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Summability in Anisotropic Musielak–Orlicz Hardy Spaces

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Abstract. Let $\varphi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be an anisotropic growth function and A a general expansive matrix on $\mathbb{R}^n$. Let $H^{\varphi}_A(\mathbb{R}^n)$ be the anisotropic Musielak–Orlicz Hardy space associated with A. In this paper, a general summability method, the so-called θ-summability is considered for multi-dimensional Fourier transforms in $H^{\varphi}_A(\mathbb{R}^n)$. Precisely, the author establishes the boundedness of maximal operators, induced by the so-called θ-means, from $H^{\varphi}_A(\mathbb{R}^n)$ to the Musielak–Orlicz space $L^{\varphi}(\mathbb{R}^n)$. As applications, some norm and almost everywhere convergence results of the θ-means, which generalize the well-known Lebesgue's theorem, are presented. Finally, the corresponding conclusions of two well-known specific summability methods, that is, Bochner–Riesz and Weierstrass means, are also obtained.

1. Introduction

To give a unified framework of the real-variable theory of both the isotropic Hardy space and the parabolic Hardy space of Calderón and Torchinsky [6], in 2003, Bownik [4] first introduced and investigated the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$ with $p \in (0,\infty)$, where A is a general expansive matrix on $\mathbb{R}^n$ (see [4, p. 5, Definition 2.1]). In addition, Ky [11] introduced the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ with

$$\varphi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$$

being a growth function (see [11, Definition 2.1]), and also obtained the atomic characterization and the dual space of $H^{\varphi}(\mathbb{R}^n)$. Here we point out that some special Musielak–Orlicz Hardy spaces appear naturally in the study of the products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see, for instance, [1, 3, 10]), and the endpoint estimates for both the div-curl lemma and the commutators of Calderón–Zygmund operators (see, for instance, [2, 9]). For more progresses about the theory of Musielak–Orlicz-type spaces, we refer the reader to [8, 25].

Based on the work of both Bownik [4] and Ky [11], Li et al. [14] introduced the anisotropic Musielak–Orlicz Hardy space $H^{\varphi}_A(\mathbb{R}^n)$ via the non-tangential grand maximal

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function, where $\varphi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is an anisotropic growth function (see Definition 2.4 below), and then characterized $H^{\varphi}_A(\mathbb{R}^n)$, respectively, in terms of radial or non-tangential maximal functions and atoms. Moreover, in [13], the authors obtained the characterizations of $H^{\varphi}_A(\mathbb{R}^n)$ via various Littlewood–Paley functions including the Lusin area function, the Littlewood–Paley g-function or g^*_λ-function. Besides these, both the molecular characterizations of $H^{\varphi}_A(\mathbb{R}^n)$ and the boundedness of integral anisotropic Calderón–Zygmund operators from $H^{\varphi}_A(\mathbb{R}^n)$ to itself [or to the Musielak–Orlicz space $L^{\varphi}(\mathbb{R}^n)$] were obtained in [12, 15].

On the other hand, it is well known that Stein, Taibleson and Weiss [22] proved for the Bochner–Riesz summability that the maximal operator σ^{θ}_* of the θ-means is bounded from the classical Hardy $H^p(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$ with the index p greater than some constant p_0. This result has been extended to many other Hardy-type and other summability methods, For more progress about this topic, we refer the reader to [16–18, 20, 22–24] and references therein. In particular, Liu and Xia [18] obtained the boundedness of maximal operators of the so-called θ-means from the isotropic Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ to the Musielak–Orlicz space $L^{\varphi}(\mathbb{R}^n)$. However, the corresponding conclusion of summability in anisotropic Musielak–Orlicz Hardy spaces $H^{\varphi}_A(\mathbb{R}^n)$ is still unknown.

In this paper, under some conditions on θ and $\vec{p}$, we show that the maximal operator σ^{θ}_* is bounded from $H^{\varphi}_A(\mathbb{R}^n)$ to $L^{\varphi}(\mathbb{R}^n)$. As applications, we present some norm and almost everywhere convergence results for the θ-means. Moreover, sa special cases, we obtain the corresponding results for Bochner–Riesz and Weierstrass summations.

This paper is organized as follows: Section 2 is devoted to recalling some definitions of expansive matrices, Musielak–Orlicz spaces $L^{\varphi}(\mathbb{R}^n)$ and anisotropic Musielak–Orlicz Hardy spaces $H^{\varphi}_A(\mathbb{R}^n)$. In Section 3, via borrowing some ideas from [23, Theorem 3] and [18, Theorem 3.1], we show our main result by using the finite atomic decomposition for a new dense subspace of $H^{\varphi}_A(\mathbb{R}^n)$. Section 4 is aimed to consider two special summability methods, that is, the Bochner–Riesz and Weierstrass summations.

Finally, we make some conventions on notation. We always use C to denote a positive constant which is independent of the main parameters, but its value may change from line to line. In addition, the symbol $f \lesssim g$ means $f \le Cg$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. Let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\mathbf{0}$ be the *origin* of $\mathbb{R}^n$. For any $\gamma := (\gamma_1, \dots, \gamma_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}^n_+$, let $|\gamma| := \gamma_1 + \cdots + \gamma_n$ and $\partial^\gamma := \left(\frac{\partial}{\partial x_1}\right)^{\gamma_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\gamma_n}$. For each $r \in [1,\infty]$, we denote by r' its *conjugate index*, that is, $1/r + 1/r' = 1$. For any $t \in \mathbb{R}$, the symbol $\lfloor t \rfloor$ denotes the *largest integer not greater than* t. Moreover, for a given set $\Omega \subset \mathbb{R}^n$, we denote its *characteristic function* by $\mathbf{1}_\Omega$, the set $\mathbb{R}^n \setminus \Omega$ by $\Omega^{\complement}$ and its n*-dimensional Lebesgue measure* by $|\Omega|$.

