{(1)}, \dots, x^{(r)}) \in \mathbb{G}$. Then there exists a constant $C > 0$ such that

$$C^{-1} |x|_{\mathbb{G}} \le \rho(x) \le C |x|_{\mathbb{G}}$$

for any $x \in \mathbb{G}$ (see [1, Proposition 5.1.4 and Theorem 5.2.8]). Since the homogeneous degree of $|\cdot|_{\mathbb{G}}$ is also one, we deal with $|\cdot|_{\mathbb{G}}$ as a homogeneous norm on $\mathbb{G}$.

Stratified Lie groups $\mathbb{G}$ are locally compact Hausdorff spaces and their Haar measure of $\mathbb{G}$ is the Lebesgue measure $dx = dx_1 dx_2 \cdots dx_d$. We can see that the following relation

$$\int_{\mathbb{G}} f(\delta_\lambda(x))\, dx = \lambda^{-N} \int_{\mathbb{G}} f(x)\, dx$$

holds. Let f and h be suitable functions. Then the convolution $f * h$ of f with h on $\mathbb{G}$ is defined by

$$(f * h)(x) = \int_{\mathbb{G}} f(x') h(x'^{-1} \cdot x)\, dx' = \int_{\mathbb{G}} f(x \cdot x'^{-1}) h(x')\, dx'.$$

The convolution $*$ is non-commutative.

We set

$$L^p(\mathbb{G}) = \{ f \mid \|f\|_p < \infty \}$$

with a norm $\|\cdot\|_p$ defined by

$$\|f\|_p = \left(\int_{\mathbb{G}} |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

for $1 \le p < \infty$ and if $p = \infty$, we set

$$L^\infty(\mathbb{G}) = \{ f \mid \|f\|_\infty < \infty \}$$

with a norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \inf \big\{ M \mid \text{the measure of } \{x \in \mathbb{G} \mid |f(x)| > M\} \text{ is } 0 \big\}.$$

Let $\mathcal{C}_0(\mathbb{G})$ be the space of continuous functions on $\mathbb{G}$ with compact support and $\mathcal{D} = \mathcal{D}(\mathbb{G})$ be the space of smooth functions with compact support.

2.2. The heat kernel

We can construct a heat semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ of linear operators on $L^1(\mathbb{G}) + L^\infty(\mathbb{G})$ with the infinitesimal generator $-\mathcal{L}$ by a theorem of Hunt in [9], whose properties are summarized as follows.

Proposition 2.2. [4, 5, 9]

(i) *There exists a unique semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ of linear operators on $L^1(\mathbb{G}) + L^\infty(\mathbb{G})$ satisfying conditions:*

(a) $e^{-t\mathcal{L}} f(x) = (f * h_t)(x)$, *where* $h_t(x) = h(t, x)$ *is* $C^\infty((0,\infty) \times \mathbb{G})$. *Moreover,* $h_t \in \mathcal{S}(\mathbb{G})$.

(b) $h_t(x) \ge 0$, $\int_{\mathbb{G}} h_t(x)\, dx = 1$, $h_t(x) = h_t(x^{-1})$, $(\partial/\partial t + \mathcal{L}) h_t = 0$ *and* $h_{r^2 t}(\delta_r(x)) = r^{-N} h_t(x)$, $r > 0$, $x \in \mathbb{G}$, *where* N *is the homogeneous dimension of* $\mathbb{G}$.

(c) $\{e^{-t\mathcal{L}}\}_{t>0}$ *is a contraction semigroup and strongly continuous on* $L^p(\mathbb{G})$ *for* $1 < p < \infty$. *Furthermore, for any* $f \in \mathcal{D}(\mathbb{G})$ *and any* $p \in [1, \infty]$, *we have the convergence*

$$\left\| \frac{1}{h}(e^{-h\mathcal{L}} f - f) + \mathcal{L} f \right\|_p \to 0$$

as $h \to 0$.

(ii) *Let* $\mathcal{L}_p$ *be minus the infinitesimal generator of* $\{e^{-t\mathcal{L}}\}_{t>0}$ *on* $L^p(\mathbb{G})$. *Then*

(a) *For* $1 < p < \infty$, $\mathcal{L}_p$ *is a closed operator on* $L^p(\mathbb{G})$ *whose domain is dense in* $L^p(\mathbb{G})$.

(b) *For* $1 < p < \infty$, $\mathcal{D} \subset \mathrm{Dom}(\mathcal{L}_p)$. *Furthermore* $\mathcal{L}_p u = \mathcal{L} u$ *for* $u \in \mathcal{D}$.

(c) *$\mathcal{L}_p$ is the maximal restriction of $\mathcal{L}$ to $L^p(\mathbb{G})$, $1 < p < \infty$, that is,* $\mathrm{Dom}(\mathcal{L}_p)$ *is the set of all $f \in L^p(\mathbb{G})$ such that the distribution derivative $\mathcal{L}f$ is in $L^p(\mathbb{G})$, and $\mathcal{L}_p f = \mathcal{L}f$. Also, $\mathcal{L}_p$ is the smallest closed extension of $\mathcal{L}|_{\mathcal{D}}$ on $L^p(\mathbb{G})$.*

We call the function $h_t(x)$ as the heat kernel associated with $\mathcal{L}$ on $\mathbb{G}$. The heat kernel h_t has the following estimate.

Lemma 2.3. [18] *The heat kernel h_t associated to $\mathcal{L}$ on $\mathbb{G}$ satisfies the following estimate: For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$h_t(x) \le C_\varepsilon t^{-\frac{N}{2}} \exp\left(\frac{-\rho^2(x)}{4(1+\varepsilon)t}\right)$$

for any $x \in \mathbb{G}$ and $t > 0$, where N is the homogeneous dimension of $\mathbb{G}$.

By Young's inequality and Lemma 2.3, we have the following $L^\alpha - L^\beta$ estimate.

Proposition 2.4. *Assume $1 \le \alpha \le \beta \le \infty$. Then there exists a positive constant $C_{p,q}$ such that*

$$\|e^{-t\mathcal{L}}\varphi\|_\beta \le Ct^{-\frac{N}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)}\|\varphi\|_\alpha, \quad t > 0$$

for any $\varphi \in L^\alpha(\mathbb{G})$, where N is the homogeneous dimension of $\mathbb{G}$.

3. Sobolev spaces on $\mathbb{G}$

Here we recall the definition and basic properties of Sobolev spaces on stratified Lie groups $\mathbb{G}$ first introduced in [5] by Folland (see also [17]). In [4], Fischer and Ruzhansky gave the definition and basic properties of Sobolev spaces on graded groups, which are more general than stratified Lie groups. So we refer to [4,5]. Especially, we adopt the definition of the Sobolev spaces in [4].

