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On the Boundary Behaviour of the Squeezing Function near Weakly Pseudoconvex Boundary Points

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Abstract. The purpose of this article is to investigate the boundary behaviour of the squeezing function of a general ellipsoid.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ and $p \in \Omega$. Let us denote by $\mathrm{Aut}(D)$ the automorphism group of a domain D. For a holomorphic embedding $f\colon \Omega \to \mathbb{B}^n := \mathbb{B}(0;1)$ with $f(p) = 0$, we set

$$\sigma_{\Omega,f}(p) := \sup\{r > 0 : B(0;r) \subset f(\Omega)\},$$

where $\mathbb{B}^n(z;r) \subset \mathbb{C}^n$ denotes the Euclidean ball of radius r with center at z. Then the *squeezing function* $\sigma_\Omega\colon \Omega \to \mathbb{R}$ is defined as

$$\sigma_\Omega(p) := \sup_f \{\sigma_{\Omega,f}(p)\}$$

(see Definition in [5]). Note that the squeezing function is invariant under biholomorphisms and $0 < \sigma_\Omega(z) \leq 1$ for any $z \in \Omega$. Moreover, by definition one sees that Ω is biholomorphically equivalent to the unit ball $\mathbb{B}^n$ if $\sigma_\Omega(z) = 1$ for some $z \in \Omega$.

It is well-known that $\lim_{\Omega \ni z \to p \in \partial\Omega} \sigma_\Omega(z) = 1$ if p is a strongly pseudoconvex boundary point (cf. [6, 7, 12]). Conversely, motivated by Problem 4.1 in [8], let us consider the following problem.

Problem 1.1. If Ω is a bounded pseudoconvex domain with smooth boundary, and if $\lim_{j\to\infty} \sigma_\Omega(q_j) = 1$ for some sequence $\{q_j\} \subset \Omega$ converging to $p \in \partial\Omega$, then is the boundary of Ω strongly pseudoconvex at p?

In the case that $\partial\Omega$ is pseudoconvex of D'Angelo finite type near ξ_0, the answer to this problem is affirmative for the following cases:

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- $\{q_j\} \subset \Omega$ converges to ξ_0 along the inner normal line to $\partial\Omega$ at ξ_0 (for details, see [11] for $n = 2$ and [14] for general case).
- $\{q_j\} \subset \Omega$ converges nontangentially to ξ_0 (see [15]).
- $\{q_j\} \subset \Omega$ converges $\big(\frac{1}{m_1}, \dots, \frac{1}{m_{n-1}}\big)$-nontangentially to an h-extendible boundary point ξ_0 (see [16, Definition 3.4]), where $(1, m_1, \dots, m_{n-1})$ is the *multitype of* $\partial\Omega$ *at* ξ_0 and the *h-extendibility at* ξ_0 means that the Catlin multitype and D'Angelo multitype of $\partial\Omega$ at ξ_0 coincide (see [20, Definition 3.3]).

Now we consider the case that $\{q_j\} \subset \Omega$ is a sequence converging $\big(\frac{1}{m_1}, \dots, \frac{1}{m_{n-1}}\big)$-nontangentially to ξ_0. Then, the condition that $\lim_{j\to\infty} \sigma_\Omega(q_j) = 1$ ensures that the unit ball $\mathbb{B}^n$ is biholomorphically equivalent to some model M_P given by

$$M_P = \{z \in \mathbb{C}^n : \mathrm{Re}(z_n) + P(z') < 1\},$$

where P is a $\big(\frac{1}{m_1}, \dots, \frac{1}{m_{n-1}}\big)$-homogeneous polynomial on $\mathbb{C}^{n-1}$ (see [20, Definition 3.1]). Therefore, $m_1 = m_2 = \cdots = m_{n-1} = 1$, or ξ_0 is strongly pseudoconvex (see [16]). Unfortunately, the point ξ_0 may not be strongly pseudoconvex when $\{q_j\} \subset \Omega$ does not converge $\big(\frac{1}{m_1}, \dots, \frac{1}{m_{n-1}}\big)$-nontangentially to ξ_0. For instance, the following example points out that $\lim_{j\to\infty} \sigma_\Omega(q_j) = 1$ for some sequence $\{q_j\} \subset \Omega$ converging to a weakly pseudoconvex boundary point (see also Example 3.1 for general case).

Example 1.2. Let $E_{1,2} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 + |z_1|^4 < 1\}$. Consider the sequence $a_n = \Big(\sqrt[4]{\frac{2}{n} - \frac{2}{n^2}}, 1 - \frac{1}{n}\Big) \to (0, 1)$ as $n \to \infty$. Denote by $\rho(z) := |z_2|^2 - 1 + |z_1|^4$ a defining function for $E_{1,2}$ and denote by $\sigma(z_1) = |z_1|^4$ a $\big(\frac{1}{4}\big)$-weighted homogeneous polynomial. Then, a computation shows that

$$\rho(a_n) = \left|1 - \frac{1}{n}\right|^2 - 1 + \left|\sqrt[4]{\frac{2}{n} - \frac{2}{n^2}}\right|^4 = -\frac{2}{n} + \frac{1}{n^2} + \frac{2}{n} - \frac{2}{n^2} = -\frac{1}{n^2} < 0.$$

Therefore, $\mathrm{dist}(a_n, \partial E_{1,2}) \approx |\rho(a_n)| = \frac{1}{n^2}$, $|\mathrm{Re}(a_{n2}) - 1| = \big|-\frac{1}{n}\big| = \frac{1}{n}$, and $\sigma(a_{n1}) = \sigma\Big(\sqrt[4]{\frac{2}{n} - \frac{2}{n^2}}\Big) = \Big(\sqrt[4]{\frac{2}{n} - \frac{2}{n^2}}\Big)^4 = \frac{2}{n} - \frac{2}{n^2} \approx \frac{2}{n}$. Here and in what follows, $\lesssim$ and $\gtrsim$ denote inequality up to a positive constant. Moreover, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

This implies that $\{a_n\}$ does not converge $\big(\frac{1}{4}\big)$-nontangentially to the boundary point $p = (0, 1)$.