2. Preliminaries

In this section, we recall the notions of expansive matrices, Musielak–Orlicz spaces and anisotropic Musielak–Orlicz Hardy spaces.

We begin with the following notions of expansive matrices and homogeneous quasi-norms introduced by Bownik in [4].

Definition 2.1. A real $n \times n$ matrix A is called an *expansive matrix* (shortly, a *dilation*) if

$$\min_{\lambda\in\sigma(A)} |\lambda| > 1,$$

here and thereafter, $\sigma(A)$ denotes the *collection of all eigenvalues of* A.

Definition 2.2. Let A be a dilation. A measurable mapping $\rho\colon \mathbb{R}^n \to [0,\infty)$ is called a *homogeneous quasi-norm*, respect to A, if

(i) $x \neq \mathbf{0} \implies \rho(x) \in (0,\infty)$;

(ii) for every $x \in \mathbb{R}^n$, $\rho(Ax) = b\rho(x)$, where $b := |\det A|$;

(iii) there exists a positive constant C such that, for any $x, y \in \mathbb{R}^n$,

$$\rho(x+y) \le C[\rho(x)+\rho(y)].$$

For any given dilation A, let λ_-, $\lambda_+ \in (1,\infty)$ be two *numbers* such that

$$\lambda_- \le \min\{|\lambda| : \lambda \in \sigma(A)\} \le \max\{|\lambda| : \lambda \in \sigma(A)\} \le \lambda_+.$$

It follows from [4, p. 5, Lemma 2.2] that there exists an open set $\Delta \subset \mathbb{R}^n$ which has the following property: $|\Delta| = 1$, and we can find a constant $r \in (1,\infty)$ such that $\Delta \subset r\Delta \subset A\Delta$. For any $k \in \mathbb{Z}$, define $B_k := A^k\Delta$. Then $\{B_k\}_{k\in\mathbb{Z}}$ is a family of open sets around $\mathbf{0}$, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$. Moreover, let

$$\mathfrak{B} := \left\{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\right\} \tag{2.1}$$

and

$$\omega := \inf\left\{i \in \mathbb{Z} : r^i \ge 2\right\}.$$

Recall also that the following classes of uniform anisotropic Muckenhoupt weights respect to A were introduced in [14].

Definition 2.3. Let $q \in [1, \infty)$. The *class of uniform anisotropic Muckenhoupt weights* $\mathcal{A}_q(A) := \mathcal{A}_q(\mathbb{R}^n; A)$ is defined to be the set of all measurable functions $\varphi \colon \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ satisfying, if $q \in (1, \infty)$,

$$\sup_{t\in(0,\infty)} \sup_{B\in\mathfrak{B}} \left\{ \frac{1}{|B|} \int_B \varphi(y,t)\, dy \right\} \left\{ \frac{1}{|B|} \int_B [\varphi(y,t)]^{-\frac{1}{q-1}}\, dy \right\}^{q-1} < \infty$$

and, if $q = 1$,

$$\sup_{t\in(0,\infty)} \sup_{B\in\mathfrak{B}} \left\{ \frac{1}{|B|} \int_B \varphi(y,t)\, dy \right\} \left\{ \operatorname*{ess\,sup}_{y\in B} [\varphi(y,t)]^{-1} \right\} < \infty,$$

where $\mathfrak{B}$ is as in (2.1). Moreover, let $\mathcal{A}_\infty(A) := \bigcup_{q\in[1,\infty)} \mathcal{A}_q(A)$.

For any $\varphi \in \mathcal{A}_\infty(A)$, let

$$q(\varphi) := \inf\{q \in [1, \infty) : \varphi \in \mathcal{A}_q(A)\}. \tag{2.2}$$

A function $\Phi \colon [0, \infty) \to [0, \infty)$ is called an *Orlicz function* if Φ is non-decreasing, $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$ and, for any $t \in (0, \infty)$, $\Phi(t) \in (0, \infty)$ (see, for instance, [11]). For a given function $\varphi \colon \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ satisfying, for any given $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is an Orlicz function, φ is said to be of *uniformly upper* (resp., *lower*) *type* p for some $p \in (-\infty, \infty)$ if there exists a positive constant C such that, for almost every $x \in \mathbb{R}^n$, $s \in [1, \infty)$ (resp., $s \in (0, 1)$) and $t \in [0, \infty)$, $\varphi(x, st) \le C s^p \varphi(x, t)$. Let $i(\varphi)$ denote the *critical uniformly lower type index* of φ, that is,

$$i(\varphi) := \sup\{p \in (-\infty, \infty) : \varphi \text{ is of uniformly lower type } p\}. \tag{2.3}$$

The next notion of anisotropic growth functions is just [14, Definition 3].

Definition 2.4. A function $\varphi \colon \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is called an *anisotropic growth function* if it satisfies the following conditions:

(i) φ is a *Musielak–Orlicz function*, namely,

- (i)$_1$ for each given $x \in \mathbb{R}^n$, $\varphi(x, \cdot) \colon [0, \infty) \to [0, \infty)$ is an Orlicz function;
- (i)$_2$ for each given $t \in [0, \infty)$, $\varphi(\cdot, t)$ is a Lebesgue measurable function on $\mathbb{R}^n$.

(ii) $\varphi \in \mathcal{A}_\infty(A)$.

(iii) φ is of uniformly lower type p for some $p \in (0, 1]$ and of uniformly upper type 1.