At first, we recall the definition of fractional powers of the sublaplacian $\mathcal{L}$.

Definition 3.1. [4, 5, 17] Assume that $1 < p < \infty$, $s > 0$ and $k = [s] + 1$. Then the operator $\mathcal{L}_p^s$ is defined by

$$\mathcal{L}_p^s f = \lim_{\varepsilon\to 0} \frac{1}{\Gamma(k-s)} \int_\varepsilon^\infty \nu^{k-s-1}\mathcal{L}^k e^{-\nu\mathcal{L}} f\, d\nu$$

on the domain of all $f \in L^p(\mathbb{G})$ such that the indicated limit exists in $L^p(\mathbb{G})$. The operator $\mathcal{L}_p^{-s}$ is defined by

$$\mathcal{L}_p^{-s} f = \lim_{\eta\to\infty} \frac{1}{\Gamma(s)} \int_0^\eta \nu^{s-1} e^{-\nu\mathcal{L}} f\, d\nu$$

on the domain of all $f \in L^p(\mathbb{G})$ such that the indicated limit exists in $L^p(\mathbb{G})$. The operator $(\mathrm{Id} + \mathcal{L}_p)^s$ is defined by

$$(\mathrm{Id} + \mathcal{L}_p)^s f = \lim_{\varepsilon\to 0} \frac{1}{\Gamma(k-s)} \int_\varepsilon^\infty \nu^{k-s-1}(\mathrm{Id} + \mathcal{L})^k e^{-\nu} e^{-\nu\mathcal{L}} f\, d\nu$$

on the domain of all $f \in L^p(\mathbb{G})$ such that the indicated limit exists in $L^p(\mathbb{G})$. Also, we define the operator $(\mathrm{Id} + \mathcal{L}_p)^{-s}$ by

$$(\mathrm{Id} + \mathcal{L}_p)^{-s} f = \frac{1}{\Gamma(s)} \int_0^\infty \nu^{s-1} e^{-\nu} e^{-\nu \mathcal{L}} f \, d\nu.$$

The operator $(\mathrm{Id} + \mathcal{L}_p)^{-s}$ is a bounded operator on L^p.

Proposition 3.2. [4, 5] *Let $1 < p < \infty$ and $\mathcal{M}_p$ denote either $\mathcal{L}_p$ or $\mathrm{Id} + \mathcal{L}_p$. Then,*

(i) *$\mathcal{M}_p^s$ is a closed operator on $L^p(\mathbb{G})$ for all $s \in \mathbb{R}$ and injective with $(\mathcal{M}_p^s)^{-1} = \mathcal{M}_p^{-s}$.*

(ii) *If $f \in \mathrm{Dom}(\mathcal{M}_p^\beta) \cap \mathrm{Dom}(\mathcal{M}_p^{\alpha+\beta})$, then $\mathcal{M}_p^\beta f \in \mathrm{Dom}(\mathcal{M}_p^\alpha)$ and $\mathcal{M}_p^\alpha \mathcal{M}_p^\beta f = \mathcal{M}_p^{\alpha+\beta} f$. $\mathcal{M}_p^{\alpha+\beta}$ becomes the smallest closed extension of $\mathcal{M}_p^\alpha \mathcal{M}_p^\beta$.*

(iii) *When $s > 0$, if $f \in \mathrm{Dom}(\mathcal{M}_p^s) \cap L^q(\mathbb{G})$, then $f \in \mathrm{Dom}(\mathcal{M}_q^s)$ if and only if $\mathcal{M}_p^s f \in L^q(\mathbb{G})$, in which case $\mathcal{M}_p^s = \mathcal{M}_q^s$.*

(iv) *If $s > 0$, then $\mathrm{Dom}(\mathcal{L}_p^s) = \mathrm{Dom}((\mathrm{Id} + \mathcal{L}_p)^s)$.*

By Proposition 3.2(iii), $\mathcal{L}_p^s$ (resp. $(\mathrm{Id} + \mathcal{L}_p)^s$) agrees with $\mathcal{L}_q^s$ (resp. $(\mathrm{Id} + \mathcal{L}_q)^s$) on their common domains for $s \in \mathbb{R}$ and $1 < p, q < \infty$. So we omit the subscripts on these operators except when we wish to specify the domains.

Next, we recall the definition of Sobolev spaces $L_s^p(\mathbb{G})$ and $\dot{L}_s^p(\mathbb{G})$. We adopt the definition in [4] (see also [5]).

Definition 3.3. [4]

(i) (the inhomogeneous Sobolev space) Let $s \in \mathbb{R}$ and $1 < p < \infty$. We denote by $L_s^p(\mathbb{G})$ the space of tempered distributions obtained by the completion of the Schwartz class $\mathcal{S}(\mathbb{G})$ with respect to the Sobolev norm

$$\|f\|_{L_s^p} := \|(\mathrm{Id} + \mathcal{L})^{\frac{s}{2}} f\|_p$$

for $f \in \mathcal{S}(\mathbb{G})$.

(ii) (the homogeneous Sobolev space) Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then we denote by $\dot{L}_s^p(\mathbb{G})$ the space of tempered distributions obtained by the completion of $\mathcal{S}(\mathbb{G}) \cap \mathrm{Dom}(\mathcal{L}^{\frac{s}{2}})$ with respect to the norm

$$\|f\|_{\dot{L}_s^p} := \|\mathcal{L}^{\frac{s}{2}} f\|_p$$

for $f \in \mathcal{S}(\mathbb{G}) \cap \mathrm{Dom}(\mathcal{L}_p^{\frac{s}{2}})$, especially, for $f \in \mathcal{S}(\mathbb{G})$ if $s > 0$.

The Sobolev spaces $L_s^p(\mathbb{G})$ and $\dot{L}_s^p(\mathbb{G})$ have the following basic properties.

Proposition 3.4. [4]

(i) *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then $L^p_s(\mathbb{G})$ and $\dot{L}^p_s(\mathbb{G})$ are Banach space satisfying*

$$\mathcal{S}(\mathbb{G}) \subsetneq L^p_s(\mathbb{G}) \subset \mathcal{S}'(\mathbb{G}) \quad \textit{and} \quad (\mathcal{S}(\mathbb{G}) \cap \mathrm{Dom}(\mathcal{L}_p^{\frac{s}{2}})) \subsetneq \dot{L}^p_s(\mathbb{G}) \subsetneq \mathcal{S}'(\mathbb{G}),$$

respectively.

