TAIWANESE JOURNAL OF MATHEMATICS
Vol. 28, No. 6, pp. 1137–1159, December 2024
DOI: 10.11650/tjm/240601

Multiple Nontrivial Solutions for Biharmonic Systems Involving Rellich-type Potentials and Homogeneous Terms

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Abstract. In this work, a biharmonic elliptic system is investigated, which involves a Rellich-type potential and a homogeneous term of a critical Rellich–Sobolev degree. By using the variational methods and analytic techniques, we obtain multiplicity result of nontrivial solutions for the system.

1. Introduction

In this paper, we study the following biharmonic elliptic system

$$\begin{cases} \Delta^2 u - \gamma \frac{u}{|x|^4} = \lambda f(x) \frac{|u|^{q-2}u}{|x|^\alpha} + \frac{1}{2^*(\beta)} \frac{F_u(u,v)}{|x|^\beta} & \text{in } \Omega, \\ \Delta^2 v - \gamma \frac{v}{|x|^4} = \mu g(x) \frac{|v|^{q-2}v}{|x|^\alpha} + \frac{1}{2^*(\beta)} \frac{F_v(u,v)}{|x|^\beta} & \text{in } \Omega, \\ u = v = \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \in \Omega$ is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 5)$, Δ^2 denotes the biharmonic operator, $\frac{\partial}{\partial n}$ is the outward normal derivative, $1 \leq q < 2$, $0 \leq \alpha, \beta < 4$, $0 \leq \gamma < \overline{\gamma} := \frac{1}{16} N^2 (N-4)^2$, $2^*(\beta) := \frac{2(N-\beta)}{N-4}$ is the critical Rellich–Sobolev exponent, $\lambda, \mu > 0$ are parameters, $F \in C^1(\mathbb{R}^2, \mathbb{R}^+)$ is a homogeneous function of degree $2^*(\beta)$ and the functions $f, g \colon \Omega \to \mathbb{R}$ are sign-changing and satisfy the following conditions:

(H_1) $f, g \in L^p(\Omega, |x|^{-\alpha}\, dx)$ and $f^\pm = \max\{\pm f, 0\} \neq 0$, $g^\pm = \max\{\pm g, 0\} \neq 0$, where $p := \frac{2^*(\alpha)}{2^*(\alpha) - q}$.

(H_2) There exist $a_0, r_0 > 0$ such that $B_{r_0}(0) \subset \Omega$ and $f(x), g(x) \geq a_0$ for all $x \in B_{r_0}(0)$.

Here $\overline{\gamma}$ is the best constant in the Rellich inequality (see [16])

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^4}\, dx \leq \frac{1}{\overline{\gamma}} \int_{\mathbb{R}^N} |\Delta u|^2\, dx, \quad \forall\, u \in D^{2,2}(\mathbb{R}^N), \tag{1.2}$$

Received March 17, 2024; Accepted June 1, 2024.
Communicated by François Hamel.
2020 *Mathematics Subject Classification.* 35J60, 31B30, 35B33, 35B25.
Key words and phrases. multiple nontrivial solutions, biharmonic systems, Rellich-type potentials, homogeneous terms.
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where $D^{2,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $\big(\int_{\mathbb{R}^N} |\Delta\,\cdot\,|^2\,dx\big)^{1/2}$. Moreover, there exists a constant $C(\beta) > 0$ such that the following Rellich–Sobolev inequality holds (see [3,9,14]):

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*(\beta)}}{|x|^\beta}\,dx\right)^{2/2^*(\beta)} \le C(\beta) \int_{\mathbb{R}^N} |\Delta u|^2\,dx, \quad \forall\, u \in D^{2,2}(\mathbb{R}^N). \tag{1.3}$$

In this paper, for $0 \le \gamma < \overline{\gamma}$, we use $H_0^2(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_\gamma = \left(\int_\Omega \left(|\Delta u|^2 - \gamma \frac{u^2}{|x|^4}\right)\,dx\right)^{1/2}, \quad u \in H_0^2(\Omega).$$

According to (1.2), (1.3) and [12], we have that the following best Rellich–Sobolev constant is well defined for all $\gamma < \overline{\gamma}$, and $0 \le \beta < 4$:

$$\Lambda_\Omega(\beta) := \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\|u\|_\gamma^2}{\big(\int_\Omega \frac{|u|^{2^*(\beta)}}{|x|^\beta}\,dx\big)^{2/2^*(\beta)}}.$$

It is well-known that $\Lambda_\Omega(\beta)$ is independent of $\Omega \subset \mathbb{R}^N$. Thus, we will simply denote that $\Lambda_\Omega(\beta) = \Lambda_{\mathbb{R}^N}(\beta) = \Lambda(\beta)$.

Energy functional of (1.1) is defined on the product space $W := H_0^2(\Omega) \times H_0^2(\Omega)$ by

$$J_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|_W^2 - \frac{1}{2^*(\beta)} \int_\Omega \frac{F(u,v)}{|x|^\beta}\,dx - \frac{1}{q} Q_{\lambda,\mu}(u,v),$$

where

$$\|(u,v)\|_W^2 := \|u\|_\gamma^2 + \|v\|_\gamma^2 \quad \text{and} \quad Q_{\lambda,\mu}(u,v) := \lambda \int_\Omega \frac{f(x)|u|^q}{|x|^\alpha}\,dx + \mu \int_\Omega \frac{g(x)|v|^q}{|x|^\alpha}\,dx.$$

Then $J_{\lambda,\mu} \in C^1(W,\mathbb{R})$ and $(u,v) \in W$ is said to be a solution of (1.1) if

$$(u,v) \ne (0,0), \quad \langle J'_{\lambda,\mu}(u,v), (\phi,\psi)\rangle = 0, \quad \forall\, (\phi,\psi) \in W,$$

where $J'_{\lambda,\mu}(u,v)$ denotes the Fréchet derivative of $J_{\lambda,\mu}$ at the point (u,v).

The starting point on the study of the system (1.1) is its scalar version:

$$\begin{cases} \Delta^2 u - \gamma \frac{u}{|x|^4} = \lambda f(x) \frac{|u|^{q-2}u}{|x|^\alpha} + \frac{|u|^{2^*(\beta)-2}u}{|x|^\beta} & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In recent years, much attention has been paid to the biharmonic equation (1.4) involving the Rellich–Sobolev inequality and critical nonlinearities (e.g., [4,8,11,12] and the references therein). For example, Kang and Xu [12] showed that the existence of nontrivial solutions to the equation (1.4) with $\gamma \in [0, \overline{\gamma})$, $\lambda > 0$, $f(x) \equiv 1$, $\alpha = 0$, $2 \le q < 2^*$, $0 \le \beta < 4$ and $N \ge 5$. Recently, Hsu and Sang [8] obtained the existence and multiplicity of nontrivial solutions to the equation (1.4) with $\gamma \in [0, \overline{\gamma})$, $\lambda > 0$, $f(x) \equiv 1$, $0 \le \alpha, \beta < 4$, $1 \le q < 2$ and $N \ge 5$.

However, the biharmonic systems involving the Rellich–Sobolev inequality and critical nonlinearities have been seldom studied, only a few related results are found in [5,8,10,13]. Biharmonic equations and systems can build some models to investigate traveling waves in suspension bridges, thus they are important to physics and engineering mechanics. Many attractive and challenging topics on singular biharmonic systems remain unsolved. In particular, Hsu and Sang in [8] concerned the case $F(u,v) = \sum_{i=1}^m \eta_i |u|^{s_i} |v|^{t_i}$, where $s_i, t_i > 1$ satisfying $s_i + t_i = 2^*(\beta)$ for $1 \le i \le m$, i.e., the following biharmonic system

$$(1.5) \qquad \begin{cases} \Delta^2 u - \gamma \frac{u}{|x|^4} = \lambda \frac{|u|^{q-2}u}{|x|^\alpha} + \sum_{i=1}^m \frac{\eta_i s_i}{2^*(\beta)} \frac{|u|^{s_i-2}u|v|^{t_i}}{|x|^\beta} & \text{in } \Omega, \\ \Delta^2 v - \gamma \frac{v}{|x|^4} = \mu \frac{|v|^{q-2}u}{|x|^\alpha} + \sum_{i=1}^m \frac{\eta_i t_i}{2^*(\beta)} \frac{|u|^{s_i}|v|^{t_i-2}v}{|x|^\beta} & \text{in } \Omega, \\ u = v = \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the Nehari manifold methods, the authors in [8] obtained the existence of two nontrivial nonnegative solutions of system (1.5) with the sublinear perturbation of $1 < q < 2$.

In this paper, we extended the results of [8] to the biharmonic case (1.1) containing a general class of homogeneous critical nonlinearities and Rellich-type potentials. To the best of our knowledge, problem (1.1) has not been considered before. Thus it is necessary for us to investigate the related biharmonic systems (1.1) deeply.

The following assumptions are used in this article.

(F_0) $F \in C^1(\mathbb{R}^2, \mathbb{R}^+)$ and $F(tu, tv) = t^{2^*(\beta)} F(u,v)$, $\forall\, t \ge 0$, $(u,v) \in \mathbb{R}^2$;

(F_1) $F(u,0) = F(0,v) = F_u(u,0) = F_v(0,v) = 0$, $\forall\, u, v \in \mathbb{R}$;

(F_2) $F_u(u,v)$, $F_v(u,v)$ are strictly increasing functions about u, v for all $(u,v) \in \mathbb{R}^2$.