For any φ as in Definition 2.4(i), the Musielak–Orlicz space $L^{\varphi}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f with their *quasi-norms*

$$\|f\|_{L^{\varphi}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda)\, dx \le 1\right\} < \infty.$$

Due to [4, p. 5, Lemma 2.4], we may use the step homogeneous quasi-norm defined by setting, for any $x \in \mathbb{R}^n$,

$$\rho(x) := \begin{cases} b^k & \text{if } x \in B_{k+1} \setminus B_k, \\ 0 & \text{if } x = \mathbf{0} \end{cases}$$

for convenience.

Hereinafter, the symbol $\mathcal{S}(\mathbb{R}^n)$ denotes the space of all Schwartz functions, that is, the set of all $C^{\infty}(\mathbb{R}^n)$ functions ϕ satisfying that, for any $i \in \mathbb{Z}_+$ and multi-index $\gamma \in \mathbb{Z}_+^n$,

$$\|\phi\|_{\gamma,i} := \sup_{x\in\mathbb{R}^n} [\rho(x)]^i |\partial^{\gamma}\phi(x)| < \infty$$

with the topology determined by $\{\|\cdot\|_{\gamma,\,i}\}_{\gamma\in\mathbb{Z}_+^n,\, i\in\mathbb{Z}_+}$. Furthermore, denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$, that is, the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology. For any $N \in \mathbb{Z}_+$, denote by $\mathcal{S}_N(\mathbb{R}^n)$ the following set:

$$\left\{\phi \in \mathcal{S}(\mathbb{R}^n) : \|\phi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{\gamma\in\mathbb{Z}_+^n,\, |\gamma|\le N}\ \sup_{x\in\mathbb{R}^n} \left[|\partial^{\gamma}\phi(x)| \max\{1, [\rho(x)]^N\}\right] \le 1\right\}.$$

The following definition of anisotropic Musielak–Orlicz Hardy spaces was first introduced by Li et al. [14].

Definition 2.5. (i) Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *non-tangential maximal function* $M_{\phi}(f)$ with respect to ϕ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_{\phi}(f)(x) := \sup_{y\in x+B_k,\, k\in\mathbb{Z}} |f * \phi_k(y)|,$$

where $\phi_k(\cdot) := b^k\phi(A^k\,\cdot)$. Moreover, for any given $N \in \mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N(f)(x) := \sup_{\phi\in\mathcal{S}_N(\mathbb{R}^n)} M_{\phi}(f)(x).$$

(ii) Let $N \in \mathbb{N}$ and φ be an anisotropic growth function as in Definition 2.4. The *anisotropic Musielak–Orlicz Hardy space* $H^{\varphi}_{N,A}(\mathbb{R}^n)$ is defined as

$$H^{\varphi}_{N,A}(\mathbb{R}^n) := \left\{f \in \mathcal{S}'(\mathbb{R}^n) : M_N(f) \in L^{\varphi}(\mathbb{R}^n)\right\}$$

and, for any $f \in H^{\varphi}_{N,A}(\mathbb{R}^n)$, let $\|f\|_{H^{\varphi}_{N,A}(\mathbb{R}^n)} := \|M_N(f)\|_{L^{\varphi}(\mathbb{R}^n)}$.

Remark 2.6. (i) In [14, Theorem 33], it was proved that the space $H^{\varphi}_{N,A}(\mathbb{R}^n)$ is independent of the choice of N as long as $N \in \mathbb{N} \cap [m(\varphi), \infty)$ with

$$m(\varphi) := \left\lfloor \left[\frac{q(\varphi)}{i(\varphi)} - 1\right] \frac{\ln b}{\ln \lambda_-} \right\rfloor, \tag{2.4}$$

where $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2) and (2.3). Therefore, we always denote simply by $H^{\varphi}_{A}(\mathbb{R}^n)$ the anisotropic Musielak–Orlicz Hardy space.

(ii) When $A := d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1, \infty)$, where $\mathrm{I}_{n\times n}$ denotes the $n \times n$ *unit matrix*, the space $H^{\varphi}_{A}(\mathbb{R}^n)$ becomes the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ of Ky [11], which includes the classical Hardy space, the classical weighted Hardy space and the classical Orlicz–Hardy space as special cases. In addition, if, for any $p \in (0, 1]$, $x \in \mathbb{R}^n$ and $t \in (0, \infty)$,

$$\varphi(x, t) := w(x)t^p$$

with w being an anisotropic Muckenhoupt weight, then the space $H^{\varphi}_{A}(\mathbb{R}^n)$ coincides with the weighted anisotropic Hardy space $H^p_w(\mathbb{R}^n; A)$ from [5], which includes the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$ of Bownik [4] as a special case.

3. Summability in $H^{\varphi}_{A}(\mathbb{R}^n)$

In this section, we study the so-called θ-summability for multi-dimensional Fourier transforms in the anisotropic Musielak–Orlicz Hardy space $H^{\varphi}_{A}(\mathbb{R}^n)$.