(ii) *If $s = 0$ and $1 < p < \infty$, then $\dot{L}^p_0(\mathbb{G}) = L^p_0(\mathbb{G}) = L^p(\mathbb{G})$ with $\|\cdot\|_{\dot{L}^p_0} = \|\cdot\|_{L^p_0} = \|\cdot\|_p$.*

(iii) *If $s > 0$ and $1 < p < \infty$, then we have*

$$L^p_s(\mathbb{G}) = \dot{L}^p_s(\mathbb{G}) \cap L^p(\mathbb{G}) \quad \textit{and} \quad \|\cdot\|_{L^p_s} \sim \|\cdot\|_p + \|\cdot\|_{\dot{L}^p_s}.$$

We summarize the useful properties of the Sobolev spaces on stratified Lie groups $\mathbb{G}$ as follows. At first, we introduce the Sobolev embedding theorem on $\mathbb{G}$ (for example, see [4, Theorem 4.4.28] and see also [5]).

Proposition 3.5. [4, 5] *Let $\mathbb{G}$ be stratified Lie groups with homogeneous dimension N.*

(i) *If $a, b \in \mathbb{R}$ with $a < b$ and $1 < p < \infty$, then the continuous inclusions $L^p_b \subset L^p_a$ holds.*

(ii) *If $1 < p < q < \infty$ and $a, b \in \mathbb{R}$ with $b - a = N\left(\frac{1}{p} - \frac{1}{q}\right)$, then we have the continuous inclusion*

$$L^p_b(\mathbb{G}) \subset L^q_a(\mathbb{G}),$$

that is, there exists a constant $C > 0$ such that

$$\|f\|_{L^q_a} \leq C\|f\|_{L^p_b},$$

where C depends only a, b, p, q, independent of $f \in L^p_b(\mathbb{G})$. If $1 < p < \infty$ and $s > N/p$, then we have $L^p_s(\mathbb{G}) \subset L^\infty(\mathbb{G})$. Furthermore, there exists a constant $C_{s,p} > 0$ independent of f such that

$$\|f\|_\infty \leq C\|f\|_{L^p_s}.$$

Secondly, we introduce the $\dot{L}^p_{s+\theta} - \dot{L}^p_s$ estimate on $\mathbb{G}$ (for example, see [8, Lemma 3.9]).

Proposition 3.6. [8] *Assume $1 < p < \infty$, $s \in \mathbb{R}$ and $\theta \geq 0$. Then there exists a positive constant C (independent of t) such that*

$$\|e^{-t\mathcal{L}}\varphi\|_{\dot{L}^p_{s+\theta}} \leq Ct^{-\frac{\theta}{2}}\|\varphi\|_{\dot{L}^p_s}, \quad t > 0$$

for any $\varphi \in \dot{L}^p_s(\mathbb{G})$, where $e^{-t\mathcal{L}}f(x) = (f * h_t)(x)$ and h_t is the heat kernel associated to a sublaplacian $\mathcal{L}$ on $\mathbb{G}$.

4. Proof of Theorem 1.2

By Proposition 3.5 (the Sobolev embedding theorem), we have the inclusion relationship $L^p_{s_c}(\mathbb{G}) \subset L^{p_c}(\mathbb{G})$. On the other hand, $L^p_{s_c}(\mathbb{G}) = L^p(\mathbb{G}) \cap \dot{L}^p_{s_c}(\mathbb{G}) \subset L^p(\mathbb{G})$. Hence we prove Theorem 1.2 in the following three steps. In Step 1, we show the existence and uniqueness of global solutions to (1.5) for small initial data in the critical space $L^{p_c}(\mathbb{G})$. In Step 2, we give the time decay estimate for the global solution in $L^\gamma(\mathbb{G})$ with $\gamma \in [p_c, \infty)$. Finally, in Step 3, we show the regularity of the unique global solution for initial data in the critical Sobolev space $L^p_{s_c}(\mathbb{G})$.

Step 1. To prove Theorem 1.2, at first, we prove the following Lemma 4.1 concerning the existence and uniqueness of global solutions to (1.5) for small initial data in the critical space $L^{p_c}(\mathbb{G})$.

Lemma 4.1. *Assume* (1.2) *holds and* $p_c > 1$. *There exists* $\varepsilon > 0$ *such that the following holds: For any initial data* $u_0 \in L^{p_c}(\mathbb{G})$ *with* $\|u_0\|_{L^{p_c}} \le \varepsilon$, *then there exists a unique global solution* $u(t,x)$ *to* (1.5) *in* $C([0,\infty); L^{p_c}(\mathbb{G}))$ *satisfying* $u \in C((0,\infty); L^q(\mathbb{G}))$ *and*

$$\sup_{t>0} \|t^{b(q)} u(t)\|_q \le 2C_0 \|u_0\|_{L^{p_c}}$$

for some $C_0 > 0$ *and for any* $q > \alpha$ *with* $0 < b(q) < 1/\alpha$, *where* $b(q) = N(1/p_c - 1/q)/2$.

Proof. Let $\alpha > 1$ and $1 < q/\alpha < p_c < q$. Assume that $u_0 \in L^{p_c}(\mathbb{G})$ and $\|u_0\|_{p_c}$ is sufficiently small. We will show that there exists a unique global solution $u(t)$ to (1.5) in $C([0,\infty); L^{p_c}) \cap C((0,\infty); L^q)$ with $u(0) = u_0$ satisfying $\sup_{t>0} \|t^{b(q)} u(t)\|_q \le 2C_0 \|u_0\|_{p_c}$, where $b(q) = N(1/p_c - 1/q)/2 > 0$.

By Proposition 2.4 ($L^\alpha - L^\beta$ estimate), we have

$$\|t^{b(q)} e^{-t\mathcal{L}} u_0\|_q \le C_0 \|u_0\|_{p_c}$$

for any $t \in [0,\infty)$. Furthermore, since $\mathcal{C}_0(\mathbb{G})$ is dense in $L^{p_c}(\mathbb{G})$, there exists a sequence $\{a_i\} \subset \mathcal{C}_0(\mathbb{G})$ such that a_i converges to u_0 in $L^{p_c}(\mathbb{G})$. Note that $\mathcal{C}_0(\mathbb{G})$ is also dense in $L^q(\mathbb{G})$. By Proposition 2.4 ($L^\alpha - L^\beta$ estimate), we have for any $\delta > 0$,

$$\begin{aligned} \|t^{b(q)} e^{-t\mathcal{L}} u_0\|_q &\le \|t^{b(q)} e^{-t\mathcal{L}} (u_0 - a_i)\|_q + \|t^{b(q)} e^{-t\mathcal{L}} a_i\|_q \\ &\le C\|u_0 - a_i\|_{p_c} + t^{b(q)} \|a_i\|_q \\ &\le C\delta + t^{b(q)} \|a_i\|_q. \end{aligned}$$