Let us consider the automorphism $\psi_n \in \mathrm{Aut}(E_{1,2})$, given by

$$\psi_n(z) = \left(\frac{(1 - |a_{n2}|^2)^{1/4}}{(1 - \overline{a}_{n2} z_2)^{1/2}} z_1, \frac{z_2 - a_{n2}}{1 - \overline{a}_{n2} z_2}\right),$$

and hence $\psi_n(a_n) = (b_n, 0)$, where $b_n = \frac{a_{n1}}{(1-|a_{n2}|^2)^{1/4}} = \frac{\sqrt[4]{\frac{2}{n}-\frac{2}{n^2}}}{\sqrt[4]{\frac{2}{n}-\frac{1}{n^2}}} \to 1$ as $n \to \infty$. Since $\psi_n(a_n)$ converges to the strongly pseudoconvex boundary point $(1,0)$ of $\partial E_{1,2}$, by [12, Theorrem 3.1] it follows that $\sigma_{E_{1,2}}(a_n) = \sigma_{E_{1,2}}(\psi_n(a_n)) \to 1$ as $n \to \infty$. However, the point $(0,1)$ is weakly pseudoconvex.

To give a statement of our result, let us fix positive integers $m_1, \ldots, m_{n-1}$ and let $P(z')$ be a $(1/m_1, \ldots, 1/m_{n-1})$-homogeneous polynomial given by

$$P(z') = \sum_{\mathrm{wt}(K)=\mathrm{wt}(L)=1/2} a_{KL} {z'}^K \overline{z}'^L,$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \overline{a}_{LK}$, satisfying that $P(z') > 0$ whenever $z' \neq 0$. Here and in what follows, $z' := (z_1, \ldots, z_{n-1})$ and $\mathrm{wt}(K) := \sum_{j=1}^{n-1} \frac{k_j}{2m_j}$ denotes the weight of any multi-index $K = (k_1, \ldots, k_{n-1}) \in \mathbb{N}^{n-1}$ with respect to $\Lambda := (1/m_1, \ldots, 1/m_{n-1})$. Then the general ellipsoid D_P in $\mathbb{C}^n$ $(n \geq 1)$, defined in [18] by

$$D_P := \{(z', z_n) \in \mathbb{C}^n : |z_n|^2 + P(z') < 1\}.$$

We note that

$$P(a^{1/m_1} z_1, a^{1/m_2} z_2, \ldots, a^{1/m_{n-1}} z_{n-1}) = |a|^2 P(z'), \quad \forall\, z' \in \mathbb{C}^{n-1},\ \forall\, a \in \mathbb{C} \setminus \{0\}. \tag{1.1}$$

Therefore, $\mathrm{Aut}(D_P)$ contains the automorphisms $\phi_a \in \mathrm{Aut}(D_P)$, $a \in \Delta := \{z \in \mathbb{C} : |z| < 1\}$, defined by

$$(z', z_n) \mapsto \left(\frac{(1-|a|^2)^{1/2m_1}}{(1+\overline{a}z_n)^{1/m_1}} z_1, \ldots, \frac{(1-|a|^2)^{1/2m_{n-1}}}{(1+\overline{a}z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n + a}{1+\overline{a}z_n} \right),$$

These automorphisms play a crucial role in the proofs of Theorems 1.6 and 1.10 below.

It was shown in [3] (see also [18, Theorem 5]) that D_P is biholomorphically equivalent to the domain

$$Q_P := \{(z', z_n) \in \mathbb{C}^n : \mathrm{Re}(z_n) + P(z') < 1\}.$$

Furthermore, as in [9, 10], Q_P is called *homogeneous finite diagonal type* if there exists a small positive number $\delta > 0$ such that

$$P(z') - \delta(|z_1|^{2m_1} + \cdots + |z_{n-1}|^{2m_{n-1}}) \text{ is plurisubharmonic in } \mathbb{C}^{n-1}, \tag{1.2}$$

i.e., P is strictly plurisubharmonic away from the union of all coordinates axes. In addition, by following the proofs of [2, Theorem 4.1] and [1, Theorem 4.2], the condition (1.2) yields the existence of a peak function at $0 = (0, 0, \ldots, 0)$ for $\mathcal{O}(Q_P)$. This condition means in fact that Q_P (or D_P) is a WB-domain in the sense of [1].

In this paper, we need the following definition.

Definition 1.3. The domain D_P is called a $\widetilde{WB}$-domain if D_P is strongly pseudoconvex at every boundary point outside the set $\{(0', e^{i\theta}) : \theta \in \mathbb{R}\}$.

Remark 1.4. It is note that the Ellipsoid $E_{1m} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\}$ with $m \in \mathbb{Z}_{\geq 1}$ is a $\widetilde{WB}$-domain. Although the domain $\Omega := \left\{z \in \mathbb{C}^3 : |z_1|^6 + |z_2|^4 + |z_3|^2 < 1\right\}$ is a WB-domain, but it is not a $\widetilde{WB}$-domain since the boundary point $(1, 0, 0) \in \partial\Omega \setminus \{(0, 0, e^{i\theta}) : \theta \in \mathbb{R}\}$ is not strongly pseudoconvex. Therefore, the notion of $\widetilde{WB}$-domains is more restrictive than the that of WB-domains, in particular in higher dimension. However, for a $\widetilde{WB}$-domain Ω in $\mathbb{C}^n$ we may have $\lim_{j\to\infty} \sigma_\Omega(q_j) = 1$ for some sequence $\{q_j\} \subset \Omega$ converging to a weakly pseudoconvex boundary point (cf. Theorem 1.10 below).