Recall that, for any $p \in (0, \infty]$ and any measurable set $\Omega \subset \mathbb{R}^n$, the Lebesgue space $L^p(\Omega)$ is defined to be the set of all the measurable functions f on Ω such that, if $p \in (0, \infty)$,

$$\|f\|_{L^p(\Omega)} := \left[\int_{\Omega} |f(x)|^p\, dx\right]^{1/p} < \infty \quad \text{and} \quad \|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty.$$

The *Fourier transform* of $f \in L^1(\mathbb{R}^n)$, denoted by $\widehat{f}$, is defined by setting, for any $v \in \mathbb{R}^n$,

$$\widehat{f}(v) := \int_{\mathbb{R}^n} f(x)e^{-2\pi\imath x\cdot v}\, dx,$$

where $\imath := \sqrt{-1}$ and, for any $x := (x_1, \ldots, x_n)$, $v := (v_1, \ldots, v_n) \in \mathbb{R}^n$, $x \cdot v := \sum_{k=1}^n x_k v_k$. Let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, 2]$. Then the Fourier inversion formula, that is, for any $x \in \mathbb{R}^n$,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(t)e^{2\pi\imath x\cdot t}\, dt$$

holds true if $\widehat{f} \in L^1(\mathbb{R}^n)$. This motivates the following definition of θ-summability of Fourier transforms, which was considered in a great number of monographs and articles;

see, for instance, [18, 20–24] for the classical case and [16, 17] for the anisotropic case. We always assume that

$$\theta \in C_0(\mathbb{R}), \quad \theta(|\cdot|) \in L^1(\mathbb{R}^n), \quad \theta(0) = 1 \quad \text{and} \quad \theta \text{ is even,} \tag{3.1}$$

where $C_0(\mathbb{R})$ denotes the *set of all continuous functions* f *satisfying* $\lim_{|x|\to\infty} |f(x)| = 0$.

Let A^* be the transposed matrix of A. The *m-th anisotropic* θ*-mean* of the function $f \in L^p(\mathbb{R}^n)$, with $p \in [1, 2]$, is defined by setting, for any $m \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\sigma_m^\theta f(x) := \int_{\mathbb{R}^n} \theta(|(A^*)^{-m}u|)\widehat{f}(u)e^{2\pi\imath x\cdot u}\, du.$$

Let $\theta_0(x) := \theta(|x|)$ for any $x \in \mathbb{R}^n$ and assume that

$$\widehat{\theta_0} \in L^1(\mathbb{R}^n). \tag{3.2}$$

It was proved in [17] that, for any $m \in \mathbb{Z}$, $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we can rewrite $\sigma_m^\theta f$ as

$$\sigma_m^\theta f(x) = b^m \int_{\mathbb{R}^n} f(t)\widehat{\theta_0}(A^m(x-t))\, dt.$$

Moreover, we can extend the definition of the anisotropic θ-means to any $f \in L^\varphi(\mathbb{R}^n)$ by setting, for any $x \in \mathbb{R}^n$,

$$\sigma_m^\theta f(x) := b^m \int_{\mathbb{R}^n} f(x-t)\widehat{\theta_0}(A^m t)\, dt.$$

Then we define the *maximal* θ*-operator* σ_*^θ by setting, for any $f \in L^\varphi(\mathbb{R}^n)$,

$$\sigma_*^\theta f := \sup_{m\in\mathbb{Z}} |\sigma_m^\theta f|.$$

Now we state the main result of this paper as follows, which shows the boundedness of maximal θ-operators from $H_A^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$.

Theorem 3.1. *Let* θ *and* θ_0 *be, respectively, as in* (3.1) *and* (3.2) *satisfying that there exists a positive constant* $\beta \in (1, \infty)$ *such that, for any* $\alpha \in (\mathbb{Z}_+)^n$ *and* $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\left|\partial^\alpha \widehat{\theta_0}(x)\right| \le C_{(\alpha,\beta)}|x|^{-\beta},$$

where the positive constant $C_{(\alpha,\beta)}$ *is independent of* x. *Suppose further that* φ *is a growth function such that*

$$\frac{i(\varphi)}{q(\varphi)} \in \left(\frac{\ln b}{\beta \ln \lambda_-}, \infty\right) \quad \text{with } \beta \in \left(\frac{\ln b}{\ln \lambda_-}, \infty\right), \tag{3.3}$$

where $i(\varphi)$ *and* $q(\varphi)$ *are, respectively, as in* (2.3) *and* (2.2). *Then there exists a positive constant* $C_{(i(\varphi),q(\varphi))}$ *such that, for any* $f \in H_A^\varphi(\mathbb{R}^n)$,

$$\|\sigma_*^\theta f\|_{L^\varphi(\mathbb{R}^n)} \le C_{(i(\varphi),q(\varphi))}\|f\|_{H_A^\varphi(\mathbb{R}^n)}.$$

To show Theorem 3.1, we need some technical lemmas. Recall that, for any locally integrable function f, the *Hardy–Littlewood maximal function* $M_{\rm HL}(f)$ is defined by setting, for any $x\in\mathbb{R}^n$,

$$M_{\rm HL}(f)(x):=\sup_{k\in\mathbb{Z}}\sup_{y\in x+B_k}\frac{1}{|B_k|}\int_{y+B_k}|f(z)|\,dz=\sup_{x\in B\in\mathfrak{B}}\frac{1}{|B|}\int_B|f(z)|\,dz,$$

where $\mathfrak{B}$ is as in (2.1).

The succeeding boundedness of $M_{\rm HL}$ on the space $L^\varphi(\mathbb{R}^n)$ comes from [13, Lemma 3.6].

Lemma 3.2. *Let φ be a Musielak–Orlicz function with uniformly lower type p_φ^- and uniformly upper type p_φ^+ satisfying $q(\varphi)<p_\varphi^-\le p_\varphi^+<\infty$, where $q(\varphi)$ is as in (2.2). Then there exists a positive constants C such that, for any $f\in L^\varphi(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n}\varphi(x,M_{\rm HL}f(x))\,dx\le C\int_{\mathbb{R}^n}\varphi(x,|f(x)|)\,dx.$$

The following items are just, respectively, [14, Lemma 10] and [11, Lemma 4.1(i) and Lemma 4.3(i)].