Since the second term $t^{b(q)}\|a_i\|_q$ converges to 0 as $t \to 0^+$, we have

$$\lim_{t\to 0^+} \|t^{b(q)} e^{-t\mathcal{L}} u_0\|_q = 0. \tag{4.1}$$

We denote by X the class of all functions $u\colon [0,\infty) \to L^{p_c}(\mathbb{G})$ satisfying $u \in C((0,\infty); L^{p_c}) \cap L^\infty([0,\infty); L^{p_c})$ and $u \in C((0,\infty); L^q)$. We define the norm $\|\cdot\|_X$ by

$$\|u\|_X := \max\left\{\sup_{t>0} \|u(t)\|_{p_c}, \sup_{t>0} t^{b(q)} \|u(t)\|_q\right\}.$$

Then X is a Banach space. In addition, we define the closed subset $Y \subset X$ as

$$Y = \left\{u \in X \;\middle|\; \sup_{t>0} \|t^{b(q)} u(t)\|_q \le 2C_0 \|u_0\|_{p_c}\right\}.$$

For $u \in Y$, we set $\Phi[u](t)$ by

$$\Phi[u](t) := e^{-t\mathcal{L}} u_0 + \int_0^t e^{-(t-\tau)\mathcal{L}} F(u(\tau))\, d\tau.$$

Now we show that the mapping Φ is a contraction mapping from Y into Y. Indeed, by Proposition 2.4 ($L^\alpha - L^\beta$ estimate) and by the condition (1.2), we have for $t > 0$ that

$$\begin{aligned}
\|\Phi[u](t)\|_{p_c} &\le \|e^{-t\mathcal{L}} u_0\|_{p_c} + \int \|e^{-(t-\tau)\mathcal{L}} F(u(\tau))\|_{p_c}\, d\tau \\
&\le \|u_0\|_{p_c} + C \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{p_c}\right)} \|F(u(\tau))\|_{\frac{q}{\alpha}}\, d\tau \\
&\le \|u_0\|_{p_c} + C \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{p_c}\right)} \|u(\tau)\|_q^\alpha\, d\tau \\
&\le \|u_0\|_{p_c} + C \left(\sup_{\tau>0} \tau^{b(q)} \|u(\tau)\|_q\right)^\alpha \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{p_c}\right)} \tau^{-b(q)\alpha}\, d\tau \\
&\le \|u_0\|_{p_c} + (2C_0)^\alpha C B \|u_0\|_{p_c}^\alpha,
\end{aligned} \tag{4.2}$$

where $B = \int_0^1 (1-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{p_c}\right)} \tau^{-b(q)\alpha}\, d\tau$, since

$$1 - \frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{p_c}\right) - b(q)\alpha = 0.$$

Furthermore by the condition (1.2), we obtain for $t > 0$,

$$\begin{aligned}
\|t^{b(q)} \Phi[u](t)\|_q &\le \|t^{b(q)} e^{-t\mathcal{L}} u_0\|_q + t^{b(q)} \int_0^t \|e^{-(t-\tau)\mathcal{L}} F(u(\tau))\|_q\, d\tau \\
&\le C_0 \|u_0\|_{p_c} + C_1 t^{b(q)} \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{q}\right)} \|F(u(\tau))\|_{\frac{q}{\alpha}}\, d\tau \\
&\le C_0 \|u_0\|_{p_c} + C_1 t^{b(q)} \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{q}\right)} \|u(\tau)\|_q^\alpha\, d\tau \\
&\le C_0 \|u_0\|_{p_c} + C_1 B_1 \left(\sup_{t>0} \tau^{b(q)} \|u(\tau)\|_q\right)^\alpha \\
&\le C_0 \|u_0\|_{p_c} + (2C_0)^\alpha C_1 B_1 \|u_0\|_{p_c}^\alpha,
\end{aligned} \tag{4.3}$$

where $B_1 = \int_0^1 (1-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q}-\frac{1}{q}\right)} \tau^{-b(q)\alpha}\, d\tau$, since

$$1 - \frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{q}\right) - b(q)\alpha = -b(q).$$

If we take $\|u_0\|_{p_c}$ sufficiently small such that $2^\alpha C_0^{\alpha-1} C_1 B_1 \|u_0\|_{p_c}^{\alpha-1} \leq 1$, then we have $\sup_{t>0} \|t^{b(q)}\Phi[u](t)\|_q \leq 2C_0\|u_0\|_{p_c}$. On the other hand, we can show $\Phi[u] \in C((0,\infty); L^{p_c}) \cap C((0,\infty); L^q)$ as in [19, Lemma 2.1]. Hence the mapping Φ is a mapping from Y into Y. If we take u, v in Y, similarly as (4.2), we have

$$\|\Phi[u] - \Phi[v]\|_X \leq 2^\alpha C_0^{\alpha-1} C \max\{B, B_1\} \|u_0\|_{p_c}^{\alpha-1} \|u - v\|_X.$$

If we take $\|u_0\|_{p_c}$ sufficiently small such that $2^\alpha C_0^{\alpha-1} C \max\{B, B_1\}\|u_0\|_{p_c}^{\alpha-1} \leq 1/2$, the mapping Φ from Y into Y is a contraction. Therefore by Banach's fixed point theorem, there exists $u \in Y$ such that $u = \Phi[u]$.

Next, we will show the continuity of $u(t)$ at $t = 0$. Let $T > 0$ be small. Similar to (4.3), we obtain

$$\|t^{b(q)}u(t)\|_q \leq \|t^{b(q)}e^{-t\mathcal{L}}u_0\|_q + (2C_0)^{\alpha-1}C_1B_1\|u_0\|_{p_c}^{\alpha-1} \sup_{0<t\leq T} \|t^{b(q)}u(t)\|_q.$$

If we take $\|u_0\|_{p_c}$ sufficiently small such that $(2C_0)^{\alpha-1}C_1B_1\|u_0\|_{p_c}^{\alpha-1} \leq 1/2$, we have

$$\sup_{0<t\leq T} \|t^{b(q)}u(t)\|_q \leq 2 \sup_{0<t\leq T} \|t^{b(q)}e^{-t\mathcal{L}}u_0\|_q. \tag{4.4}$$

By (4.1) and (4.4), we obtain

$$\lim_{t\to 0^+} \|t^{b(q)}u(t)\|_q = 0. \tag{4.5}$$

Hence we have the equality

$$\lim_{t\to 0^+} \left\| \int_0^t e^{-(t-\tau)\mathcal{L}} F(u(\tau))\, d\tau \right\|_{p_c} = 0. \tag{4.6}$$