Set

$$(1.6) \qquad S_F(\beta) := \inf_{(u,v) \in W \setminus \{(0,0)\}} \frac{\|(u,v)\|_W^2}{\left(\int_\Omega \frac{F(u,v)}{|x|^\beta} \, dx \right)^{2/2^*(\beta)}}.$$

To establish a relation between $\Lambda(\beta)$ and $S_F(\beta)$, which shall be useful later, we need some definition and comments. Let us define

$$M_F = \max \left\{ F(|u|, |v|)^{2/2^*(\beta)} : (u,v) \in \mathbb{R}^2 \text{ and } |u|^2 + |v|^2 = 1 \right\}.$$

Then we have the following result.

Theorem 1.1. *Suppose that* (F_0)–(F_2) *hold. Then* $S_F(\beta) = M_F^{-1}\Lambda(\beta)$ *and* $S_F(\beta)$ *has the minimizers* $(\theta_1 U_\varepsilon(x), \theta_2 U_\varepsilon(x))$, *where* $U_\varepsilon(x)$ *are defined as in* (2.6) *and* θ_1, θ_2 *are constants given in* (2.2).

In order to present the existence results, we set

$$\Lambda_0 = \left(\frac{2^*(\beta)-2}{2^*(\beta)-q}\right)^{\frac{2}{2-q}} \left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2}{2^*(\beta)-2}} \Lambda(\alpha)^{\frac{q}{2-q}} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}. \tag{1.7}$$

Then we have the following theorem.

Theorem 1.2. *Suppose that* (H_1)–(H_2) *and* (F_0)–(F_2) *hold. Then we have the following results.*

(i) *If* $\lambda, \mu > 0$ *satisfy* $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, *then problem* (1.1) *has at least one nonnegative solution* $(u^{(1)}, v^{(1)}) \in W$ *and* $u^{(1)} \not\equiv 0$, $v^{(1)} \not\equiv 0$.

(ii) *If* $\lambda, \mu > 0$ *satisfy* $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_1 = \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0$, *then problem* (1.1) *has at least two nonnegative solutions* $(u^{(i)}, v^{(i)}) \in W$ *and* $u^{(i)} \not\equiv 0$, $v^{(i)} \not\equiv 0$ *for* $i = 1, 2$.

In this paper, we study the existence and multiplicity of nontrivial solutions for problem (1.1) by using the variational methods. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all, and we will be faced with serious difficulties. For example, the explicit formula of the ground states of limiting problem (2.7) is not clear, we will get around the difficulty by working with certain asymptotic estimates for this solution at zero and infinity. Secondly, the energy functional $J_{\lambda,\mu}$ corresponding to (1.1) does not satisfy Palais–Smale condition due to the lack of compactness of the embedding $H_0^2(\Omega) \hookrightarrow L^{2^*(\beta)}(\Omega, |x|^{-\beta}\,dx)$, so the standard variational argument is not applicable directly. In order to construct suitable Palais–Smale compact sequences, we need to locate the energy range where $J_{\lambda,\mu}$ satisfies Palais–Smale condition. Thus, our conclusion are new for the biharmonic system with critical Rellich–Sobolev exponent and homogeneous nonlinearity.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.1 and present some norm estimates about the ground states of limiting problem. Some properties about the Nehari manifold and fibering maps are established in Section 3, and Theorem 1.2 is proved in Section 4.

2. Preliminaries

Throughout this paper, $\|(u,v)\| := \|(u,v)\|_W = \left(\|u\|_\gamma^2 + \|v\|_\gamma^2\right)^{1/2}$ means the norm of the space $W := H_0^2(\Omega) \times H_0^2(\Omega)$; $z = (u,v)$ is said to be nonnegative in Ω if $u \geq 0$ and $v \geq 0$

in Ω; $\|\cdot\|_{L^p}$ denotes the norm of the Lebesgue space $L^p(\Omega)$; W^{-1} is the dual space of W; $L^q(\Omega, |x|^\alpha\, dx)$ denotes the space $L^q(\Omega)$ with the weight $|x|^\alpha$ which norm denote by $\|\cdot\|_{L^q(\Omega,|x|^\alpha)}$; $O(\varepsilon^t)$ denotes $\frac{|O(\varepsilon^t)|}{\varepsilon^t} \le C$ and $o(\varepsilon^t)$ means $\frac{|o(\varepsilon^t)|}{\varepsilon^t} \to 0$ as $\varepsilon \to 0$, $O_1(\varepsilon^t)$ ($\varepsilon \to \varepsilon_0$) means that there exist the constants $C_1, C_2 > 0$ such that $C_1\varepsilon^t \le O_1(\varepsilon^t) \le C_2\varepsilon^t$ as $\varepsilon \to \varepsilon_0$ for $t \ge 0$; $o_n(1)$ means $o_n(1) \to 0$ as $n \to \infty$. We always denote positive constants as C, C_i, c which may vary from line to line.

The following properties about homogeneous function are important and well known:

Lemma 2.1. *Let $\sigma \ge 1$ and G be a differentiable σ-homogeneous function, then*

(i) *for all $(u,v) \in \mathbb{R}^2$, $uG_u(u,v) + vG_v(u,v) = \sigma G(u,v)$;*

(ii) *there exists $M_G > 0$ such that $|G(u,v)| \le M_G(|u|^\sigma + |v|^\sigma)$ for all $(u,v) \in \mathbb{R}^2$, where*

$$M_G = \max\{G(u,v) : (u,v) \in \mathbb{R}^2, |u|^\sigma + |v|^\sigma = 1\};$$

(iii) *the maximum M_G is attained for some $(u_0,v_0) \in \mathbb{R}^2$, i.e., $|u_0|^\sigma + |v_0|^\sigma = 1$ and $|G(u_0,v_0)| = M_G$;*

(iv) *$G_u(u,v)$, $G_v(u,v)$ are $(\sigma - 1)$ homogenous.*

By (F_0) and Lemma 2.1, we have

$$uF_u(u,v) + vF_v(u,v) = 2^*(\beta)F(u,v)$$

and

$$F(u,v) \le \big(M_F(|u|^2 + |v|^2)\big)^{2^*(\beta)/2}, \tag{2.1}$$

where

$$M_F = \max\big\{(F(u,v))^{2/2^*(\beta)} : (u,v) \in \mathbb{R}^2, |u|^2 + |v|^2 = 1\big\}.$$

Moreover, from Lemma 2.1(iii), there exists $(\theta_1, \theta_2) \in \mathbb{R}^2$ such that

$$\theta_1^2 + \theta_2^2 = 1 \quad \text{and} \quad M_F = F(\theta_1,\theta_2)^{2/2^*(\beta)}. \tag{2.2}$$

Now, we will study $S_F(\beta)$ and verify Theorem 1.1.

Proof of Theorem 1.1. Let $\{w_n\} \subset H_0^2(\Omega)$ be a minimizing sequence for $\Lambda(\beta)$ and (θ_1, θ_2) be defined as in (2.2). Choosing $(u_n, v_n) = (\theta_1 w_n, \theta_2 w_n)$ in (1.6), from (F_0) we have

$$\frac{(\theta_1^2 + \theta_2^2)\|w_n\|_\gamma^2}{F(\theta_1,\theta_2)^{2/2^*(\beta)}\big(\int_\Omega \frac{|w_n|^{2^*(\beta)}}{|x|^\beta}\, dx\big)^{2/2^*(\beta)}} \ge S_F(\beta). \tag{2.3}$$

Taking $n \to \infty$ in (2.3), we have

$$S_F(\beta) \le M_F^{-1}\Lambda(\beta). \tag{2.4}$$

On the other hand, let $\{(u_n, v_n)\} \subset W \setminus \{(0,0)\}$ be a minimizing sequence for $S_F(\beta)$, from Proposition 1 in [15], we have

$$\begin{aligned}\int_\Omega \frac{F(u_n, v_n)}{|x|^\beta}\,dx &= \int_\Omega F\left(\frac{u_n}{|x|^{\beta/2^*(\beta)}}, \frac{v_n}{|x|^{\beta/2^*(\beta)}}\right)dx \\ &\le F\left(\left\|\frac{u_n}{|x|^{\beta/2^*(\beta)}}\right\|_{L^{2^*(\beta)}}, \left\|\frac{v_n}{|x|^{\beta/2^*(\beta)}}\right\|_{L^{2^*(\beta)}}\right).\end{aligned}$$

Set

$$\mathcal{A} = \frac{1}{\sqrt{\left\|\frac{u_n}{|x|^{\beta/2^*(\beta)}}\right\|^2_{L^{2^*(\beta)}} + \left\|\frac{v_n}{|x|^{\beta/2^*(\beta)}}\right\|^2_{L^{2^*(\beta)}}}}.$$