Lemma 3.3. *Let φ be a growth function as in Definition 2.4.*

(i) *Let $q\in[1,\infty)$ and $\varphi\in\mathcal{A}_q(A)$. Then there exists a positive constant C such that, for any dilated ball $B\subset\mathfrak{B}$, subset $E\subset B$ and $t\in(0,\infty)$,*

$$\frac{\varphi(B,t)}{\varphi(E,t)}\le C\left[\frac{|B|}{|E|}\right]^q.$$

(ii) *There exists a positive constant C such that, for any $\{(x,r_k)\}_{k\in\mathbb{N}}\subset\mathbb{R}^n\times[0,\infty)$,*

$$\varphi\left(x,\sum_{k\in\mathbb{N}}r_k\right)\le C\sum_{k\in\mathbb{N}}\varphi(x,t_k).$$

(iii) *For any $r\in(0,\infty)$ and measurable function f on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n}\varphi\left(x,\frac{|f(x)|}{r}\right)dx\lesssim 1\quad\textit{implies}\quad\|f\|_{L^\varphi(\mathbb{R}^n)}\lesssim r,$$

where the positive equivalence constants are independent of r and f.

Recall also that, for any given growth function φ and any measurable subset $E\subset\mathbb{R}^n$, the *space* $L^q_\varphi(E)$ is defined to be the set of all the measurable functions f on E such that

$$\|f\|_{L^q_\varphi(E)}:=\begin{cases}\sup_{t\in(0,\infty)}\left[\frac{1}{\varphi(E,t)}\int_E|f(x)|^q\varphi(x,t)\,dx\right]^{1/q}<\infty & \text{when } q\in[1,\infty),\\ \|f\|_{L^\infty(E)}<\infty & \text{when } q=\infty.\end{cases}$$

The next definitions of anisotropic Musielak–Orlicz atoms and finite atomic Hardy spaces are from [14].

Definition 3.4. Let φ be as in Definition 2.4 and $q(\varphi)$ as in (2.2).

(i) An anisotropic triplet (φ, q, s) is said to be *admissible* if $q \in (q(\varphi), \infty]$ and $s \in \mathbb{Z}_+ \cap [m(\varphi), \infty)$, where $m(\varphi)$ is as in (2.4).

(ii) For any given anisotropic admissible triplet (φ, q, s), a measurable function a on $\mathbb{R}^n$ is called an *anisotropic Musielak–Orlicz* (φ, q, s)*-atom* (shortly, a (φ, q, s)*-atom*) if

(ii)$_1$ $\operatorname{supp} a \subset B$, where $B \in \mathfrak{B}$ and $\mathfrak{B}$ is as in (2.1);

(ii)$_2$ $\|a\|_{L^q_\varphi(\mathbb{R}^n)} \le \|\mathbf{1}_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;

(ii)$_3$ for any multi-index $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \le s$, $\int_{\mathbb{R}^n} a(x) x^\gamma \, dx = 0$.

(iii) For any given anisotropic admissible triplet (φ, q, s), the *anisotropic Musielak–Orlicz finite atomic Hardy space* $H^{\varphi,q,s}_{A,\mathrm{fin}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist $I \in \mathbb{N}$, $\{\lambda_k\}_{k\in[1,I]\cap\mathbb{N}} \subset \mathbb{C}$ and a sequence of (φ, q, s)-atoms, $\{a_k\}_{k\in[1,I]\cap\mathbb{N}}$, supported, respectively, in $\{B^{(k)}\}_{k\in[1,I]\cap\mathbb{N}} \subset \mathfrak{B}$ such that $f = \sum_{k=1}^I \lambda_k a_k$ in $\mathcal{S}'(\mathbb{R}^n)$. Furthermore, for any $f \in H^{\varphi,q,s}_{A,\mathrm{fin}}(\mathbb{R}^n)$, let

$$\|f\|_{H^{\varphi,q,s}_{A,\mathrm{fin}}(\mathbb{R}^n)} := \inf\left\{\Lambda(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}})\right\},$$

where the infimum is taken over all the finite decompositions of f as above and

$$\Lambda(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}}) := \inf\left\{\lambda \in (0,\infty) : \sum_{k\in[1,I]\cap\mathbb{N}} \varphi\left(B^{(k)}, \frac{|\lambda_k|}{\lambda\|\mathbf{1}_{B^{(k)}}\|_{L^\varphi(\mathbb{R}^n)}}\right) \le 1\right\}.$$

The following conclusions are from [15]. The space $L^\infty_{c,s}(\mathbb{R}^n)$, with $s \in \mathbb{N}$, is defined to be the set of all the functions $f \in L^\infty(\mathbb{R}^n)$ with compact support satisfying that, for any $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le s$, $\int_{\mathbb{R}^n} f(x) x^\alpha \, dx = 0$ holds true.

Lemma 3.5. *Let φ be as in Definition 2.4, $q \in (q(\varphi), \infty)$ and $s \in \mathbb{N} \cap [m(\varphi), \infty)$, where $q(\varphi)$ and $m(\varphi)$ are as in (2.4). Then,*

(i) *for any $f \in L^\infty_{c,s}(\mathbb{R}^n)$, there exist an $I \in \mathbb{N}$, a sequence $\{\lambda_k\}_{k\in[1,I]\cap\mathbb{N}} \subset \mathbb{C}$ and a sequence of (φ, q, s)-atoms, $\{a_k\}_{k\in[1,I]\cap\mathbb{N}}$, such that $f = \sum_{k=1}^I \lambda_k a_k$ holds true both in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere, and*

$$\Lambda\big(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}}\big) \lesssim \|f\|_{H^\varphi_A(\mathbb{R}^n)}.$$

(ii) *$L^\infty_{c,s}(\mathbb{R}^n)$ is dense in $H^\varphi_A(\mathbb{R}^n)$.*

We also need the succeeding Lemma 3.6, which can be deduce from [7, Lemma 3.2(ii)] and an argument similar to [20, (5.19)]; the details are omitted.