Indeed, since similarly as (4.2), we have for any $t \in (0, T]$,

$$\left\| \int_0^t e^{-(t-\tau)\mathcal{L}} F(u(\tau))\, d\tau \right\|_{p_c} \leq CB \left(\sup_{0<t\leq T} t^{b(q)}\|u(t)\|_q \right)^\alpha, \tag{4.7}$$

for $(2C_0)^{\alpha-1}CB\|u_0\|_{p_c}^{\alpha-1} \leq 1/2$, by (4.5) and (4.7), we obtain (4.6). On the other hand, since $\{e^{-t\mathcal{L}}\}$ is a strongly continuous semigroup by Proposition 2.2, we can see

$$\lim_{t\to 0^+} \|e^{-t\mathcal{L}}u_0 - u_0\|_{p_c} = 0. \tag{4.8}$$

By (4.6) and (4.8), we have

$$\lim_{t\to 0^+} \|u(t) - u_0\|_{p_c} = 0.$$

So we can see $u(t) \in C([0,\infty); L^{p_c})$.

Finally, we will show the uniqueness of the solution constructed above. Let $T_0 > 0$ be arbitrary. For $T \in (0, T_0]$ and $u \in C((0,\infty); L^q)$, we put

$$\|u\|_{Z_T} := \sup_{0\le t\le T} t^{b(q)} \|u(t)\|_q.$$

Let $u, v \in C([0,\infty); L^{p_c}) \cap C((0,\infty); L^q)$ be solutions to (1.5) with $u(0) = v(0) = u_0$. By the same argument as in (4.3), we have

$$\|u - v\|_{Z_T} \le 2C_1 B_1 \big(\|u\|_{Z_T}^{\alpha-1} + \|v\|_{Z_T}^{\alpha-1}\big) \|u - v\|_{Z_T}.$$

Since $\lim_{T\to 0} \|u\|_{Z_T} = \lim_{T\to 0} \|v\|_{Z_T} = 0$ by (4.5), if we take $T > 0$ sufficiently small, we obtain

$$\|u - v\|_{Z_T} \le \frac{1}{2} \|u - v\|_{Z_T}.$$

This implies $u(t) = v(t)$ on $[0, T]$. By finitely repeating the same argument, we can see that $u(t) = v(t)$ on $[0, T_0]$. □

Step 2. Next, we prove the time decay estimate of the unique global solution constructed in Lemma 4.1 in $L^\gamma(\mathbb{G})$ with $\gamma \in [p_c, \infty)$.

Lemma 4.2. *Assume that* $\|u_0\|_{p_c} \le \varepsilon$*, where* $\varepsilon > 0$ *is sufficiently small, and let* $u(t,x)$ *be the solution to* (1.5) *constructed in Lemma* 4.1*. Then we have for any* $t > 0$*,*

$$\|u(t)\|_\gamma \le 2Ct^{-b(\gamma)} \|u_0\|_{p_c} \tag{4.9}$$

for any $\gamma \in [p_c, \infty)$*, where* $b(\gamma) = N(1/p_c - 1/\gamma)/2$.

Proof. Let $q > p_c$ such that $0 < b(q) < 1/\alpha$ and $\alpha < q$. At first, for any $\gamma \in [p_c, q]$, we will show that (4.9) holds. Let $u(t)$ be the solution constructed in Lemma 4.1. Then similarly as (4.2), we have for any $t > 0$,

$$\|u(t)\|_{p_c} \le 2C\|u_0\|_{p_c}. \tag{4.10}$$

For $p_c < \gamma \le q$, similarly as (4.3), we have for any $t > 0$,

$$\begin{aligned}
\|t^{b(\gamma)} u(t)\|_\gamma &\le \|t^{b(\gamma)} e^{-t\mathcal{L}} u_0\|_\gamma + t^{b(\gamma)} \int_0^t \|e^{-(t-\tau)\mathcal{L}} F(u(\tau))\|_\gamma \, d\tau \\
&\le C_0 \|u_0\|_{p_c} + Ct^{b(\gamma)} \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q} - \frac{1}{\gamma}\right)} \|F(u(\tau))\|_{\frac{q}{\alpha}} \, d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq C_0\|u_0\|_{p_c} + Ct^{b(\gamma)}\int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q}-\frac{1}{\gamma}\right)}\|u(\tau)\|_q^\alpha\, d\tau \\
&\leq C_0\|u_0\|_{p_c} + Ct^{b(\gamma)}\left(\sup_{0<\tau<t}\tau^{b(q)}\|u(\tau)\|_q\right)^\alpha\int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q}-\frac{1}{\gamma}\right)}\tau^{-b(q)\alpha}\, d\tau \\
&\leq C_0\|u_0\|_{p_c} + CB_3(2C_0\|u_0\|_{p_c})^\alpha,
\end{aligned}
$$

where $B_3 = \int_0^1 (1-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q}-\frac{1}{\gamma}\right)}\tau^{-b(q)\alpha}\, d\tau$, since $1 - N(\alpha/q - 1/\gamma)/2 - b(q)\alpha = -b(\gamma)$. Since $\|u_0\|_{p_c}$ is sufficiently small, we obtain

$$
\|u(t)\|_\gamma \leq 2Ct^{-b(\gamma)}\|u_0\|_{p_c}. \tag{4.11}
$$

By (4.10) and (4.11), for $p_c \leq \gamma \leq q$, we have

$$
\|u(t)\|_\gamma \leq 2Ct^{-b(\gamma)}\|u_0\|_{p_c}
$$

for any $t > 0$.