To state our main results, let us introduce several classes of domains. Indeed, for any $s, r \in (0, 1]$ and $\alpha \in [0, 2)$, inspired by [13, Lemma 2.5] we define D_P^s, $D_{P,r}^s$, $D_{P,r}$ and $D_P^s(\alpha)$, respectively, by

$$
\begin{aligned}
D_P^s &:= \{z \in \mathbb{C}^n : |z_n - b|^2 + sP(z') < s^2\},\\
D_{P,r}^s &:= \left\{z \in \mathbb{C}^n : |z_n - b|^2 + \frac{s}{r}P(z') < s^2\right\},\\
D_{P,r} := D_{P/r} &= \left\{z \in \mathbb{C}^n : |z_n|^2 + \frac{1}{r}P(z') < 1\right\},\\
D_P^s(\alpha) &= \Bigg\{z \in \mathbb{C}^n : \left|z_n + \frac{(1-s)\alpha}{2s(1-\alpha)+\alpha}\right|^2 + \frac{s(2-\alpha)}{2s(1-\alpha)+\alpha}P(z')\\
&\qquad < \frac{2s-\alpha}{2s(1-\alpha)+\alpha} + \left|\frac{(1-s)\alpha}{2s(1-\alpha)+\alpha}\right|^2\Bigg\},
\end{aligned}
$$

where $b = 1 - s$.

We note that $D_P^s(0) = D_P$, and $D_{P,1}^s = D_P^s$. Moreover, since $P(z') > 0$ whenever $z' \neq 0$, it is easy to see that $D_{P,r}^s \subset D_P^s$. Moreover, we also have $D_P^s \subset D_P$. Indeed, let $z \in D_P^s$ be arbitrary. Then, we have

$$|z_n - 1|^2 + 2s\,\mathrm{Re}(z_n - 1) + sP(z') < 0,$$

or equivalently

$$\frac{1}{s}|z_n - 1|^2 + 2\,\mathrm{Re}(z_n - 1) + P(z') < 0.$$

Since $0 < s < 1$, it follows that

$$|z_n - 1|^2 + 2\,\mathrm{Re}(z_n - 1) + P(z') \leq \frac{1}{s}|z_n - 1|^2 + 2\,\mathrm{Re}(z_n - 1) + P(z') < 0,$$

which implies that $z \in D_P$.

In what follows, let us denote by Δ the unit disc in $\mathbb{C}$ and for a sequence $\{a_j\} \subset \Delta$ converging to $1 \in \partial\Delta$ we always denote by $x_j := 1 - \mathrm{Re}(a_j)$ and $y_j := \mathrm{Im}(a_j)$ for $j \geq 1$. Suppose that $\{q_j = (q_j', a_j)\} \subset D_P^s$ for some $0 < s < 1$. Then one sees that

$$|a_j - 1|^2 + 2s\,\mathrm{Re}(a_j - 1) + sP(q_j') < 0,$$

which implies that

$$|a_j - 1|^2 < -2s\,\mathrm{Re}(a_j - 1) \quad \text{for } j \geq 1,$$

or equivalently $x_j^2 + y_j^2 < 2sx_j$ for $j \geq 1$. Therefore, passing to a subsequence if necessary, we can assume that there exists

$$0 \leq \alpha := \lim_{j\to\infty} \frac{y_j^2}{x_j} \leq 2s < 2.$$

In addition, to each sequence $\{a_j\} \subset \Delta$ we associate a sequence $\phi_j := \phi_{a_j} \in \mathrm{Aut}(D_P)$, i.e.,

$$(1.3)\quad \phi_j(z', z_n) = \left(\frac{(1-|a_j|^2)^{1/2m_1}}{(1+\bar{a}_j z_n)^{1/m_1}} z_1, \ldots, \frac{(1-|a_j|^2)^{1/2m_{n-1}}}{(1+\bar{a}_j z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n + a_j}{1+\bar{a}_j z_n} \right), \quad j \geq 1.$$

We now recall that a sequence $\{q_j\} \subset D_P$ converges Λ-nontangentically to $p = (0', 1)$ if $|q_{jk}|^{m_k} \lesssim \mathrm{dist}(q_j, \partial D_P)$, $1 \leq k \leq n-1$; $|\mathrm{Im}(q_{jn})| \lesssim \mathrm{dist}(q_j, \partial D_P)$ (cf. [16, Definition 3.4]). In particular, the sequence $\{q_{jn}\} \subset \Delta$ converges nontangentially to the point $1 \in \partial\Delta$. However, in this paper we shall focus attention on the behaviour of $\{q_{jn}\} \subset \Delta$ converging to $1 \in \partial\Delta$. Namely, we need the following definition.

Definition 1.5. We say that $\{q_j\} \subset D_P \cap U$ converges Λ^α-*nontangentially* to $p = (0', 1)$ if there exists $0 < r < 1$ such that $q_j \in D_{P,r}$ for all $j \geq 1$, $\lim_{j\to\infty} q_j = (0', 1)$, and $\lim_{j\to\infty} \frac{y_j^2}{x_j} = \alpha \in [0, 2)$, where $q_{jn} = 1 - x_j + iy_j$, $j \geq 1$.

The first aim of this paper is to prove the following theorem.

Theorem 1.6. *Let Ω be a subdomain of D_P such that $D_P^s \subset \Omega \subset D_P$ for some $s \in (0, 1]$. Let $\{q_j\} \subset D_{P,r}^s$ be a sequence that converges Λ^α-nontangentially to $(0', 1)$ in D_P for some $0 < r < 1$. Then, there exists $\gamma_1 > 0$ depending on s, α, P, r such that*

$$\liminf_{j\to\infty} \sigma_\Omega(q_j) \geq \gamma_1.$$

Remark 1.7. Let $\{q_j = (q_j', q_{nj})\} \subset D_{P,r}^s$ be as in the statement of Theorem 1.6. Then Lemma 2.1 ensures that $\lim_{j\to\infty} \psi_j^{-1}(D_{P,r}^s) = D_{P,r}^s(\alpha)$ and $\lim_{j\to\infty} \psi_j^{-1}(D_P^s) = D_P^s(\alpha)$. Therefore, the proof of Theorem 1.6 follows from the invariance of the squeezing function under biholomorphisms.