Then, we obtain

$$\left\|\frac{\mathcal{A}u_n}{|x|^{\beta/2^*(\beta)}}\right\|^2_{L^{2^*(\beta)}} + \left\|\frac{\mathcal{A}v_n}{|x|^{\beta/2^*(\beta)}}\right\|^2_{L^{2^*(\beta)}} = 1,$$

and

$$\begin{aligned}\frac{\|(u_n, v_n)\|^2}{\big(\int_\Omega \frac{F(u_n,v_n)}{|x|^\beta}\,dx\big)^{2/2^*(\beta)}} &\ge \frac{\Lambda(\beta)\big(\int_\Omega \frac{|u_n|^{2^*(\beta)}}{|x|^\beta}\,dx\big)^{2/2^*(\beta)} + \Lambda(\beta)\big(\int_\Omega \frac{|v_n|^{2^*(\beta)}}{|x|^\beta}\,dx\big)^{2/2^*(\beta)}}{\big(F\big(\big\|\frac{u_n}{|x|^{\beta/2^*(\beta)}}\big\|_{L^{2^*(\beta)}}, \big\|\frac{v_n}{|x|^{\beta/2^*(\beta)}}\big\|_{L^{2^*(\beta)}}\big)\big)^{2/2^*(\beta)}} \\ &= \Lambda(\beta)\frac{\big\|\frac{u_n}{|x|^{\beta/2^*(\beta)}}\big\|^2_{L^{2^*(\beta)}} + \big\|\frac{v_n}{|x|^{\beta/2^*(\beta)}}\big\|^2_{L^{2^*(\beta)}}}{\big(F\big(\big\|\frac{u_n}{|x|^{\beta/2^*(\beta)}}\big\|_{L^{2^*(\beta)}}, \big\|\frac{v_n}{|x|^{\beta/2^*(\beta)}}\big\|_{L^{2^*(\beta)}}\big)\big)^{2/2^*(\beta)}} \\ &= \Lambda(\beta)\frac{\big\|\frac{\mathcal{A}u_n}{|x|^{\beta/2^*(\beta)}}\big\|^2_{L^{2^*(\beta)}} + \big\|\frac{\mathcal{A}v_n}{|x|^{\beta/2^*(\beta)}}\big\|^2_{L^{2^*(\beta)}}}{\big(F\big(\big\|\frac{\mathcal{A}u_n}{|x|^{\beta/2^*(\beta)}}\big\|_{L^{2^*(\beta)}}, \big\|\frac{\mathcal{A}v_n}{|x|^{\beta/2^*(\beta)}}\big\|_{L^{2^*(\beta)}}\big)\big)^{2/2^*(\beta)}} \\ &\ge M_F^{-1}\Lambda(\beta).\end{aligned}$$

Passing to the limit in the above inequality, we have

$$S_F(\beta) \ge M_F^{-1}\Lambda(\beta). \tag{2.5}$$

Hence, (2.4) and (2.5) give the proof of Theorem 1.1. □

For the best constant $\Lambda(\beta)$, from [12], we know that, for all $0 \le \beta < 4$ and $0 \le \gamma < \overline{\gamma}$, the best constant $\Lambda(\beta)$ is achieved by the form

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-4}{2}} U\left(\frac{x}{\varepsilon}\right), \quad \forall\, \varepsilon > 0, \tag{2.6}$$

where $U \in C^1(\mathbb{R}^N \setminus \{0\})$ is a positive, radially symmetric, radially decreasing ground state solution of the limit problem

$$\begin{cases} \Delta^2 u - \gamma \frac{u}{|x|^4} = \frac{u^{2^*(\beta)-1}}{|x|^\beta} & \text{in } \mathbb{R}^N, \\ u \geq 0, \quad u \not\equiv 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.7}$$

Moreover, by setting $\rho = |x|$ there holds that

$$U(\rho) = O_1(\rho^{-a(\gamma)}) \quad \text{as } \rho \to 0,$$
$$U(\rho) = O_1(\rho^{-b(\gamma)}), \quad U'(\rho) = O_1(\rho^{-b(\gamma)-1}) \quad \text{as } \rho \to +\infty,$$

where $a(\gamma) := \frac{N-4}{2} h(\gamma)$, $b(\gamma) := \frac{N-4}{2}(2 - h(\gamma))$ and $h\colon [0, \overline{\gamma}] \to [0,1]$ is defined as

$$h(\gamma) := 1 - \frac{\sqrt{N^2 - 4N + 8 - 4\sqrt{(N-2)^2 + \gamma}}}{N-4}, \quad \gamma \in [0, \overline{\gamma}].$$

We also have

$$0 \leq a(\gamma) < \delta < b(\gamma) \leq 2\delta, \quad \delta := \frac{N-4}{2}, \tag{2.8}$$

and there exist positive constants $\mathcal{C}_1(\gamma)$ and $\mathcal{C}_2(\gamma)$ such that

$$0 < \mathcal{C}_1(\gamma) \leq U(x) \big(|x|^{a(\gamma)/\delta} + |x|^{b(\gamma)/\delta}\big)^\delta \leq \mathcal{C}_2(\gamma), \quad \forall\, x \in \mathbb{R}^N \setminus \{0\}.$$

By a direct computation, we get

$$\int_{\mathbb{R}^N} \left(|\Delta U_\varepsilon(x)|^2 - \gamma \frac{|U_\varepsilon(x)|^2}{|x|^4} \right) dx = \int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^{2^*(\beta)}}{|x|^\beta}\, dx = \Lambda(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}.$$

Take $R \in (0, r_0)$ small enough such that $B_R(0) \subset \Omega$, $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$, where r_0 is given in (H_2). Choose the radial cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $0 \leq \varphi(x) \leq 1$ in $B_R(0)$, $\varphi(x) = 1$ in $B_{R/2}(0)$ and $\varphi(x) = 0$ in $B_R(0)^c$. Set

$$u_\varepsilon(x) = \varphi(x) U_\varepsilon(x), \quad \forall\, \varepsilon > 0. \tag{2.9}$$

Proposition 2.2. *Suppose that $N \geq 5$, $\gamma \in [0, \overline{\gamma})$, $0 \leq \alpha, \beta < 4$ and $1 \leq q < 2^*(\alpha)$. Then, the following estimations hold as $\varepsilon \to 0^+$:*

$$\|u_\varepsilon\|_\gamma^2 = \Lambda(\beta)^{\frac{N-\beta}{4-\beta}} + O(\varepsilon^{2(b(\gamma)-\delta)}); \tag{2.10}$$

$$\int_\Omega \frac{|u_\varepsilon|^{2^*(\beta)}}{|x|^\beta}\, dx = \Lambda(\beta)^{\frac{N-\beta}{4-\beta}} + O(\varepsilon^{2^*(\beta)(b(\gamma)-\delta)}); \tag{2.11}$$

$$\int_\Omega \frac{|u_\varepsilon|^q}{|x|^\alpha}\, dx = \begin{cases} O_1(\varepsilon^{N-\alpha-q\delta}) & \text{if } q > \frac{N-\alpha}{b(\gamma)}, \\ O_1(\varepsilon^{N-\alpha-q\delta}) |\ln \varepsilon| & \text{if } q = \frac{N-\alpha}{b(\gamma)}, \\ O_1(\varepsilon^{q(b(\gamma)-\delta)}) & \text{if } q < \frac{N-\alpha}{b(\gamma)}. \end{cases} \tag{2.12}$$

Proof. See Kang–Xu [12, Lemma 3.2]. □

By the Hölder and Sobolev–Hardy inequalities, for all $u \in H_0^2(\Omega)$, we get

$$
\begin{aligned}
\int_\Omega \frac{f(x)|u|^q}{|x|^\alpha}\,dx &= \int_\Omega \left(\frac{|u|^q}{|x|^{\frac{q}{2^*(\alpha)}\alpha}} \cdot \frac{f(x)}{|x|^{\left(1-\frac{q}{2^*(\alpha)}\right)\alpha}} \right) dx \\
&\le \left(\int_\Omega \frac{|f|^{\frac{2^*(\alpha)}{2^*(\alpha)-q}}}{|x|^\alpha}\,dx \right)^{\frac{2^*(\alpha)-q}{2^*(\alpha)}} \left(\int_\Omega \frac{|u|^{2^*(\alpha)}}{|x|^\alpha}\,dx \right)^{\frac{q}{2^*(\alpha)}} \\
&= \|f\|_{L^p(\Omega,|x|^{-\alpha})} \Lambda(\alpha)^{-\frac{q}{2}} \|u\|_\gamma^q,
\end{aligned} \tag{2.13}
$$

where $p := \frac{2^*(\alpha)}{2^*(\alpha)-q}$. Then

$$
\begin{aligned}
Q_{\lambda,\mu}(u,v) &\le \Lambda(\alpha)^{-\frac{q}{2}} \left(\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})} \|u\|_\gamma^q + \mu \|g\|_{L^p(\Omega,|x|^{-\alpha})} \|v\|_\gamma^q \right) \\
&= \left(\left[\frac{2}{q} \left(\frac{1}{2} - \frac{1}{2^*(\beta)} \right) \left(\frac{1}{q} - \frac{1}{2^*(\beta)} \right)^{-1} \right]^{\frac{q}{2}} \|u\|_\gamma^q \right) \\
&\quad \times \left(\left[\frac{2}{q} \left(\frac{1}{2} - \frac{1}{2^*(\beta)} \right) \left(\frac{1}{q} - \frac{1}{2^*(\beta)} \right)^{-1} \right]^{-\frac{q}{2}} \Lambda(\alpha)^{-\frac{q}{2}} \lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})} \right) \\
&\quad + \left(\left[\frac{2}{q} \left(\frac{1}{2} - \frac{1}{2^*(\beta)} \right) \left(\frac{1}{q} - \frac{1}{2^*(\beta)} \right)^{-1} \right]^{\frac{q}{2}} \|v\|_\gamma^q \right) \\
&\quad \times \left(\left[\frac{2}{q} \left(\frac{1}{2} - \frac{1}{2^*(\beta)} \right) \left(\frac{1}{q} - \frac{1}{2^*(\beta)} \right)^{-1} \right]^{-\frac{q}{2}} \Lambda(\alpha)^{-\frac{q}{2}} \mu \|g\|_{L^p(\Omega,|x|^{-\alpha})} \right) \\
&\le \left(\frac{1}{2} - \frac{1}{2^*(\beta)} \right) \left(\frac{1}{q} - \frac{1}{2^*(\beta)} \right)^{-1} \|(u,v)\|_W^2 \\
&\quad + C_* \left((\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} \right),
\end{aligned}
$$

where

$$
C_* = \frac{2-q}{2} \left(\frac{2^*(\beta)-q}{2^*(\beta)-2} \right)^{\frac{q}{2-q}} \Lambda(\alpha)^{-\frac{q}{2-q}} > 0.
$$