Next, for any $\gamma \in [q, \infty)$, we will show that (4.9) holds. Let $q_0 = q$ and $\{q_l\}_{l=0,1,2,\ldots}$ be the sequence such that

$$
0 < \delta_l = \frac{N}{2}\left(\frac{1}{q_l} - \frac{1}{q_{l+1}}\right) < \frac{1}{\alpha}. \tag{4.12}
$$

Note that the sequence $\{q_l\}_{l=0,1,2,\ldots}$ is increasing. Furthermore, we put

$$
I(q_l, q_{l+1}) = \int_0^1 (1-\eta)^{-\frac{N}{2}\left(\frac{\alpha}{q_{l+1}} - \frac{1}{q_{l+1}}\right)}\eta^{-\delta_l\alpha}\, d\eta.
$$

Then for all $l \geq 0$,

$$
I(q_l, q_{l+1}) < \infty. \tag{4.13}
$$

Now we take $t_0 > 0$ and consider the solution $v(t,x)$ to the initial value problem

$$
\begin{aligned}
&v(t,x) = e^{-t\mathcal{L}}v_0(x) + \int_0^t e^{-(t-\tau)\mathcal{L}}F(v(\tau,x))\, d\tau, \\
&v(0,x) = v_0(x) = u(t_0,x).
\end{aligned}
$$

Since $0 < b(q_0) < 1/\alpha$ and $\alpha < q_0$, by Lemma 4.1, it follows that $v_0 \in L^{p_c}(\mathbb{G}) \cap L^{q_0}(\mathbb{G})$ with

$$
\|v_0\|_{q_0} \leq 2Ct_0^{-b(q_0)}\|u_0\|_{p_c}. \tag{4.14}
$$

Furthermore, since $v_0 \in L^{q_0}(\mathbb{G})$, $q_0 > p_c$, by Remark 1.3 in [8], the sequence $\{v^j\}$ such that

$$
v^0 = e^{-t\mathcal{L}}v_0, \quad v^{j+1} = v^0 + \int_0^t e^{-(t-\tau)\mathcal{L}}F(v^j(\tau))\, d\tau
$$

converges strongly to $v(t,x)$ in $C([0,T];L^{q_0})$ for some $T>0$. Now we show that for any $j\geq 0$, we have the following estimate

$$\|v^j(t)\|_{q_1}\leq 2Ct^{-\delta_0}\|v_0\|_{q_0}$$

for any $t\in(0,T]$. Indeed, by Proposition 2.4 ($L^\alpha-L^\beta$ estimate), we have

$$\|v^0(t)\|_{q_1}\leq 2Ct^{-\delta_0}\|v_0\|_{q_0}.$$

Assume that for $j=k$, the following estimate

$$\|v^k(t)\|_{q_1}\leq 2Ct^{-\delta_0}\|v_0\|_{q_0}$$

holds. Then by Proposition 2.4 ($L^\alpha-L^\beta$ estimate), we obtain

$$\begin{aligned}\|v^{k+1}(t)\|_{q_1}&\leq\|e^{-t\mathcal{L}}v_0\|_{q_1}+\int_0^t\|e^{-(t-\tau)\mathcal{L}}F(v^k(\tau))\|_{q_1}\,d\tau\\&\leq Ct^{-\delta_0}\|v_0\|_{q_0}+C\int_0^t(t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q_1}-\frac{1}{q_1}\right)}\|F(v^k(\tau))\|_{\frac{q_1}{\alpha}}\,d\tau\\&\leq Ct^{-\delta_0}\|v_0\|_{q_0}+C\int_0^t(t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q_1}-\frac{1}{q_1}\right)}\|v^k(\tau)\|_{q_1}^\alpha\,d\tau\\&\leq Ct^{-\delta_0}\|v_0\|_{q_0}+2^\alpha C^{\alpha+1}\|v_0\|_{q_0}^\alpha\int_0^t(t-\tau)^{-\frac{N}{2}\left(\frac{\alpha}{q_1}-\frac{1}{q_1}\right)}\tau^{-\delta_0\alpha}\,d\tau.\end{aligned}\tag{4.15}$$

Since $1-N(\alpha/q_1-1/q_1)/2-\delta_0\alpha=1-N(\alpha/q_0-1/q_1)/2$, we have

$$\begin{aligned}(4.15)&\leq Ct^{-\delta_0}\|v_0\|_{q_0}+2^\alpha C^{\alpha+1}I(q_0,q_1)\|v_0\|_{q_0}^\alpha t^{1-N(\alpha/q_0-1/q_1)/2}\\&\leq 2Ct^{-\delta_0}\|v_0\|_{q_0}\left(\frac{1}{2}+2^{\alpha-1}C^\alpha I(q_0,q_1)\|v_0\|_{q_0}^{\alpha-1}t^{1-\frac{p_c}{q_0}}\right).\end{aligned}\tag{4.16}$$

If we take T such that

$$2^{\alpha-1}C^\alpha I(q_0,q_1)\|v_0\|_{q_0}^{\alpha-1}T^{1-\frac{p_c}{q_0}}\leq\frac{1}{2},\tag{4.17}$$

by (4.16), we have

$$\|v^{k+1}(t)\|_{q_1}\leq 2Ct^{-\delta_0}\|v_0\|_{q_0}.$$

By induction, for all $j\geq 0$, we have

$$\|v^j(t)\|_{q_1}\leq 2Ct^{-\delta_0}\|v_0\|_{q_0}.$$

Hence we obtain

$$\|v(t)\|_{q_1}\leq 2Ct^{-\delta_0}\|v_0\|_{q_0}\tag{4.18}$$

for any $t\in(0,T]$.

By using the uniqueness result in Remark 1.1, $v(t,x) = u(t+t_0,x)$. So by (4.18), we have

$$\|u(t+t_0)\|_{q_1} \le 2Ct^{-\delta_0}\|u(t_0)\|_{q_0} \tag{4.19}$$

for any $t \in (0,T]$, where T satisfies the condition (4.17). If $\|u_0\|_{p_c}$ is small enough, one can always take $T = t_0/2$ (for details, see [13]). Namely, by (4.14) and (4.17), we choose u_0 such as

$$2^{\alpha-1}C^{\alpha}I(q_0,q_1)(2Ct_0^{-b(q_0)}\|u_0\|_{p_c})^{\alpha-1}\left(\frac{t_0}{2}\right)^{1-\frac{p_c}{q_0}} \le \frac{1}{2},$$

which is equivalent to

$$\|u_0\|_{p_c}^{\alpha-1} \le \frac{C}{I(q_0,q_1)}.$$

Hence, by (4.14) and (4.19), we have for $t = t_0/4$

$$\left\|u\left(\frac{5}{4}t_0\right)\right\|_{q_1} \le 2C\left(\frac{t_0}{4}\right)^{-\delta_0}\|u(t_0)\|_{q_0} \le 2Ct_0^{-\delta_0}t_0^{-b(q_0)}\|u_0\|_{p_c}.$$

Since $t_0 > 0$ is arbitrary, we have

$$\|u(t)\|_{q_1} \le 2Ct^{-b(q_1)}\|u_0\|_{p_c}$$

for any $t > 0$. On the other hand, since the sequence $\{q_l\}_{l=0,1,2,\dots}$ satisfies the condition (4.12) if we choose u_0 such as

$$\|u_0\|_{p_c}^{\alpha-1} \le \frac{C}{I(q_l,q_{l+1})},$$

we obtain

$$\|u(t)\|_{q_l} \le 2Ct^{-b(q_l)}\|u_0\|_{p_c} \tag{4.20}$$

for any $t > 0$. We note that

$$\inf_{l\ge 0}\frac{1}{I(q_l,q_{l+1})} > 0$$

by (4.13). Therefore if we put

$$\varepsilon := \left(\inf_{l\ge 0}\frac{C}{I(q_l,q_{l+1})}\right)^{\frac{1}{\alpha-1}}$$

and choose u_0 such as $\|u_0\|_{p_c} \le \varepsilon$, then we obtain (4.20) for any $l \ge 0$.