Now let us denote the cone with vertex at $p = (0', 1)$ by

$$\Gamma_c := \left\{ (z', z_n) \in \mathbb{C}^n : |\mathrm{Im}(z_n)| \leq c|1 - \mathrm{Re}(z_n)| \right\}$$

for some $c > 0$. Then for any sequence $\{q_j\} \subset D_{P,r}^s \cap \Gamma_c$ converging to $(0', 1)$, we always have $\alpha = \lim_{j\to\infty} \frac{y_j^2}{x_j} = 0$. Therefore, again by Lemma 2.1, $\lim_{j\to\infty} \psi_j^{-1}(D_{P,r}^s) = D_{P,r}$ for any $0 < r \leq 1$. Moreover, we obtain the following corollary, which is a generalization of [17, Theorem 1.3].

Corollary 1.8. *Let Ω be a subdomain of D_P such that $D_P^s \subset \Omega \subset D_P$ for some $s \in (0,1]$. Then, for any $r \in (0,1)$, $c > 0$ there exist $\epsilon_0, \gamma_2 > 0$ depending on r and c such that*

$$\sigma_\Omega(q) \geq \gamma_2, \quad \forall\, q \in D^s_{P,r} \cap \Gamma_c \cap B(p, \epsilon_0).$$

In contrast to the Λ^α-nontangential convergence ($0 \leq \alpha < 2$), we have the following definition.

Definition 1.9. We say that $\{q_j\} \subset D_P \cap U$ converges Λ-*tangentially* to $p = (0', 1)$ if $\lim_{j\to\infty} q_j = (0', 1)$ and for any $0 < r < 1$ there exists $j_r \in \mathbb{N}$ such that $q_j \notin D_{P,r}$ for all $j \geq j_r$.

With the notion of Λ-tangential convergence, the second aim of this paper is to prove the following theorem.

Theorem 1.10. *Let $\{\Omega_j\}$ be a sequence of subdomains of D_P such that $\Omega_j \cap U = D_P \cap U$, $j \geq 1$, for a fixed neighborhood U of $(0', 1)$ in $\mathbb{C}^n$. Let $\{q_j\} \subset D_P \cap U$ be a sequence that converges Λ-tangentially to $(0', 1)$ in D_P. If D_P is a $\widetilde{WB}$-domain, then $\lim_{j\to\infty} \sigma_{\Omega_j}(q_j)$ $= 1$.*

We note that D_P is holomorphically homogeneous regular (cf. [17, Theorem 1.1]). Furthermore, we prove the following proposition, which provides a uniform lower bound for the squeezing function near $(0', 1) \in \partial D_P$.

Proposition 1.11. *Let Ω be a subdomain of D_P and $\Omega \cap U = D_P \cap U$ for a fixed neighborhood U of $p = (0', 1)$ in $\mathbb{C}^n$. If D_P is a $\widetilde{WB}$-domain, then there exist $\epsilon_0, \gamma_0 > 0$ depending only on D_P such that*

$$\sigma_\Omega(z) > \gamma_0, \quad \forall\, z \in D_P \cap B(p; \epsilon_0).$$

The organization of this paper is as follows. In Section 2, we introduce several technical lemmas needed later. Then, the proofs of Theorems 1.6 and 1.10 are given in Section 3.

2. Several technical lemmas

In this section, we first prove the following lemma.

Lemma 2.1. *Let $\{a_j = 1 - x_j + iy_j\} \subset \Delta$ be a given sequence satisfying that $\lim_{j\to\infty} a_j = 1$ and $\lim_{j\to\infty} \frac{y_j^2}{x_j} = \alpha \in [0, 2)$. Then, for any $s \in (0,1)$ we have that $\psi_j^{-1}(D_P^s)$ converges to $D_P^s(\alpha)$, where the sequence $\{\psi_j\}$ is given in (1.3).*

Remark 2.2. In the case that $\alpha = 0$, one sees that $D_P^s(0) = D_P$ and therefore $\psi_j^{-1}(D_P^s)$ converges to D_P. In addition, Liu [13, Lemma 2.5] restricted himself to the case that

$\mathrm{Im}(a_j)=0$ and $P(z')=|z'|^2$, i.e., D_P is the unit ball $\mathbb{B}^n$. Instead of D_P^s, he considered the ball $\mathcal{B}_s$ center at $(0',b)$ with radius $s=1-b$. However, the limit of $\psi_j^{-1}(\mathcal{B}_s)$ is exactly the ellipsoid $\big\{|z_n|^2+\frac{1}{1-b}|z'|^2<1\big\}$, which is contained in the unit ball $\mathbb{B}^n$. Of course, according to Lemma 2.1 the limit of $\psi_j^{-1}(D_P^s)$ is $\mathbb{B}^n$.

To give a proof of Lemma 2.1, we need the following lemma.

Lemma 2.3. *Let $\{a_j\}$ be a sequence in Δ such that $\lim_{j\to\infty}\frac{\left(\mathrm{Im}(a_j)\right)^2}{1-\mathrm{Re}(a_j)}=\alpha\in[0,2)$ and $\lim_{j\to\infty}a_j=1$. Then we have*

$$\text{(i)}\ \lim_{j\to\infty}\frac{1-\mathrm{Re}(a_j)}{1-|a_j|^2}=\frac{1}{2-\alpha};\quad \text{(ii)}\ \lim_{j\to\infty}\frac{(1-\overline{a}_j)^2}{1-|a_j|^2}=\frac{-\alpha}{2-\alpha};\quad \text{(iii)}\ \lim_{j\to\infty}\frac{|1-a_j|^2}{1-|a_j|^2}=\frac{\alpha}{2-\alpha}.$$