3. Nehari manifold

In this section, we will give some properties about the Nehari manifold and fibering maps. Since $J_{\lambda,\mu}$ is not bounded from below on W, we need to study $J_{\lambda,\mu}$ on the Nehari manifold

$$
\mathcal{N}_{\lambda,\mu} = \{(u,v) \in W \setminus \{(0,0)\} : \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\}.
$$

Note that $\mathcal{N}_{\lambda,\mu}$ contains every nonzero solution of (1.1), and $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\|(u,v)\|_W^2 - \int_\Omega \frac{F(u,v)}{|x|^\beta}\,dx - Q_{\lambda,\mu}(u,v) = 0. \tag{3.1}$$

Lemma 3.1. *$J_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Letting $(u,v) \in \mathcal{N}_{\lambda,\mu}$, by (2.13) and the Sobolev inequality, Hölder inequality, we find

$$Q_{\lambda,\mu}(u,v) \le C_1 \|(u,v)\|_W^q, \tag{3.2}$$

where

$$C_1 = \left[\left(\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})}\right)^{\frac{2}{2-q}} + \left(\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})}\right)^{\frac{2}{2-q}} \right]^{\frac{2-q}{2}} \Lambda(\alpha)^{-\frac{q}{2}} > 0.$$

From (3.1) and (3.2), we get

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \left(\frac{1}{2} - \frac{1}{2^*(\beta)}\right) \|(u,v)\|_W^2 - \left(\frac{1}{q} - \frac{1}{2^*(\beta)}\right) Q_{\lambda,\mu}(u,v) \\ &\ge \frac{2^*(\beta)-2}{2\cdot 2^*(\beta)} \|(u,v)\|_W^2 - \frac{2^*(\beta)-q}{q\cdot 2^*(\beta)} C_1 \|(u,v)\|_W^q. \end{aligned}$$

As $1 \le q < 2$, $J_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. □

Define $\Psi_{\lambda,\mu}(u,v) := \langle J'_{\lambda,\mu}(u,v), (u,v)\rangle$, then for all $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we get

$$\begin{aligned} \langle \Psi'_{\lambda,\mu}(u,v), (u,v)\rangle &= 2\|(u,v)\|_W^2 - qQ_{\lambda,\mu}(u,v) - 2^*(\beta) \int_\Omega \frac{F(u,v)}{|x|^\beta}\,dx \\ &= (2-q)\|(u,v)\|_W^2 - (2^*(\beta)-q) \int_\Omega \frac{F(u,v)}{|x|^\beta}\,dx \qquad (3.3) \\ &= (2^*(\beta)-q) Q_{\lambda,\mu}(u,v) - (2^*(\beta)-2)\|(u,v)\|_W^2. \qquad (3.4) \end{aligned}$$

We split $\mathcal{N}_{\lambda,\mu}$ into three parts

$$\begin{aligned} \mathcal{N}_{\lambda,\mu}^+ &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \langle \Psi'_{\lambda,\mu}(u,v), (u,v)\rangle > 0\}, \\ \mathcal{N}_{\lambda,\mu}^0 &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \langle \Psi'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\}, \\ \mathcal{N}_{\lambda,\mu}^- &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \langle \Psi'_{\lambda,\mu}(u,v), (u,v)\rangle < 0\}. \end{aligned}$$

We now present some important properties of $\mathcal{N}_{\lambda,\mu}$, $\mathcal{N}_{\lambda,\mu}^+$, $\mathcal{N}_{\lambda,\mu}^-$.

Lemma 3.2. *Assume that (u_0,v_0) is a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and $(u_0,v_0) \notin \mathcal{N}_{\lambda,\mu}^0$. Then $J'_{\lambda,\mu}(u_0,v_0) = 0$ in W^{-1}.*

Proof. The proof is similar to that of Theorem 2.3 in [2] and the details are omitted. □

Lemma 3.3. *We have* $\mathcal{N}^0_{\lambda,\mu} = \emptyset$ *for all* $\lambda, \mu > 0$ *satisfying* $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, *where* Λ_0 *is the same as that in* (1.7).

Proof. We argue by contradiction. Assume that there exist $\lambda, \mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$ such that $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. Then by (3.3) and (3.4), for $(u, v) \in \mathcal{N}^0_{\lambda,\mu}$, we have

$$\|(u,v)\|_W^2 = \frac{2^*(\beta) - q}{2 - q} \int_\Omega \frac{F(u,v)}{|x|^\beta}\, dx \tag{3.5}$$

and

$$\|(u,v)\|_W^2 = \frac{2^*(\beta) - q}{2^*(\beta) - 2} Q_{\lambda,\mu}(u,v). \tag{3.6}$$

According to (2.1) and the Minkowski inequality, we obtain that

(3.7)
$$\begin{aligned}
\int_\Omega \frac{F(u,v)}{|x|^\beta}\, dx &\le M_F^{2^*(\beta)/2} \int_\Omega \frac{(|u|^2 + |v|^2)^{2^*(\beta)/2}}{|x|^\beta}\, dx \\
&\le M_F^{2^*(\beta)/2} \left[\left(\int_\Omega \frac{|u|^{2^*(\beta)}}{|x|^\beta}\, dx \right)^{2/2^*(\beta)} + \left(\int_\Omega \frac{|v|^{2^*(\beta)}}{|x|^\beta}\, dx \right)^{2/2^*(\beta)} \right]^{2^*(\beta)/2} \\
&\le M_F^{2^*(\beta)/2} \big(\Lambda(\beta)^{-1}\|u\|_\gamma^2 + \Lambda(\beta)^{-1}\|v\|_\gamma^2 \big)^{2^*(\beta)/2} \\
&= \left(\frac{\Lambda(\beta)}{M_F} \right)^{-2^*(\beta)/2} \big(\|u\|_\gamma^2 + \|v\|_\gamma^2 \big)^{2^*(\beta)/2} \\
&= S_F(\beta)^{-2^*(\beta)/2} \|(u,v)\|_W^{2^*(\beta)},
\end{aligned}$$

which and (3.5) lead that

$$\|(u,v)\|_W \ge \left(\frac{2 - q}{2^*(\beta) - q} S_F(\beta)^{2^*(\beta)/2} \right)^{\frac{1}{2^*(\beta) - 2}}. \tag{3.8}$$

On the other hand, by (3.2) and (3.6), we find that

$$\begin{aligned}
\|(u,v)\|_W &\le \left(\frac{2^*(\beta) - q}{2^*(\beta) - 2} \right)^{\frac{1}{2-q}} \Lambda(\alpha)^{-\frac{q}{2(2-q)}} \\
&\quad \times \left((\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} \right)^{1/2}.
\end{aligned} \tag{3.9}$$

Consequently, (3.8) and (3.9) imply

$$(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} \ge \Lambda_0,$$

which contradicts to $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$. This completes the proof of Lemma 3.3. □

By Lemmas 3.1 and 3.3, for each $\lambda, \mu > 0$ with $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, we can write $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$. Define

$$c_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} J_{\lambda,\mu}(u,v), \quad c^+_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v), \quad c^-_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).$$

Then, we have the following results.

Lemma 3.4. *The following results hold.*

(i) *If* $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, *then* $c_{\lambda,\mu} \le c^+_{\lambda,\mu} < 0$.

(ii) *There exists*

$$\Lambda_1 := \left(\frac{q(2^*(\beta)-2)}{2(2^*(\beta)-q)}\right)^{\frac{2}{2-q}} \left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2}{2^*(\beta)-2}} \Lambda(\alpha)^{\frac{q}{2-q}} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}} = \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0$$

such that for all $\lambda, \mu > 0$ *and* $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_1$, *then* $c^-_{\lambda,\mu} \ge c_0$ *for some* $c_0 > 0$.

Proof. (i) Suppose $(u,v) \in \mathcal{N}^+_{\lambda,\mu}$. By (3.3), we get

$$(2-q)\|(u,v)\|^2_W > (2^*(\beta)-q) \int_\Omega \frac{F(u,v)}{|x|^\beta}\, dx. \tag{3.10}$$

According to (3.1) and (3.10), we get

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|(u,v)\|^2_W - \left(\frac{1}{2^*(\beta)}-\frac{1}{q}\right)\int_\Omega \frac{F(u,v)}{|x|^\beta}\, dx \\ &\le \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{2^*(\beta)}\right)\frac{2-q}{2^*(\beta)-q}\right]\|(u,v)\|^2_W \\ &= -\frac{(2-q)(2^*(\beta)-2)}{2q\cdot 2^*(\beta)}\|(u,v)\|^2_W \\ &< 0. \end{aligned}$$

Then, by the definitions of $c_{\lambda,\mu}$ and $c^+_{\lambda,\mu}$, we can deduce that $c_{\lambda,\mu} \le c^+_{\lambda,\mu} < 0$.