Lemma 3.6. *Let θ and θ_0 be, respectively, as in* (3.1) *and* (3.2). *Let $q \in (1,\infty]$ and $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any locally integrable function f and $t \in (0,\infty)$,*

$$\int_{\mathbb{R}^n} [\sigma_*^\theta f(x)]^q \varphi(x,t)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)\,dx.$$

We now show Theorem 3.1.

Proof of Theorem 3.1. Let all notation be as in Theorem 3.1 and (φ,q,s) an anisotropic admissible triplet. We next prove the present theorem by three steps.

Step 1. This step is devoted to showing that, for any $\lambda \in (0,\infty)$ and (φ,q,s)-atom a supported in $B \subset \mathfrak{B}$,

$$\int_{\mathbb{R}^n} \varphi(x,\lambda\sigma_*^\theta(a)(x))\,dx \le C_{(\varphi,q,s)}\varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big) \tag{3.4}$$

holds true, where $C_{(\varphi,q,s)}$ is a positive constant depending on φ, q and s. For this purpose, we rewrite

$$\int_{\mathbb{R}^n} \varphi(x,\lambda\sigma_*^\theta(a)(x))\,dx = \int_{A^\omega B} \varphi(x,\lambda\sigma_*^\theta(a)(x))\,dx + \int_{(A^\omega B)^\complement} \cdots =: \mathrm{L}_1 + \mathrm{L}_2.$$

When $q \in (q(\varphi),\infty)$, note that φ is non-decreasing and of uniformly upper type 1. From Lemma 3.3(i), the Hölder inequality, Lemma 3.6 and Definition 3.4(ii), we deduce that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
\mathrm{L}_1 &\lesssim \int_{A^\omega B} \left[\frac{\sigma_*^\theta(a)(x)}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}} + 1\right] \varphi\big(x,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big)\,dx \\
&\lesssim \varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big) + \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)} \\
&\quad\times \left\{\int_{A^\omega B} [\sigma_*^\theta(a)(x)]^q \varphi\big(x,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big)\,dx\right\}^{1/q} \left[\varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big)\right]^{(q-1)/q} \\
&\lesssim \varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big) + \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}\|a\|_{L^q_\varphi(\mathbb{R}^n)(B)}\varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big) \\
&\lesssim \varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big).
\end{aligned} \tag{3.5}$$

For L_2, similar to [17, (5.10)] and [15, (3.6)], we obtain that, for any $x \in (A^\omega B)^\complement$,

$$\sigma_*^\theta(a)(x) \lesssim \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}[M_{\mathrm{HL}}(\mathbf{1}_B)(x)]^{\beta\ln\lambda_-/\ln b}. \tag{3.6}$$

In addition, by (3.3), we can find two numbers $\tau \in (q(\varphi),\infty)$ and $\sigma \in (0,i(\varphi))$ satisfying that $\varphi \in \mathcal{A}_\tau(\mathbb{R}^n)$, $\tau\ln b/\beta\ln\lambda_- < \sigma$ and φ is of uniformly lower type σ. This implies that

$\widetilde{\varphi}(x,t):=\varphi(x,t^{\beta\ln\lambda_-/\ln b})$ is of uniformly lower type $\sigma\beta\ln\lambda_-/\ln b$. Therefore, by (3.6), Lemma 3.2 and the inequality $\sigma\beta\ln\lambda_-/\ln b>\tau>q(\varphi)$, we conclude that

$$\begin{aligned}\mathrm{L}_2&\lesssim\int_{(A^\omega B)^\complement}\widetilde{\varphi}\big(x,\lambda^{\ln b/\beta\ln\lambda_-}\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-\ln b/\beta\ln\lambda_-}M_{\mathrm{HL}}(\mathbf{1}_B)(x)\big)\,dx\\&\lesssim\int_{\mathbb{R}^n}\widetilde{\varphi}\big(x,\lambda^{\ln b/\beta\ln\lambda_-}\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-\ln b/\beta\ln\lambda_-}\mathbf{1}_B(x)\big)\,dx\\&\sim\varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big).\end{aligned}\tag{3.7}$$

This proves (3.4) for $q\in(q(\varphi),\infty)$.

When $q=\infty$, for L_1, by an argument similar to that used in (3.5), we find that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}\mathrm{L}_1&\lesssim\int_{A^\omega B}\left[\frac{\sigma_*^\theta(a)(x)}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}}+1\right]\varphi\big(x,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big)\,dx\\&\lesssim\varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big)+\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}\|a\|_{L^\infty(\mathbb{R}^n)}\varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big)\\&\lesssim\varphi\big(B,\lambda\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\big).\end{aligned}$$

This, combined with the validity of (3.7) for $q=\infty$, finishes the proof of (3.4).