Proof. We have $x_j\to 0^+$, $y_j\to 0$, and $y_j^2/x_j\to\alpha$ as $j\to\infty$, where $x_j:=1-\mathrm{Re}(a_j)$, $y_j:=\mathrm{Im}(a_j)$. Moreover, a direct calculation yields that

$$\begin{aligned}
\frac{1-\mathrm{Re}(a_j)}{1-|a_j|^2}&=\frac{x_j}{1-(1-x_j)^2-y_j^2}=\frac{x_j}{2x_j-x_j^2-y_j^2}=\frac{1}{2-x_j-y_j^2/x_j},\\
\frac{(1-\overline{a}_j)^2}{1-|a_j|^2}&=\frac{(x_j+iy_j)^2}{1-(1-x_j)^2-y_j^2}=\frac{x_j^2-y_j^2+2ix_jy_j}{2x_j-x_j^2-y_j^2}=\frac{x_j-y_j^2/x_j+2iy_j}{2-x_j-y_j^2/x_j},\\
\frac{|1-a_j|^2}{1-|a_j|^2}&=\frac{x_j^2+y_j^2}{1-(1-x_j)^2-y_j^2}=\frac{x_j^2+y_j^2}{2x_j-x_j^2-y_j^2}=\frac{x_j+y_j^2/x_j}{2-x_j-y_j^2/x_j},\quad \forall j\geq 1.
\end{aligned}$$

Therefore, the assertions follow since $x_j\to 0^+$ and $y_j^2/x_j\to\alpha$ as $j\to\infty$. □

Proof of Lemma 2.1. The proof of this lemma is given in [19]. However, for the convenience of the reader we give a detailed proof. Indeed, recall that $b=1-s$ or $s=1-b\in(0,1)$. Then, by the property (1.1) a straightforward calculation shows that

$$\begin{aligned}
&\left|\frac{z_n+a_j}{1+\overline{a}_jz_n}-b\right|^2+sP\left(\frac{(1-|a_j|^2)^{1/2m_1}}{(1+\overline{a}_jz_n)^{1/m_1}}z_1,\ldots,\frac{(1-|a_j|^2)^{1/2m_{n-1}}}{(1+\overline{a}_jz_n)^{1/m_{n-1}}}z_{n-1}\right)<s^2\\
\Longleftrightarrow\quad&\left|\frac{z_n+a_j}{1+\overline{a}_jz_n}-b\right|^2+s\frac{1-|a_j|^2}{|1+\overline{a}_jz_n|^2}P(z')<s^2\\
\Longleftrightarrow\quad&\left|\frac{z_n+a_j-b(1+\overline{a}_jz_n)}{1+\overline{a}_jz_n}\right|^2+s\frac{1-|a_j|^2}{|1+\overline{a}_jz_n|^2}P(z')<s^2\\
\Longleftrightarrow\quad&|z_n+a_j-b(1+\overline{a}_jz_n)|^2+s(1-|a_j|^2)P(z')<s^2|1+\overline{a}_jz_n|^2\\
\Longleftrightarrow\quad&|z_n(1-\overline{a}_jb)+a_j-b|^2+s(1-|a_j|^2)P(z')<s^2|1+\overline{a}_jz_n|^2\\
\Longleftrightarrow\quad&|z_n|^2|1-\overline{a}_jb|^2+2\,\mathrm{Re}\left[(\overline{a}_j-b)(1-\overline{a}_jb)z_n\right]+|a_j-b|^2+(1-b)(1-|a_j|^2)P(z')\\
&<s^2\big(|a_j|^2|z_n|^2+2\,\mathrm{Re}[\overline{a}_jz_n]+1\big)\\
\Longleftrightarrow\quad&|z_n|^2\big(|1-\overline{a}_jb|^2-(1-b)^2|a_j|^2\big)+2\,\mathrm{Re}\left[\big((\overline{a}_j-b)(1-\overline{a}_jb)-(1-b)^2\overline{a}_j\big)z_n\right]\\
&+(1-b)(1-|a_j|^2)P(z')<(1-b)^2-|a_j-b|^2
\end{aligned}$$

$$\begin{aligned}
\iff\quad & |z_n|^2 + 2\operatorname{Re}\left[\frac{(\overline{a}_j - b)(1-\overline{a}_j b) - (1-b)^2\overline{a}_j}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2} z_n\right] \\
& + \frac{(1-b)(1-|a_j|^2)}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2} P(z') < \frac{(1-b)^2 - |a_j - b|^2}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2} \\
\iff\quad & \left| z_n + \frac{(\overline{a}_j - b)(1-\overline{a}_j b) - (1-b)^2\overline{a}_j}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2}\right|^2 + \frac{(1-b)(1-|a_j|^2)}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2} P(z') \\
& < \frac{(1-b)^2 - |a_j - b|^2}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2} + \left| \frac{(\overline{a}_j - b)(1-\overline{a}_j b) - (1-b)^2\overline{a}_j}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2}\right|^2 .
\end{aligned}$$

Moreover, by a computation one obtains

$$\begin{aligned}
(\overline{a}_j - b)(1-\overline{a}_j b) - (1-b)^2\overline{a}_j &= \overline{a}_j - b - \overline{a}_j^2 b + \overline{a}_j b^2 - \overline{a}_j + 2\overline{a}_j b - \overline{a}_j b^2 = -b(1-\overline{a}_j)^2, \\
(1-b)^2 - |a_j - b|^2 &= 1 - 2b + b^2 - |a_j|^2 + 2b\operatorname{Re}(a_j) - b^2 \\
&= 1 - |a_j|^2 - 2b(1 - \operatorname{Re}(a_j)), \\
|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2 &= 1 - 2\operatorname{Re}(a_j b) + |a_j|^2 b^2 - |a_j|^2 + 2b|a_j|^2 - b^2|a_j|^2 \\
&= 1 - |a_j|^2 - 2b\big(\operatorname{Re}(a_j) - |a_j|^2\big) \\
&= 1 - |a_j|^2 - 2b\big(\operatorname{Re}(a_j) - 1 + 1 - |a_j|^2\big) \\
&= (1-|a_j|^2)\left[1 - 2b\left(1 - \frac{1-\operatorname{Re}(a_j)}{1-|a_j|^2}\right)\right].
\end{aligned}$$