(ii) Suppose $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. From (3.3), it follows that

$$(2-q)\|(u,v)\|^2_W < (2^*(\beta)-q) \int_\Omega \frac{F(u,v)}{|x|^\beta}\, dx. \tag{3.11}$$

This and (3.7) yield

$$\|(u,v)\|_W > \left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{1}{2^*(\beta)-2}} S_F(\beta)^{\frac{2^*(\beta)}{2(2^*(\beta)-2)}}. \tag{3.12}$$

By the proof of Lemma 3.1 and (3.12), we infer that

$$\begin{aligned}J_{\lambda,\mu}(u,v) &\geq \|(u,v)\|_W^q\Bigg[\left(\frac{1}{2}-\frac{1}{2^*(\beta)}\right)\|(u,v)\|_W^{2-q}-\left(\frac{2^*(\beta)-q}{q2^*(\beta)}\right)\\&\qquad\times\left(\left(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})}\right)^{\frac{2}{2-q}}+\left(\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})}\right)^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\Lambda(\alpha)^{-\frac{q}{2}}\Bigg]\\&> \|(u,v)\|_W^q\Bigg[\frac{2^*(\beta)-2}{2\cdot 2^*(\beta)}\left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2-q}{2^*(\beta)-2}}S_F(\beta)^{\frac{2^*(\beta)(2-q)}{2(2^*(\beta)-2)}}-\left(\frac{2^*(\beta)-q}{q2^*(\beta)}\right)\\&\qquad\times\left(\left(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})}\right)^{\frac{2}{2-q}}+\left(\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})}\right)^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\Lambda(\alpha)^{-\frac{q}{2}}\Bigg].\end{aligned}$$

Then, if $(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}+(\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}<\Lambda_1$, we get $J_{\lambda,\mu}(u,v)\geq c_0$ for all $(u,v)\in\mathcal{N}^-_{\lambda,\mu}$, where $c_0=c(q,\alpha,\beta,N)$ is a positive constant. □

For $t>0$, we define the fibering maps $\Phi_{u,v}(t)=J_{\lambda,\mu}(tu,tv)$. Then

$$\Phi'_{u,v}(t)=t\|(u,v)\|_W^2-t^{2^*(\beta)-1}\int_\Omega\frac{F(u,v)}{|x|^\beta}\,dx-t^{q-1}Q_{\lambda,\mu}(u,v).$$

For $(u,v)\in\mathcal{N}_{\lambda,\mu}$, we get $\Phi'_{u,v}(1)=\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle$, which implies that $(u,v)\in\mathcal{N}_{\lambda,\mu}$ if and only if $\Phi'_{u,v}(1)=0$, and more generally $(tu,tv)\in\mathcal{N}_{\lambda,\mu}$ if and only if $\Phi'_{u,v}(t)=0$. That is, the elements in $\mathcal{N}_{\lambda,\mu}$ correspond to stationary points of the fibering maps $\Phi_{u,v}(t)$.

For each $(u,v)\in W$ with $\int_\Omega\frac{F(u,v)}{|x|^\beta}\,dx>0$, set

$$t_{\max}=\left(\frac{(2-q)\|(u,v)\|_W^2}{(2^*(\beta)-q)\int_\Omega\frac{F(u,v)}{|x|^\beta}\,dx}\right)^{\frac{1}{2^*(\beta)-2}}>0.$$

Then the following lemma holds.

Lemma 3.5. *If* $0<(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}+(\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}}<\Lambda_0$, *then for every* $(u,v)\in W\setminus\{(0,0)\}$ *with* $\int_\Omega\frac{F(u,v)}{|x|^\beta}\,dx>0$ *and* $Q_{\lambda,\mu}(u,v)>0$, *there exist unique* t_1 *and* $t_2>0$ *such that* $(t_1u,t_1v)\in\mathcal{N}^+_{\lambda,\mu}$, $(t_2u,t_2v)\in\mathcal{N}^-_{\lambda,\mu}$. *Moreover, we have* $0<t_1<t_{\max}<t_2$ *and* $J_{\lambda,\mu}(t_1u,t_1v)=\inf_{t\in[0,t_{\max}]}J_{\lambda,\mu}(tu,tv)$, $J_{\lambda,\mu}(t_2u,t_2v)=\sup_{t\in[0,\infty)}J_{\lambda,\mu}(tu,tv)$.

Proof. Fix $(u,v)\in W$ with $\int_\Omega\frac{F(u,v)}{|x|^\beta}\,dx>0$. Let

$$\phi_{u,v}(t)=t^{2-q}\|(u,v)\|_W^2-t^{2^*(\beta)-q}\int_\Omega\frac{F(u,v)}{|x|^\beta}\,dx,\quad\forall\,t\geq 0.$$

Clearly, $\lim_{t\to 0^+}\phi_{u,v}(t)=0$, $\lim_{t\to+\infty}\phi_{u,v}(t)=-\infty$ and $\phi_{u,v}$ has a unique critical point at $t_{\min}$. By a direct computation, we get $\phi'_{u,v}(t)>0$ for all $t\in(0,t_{\max})$ and $\phi'_{u,v}(t)<0$ for all $t \in (t_{\max}, +\infty)$. Then $\phi_{u,v}(t)$ is increasing in $(0, t_{\max})$ and decreasing in $(t_{\max}, +\infty)$. Moreover,

$$\begin{aligned}\phi_{u,v}(t_{\max}) &= \frac{\left(\|(u,v)\|_W^2\right)^{\frac{2^*(\beta)-q}{2^*(\beta)-2}}}{\left(\int_\Omega \frac{F(u,v)}{|x|^\beta}\,dx\right)^{\frac{2-q}{2^*(\beta)-2}}} \left(\left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2-q}{2^*(\beta)-2}} - \left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2^*(\beta)-q}{2^*(\beta)-2}}\right) \\ &\geq \|(u,v)\|_W^q \left(\left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2-q}{2^*(\beta)-2}} - \left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2^*(\beta)-q}{2^*(\beta)-2}}\right) S_F(\beta)^{\frac{2^*(\beta)(2-q)}{2(2^*(\beta)-2)}} \\ &= \|(u,v)\|_W^q \left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2-q}{2^*(\beta)-2}} \left(\frac{2^*(\beta)-2}{2^*(\beta)-q}\right) S_F(\beta)^{\frac{2^*(\beta)(2-q)}{2(2^*(\beta)-2)}}.\end{aligned}$$

On the other hand, for all $\lambda, \mu > 0$ with $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$ and from (3.2), we have

$$\begin{aligned}\phi_{u,v}(0) &= 0 < Q_{\lambda,\mu}(u,v) \\ &\leq \left(\left(\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})}\right)^{\frac{2}{2-q}} + \left(\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})}\right)^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \Lambda(\alpha)^{-\frac{q}{2}} \|(u,v)\|_W^q \\ &< \|(u,v)\|_W^q \left(\frac{2-q}{2^*(\beta)-q}\right)^{\frac{2-q}{2^*(\beta)-2}} \left(\frac{2^*(\beta)-2}{2^*(\beta)-q}\right) S_F(\beta)^{\frac{2^*(\beta)(2-q)}{2(2^*(\beta)-2)}} \\ &\leq \phi_{u,v}(t_{\max}).\end{aligned}$$

Then, there exist unique t_1 and t_2 with $0 < t_1 < t_{\max} < t_2$ such that

$$\phi_{u,v}(t_1) = \phi_{u,v}(t_2) = Q_{\lambda,\mu}(u,v), \quad \phi'_{u,v}(t_1) > 0 > \phi'_{u,v}(t_2). \tag{3.13}$$

From $\Phi'_{u,v}(t) = t^{q-1}[\phi_{u,v}(t) - Q_{\lambda,\mu}(u,v)]$ and (3.13), we get $\Phi'_{u,v}(t_1) = \Phi'_{u,v}(t_2) = 0$ and $\Phi''_{u,v}(t_1) > 0 > \Phi''_{u,v}(t_2)$. So, the fibering map $\Phi_{u,v}(t)$ has a local minimum at t_1 and a local maximum at t_2 such that $(t_1u, t_1v) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t_2u, t_2v) \in \mathcal{N}^-_{\lambda,\mu}$, and $\Phi_{u,v}(t_1) \leq \Phi_{u,v}(t) \leq \Phi_{u,v}(t_2)$ for all $t \in [t_1, t_2]$, $\Phi_{u,v}(t_1) \leq \Phi_{u,v}(t)$ for all $t \in [0, t_1]$. Thus, $J_{\lambda,\mu}(t_1u, t_1v) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda,\mu}(tu, tv)$, $J_{\lambda,\mu}(t_2u, t_2v) = \sup_{t \geq 0} J_{\lambda,\mu}(tu, tv)$. This completes the proof of Lemma 3.5. □

For each $(u,v) \in W$ with $Q_{\lambda,\mu}(u,v) > 0$, we write

$$\bar{t}_{\max} = \left(\frac{(2^*(\beta)-q)Q_{\lambda,\mu}(u,v)}{(2^*(\beta)-2)\|(u,v)\|_W^2}\right)^{\frac{1}{2-q}} > 0.$$

Then we have the following lemma.

Lemma 3.6. *If $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, then for every $(u,v) \in W \setminus \{(0,0)\}$ with $Q_{\lambda,\mu}(u,v) > 0$, there exist unique t_3 and $t_4 > 0$ such that $(t_3u, t_3v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t_4u, t_4v) \in \mathcal{N}^-_{\lambda,\mu}$. Moreover, we have $0 < t_3 < \bar{t}_{\max} < t_4$ and $J_{\lambda,\mu}(t_3u, t_3v) = \inf_{t \in [0, \bar{t}_{\max}]} J_{\lambda,\mu}(tu, tv)$, $J_{\lambda,\mu}(t_4u, t_4v) = \sup_{t \in [0,\infty)} J_{\lambda,\mu}(tu, tv)$.*

Proof. The proof is almost the same as that of Lemma 2.7 in [1] and is omitted here. □

4. Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we need the following lemma.

Proposition 4.1. *Suppose that* (H_1)–(H_2) *and* (F_0)–(F_2)*. The following facts hold.*

(i) *If* $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$*, then there exists a* $(PS)_{c_{\lambda,\mu}}$*-sequence* $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ *for* $J_{\lambda,\mu}$*.*

(ii) *If* $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_1 = \left(\frac{q}{2}\right)^{\frac{2}{2-q}}\Lambda_0$*, then there exists a* $(PS)_{c^-_{\lambda,\mu}}$*-sequence* $\{(u_n,v_v)\} \subset \mathcal{N}^-_{\lambda,\mu}$ *for* $J_{\lambda,\mu}$*.*

Proof. The proof is similar to that of Proposition 3.3 in [6] and the details are omitted. □

Now, we establish the existence of a local minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$.