Step 2. Let $q\in(q(\varphi),\infty)$. In this step, we show that, for any $f\in L^\infty_{c,s}(\mathbb{R}^n)$,

$$\|\sigma_*^\theta f\|_{L^\varphi(\mathbb{R}^n)}\le C_{(i(\varphi),q(\varphi))}\|f\|_{H^\varphi_A(\mathbb{R}^n)},\tag{3.8}$$

where $C_{(i(\varphi),q(\varphi))}$ is a positive constant independent of f. To this end, for any $f\in L^\infty_{c,s}(\mathbb{R}^n)$, it follows from Lemma 3.5(i) that there exist an $I\in\mathbb{N}$, a sequence $\{\lambda_k\}_{k\in[1,I]\cap\mathbb{N}}\subset\mathbb{C}$ and a sequence of (φ,q,s)-atoms, $\{a_k\}_{k\in[1,I]\cap\mathbb{N}}$, supported respectively in $\{B^{(k)}\}_{k\in[1,I]\cap\mathbb{N}}\subset\mathbb{R}^n$, such that $f=\sum_{k=1}^I\lambda_k a_k$ is valid almost everywhere and also in $\mathcal{S}'(\mathbb{R}^n)$, and

$$\Lambda\big(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}}\big)\lesssim\|f\|_{H^\varphi_A(\mathbb{R}^n)}\quad\text{and}\quad\sum_{k\in[1,I]\cap\mathbb{N}}\varphi\left(B^{(k)},\,\frac{|\lambda_k|\|\mathbf{1}_{B^{(k)}}\|_{L^\varphi(\mathbb{R}^n)}^{-1}}{\Lambda(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}})}\right)=1.$$

By this, Lemma 3.3(ii) and (3.4), we have

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi\left(x,\frac{\sigma_*^\theta f(x)}{\Lambda(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}})}\right)dx&\lesssim\sum_{k\in[1,I]\cap\mathbb{N}}\int_{\mathbb{R}^n}\varphi\left(x,\frac{|\lambda_k|\sigma_*^\theta(a_k)(x)}{\Lambda(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}})}\right)dx\\&\lesssim\sum_{k\in[1,I]\cap\mathbb{N}}\varphi\left(B^{(k)},\frac{|\lambda_k|\|\mathbf{1}_{B^{(k)}}\|_{L^\varphi(\mathbb{R}^n)}^{-1}}{\Lambda(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}})}\right)\lesssim 1.\end{aligned}$$

From this, Lemma 3.3(iii), we infer that

$$\|\sigma_*^\theta f\|_{L^\varphi(\mathbb{R}^n)}\lesssim\Lambda(\{\lambda_k a_k\}_{k\in[1,I]\cap\mathbb{N}})\lesssim\|f\|_{H^\varphi_A(\mathbb{R}^n)}.$$

This proves (3.8).

Step 3. This step is aimed to show that (3.8) is valid for $f \in H^{\varphi}_A(\mathbb{R}^n)$. For this purpose, let $f \in H^{\varphi}_A(\mathbb{R}^n)$. Then Lemma 3.5(ii) implies that there exists a Cauchy sequence $\{f_i\}_{i\in\mathbb{N}} \subset L^{\infty}_{c,s}(\mathbb{R}^n)$ satisfying that

$$\lim_{i\to\infty} \|f_i - f\|_{H^{\varphi}_A(\mathbb{R}^n)} = 0.$$

By this, the linearity of σ^{θ}_{*} and the validity of (3.8) on the space $L^{\infty}_{c,s}(\mathbb{R}^n)$, we conclude that, as $i, k \to \infty$,

$$\|\sigma^{\theta}_{*}(f_i) - \sigma^{\theta}_{*}(f_k)\|_{H^{\varphi}_A(\mathbb{R}^n)} = \|\sigma^{\theta}_{*}(f_i - f_k)\|_{H^{\varphi}_A(\mathbb{R}^n)} \lesssim \|f_i - f_k\|_{H^{\varphi}_A(\mathbb{R}^n)} \to 0.$$

Thus, $\{\sigma^{\theta}_{*}(f_i)\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $H^{\varphi}_A(\mathbb{R}^n)$. From this and the fact that the space $H^{\varphi}_A(\mathbb{R}^n)$ is complete(see [14, Proposition 7]), we infer that there exists some $h \in H^{\varphi}_A(\mathbb{R}^n)$ such that $h = \lim_{i\to\infty} \sigma^{\theta}_{*}(f_i)$ in $H^{\varphi}_A(\mathbb{R}^n)$. Let $\sigma^{\theta}_{*}(f) := h$. Then (3.8) implies that $\sigma^{\theta}_{*}(f)$ is well defined and, furthermore, for any $f \in H^{\varphi}_A(\mathbb{R}^n)$,

$$\begin{aligned}\|\sigma^{\theta}_{*}(f)\|_{H^{\varphi}_A(\mathbb{R}^n)} &\lesssim \limsup_{i\to\infty} \left[\|\sigma^{\theta}_{*}(f) - \sigma^{\theta}_{*}(f_i)\|_{H^{\varphi}_A(\mathbb{R}^n)} + \|\sigma^{\theta}_{*}(f_i)\|_{H^{\varphi}_A(\mathbb{R}^n)}\right] \\ &\lesssim \limsup_{i\to\infty} \|\sigma^{\theta}_{*}(f_i)\|_{H^{\varphi}_A(\mathbb{R}^n)} \lesssim \lim_{i\to\infty} \|f_i\|_{H^{\varphi}_A(\mathbb{R}^n)} \sim \|f\|_{H^{\varphi}_A(\mathbb{R}^n)}.\end{aligned}$$

Therefore, (3.8) is valid for any $f \in H^{\varphi}_A(\mathbb{R}^n)$ and the proof of Theorem 3.1 is completed. □

Remark 3.7. (i) When $A = d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, the space $H^{\varphi}_A(\mathbb{R}^n)$ becomes the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ of Ky [11]. In this case Theorem 3.1 is just [18, Theorem 3.1]. Moreover, if $p \in (0,1]$ and

$$\varphi(x,t) := t^p, \quad \forall\, x \in \mathbb{R}^n \text{ and } t \in (0,\infty),$$

then $\frac{i(\varphi)}{q(\varphi)} = p$, the space $H^{\varphi}_A(\mathbb{R}^n)$ goes back to the classical Hardy space $H^p(\mathbb{R}^n)$, and Theorem 3.1 goes back to the classical result with $\beta \in (n,\infty)$ and $p \in (n/\beta,\infty)$ (see Weisz [22]). The classical result was proved in a special case, namely, for the Bochner–Riesz means, in Stein et al. [19]. For the same case, a counterexample was also given in [19] to illustrate that the same conclusion is not true for $p \in (0, n/\beta]$.