Hence, Lemma 2.3 yields that

$$\begin{aligned}
\lim_{j\to\infty} \frac{(\overline{a}_j - b)(1-\overline{a}_j b) - (1-b)^2\overline{a}_j}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2} &= \frac{b\alpha}{(1-b)(2-\alpha) + b\alpha} = \frac{(1-s)\alpha}{2s(1-\alpha)+\alpha}, \\
\lim_{j\to\infty} \frac{(1-b)(1-|a_j|^2)}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2} &= \frac{(1-b)(2-\alpha)}{(1-b)(2-\alpha) + b\alpha} = \frac{s(2-\alpha)}{2s(1-\alpha)+\alpha}, \\
\lim_{j\to\infty} \frac{(1-b)^2 - |a_j - b|^2}{|1-\overline{a}_j b|^2 - (1-b)^2|a_j|^2} &= \frac{2-\alpha-2b}{(1-b)(2-\alpha) + b\alpha} = \frac{2s-\alpha}{2s(1-\alpha)+\alpha}.
\end{aligned}$$

Therefore, this implies that $\psi_j^{-1}(D_P^s) \to D_P^s(\alpha)$ as $j\to\infty$, as desired. □

We close this section with a technical lemma. Indeed, Lemma 2.1 in [17] easily yields the following lemma.

Lemma 2.4. *Let Ω be a bounded domain in $\mathbb{C}^n$ and K be a relative compact subset of Ω. Then, we have*

$$\inf_{z\in K} \sigma_\Omega(z) \geq \frac{\operatorname{dist}(K, \partial\Omega)}{d(\Omega)},$$

where $\operatorname{dist}(\cdot,\cdot)$ and $d(\Omega)$ denote respectively the Euclidean distance in $\mathbb{C}^n$ and the diameter of Ω.

3. Proofs of Theorems 1.6 and 1.10

This section is devoted to proofs of Theorems 1.6, 1.10 and Proposition 1.11.

Proof of Theorem 1.6. Let $\{q_j\} \subset D^s_{P,r}$ be a sequence converging to $(0',1)$ for some fixed $r \in (0,1)$. For simplicity, let us denote by $a_j = q_{jn}$ for $j \geq 1$. Let us denote by $x_j := 1 - \mathrm{Re}(a_j)$, $y_j := \mathrm{Im}(a_j)$ for convenience. Then we have $x_j \to 0^+$, $y_j \to 0$, and $y_j^2/x_j \to \alpha$ as $j \to \infty$.

We now consider the sequence of automorphisms $\{\psi_j\} \subset \mathrm{Aut}(D_P)$ given in (1.3). Then, Lemma 2.1 yields

$$\lim_{j\to\infty} \psi_j^{-1}(D^s_{P,r}) = D^s_{P,r}(\alpha), \quad \lim_{j\to\infty} \psi_j^{-1}(D^s_P) = D^s_P(\alpha). \tag{3.1}$$

Moreover, we have that $\psi_j^{-1}(q_j) = \big(\frac{q_{j1}}{\lambda_j^{1/2m_1}}, \ldots, \frac{q_{jn-1}}{\lambda_j^{1/2m_{n-1}}}, 0\big) \in D^s_{P,r}(\alpha) \cap \{z_n = 0\}$, where $\lambda_j = 1 - |a_j|^2$ and $D^s_{P,r}(\alpha) \cap \{z_n = 0\} \Subset D^s_P(\alpha)$. Therefore, by (3.1) and by Lemma 2.4 there exists $j_0 \in \mathbb{N}^*$ such that

$$\sigma_\Omega(q_j) = \sigma_{\psi_j^{-1}(\Omega)}(\psi_j^{-1}(q_j)) > \delta/d > 0, \quad \forall\, j \geq j_0,$$

where d denotes the diameter of D_P and $\delta := \mathrm{dist}(Z_{r,\alpha}(P), Z_{1,\alpha}(P))/2$ with $Z_{\rho,\alpha}(P) = \big\{z' \in \mathbb{C}^{n-1} : P(z') = \rho\frac{2s-\alpha}{s(2-\alpha)}\big\}$ for $0 < \rho \leq 1$. This finishes the proof with $\gamma_1 = \delta/d$. □

Proof of Corollary 1.8. We first consider an arbitrary sequence $\{q_j\} \subset D^s_{P,r} \cap \Gamma_c$ converging to $p = (0',1)$. Let us write $a_j = q_{jn} = 1 - x_j + iy_j$. Since $\{a_j\} \subset \Delta$, one has $x_j > 0$ for all $j \geq 1$. Therefore, we have

$$\frac{y_j^2}{x_j} = \frac{|y_j|}{|x_j|} \cdot |y_j| \leq c \cdot |y_j|, \quad j \geq 1.$$

This implies that $\alpha := \lim_{j\to\infty} \frac{y_j^2}{x_j} = 0$, and hence we obtain $\lim_{j\to\infty} \psi_j^{-1}(D^s_{P,r}) = D_{P,r}$ and $\lim_{j\to\infty} \psi_j^{-1}(D^s_P) = D_P$ by Remark 2.2, where $\psi_j \in \mathrm{Aut}(D_P)$ given in (1.3).

Next, the above argument shows that

$$\lim_{D^s_P \cap \Gamma_c \ni q \to (0',1)} \psi_a^{-1}(D^s_P) = D_P, \quad \lim_{D^s_P \cap \Gamma_c \ni q \to (0',1)} \psi_a^{-1}(D^s_{P,r}) = D_{P,r}, \tag{3.2}$$

where $\psi_a \in \mathrm{Aut}(D_P)$ given by

$$\psi_a(z) = \left(\frac{(1-|a|^2)^{1/2m_1}}{(1+\bar{a}z_n)^{1/m_1}} z_1, \ldots, \frac{(1-|a|^2)^{1/2m_{n-1}}}{(1+\bar{a}z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n + a}{1+\bar{a}z_n}\right), \quad j \geq 1,$$