Theorem 4.2. *Suppose that* (H_1)–(H_2) *and* (F_0)–(F_2)*. If* $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$*, then* $J_{\lambda,\mu}$ *has a minimizer* $(u^{(1)},v^{(1)}) \in \mathcal{N}^+_{\lambda,\mu}$ *such that* $(u^{(1)},v^{(1)})$ *is a nonnegative solution of* (1.1) *with* $u^{(1)} \not\equiv 0$*,* $v^{(1)} \not\equiv 0$ *and* $J_{\lambda,\mu}(u^{(1)},v^{(1)}) = c_{\lambda,\mu} = c^+_{\lambda,\mu} < 0$*.*

Proof. By Proposition 4.1(i), there exists a minimizing sequence $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(u_n,v_n) = c_{\lambda,\mu} + o_n(1) \quad \text{and} \quad J'_{\lambda,\mu}(u_n,v_n) = o_n(1). \tag{4.1}$$

Since $J_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$, we get that $\{(u_n,v_n)\}$ is bounded in W. Passing to a subsequence, still denoted by $\{(u_n,v_n)\}$, we can assume that there exists $(u^{(1)},v^{(1)}) \in W$ such that $(u_n,v_n) \rightharpoonup (u^{(1)},v^{(1)})$ weakly in W and

$$\begin{cases} u_n \rightharpoonup u^{(1)}, \quad v_n \rightharpoonup v^{(1)} & \text{weakly in } L^{2^*(\beta)}(\Omega,|x|^{-\beta}\,dx),\\ u_n \to u^{(1)}, \quad v_n \to v^{(1)} & \text{strongly in } L^q(\Omega,|x|^{-\alpha}\,dx),\ \forall\, 1 \le q < 2^*(\alpha),\\ u_n(x) \to u^{(1)}(x), \quad v_n(x) \to v^{(1)}(x) & \text{a.e. in } \Omega. \end{cases} \tag{4.2}$$

This implies that

$$Q_{\lambda,\mu}(u_n,v_n) = Q_{\lambda,\mu}(u^{(1)},v^{(1)}) + o_n(1). \tag{4.3}$$

First, we claim that $(u^{(1)},v^{(1)})$ is a nontrivial weak solution of (1.1). From (4.1), (4.2) and (4.3), it is easy to verify that $(u^{(1)},v^{(1)})$ is a weak solution of (1.1). Moreover, the fact $(u_n,v_n) \in \mathcal{N}_{\lambda,\mu}$ implies that

$$Q_{\lambda,\mu}(u_n,v_n) = \frac{q(2^*(\beta)-2)}{2(2^*(\beta)-q)}\|(u_n,v_n)\|_W^2 - \frac{q2^*(\beta)}{2^*(\beta)-q}J_{\lambda,\mu}(u_n,v_n). \tag{4.4}$$

Letting $n \to \infty$ in (4.4), by (4.3) and the fact $c_{\lambda,\mu} < 0$, we obtain

$$Q_{\lambda,\mu}(u^{(1)}, v^{(1)}) \geq -\frac{q2^*(\beta)}{2^*(\beta) - q} c_{\lambda,\mu} > 0.$$

Thus, $(u^{(1)}, v^{(1)}) \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial weak solution of (1.1).

Next, we prove that $(u_n, v_n) \to (u^{(1)}, v^{(1)})$ strongly in W and $J_{\lambda,\mu}(u^{(1)}, v^{(1)}) = c_{\lambda,\mu}$. From the fact $(u^{(1)}, v^{(1)}) \in \mathcal{N}_{\lambda,\mu}$, and the Fatou's lemma, it follows that

$$\begin{aligned}
c_{\lambda,\mu} &\leq J_{\lambda,\mu}(u^{(1)}, v^{(1)}) \\
&= \frac{2^*(\beta) - 2}{2 \cdot 2^*(\beta)} \|(u^{(1)}, v^{(1)})\|_W^2 - \frac{2^*(\beta) - q}{q2^*(\beta)} Q_{\lambda,\mu}(u^{(1)}, v^{(1)}) \\
&\leq \liminf_{n\to\infty} \left[\frac{2^*(\beta) - 2}{2 \cdot 2^*(\beta)} \|(u_n, v_n)\|_W^2 - \frac{2^*(\beta) - q}{q2^*(\beta)} Q_{\lambda,\mu}(u_n, v_n)\right] \\
&= \liminf_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) \\
&= c_{\lambda,\mu},
\end{aligned}$$

which implies that $c_{\lambda,\mu} = J_{\lambda,\mu}(u^{(1)}, v^{(1)})$ and $\lim_{n\to\infty} \|(u_n, v_n)\|_W^2 = \|(u^{(1)}, v^{(1)})\|_W^2$. Standard argument shows that $(u_n, v_n) \to (u^{(1)}, v^{(1)})$ strongly in W.

Now, we claim that $(u^{(1)}, v^{(1)}) \in \mathcal{N}^+_{\lambda,\mu}$. Otherwise, if $(u^{(1)}, v^{(1)}) \in \mathcal{N}^-_{\lambda,\mu}$, then by Lemma 3.5, there exist unique $t_1^+, t_1^- > 0$ such that $(t_1^+ u^{(1)}, t_1^+ v^{(1)}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t_1^- u^{(1)}, t_1^- v^{(1)}) \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t_1^+ < t_1^- = 1$. Since

$$\frac{d}{dt} J_{\lambda,\mu}(t_1^+ u^{(1)}, t_1^+ v^{(1)}) = 0, \quad \frac{d^2}{dt^2} J_{\lambda,\mu}(t_1^+ u^{(1)}, t_1^+ v^{(1)}) > 0,$$

there exists $t_1^* \in (t_1^+, t_1^-)$ such that $J_{\lambda,\mu}(t_1^+ u^{(1)}, t_1^+ v^{(1)}) < J_{\lambda,\mu}(t_1^* u^{(1)}, t_1^* v^{(1)})$. By Lemma 3.5, we have $J_{\lambda,\mu}(t_1^+ u^{(1)}, t_1^+ v^{(1)}) < J_{\lambda,\mu}(t_1^* u^{(1)}, t_1^* v^{(1)}) \leq J_{\lambda,\mu}(t_1^- u^{(1)}, t_1^- v^{(1)}) = J_{\lambda,\mu}(u^{(1)}, v^{(1)})$, which contradicts $J_{\lambda,\mu}(u^{(1)}, v^{(1)}) = c_{\lambda,\mu}$. Since $J_{\lambda,\mu}(u^{(1)}, v^{(1)}) = J_{\lambda,\mu}(|u^{(1)}|, |v^{(1)}|)$ with $(|u^{(1)}|, |v^{(1)}|) \in \mathcal{N}^+_{\lambda,\mu}$, by Lemma 3.2 we may assume that $(u^{(1)}, v^{(1)})$ is a nontrivial nonnegative solution of (1.1).

In particular $u^{(1)} \not\equiv 0$, $v^{(1)} \not\equiv 0$. Indeed, without loss of generality, we may assume that $v^{(1)} \equiv 0$. Then as $u^{(1)}$ is a nontrivial nonnegative solution of

$$\begin{cases}
\Delta^2 u - \gamma \frac{u}{|x|^4} = \lambda f(x) \frac{|u|^{q-2}u}{|x|^\alpha} & \text{in } \Omega, \\
u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega,
\end{cases}$$

and $\|(u^{(1)}, 0)\|_W^2 = Q_{\lambda,\mu}(u^{(1)}, 0) > 0$. Moreover, we may choose $v_0 \in H_0^2(\Omega) \setminus \{0\}$ and $v_0 \geq 0$ such that $\|(0, v_0)\|_W^2 = Q_{\lambda,\mu}(0, v_0) > 0$. Now

$$Q_{\lambda,\mu}(u^{(1)}, v_0) = Q_{\lambda,\mu}(u^{(1)}, 0) + Q_{\lambda,\mu}(0, v_0) > 0$$

and so by Lemma 3.6 there is unique $0 < t_3 < \bar{t}_{\max}$ such that $(t_3 u^{(1)}, t_3 v_0) \in \mathcal{N}^+_{\lambda,\mu}$. Moreover,

$$\bar{t}_{\max} = \left(\frac{(2^*(\beta) - q) Q_{\lambda,\mu}(u^{(1)}, v_0)}{(2^*(\beta) - 2) \|(u^{(1)}, v_0)\|_W^2} \right)^{\frac{1}{2-q}} = \left(\frac{2^*(\beta) - q}{2^*(\beta) - 2} \right)^{\frac{1}{2-q}} > 1$$

and

$$J_{\lambda,\mu}(t_3 u^{(1)}, t_3 v_0) = \inf_{0 \le t \le \bar{t}_{\max}} J_{\lambda,\mu}(t u^{(1)}, t v_0).$$

This implies that

$$c^+_{\lambda,\mu} \le J_{\lambda,\mu}(t_3 u^{(1)}, t_3 v_0) \le J_{\lambda,\mu}(u^{(1)}, v_0) < J_{\lambda,\mu}(u^{(1)}, 0) = c^+_{\lambda,\mu}$$

which is a contradiction. Therefore, $v^{(1)} \not\equiv 0$. Similarly, $u^{(1)} \not\equiv 0$. This completes the proof. □