Furthermore, if we consider $\theta_l \in (q_l, q_{l+1})$, by the Riesz–Thorin interpolation theorem, for all $l \ge 0$,

$$\|u(t)\|_{\theta_l} \le 2Ct^{-b(\theta_l)}\|u_0\|_{p_c}.$$

Therefore, for all $\gamma \in (q,\infty)$, we have

$$\|u(t)\|_{\gamma} \le 2Ct^{-b(\gamma)}\|u_0\|_{p_c}$$

for any $t > 0$. This completes the proof of Lemma 4.2. □

Step 3. Let $\max\{1, p_c/\alpha\} < p \le p_c$. Then $s_c \ge 0$. Assume $u_0 \in L^p_{s_c}(\mathbb{G})$. Finally, we will prove the regularity of unique global solution u constructed in Step 1. Precisely, we show that $\sup_{t\ge 0} \|u(t)\|_{L^p_{s_c}} < \infty$. To prove it, we first give the following lemma.

Lemma 4.3. *Let* $\max\{1, p_c/\alpha\} < p \le p_c$ *and* $u_0 \in L^p(\mathbb{G}) \cap L^{p_c}(\mathbb{G})$ *with* $\|u_0\|_{p_c} \le \varepsilon$, $\varepsilon > 0$ *is sufficiently small. Then the solution* $u(t,x)$ *to* (1.5) *constructed in Lemma* 4.1 *belongs to* $C([0,\infty); L^p) \cap C([0,\infty); L^{p_c})$.

Proof. Let $1 < p < p_c$. Assume $u_0 \in L^p(\mathbb{G}) \cap L^{p_c}(\mathbb{G})$ with $\|u_0\|_{p_c}$ is sufficiently small and $u(t,x)$ constructed in Lemma 4.1 to (1.5). Then it is enough to show that $u(t,x)$ belongs to $C([0,\infty); L^p)$. First, we prove that for any $T > 0$, there exists a positive constant $C(T)$ such that

$$\|u(t)\|_p \le C(T) \tag{4.21}$$

for any $t \in [0,T]$. Second, we prove that the solution $u(t,x)$ belongs to $C([0,\infty); L^p)$.

Assume

$$1 < \frac{p_c}{\alpha} < p < p_c \quad \text{or} \quad \frac{p_c}{\alpha} \le 1 < p < p_c. \tag{4.22}$$

Since $u(t,x)$ is the solution to (1.5), for any $T > 0$ and $t \in (0,T]$, we have

$$\begin{aligned} \|u(t)\|_p &\le \|e^{-t\mathcal{L}} u_0\|_p + \int_0^t \|e^{-(t-\tau)\mathcal{L}} F(u(\tau))\|_p \, d\tau \\ &\le \|u_0\|_p + \int_0^t \|F(u(\tau))\|_p \, d\tau \le \|u_0\|_p + C \int_0^t \|u(\tau)\|_{p\alpha}^{\alpha} \, d\tau. \end{aligned} \tag{4.23}$$

By the assumption (4.22), $p_c < p\alpha$. So by Lemma 4.2, we obtain the following estimate

$$\|u(t)\|_{p\alpha} \le C t^{-b(p\alpha)} \|u_0\|_{p_c} \tag{4.24}$$

for any $t \in (0,T]$. By (4.24), we have

$$\|u(t)\|_p \le \|u_0\|_p + C^{\alpha+1} \|u_0\|_{p_c}^{\alpha} \int_0^t \tau^{-b(p\alpha)\alpha} \, d\tau \le \|u_0\|_p + C^{\alpha+1} \|u_0\|_{p_c}^{\alpha} T^{1-b(p\alpha)\alpha}. \tag{4.25}$$

Hence we can see that (4.21) holds.

Next, we will show that $u(t,x)$ belongs to $C([0,\infty); L^p)$. Let $1 < \theta < p$ and θ is chosen later. By Hölder's inequality, we have for any $T > 0$ and $t \in (0,T]$,

$$\begin{aligned} \|u(t)\|_p &\le \|e^{-t\mathcal{L}} u_0\|_p + \int_0^t \|e^{-(t-\tau)\mathcal{L}} F(u(\tau))\|_p \, d\tau \\ &\le \|u_0\|_p + C \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)} \|F(u(\tau))\|_\theta \, d\tau \\ &\le \|u_0\|_p + C \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)} \|u(\tau)\|_{\theta\theta_1} \|u(\tau)\|_{\theta\theta_2(\alpha-1)}^{\alpha-1} \, d\tau \\ &\le \|u_0\|_p + C \left(\sup_{0<t\le T} \|u(t)\|_p \right) \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{\theta} - \frac{1}{p}\right)} \|u(\tau)\|_{\theta\theta_2(\alpha-1)}^{\alpha-1} \, d\tau, \end{aligned} \tag{4.26}$$

where $\theta\theta_1 = p$ and $1/\theta_1 + 1/\theta_2 = 1$. We then choose $\theta \in (1,p)$ such that $\theta\theta_2(\alpha-1) = p\theta(\alpha-1)/(p-\theta) \geq p_c$, that is, we take θ such that $1/p < 1/\theta < 1/p + \min\{(\alpha-1)/p_c, 2/N\}$. Then by Lemma 4.2, we obtain

$$\|u(\tau)\|_{\theta\theta_2(\alpha-1)} \leq C\tau^{-\frac{N}{2}\left(\frac{1}{p_c}-\frac{1}{\theta\theta_2(\alpha-1)}\right)}\|u_0\|_{p_c}. \tag{4.27}$$