where $a := q_n$. In addition, for $q \in D^s_{P,r} \cap \Gamma_c$ one has

$$\psi_a^{-1}(q) = \Big(\frac{q_1}{\lambda^{1/2m_1}}, \ldots, \frac{q_{n-1}}{\lambda^{1/2m_{n-1}}}, 0\Big) \in D_{P,r} \cap \{z_n = 0\} \Subset D_P \cap \{z_n = 0\},$$

where $\lambda = 1 - |a|^2$. Therefore, by (3.2) and by Lemma 2.4 we finally conclude that there exists $\epsilon_0 > 0$ such that

$$\sigma_\Omega(q) = \sigma_{\psi_a^{-1}(\Omega)}(\psi_a^{-1}(q)) > \delta_r/d > 0, \quad \forall\, q \in D_{P,r_0} \cap \Gamma_c \cap B(p, \epsilon_0),$$

where d denotes the diameter of D_P and $\delta_r := \mathrm{dist}(Z_r(P), Z_1(P))/2$ with $Z_r(P) = \{z' \in \mathbb{C}^{n-1} : P(z') = r\}$. Hence, the proof is complete with $\gamma_2 = \delta_r/d$. □

Proof of Theorem 1.10. Suppose that $\{q_j\}$ converges Λ-tangentially to $(0', 1)$ in D_P. For simplicity, let us denote by $a_j = \eta_{jn}$. Then we consider the sequence of automorphisms $\{\psi_j\} \subset \mathrm{Aut}(D_P)$ given in (1.3).

Let us set $b_j = (b_j', 0) := \psi_j^{-1}(q_j)$ for all $j \geq 1$. Then, a straightforward computation shows that

$$b_j = \psi_j^{-1}(q_j) = \left(\frac{\eta_{j1}}{\lambda_j^{1/2m_1}}, \ldots, \frac{\eta_{j(n-1)}}{\lambda_j^{1/2m_{n-1}}}, 0 \right) \in D_P \cap \{z_n = 0\},$$

where $\lambda_j = 1 - |a_j|^2$ for all $j \geq 1$.

Since $\{q_j\}$ converges Λ-tangentially to $(0', 1)$ in D_P, it follows that there exists a sequence $\{r_j\} \subset (0, 1)$ with $r_j \to 1$ as $j \to \infty$ such that

$$|a_j|^2 + \frac{1}{r_j} P(q_j') = |\eta_{jn}|^2 + \frac{1}{r_j} P(q_j') \geq 1, \quad \forall\, j \geq 1,$$

which implies that

$$1 > P(b_j') = \frac{1}{\lambda_j} P(q_j') = \frac{1}{1 - |a_j|^2} P(q_j') \geq r_j$$

for all $j \geq 1$. Therefore, we obtain that $P(b_j') \to 1$ as $j \to \infty$. Since D_P is a $\widetilde{WB}$-domain, by passing to a subsequence if necessary we may assume that $\psi_j^{-1}(q_j)$ converges to some strongly pseudoconvex boundary point $p \in \partial D_P \cap \{z_n = 0\}$.

Since $\psi_j(0', 0) = (0', a_j) \to (0', 1)$ as $j \to \infty$ and the boundary point $(0', 1)$ is of D'Angelo finite type, by [4, Proposition 2.1] it follows that

$$\lim_{j \to \infty} \psi_j^{-1}(\Omega_j) = \lim_{j \to \infty} \psi_j^{-1}(\Omega_j \cap U) = \lim_{j \to \infty} \psi_j^{-1}(D_P \cap U) = D_P.$$

In addition, for any $\epsilon > 0$ sufficiently small there exists $j_0 \geq 1$ such that

$$\psi_j^{-1}(\overline{\Omega_j}) \setminus B((0', -1), \epsilon) = \overline{D_P} \setminus B((0', -1), \epsilon)$$

for any $j \geq j_0$. Hence, since $\sigma_{D_P}(b_j) \to 1$ as $j \to \infty$ and by Theorem 3.1 in [12], one concludes that $\sigma_{\Omega_j}(q_j) = \sigma_{\psi_j^{-1}(\Omega_j)}(b_j) \to 1$ as $j \to \infty$. □

Proof of Proposition 1.11. Since D_P is a $\widetilde{WB}$-domain, it follows that any boundary point $p \in \{(z',0) \in D_P : P(z') = 1\}$ is strongly pseudoconvex. Therefore, by Theorem 3.1 in [12], for any $p \in \{(z',0) \in D_P : P(z') = 1\}$ we have $\lim_{z\to p} \sigma_{D_P}(z) = 1$. Then, there exists $r_0 \in (0,1)$ such that

$$\sigma_{D_P}(z',0) > 3/4, \quad \forall\, z' \in \mathbb{C}^{n-1} \text{ with } P(z') \geq r_0. \tag{3.3}$$

For $q \in D_P$, we consider the automorphism $\psi_a \in \mathrm{Aut}(D_P)$, given by

$$\psi_a(z) = \left(\frac{(1-|a|^2)^{1/2m_1}}{(1+\bar{a}z_n)^{1/m_1}} z_1, \ldots, \frac{(1-|a|^2)^{1/2m_{n-1}}}{(1+\bar{a}z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n + a}{1+\bar{a}z_n} \right),$$

where $a := q_n$. In addition, let us set $b := \psi_a^{-1}(q)$. Then, a straightforward computation shows that

$$b = (b',0) = \psi_a^{-1}(q) = \Big(\frac{q_1}{\lambda^{1/2m_1}}, \ldots, \frac{q_{n-1}}{\lambda^{1/2m_{n-1}}}, 0 \Big) \in D_P \cap \{z_n = 0\},$$

where $\lambda = 1 - |a|^2$.