Proof of Theorem 1.2(i). By Theorem 4.2, we obtain that for all $\lambda, \mu > 0$ with $0 < (\lambda \|f\|_{L^p(\Omega, |x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega, |x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$, problem (1.1) has a nonnegative solution $(u^{(1)}, v^{(1)}) \in \mathcal{N}^+_{\lambda,\mu}$ and $u^{(1)} \not\equiv 0$, $v^{(1)} \not\equiv 0$. □

Remark 4.3. From Lemma 3.4(i) and (3.2), for this nonnegative solution $(u^{(1)}, v^{(1)})$, we have

$$\begin{aligned}
0 > c_{\lambda,\mu} &= J_{\lambda,\mu}(u^{(1)}, v^{(1)}) \\
&= \left(\frac{1}{2} - \frac{1}{2^*(\beta)} \right) \|(u^{(1)}, v^{(1)})\|_W^2 - \left(\frac{1}{q} - \frac{1}{2^*(\beta)} \right) Q_{\lambda,\mu}(u^{(1)}, v^{(1)}) \\
&\ge -\frac{2^*(\beta) - q}{q \cdot 2^*(\beta)} Q_{\lambda,\mu}(u^{(1)}, v^{(1)}) \\
&\ge -\frac{2^*(\beta) - q}{q \cdot 2^*(\beta)} \Lambda(\alpha)^{-\frac{q}{2}} \|(u^{(1)}, v^{(1)})\|_W^q \\
&\quad \times \left(\left(\lambda \|f\|_{L^p(\Omega, |x|^{-\alpha})}\right)^{\frac{2}{2-q}} + \left(\mu \|g\|_{L^p(\Omega, |x|^{-\alpha})}\right)^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}}.
\end{aligned}$$

This implies that $J_{\lambda,\mu}(u^{(1)}, v^{(1)}) \to 0$ as $\lambda \to 0^+$ and $\mu \to 0^+$.

Next, we establish the existence of a local minimum for $J_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$.

Lemma 4.4. *Under the assumptions of Theorem* 1.2*. If* $\{(u_n, v_n)\} \subset W$ *is a* $(PS)_c$ *sequence for* $J_{\lambda,\mu}$ *with* $c \in \left(0, \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}\right)$*. Then there exists a subsequence of* $\{(u_n\, v_n)\}$ *converging weakly to a nontrivial solution of* (1.1)*.*

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$ sequence for $J_{\lambda,\mu}$ with $c \in (-\infty, c_\infty)$. Similar to the proof of [7, Lemma 2.3], it is easy to see that $\{(u_n, v_n)\}$ is bounded in W. Then there exist a subsequence still denoted by $\{(u_n, v_n)\}$ and $(u, v) \in W$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in W, and

$$(4.5) \quad \begin{cases} u_n \rightharpoonup u, \quad v_n \rightharpoonup v & \text{weakly in } L^{2^*(\beta)}(\Omega, |x|^{-\beta}\, dx), \\ u_n \to u, \quad v_n \to v & \text{strongly in } L^q(\Omega, |x|^{-\alpha}\, dx),\ \forall\, 1 \le q < 2^*(\alpha), \\ u_n(x) \to u(x), \quad v_n(x) \to v(x) & \text{a.e. in } \Omega. \end{cases}$$

Hence, from (4.5), it is easy to see that $J'_{\lambda,\mu}(u, v) = 0$ and

$$(4.6) \qquad Q_{\lambda,\mu}(u_n, v_n) = Q_{\lambda,\mu}(u, v) + o_n(1) \quad \text{as } n \to \infty.$$

Next, we verify that $(u, v) \neq (0, 0)$. Arguing by contradiction we assume that $(u, v) = (0, 0)$. Since $J'_{\lambda,\mu}(u_n, v_n) = o_n(1)$ and $\{(u_n, v_n)\}$ is bounded in W, then by (4.6), we conclude that

$$(4.7) \qquad \begin{aligned} c &= J_{\lambda,\mu}(u_n, v_n) + o_n(1) \\ &= \frac{1}{2}\|(u_n, v_n)\|_W^2 - \frac{1}{2^*(\beta)} \int_\Omega \frac{F(u_n, v_n)}{|x|^\beta}\, dx - \frac{1}{q} Q_{\lambda,\mu}(u, v) + o_n(1) \\ &= \frac{1}{2}\|(u_n, v_n)\|_W^2 - \frac{1}{2^*(\beta)} \int_\Omega \frac{F(u_n, v_n)}{|x|^\beta}\, dx + o_n(1), \end{aligned}$$

where

$$(4.8) \qquad o_n(1) = \langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) \rangle = \|(u_n, v_n)\|_W^2 - \int_\Omega \frac{F(u_n, v_n)}{|x|^\beta}\, dx.$$

Thus, we may assume that

$$(4.9) \qquad \int_\Omega \frac{F(u_n, v_n)}{|x|^\beta}\, dx \to l, \quad \|(u_n, v_n)\|_W^2 \to l \ge 0 \quad \text{as } n \to \infty.$$

If $l = 0$, then from (4.7) we get $c = 0$, which contradicts $c > 0$. Thus we conclude that $l > 0$, then from (4.8) we have

$$S_F(\beta) l^{2/2^*(\beta)} = S_F(\beta) \left(\lim_{n\to\infty} \int_\Omega \frac{F(u_n, v_n)}{|x|^\beta}\, dx \right)^{2/2^*(\beta)} \le \lim_{n\to\infty} \|(u_n, v_n)\|_W^2 = l$$

which implies that $l \ge S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}$. Hence, from (4.7) and (4.9), we conclude

$$c = \left(\frac{1}{2} - \frac{1}{2^*(\beta)} \right) l + o_n(1) \ge \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}},$$

which contradicts $c < \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}$. The proof is completed. □

Lemma 4.5. *Under the assumptions of Theorem* 1.2, *there exist a nonnegative function* $(u_0, v_0) \in W \setminus \{(0,0)\}$ *such that for all* $\lambda, \mu > 0$,

$$\sup_{t\geq 0} J_{\lambda,\mu}(tu_0, tv_0) < \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}. \tag{4.10}$$

In particular, $0 < c^-_{\lambda,\mu} < \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}$ *for all* $\lambda, \mu > 0$ *with* $0 < (\lambda\|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu\|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0$.

Proof. Now, we first consider the functional $I \colon W \to \mathbb{R}$ defined by

$$I(u,v) = \frac{1}{2}\|(u,v)\|_W^2 - \frac{1}{2^*(\beta)} \int_\Omega \frac{F(u,v)}{|x|^\beta}\, dx,$$

and define a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $0 \leq \varphi(x) \leq 1$ in $B_R(0)$, $\varphi(x) = 1$ in $B_{R/2}(0)$ and $\varphi(x) = 0$ in $B_R(0)^c$, where $0 < R < r_0$ and r_0 given in condition (H_1).

From Lemma 2.1, there exists $(e_1, e_2) \in \mathbb{R}^2$ such that $e_1^2 + e_2^2 = 1$ and $M_F = F(e_1, e_2)^{2/2^*(\beta)}$. Set $u_0 = e_1 u_\varepsilon$, $v_0 = e_2 u_\varepsilon$, where $u_\varepsilon(x) = \varphi(x) U_\varepsilon(x)$, $\varepsilon > 0$, given by (2.9). Then, by $S_F(\beta) = M_F^{-1}\Lambda(\beta)$, (2.10), (2.11) and the fact

$$\max_{t\geq 0}\left(\frac{t^2}{2}B_1 - \frac{t^{2^*(\beta)}}{2^*(\beta)}B_2\right) = \frac{2^*(\beta)-2}{2\cdot 2^*(\beta)}\left(B_1 B_2^{-2/2^*(\beta)}\right)^{\frac{2^*(\beta)}{2^*(\beta)-2}}, \quad B_1, B_2 > 0,$$

we conclude that

$$\begin{aligned}
&\sup_{t\geq 0} I(tu_0, tv_0)\\
&\leq \frac{2^*(\beta)-2}{2\cdot 2^*(\beta)}\left(\frac{(e_1^2+e_2^2)\|u_\varepsilon\|_\gamma^2}{\left(\int_\Omega \frac{F(e_2u_\varepsilon, e_2u_\varepsilon)}{|x|^\beta}\, dx\right)^{2/2^*(\beta)}}\right)^{\frac{2^*(\beta)}{2^*(\beta)-2}}\\
&= \frac{2^*(\beta)-2}{2\cdot 2^*(\beta)}\left(\frac{(e_1^2+e_2^2)\|u_\varepsilon\|_\gamma^2}{\left(M_F^{2^*(\beta)/2}\int_\Omega \frac{|u_\varepsilon|^{2^*(\beta)}}{|x|^\beta}\, dx\right)^{2/2^*(\beta)}}\right)^{\frac{2^*(\beta)}{2^*(\beta)-2}} \qquad (4.11)\\
&\leq \frac{2^*(\beta)-2}{2\cdot 2^*(\beta)}\left(\frac{1}{M_F}\right)^{\frac{2^*(\beta)}{2^*(\beta)-2}}\left[\frac{\Lambda(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}} + O(\varepsilon^{2(b(\gamma)-\delta)})}{\left(\Lambda(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}} + O(\varepsilon^{2^*(\beta)(b(\gamma)-\delta)})\right)^{2/2^*(\beta)}}\right]^{\frac{2^*(\beta)}{2^*(\beta)-2}}\\
&\leq \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}} + O(\varepsilon^{2(b(\gamma)-\delta)}).
\end{aligned}$$

Using the definitions of $J_{\lambda,\mu}(u,v)$, f, g, u_0 and v_0, we get

$$J_{\lambda,\mu}(tu_0, tv_0) \leq \frac{t^2}{2}\|(u_0,v_0)\|_W^2, \quad \forall\, t \geq 0,\ \lambda, \mu > 0.$$