Hence by (4.26) and (4.27), we obtain

$$\begin{aligned}
&\|u(t)\|_p \\
&\leq \|u_0\|_p + C^\alpha \|u_0\|_{p_c}^{\alpha-1}\left(\sup_{0<t\leq T}\|u(t)\|_p\right)\int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{\theta}-\frac{1}{p}\right)}\tau^{-\frac{N}{2}\left(\frac{\alpha-1}{p_c}-\frac{1}{\theta\theta_2}\right)}\,d\tau \\
&\leq \|u_0\|_p + B_4 C^\alpha \|u_0\|_{p_c}^{\alpha-1}\left(\sup_{0<t\leq T}\|u(t)\|_p\right),
\end{aligned} \tag{4.28}$$

where $B_4 = \int_0^1 (1-\tau)^{-\frac{N}{2}\left(\frac{1}{\theta}-\frac{1}{p}\right)}\tau^{-\frac{N}{2}\left(\frac{\alpha-1}{p_c}-\frac{1}{\theta\theta_2}\right)}\,d\tau$, since

$$1 - \frac{N}{2}\left(\frac{1}{\theta}-\frac{1}{p}\right) - \frac{N}{2}\left(\frac{\alpha-1}{p_c}-\frac{1}{\theta\theta_2}\right) = 0.$$

For $\|u_0\|_{p_c}$ sufficiently small, we have

$$1 - B_4 C^\alpha \|u_0\|_{p_c}^{\alpha-1} \geq \frac{1}{2}. \tag{4.29}$$

Hence by (4.28) and (4.29), we obtain for any $T > 0$ that

$$\sup_{0\leq t\leq T}\|u(t)\|_p \leq 2\|u_0\|_p,$$

that is, we can see that $u \in L^\infty([0,\infty); L^p)$.

Finally, we will show that $u \in C([0,\infty); L^p)$. To do this, it is enough to show

$$\lim_{t\to 0^+}\left\|\int_0^t e^{-(t-\tau)\mathcal{L}}F(u(\tau))\,d\tau\right\|_p = 0. \tag{4.30}$$

Indeed, by the same argument as in (4.23), (4.24) and (4.25), we have for $t > 0$,

$$\left\|\int_0^t e^{-(t-\tau)\mathcal{L}}F(u(\tau))\,d\tau\right\|_p \leq C^{\alpha+1}\|u_0\|_{p_c}^\alpha t^{1-b(p\alpha)\alpha}. \tag{4.31}$$

Note that $1 - b(p\alpha)\alpha > 0$ by $p < p_c$. So, by (4.31), we can see that (4.30) holds. This completes the proof of Lemma 4.3. □

The rest of the proof of Theorem 1.2. Assume that $u_0 \in L^p_{s_c}(\mathbb{G})$ and $\|u_0\|_{L^{p_c}}$ is sufficiently small. By Proposition 3.5 (the Sobolev embedding theorem), we have the inclusion relationship $L^p_{s_c}(\mathbb{G}) \subset L^{p_c}(\mathbb{G})$. So we have the solution $u(t,x)$ to (1.5) constructed in

Lemma 4.1. On the other hand, since $L^p_{s_c}(\mathbb{G}) = L^p(\mathbb{G}) \cap \dot{L}^p_{s_c}(\mathbb{G}) \subset L^p(\mathbb{G})$, we have $u_0 \in L^p(\mathbb{G}) \cap L^{p_c}(\mathbb{G})$ and $\|u_0\|_{p_c}$ is sufficiently small. By Lemma 4.3, the solution $u(t,x)$ to (1.5) belongs to $C([0,\infty); L^p)$. Since $\|u(t)\|_{L^p_{s_c}} \sim \|u(t)\|_p + \|u(t)\|_{\dot{L}^p_{s_c}}$ by Proposition 3.4, to show $\sup_{t\geq 0}\|u(t)\|_{L^p_{s_c}} < \infty$, it is enough to show $\sup_{t\geq 0}\|u(t)\|_{\dot{L}^p_{s_c}} < \infty$.

Indeed, by Proposition 3.6 ($\dot{L}^p_{s+\theta} - \dot{L}^p_s$ estimate), we have

$$
\begin{aligned}
\|u(t)\|_{\dot{L}^p_{s_c}} &\leq \|e^{-t\mathcal{L}}u_0\|_{\dot{L}^p_{s_c}} + \int_0^t \|e^{-(t-\tau)\mathcal{L}}F(u(\tau))\|_{\dot{L}^p_{s_c}}\,d\tau \\
&\leq C\|u_0\|_{\dot{L}^p_{s_c}} + C_1\int_0^t (t-\tau)^{-\frac{s_c}{2}}\|e^{-\frac{t-\tau}{2}\mathcal{L}}F(u(\tau))\|_p\,d\tau \\
&\leq C\|u_0\|_{L^p_{s_c}} + C_1\int_0^t (t-\tau)^{-\frac{s_c}{2}-\frac{N}{2}\left(\frac{\alpha}{\widetilde{q}}-\frac{1}{p}\right)}\|F(u(\tau))\|_{\frac{\widetilde{q}}{\alpha}}\,d\tau \qquad (4.32)\\
&\leq C\|u_0\|_{L^p_{s_c}} + C_1\int_0^t (t-\tau)^{-\frac{s_c}{2}-\frac{N}{2}\left(\frac{\alpha}{\widetilde{q}}-\frac{1}{p}\right)}\|u(\tau)\|^{\alpha}_{\widetilde{q}}\,d\tau \\
&\leq C\|u_0\|_{L^p_{s_c}} + C_1\left(\sup_{\tau>0}\tau^{b(\widetilde{q})}\|u(\tau)\|_{\widetilde{q}}\right)^{\alpha}\int_0^t (t-\tau)^{-\frac{s_c}{2}-\frac{N}{2}\left(\frac{\alpha}{\widetilde{q}}-\frac{1}{p}\right)}\tau^{-b(\widetilde{q})\alpha}\,d\tau
\end{aligned}
$$

for any $\widetilde{q} \in (p_c, p\alpha)$ such that $\alpha < \widetilde{q}$. Since

$$1 - \frac{s_c}{2} - \frac{N}{2}\left(\frac{\alpha}{\widetilde{q}} - \frac{1}{p}\right) - b(\widetilde{q})\alpha = 0,$$

by (4.32) and Lemma 4.2, we have

$$
\begin{aligned}
\sup_{t>0}\|u(t)\|_{\dot{L}^p_{s_c}} &\leq C\|u_0\|_{L^p_{s_c}} + C_1B_2\left(\sup_{\tau>0}\tau^{b(\widetilde{q})}\|u(\tau)\|_{\widetilde{q}}\right)^{\alpha} \\
&\leq C\|u_0\|_{L^p_{s_c}} + (2CC_0)^{\alpha}C_1B_2\|u_0\|^{\alpha}_{L^p_{s_c}} < \infty,
\end{aligned}
$$

where $B_2 = \int_0^1 (1-\tau)^{-\frac{s_c}{2}-\frac{N}{2}\left(\frac{\alpha}{\widetilde{q}}-\frac{1}{p}\right)}\tau^{-b(\widetilde{q})\alpha}\,d\tau$. □

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