Now we consider the following two cases:

Case 1*:* $q \in D_{P,r_0}$. In this case,we have

$$|a|^2 + \frac{1}{r_0} P(q') = |q_n|^2 + \frac{1}{r_0} P(q') < 1,$$

which implies that

$$P(b') = \frac{1}{\lambda} P(q') = \frac{1}{1-|a|^2} P(q') < r_0.$$

Since $\psi_a(0',0) = (0',a) \to (0',1)$ as $a \to 1$ and the boundary point $(0',1)$ is of D'Angelo finite type, again by [4, Proposition 2.1] it follows that

$$\lim_{a\to 1} \psi_a^{-1}(\Omega) = \lim_{a\to 1} \psi_a^{-1}(\Omega \cap U) = \lim_{a\to 1} \psi_a^{-1}(D_P \cap U) = D_P.$$

Therefore, by Lemma 2.4 there exists $\epsilon_0 > 0$ such that

$$\sigma_\Omega(q) = \sigma_{\psi_a^{-1}(\Omega)}(\psi_a^{-1}(q)) > \frac{\delta_{r_0}}{d} > 0, \quad \forall\, q \in D_{P,r_0} \cap B(p,\epsilon_0),$$

where d denotes the diameter of D_P and $\delta_{r_0} := \mathrm{dist}(Z_{r_0}(P), Z_1(P))/2$ with $Z_{r_0}(P) = \{z' \in \mathbb{C}^{n-1} : P(z') = r_0\}$.

Case 2*:* $q \in D_P \setminus D_{P,r_0}$. Then we have

$$|a|^2 + \frac{1}{r_0} P(q') = |q_n|^2 + \frac{1}{r_0} P(q') \geq 1,$$

which implies that

$$P(b') = \frac{1}{\lambda} P(q') = \frac{1}{1-|a|^2} P(q') \geq r_0.$$

As in Case 1 and by (3.3), there exists $\epsilon_0 > 0$ such that

$$\sigma_\Omega(q) = \sigma_{\psi_a^{-1}(\Omega)}(\psi_a^{-1}(q)) > \frac{1}{2}, \quad \forall\, q \in (D_P \setminus D_{P,r_0}) \cap B(p, \epsilon_0).$$

Hence, altogether, the proof is complete with $\gamma_0 = \min\{\frac{\delta_{r_0}}{d}, \frac{1}{2}\}$. □

We close this section with an example, which is a generalization of Example 1.2.

Example 3.1. Fix positive integers $m_1, \dots, m_{n-1}$ and denote by $\Lambda := (1/m_1, \dots, 1/m_{n-1})$. Let us consider a general ellipsoid D_P in $\mathbb{C}^n$ $(n \geq 2)$ defined by

$$D_P := \{(z', z_n) \in \mathbb{C}^n : |z_n|^2 + P(z') < 1\},$$

where $P(z')$ is a $(1/m_1, \dots, 1/m_{n-1})$-homogeneous polynomial given by

$$P(z') = \sum_{\mathrm{wt}(K)=\mathrm{wt}(L)=1/2} a_{KL} {z'}^K \overline{z'}^L,$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$, satisfying that $P(z') > 0$ whenever $z' \neq 0$. Moreover, suppose that the domain D_P is a $\widetilde{WB}$-domain.

Now let us denote by $\rho(z) := |z_n|^2 - 1 + P(z')$ a local defining function for D_P and consider a sequence $\{a_j = (a_j', a_{jn})\} \subset D_P$ which converges Λ-tangentially to $p := (0', 1)$. Since D_P is invariant under the map $z' \mapsto z'$; $z_n \mapsto e^{i\theta} z_n$ and σ_{D_P} is invariant under biholomorphisms, we may assume that $\mathrm{Im}(a_{jn}) = 0$ for all j. Since $\rho(z)$ is the defining function for D_P, it follows that $\mathrm{dist}(a_j, \partial D_P) \approx -\rho(a_j) = 1 - |a_{jn}|^2 - P(a_j')$. Moreover, since $\{a_j\}$ converges Λ-tangentially to p, we have that $P(a_j') \geq c_j \,\mathrm{dist}(a_j, \partial D_P)$ for some sequence $\{c_j\} \subset \mathbb{R}$ with $0 < c_j \to +\infty$. This implies that $P(a_j') \geq c_j'(1 - |a_{jn}|^2 - P(a_j'))$ for some sequence $\{c_j'\} \subset \mathbb{R}$ with $0 < c_j' \to +\infty$ and hence

$$P(a_j') \geq \frac{c_j'}{1 + c_j'}(1 - |a_{jn}|^2), \quad \forall\, j \geq 1.$$

Let us denote by $\widetilde{\psi}_j$ the automorphism of D_P given by

$$\widetilde{\psi}_j(z) = \left(\frac{(1 - |a_{jn}|^2)^{1/2m_1}}{(1 - \bar{a}_{jn} z_n)^{1/m_1}} z_1, \dots, \frac{(1 - |a_{jn}|^2)^{1/2m_{n-1}}}{(1 - \bar{a}_{jn} z_n)^{1/m_{n-1}}} z_{n-1}, \frac{z_n - a_{jn}}{1 - \bar{a}_{jn} z_n} \right),$$

and hence $\widetilde{\psi}_j(a_j) = (b_j', 0)$, where

$$b_j' = \left(\frac{a_{j1}}{(1 - |a_{jn}|^2)^{1/2m_1}}, \dots, \frac{a_{j(n-1)}}{(1 - |a_{jn}|^2)^{1/2m_{n-1}}} \right).$$

Thanks to the boundedness of $\{b_j'\}$, without loss of generality we may assume that $b_j' \to b' \in \mathbb{C}^{n-1}$ as $j \to \infty$. In addition, we have that $P(b_j') = \frac{1}{1 - |a_{jn}|^2} P(a_j') \geq \frac{c_j'}{1 + c_j'}$, $\forall\, j \geq 1$.

Therefore, we arrive at the situation that $b'_j \to b'$ with $P(b') = 1$ and since D_P is a $\widetilde{WB}$-domain, it follows that $\widetilde{\psi}_j(a_j)$ converges to the strongly pseudoconvex boundary point $(b', 0)$ of ∂D_P, which implies by [12, Theorrem 3.1] that $\sigma_{D_P}(a_j) = \sigma_{D_P}(\widetilde{\psi}_j(a_j)) \to 1$ as $j \to \infty$ even the boundary point p is weakly pseudoconvex.

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