Combining this with (2.10), letting $\varepsilon \in (0,1)$, then there exists $t_0 \in (0,1)$ independent of ε such that

$$\sup_{t\in[0,t_0]} J_{\lambda,\mu}(tu_0, tv_0) < \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}} \quad \text{for all } \lambda, \mu > 0 \text{ and } \varepsilon \in (0,1). \tag{4.12}$$

Next, we prove that $\sup_{t\in[t_0,\infty)} J_{\lambda,\mu}(tu_0, tv_0) < \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}$. Since $f(x), g(x) \geq a_0$ for all $x \in B_{r_0}(0) \subset \Omega$, we have

$$\begin{aligned} Q_{\lambda,\mu}(u_0, v_0) &= \lambda e_1^q \int_\Omega \frac{f(x)|u_\varepsilon|^q}{|x|^\alpha}\,dx + \mu e_2^q \int_\Omega \frac{g(x)|u_\varepsilon|^q}{|x|^\alpha}\,dx \\ &\geq a_0 M(\lambda+\mu) \int_{B_{R/2}(0)} \frac{|u_\varepsilon|^q}{|x|^\alpha}\,dx, \end{aligned} \tag{4.13}$$

where $M = \min\{e_1^q, e_2^q\}$. Combining (4.11), (4.13) and (2.12), for all $t \geq t_0$, we get

$$\begin{aligned} \sup_{t\geq t_0} J_{\lambda,\mu}(tu_0, tv_0) &= \sup_{t\geq t_0} \left(I(tu_0, tv_0) - \frac{t^q}{q} Q_{\lambda,\mu}(u_0, v_0) \right) \\ &\leq \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}} + O(\varepsilon^{2(b(\gamma)-\delta)}) \\ &\quad - \frac{t_0^q}{q} a_0 M(\lambda+\mu) \int_{B_{R/2}(0)} \frac{|u_\varepsilon|^q}{|x|^\alpha}\,dx. \end{aligned} \tag{4.14}$$

Now, we need to distinguish two cases:

Case (i): $1 \leq q < \frac{N-\alpha}{b(\gamma)}$ and $q < 2$. From (2.8) and (2.12) we have that as $\varepsilon \to 0$,

$$\int_{B_{R/2}(0)} \frac{|u_\varepsilon|^q}{|x|^\alpha} = O_1(\varepsilon^{q(b(\mu)-\delta)}) > O(\varepsilon^{2(b(\mu)-\delta)}).$$

Combining this with (4.12) and (4.14), for any $\lambda, \mu > 0$, we can choose $\varepsilon_{\lambda,\mu}$ small enough such that $(u_0, v_0) = (e_1 u_{\varepsilon_{\lambda,\mu}}, e_2 u_{\varepsilon_{\lambda,\mu}})$ and

$$\sup_{t\geq 0} J_{\lambda,\mu}(tu_0, tv_0) < \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}.$$

Case (ii): $\frac{N-\alpha}{b(\gamma)} \leq q < 2$. By (2.12) we have that

$$\int_{B_{R/2}(0)} \frac{|u_\varepsilon|^q}{|x|^\alpha} = \begin{cases} O_1(\varepsilon^{N-\alpha-q\delta}) & \text{if } q > \frac{N-\alpha}{b(\gamma)}, \\ O_1(\varepsilon^{N-\alpha-q\delta}|\ln\varepsilon|) & \text{if } q = \frac{N-\alpha}{b(\gamma)}. \end{cases}$$

Moreover, it follows from $b(\gamma) > \delta$ and $q \geq \frac{N-\alpha}{b(\gamma)}$ that

$$2(b(\gamma)-\delta) > q(b(\gamma)-\delta) \geq N-\alpha-q\delta.$$

Combining this with (4.12) and (4.14), for any $\lambda, \mu > 0$, we can choose $\varepsilon_{\lambda,\mu}$ small enough such that $(u_0, v_0) = (e_1 u_{\varepsilon_{\lambda,\mu}}, e_2 u_{\varepsilon_{\lambda,\mu}})$ and

$$\sup_{t\geq 0} J_{\lambda,\mu}(tu_0, tv_0) < \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}.$$

From Cases (i) and (ii), (4.10) holds by taking $(u_0, v_0) = (e_1 u_{\varepsilon_{\lambda,\mu}}, e_2 u_{\varepsilon_{\lambda,\mu}})$.

It is easy to see that

$$\int_\Omega \frac{F(u_0, v_0)}{|x|^\beta}\,dx > 0 \quad \text{and} \quad Q_{\lambda,\mu}(u_0, v_0) > 0.$$

Combining this with Lemma 3.5, from the definition of $c^-_{\lambda,\mu}$ and (4.10), for all $\lambda, \mu > 0$ with $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0$, we obtain that there exists $t^- > 0$ such that $(t^- u_0, t^- v_0) \in \mathcal{N}^-_{\lambda,\mu}$ and

$$0 < c^-_{\lambda,\mu} \leq J_{\lambda,\mu}(t^- u_0, t^- v_0) \leq \sup_{t\geq 0} J_{\lambda,\mu}(tu_0, tv_0) < \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}.$$

The proof is thus complete. □

Now, we establish the existence of a local minimum of $J_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$.

Theorem 4.6. *Under the assumptions of Theorem* 1.2*, the problem* (1.1) *has a nonnegative solution* $(u^{(2)}, v^{(2)}) \in \mathcal{N}^-_{\lambda,\mu}$ *with* $u^{(2)} \not\equiv 0$, $v^{(2)} \not\equiv 0$ *and* $J_{\lambda,\mu}(u^{(2)}, v^{(2)}) = c^-_{\lambda,\mu}$ *for all* $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0$.

Proof. If $\lambda, \mu > 0$ with $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0 < \Lambda_0$, then by Proposition 4.1(ii), Lemmas 3.4(ii) and 4.5, there exists a $(PS)_{c^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ for $J_{\lambda,\mu}$ with $c^-_{\lambda,\mu} \in \left(0, \frac{4-\beta}{2(N-\beta)} S_F(\beta)^{\frac{2^*(\beta)}{2^*(\beta)-2}}\right)$. Since $J_{\lambda,\mu}$ is coercive on $\mathcal{N}^-_{\lambda,\mu}$, we get that $\{(u_n, v_n)\}$ is bounded in W. Therefore, there exist a subsequence still denoted by $\{(u_n, v_n)\}$ and $(u^{(2)}, v^{(2)}) \in W \backslash \{(0,0)\}$ such that $(u_n, v_n) \rightharpoonup (u^{(2)}, v^{(2)})$ weakly in W. Arguing as in the proof of Theorem 4.2, we obtain $(u_n, v_n) \to (u^{(2)}, v^{(2)})$ strongly in W and $(u^{(2)}, v^{(2)})$ is a nonnegative solution of (1.1) for all $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0$.

Finally, we prove that $(u^{(2)}, v^{(2)}) \in \mathcal{N}^-_{\lambda,\mu}$. Arguing by contradiction, we assume $(u^{(2)}, v^{(2)}) \in \mathcal{N}^+_{\lambda,\mu}$. Since $\mathcal{N}^-_{\lambda,\mu}$ is closed in W, we have $\|(u^{(2)}, v^{(2)})\|_W < \liminf_{n\to\infty} \|(u_n, v_n)\|_W$. Moreover, by Lemma 3.5, there exists a unique t_2^- such that $(t_2^- u^{(2)}, t_2^- v^{(2)}) \in \mathcal{N}^-_{\lambda,\mu}$. This and $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}$ deduce that

$$c^-_{\lambda,\mu} \leq J_{\lambda,\mu}(t_2^- u^{(2)}, t_2^- v^{(2)}) < \lim_{n\to\infty} J_{\lambda,\mu}(t_2^- u_n, t_2^- v_n) \leq \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = c^-_{\lambda,\mu}.$$

This is a contradiction. So, $(u^{(2)}, v^{(2)}) \in \mathcal{N}^-_{\lambda,\mu}$. From (3.11), we have

$$\int_\Omega \frac{F(u^{(2)}, v^{(2)})}{|x|^\beta}\, dx > \frac{2-q}{2^*(\beta)-q} \|(u^{(2)}, v^{(2)})\|_W^2 > 0.$$

This implies that $u^{(2)} \not\equiv 0$, $v^{(2)} \not\equiv 0$ and the proof of Theorem 4.6 is completed. □

Proof of Theorem 1.2(ii). By Theorem 4.2, the system (1.1) has a nonnegative solution $(u^{(1)}, v^{(1)}) \in \mathcal{N}^+_{\lambda,\mu}$ with $u^{(1)} \not\equiv 0$, $v^{(1)} \not\equiv 0$ for all $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \Lambda_0$. On the other hand, from Theorem 4.6, we can get the second nonnegative solution $(u^{(2)}, v^{(2)}) \in \mathcal{N}^-_{\lambda,\mu}$ with $u^{(2)} \not\equiv 0$, $v^{(2)} \not\equiv 0$ for all $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0$. Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$ and $\left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0 < \Lambda_0$, we get that $(u^{(1)}, v^{(1)})$ and $(u^{(2)}, v^{(2)})$ are distinct nonnegative solutions of (1.1) for all $0 < (\lambda \|f\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} + (\mu \|g\|_{L^p(\Omega,|x|^{-\alpha})})^{\frac{2}{2-q}} < \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Lambda_0$. This completes the proof of Theorem 1.2(ii). □

Acknowledgments

This research is supported by Fundamental Research Program of Shanxi Province under grant number 202103021224198.

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