(ii) If, for any $p \in (0,1]$, $x \in \mathbb{R}^n$ and $t \in (0,\infty)$,

$$\varphi(x,t) := w(x)t^p$$

with w being an anisotropic Muckenhoupt weight, then the space $H^{\varphi}_A(\mathbb{R}^n)$ coincides with the weighted anisotropic Hardy space $H^p_w(\mathbb{R}^n; A)$ from [5]. We should point out that Theorem 3.1 is new even for this case.

As applications of Theorem 3.1, we give two convergence results, whose proofs are omitted.

Corollary 3.8. *With the same assumptions as in Theorem* 3.1, *if* $f \in H_A^{\varphi}(\mathbb{R}^n)$, *then* $\sigma_m^{\theta} f$ *converges almost everywhere as well as in the* $L^{\varphi}(\mathbb{R}^n)$ *quasi-norm as* $m \to \infty$.

Corollary 3.9. *With the same assumptions as in Theorem* 3.1, *if* $f \in H_A^{\varphi}(\mathbb{R}^n)$ *and there exists a subset* $I \subset \mathbb{R}^n$ *such that the restriction* $f|_I \in L^{\Phi}(I)$, *where* Φ *is some growth function with* $\frac{i(\Phi)}{q(\Phi)} \in [1, \infty)$, *then*

$$\lim_{m\to\infty} \sigma_m^{\theta} f(x) = f(x) \quad \textit{for almost every } x \in I \textit{ as well as in the } L^{\varphi}(I) \textit{ quasi-norm.}$$

4. Two specific summability methods

As special cases, we consider two specific summability methods.

4.1. Bochner–Riesz summation

For any $\gamma \in (0, \infty)$ and $\alpha \in \mathbb{N}$, the *Bochner–Riesz summation* is defined by setting, for any $u \in \mathbb{R}^n$,

$$\theta_0(u) := \begin{cases} (1 - |u|^{\alpha})^{\gamma} & \text{if } |u| \in [0, 1), \\ 0 & \text{if } |u| \in [1, \infty). \end{cases} \tag{4.1}$$

The next result can be found in Weisz [22].

Lemma 4.1. *Let* θ_0 *be as in* (4.1). *Then the conditions* (3.1) *and* (3.2) *are satisfied if* $\gamma \in \left(\frac{n-1}{2}, \infty\right)$ *and, for any* $\beta \in \mathbb{Z}_+^n$ *and* $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\left|\partial^{\beta} \widehat{\theta_0}(x)\right| \le C_{(\alpha,\beta)} |x|^{-n/2-\gamma-1/2},$$

where $C_{(\gamma,\beta)}$ *is a positive constant independent of* x.

The following conclusion is easily deduced from Lemma 4.1 and Theorem 3.1; the details are omitted.

Theorem 4.2. *Let* φ *be an anisotropic growth function and* θ_0 *as in* (4.1). *If*

$$\gamma \in \left(\max\left\{\frac{n-1}{2}, \frac{\ln b}{\ln \lambda_-} - \frac{n+1}{2}\right\}, \infty\right) \quad \textit{and} \quad \frac{i(\varphi)}{q(\varphi)} \in \left(\frac{\ln b}{\ln \lambda_-(n/2+\gamma+1/2)}, \infty\right),$$

then there exists a positive constant $C_{(i(\varphi),q(\varphi))}$ *such that, for any* $f \in H_A^{\varphi}(\mathbb{R}^n)$,

$$\|\sigma_*^{\theta} f\|_{L^{\varphi}(\mathbb{R}^n)} \le C_{(i(\varphi),q(\varphi))} \|f\|_{H_A^{\varphi}(\mathbb{R}^n)}.$$

Remark 4.3. Let θ_0 be as in (4.1). Then, in this special case, the corresponding conclusions in Corollaries 3.8 and 3.9 are true as well.

4.2. Weierstrass summation

The *Weierstrass summation* is defined as

$$\theta_0(u) := e^{-|u|^2/2}, \quad \forall\, u \in \mathbb{R}^n. \tag{4.2}$$

Observe that, for any $t \in \mathbb{R}^n$, $\widehat{\theta_0}(t) = e^{-|t|^2/2}$

The following Lemma 4.4 is just [17, Lemma 2.27].

Lemma 4.4. *Let θ_0 be as in* (4.2). *Then the conditions* (3.1) *and* (3.2) *are satisfied and, for any $\beta \in (1, \infty)$, $\gamma \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*

$$\left|\partial^\gamma \widehat{\theta_0}(x)\right| \le C_{(\gamma,\beta)} |x|^{-\beta},$$

where $C_{(\gamma,\beta)}$ is a positive constant independent of x.

This lemma and Theorem 3.1 imply the following Theorem 4.5 immediately; the details are omitted.

Theorem 4.5. *Let θ_0 be as in* (4.2). *If φ is an anisotropic growth function, then there exists a positive constant $C_{(i(\varphi),q(\varphi))}$ such that, for any $f \in H_A^\varphi(\mathbb{R}^n)$,*

$$\|\sigma_*^\theta f\|_{L^\varphi(\mathbb{R}^n)} \le C_{(i(\varphi),q(\varphi))} \|f\|_{H_A^\varphi(\mathbb{R}^n)}.$$

Moreover, the corresponding conclusions in Corollaries 3.8 *and* 3.9 *are true as well.*

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