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# Existence of Solutions for Fractional $(p, q)$-Laplacian Problems Involving Critical Hardy–Sobolev Nonlinearities

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Abstract. This paper is devoted to studying a class of fractional $(p, q)$-Laplacian problems with subcritical and critical Hardy potentials:

$$\begin{cases} (-\Delta)_p^{s_1} u + \nu(-\Delta)_q^{s_2} u = \lambda \frac{|u|^{r-2}u}{|x|^a} + \frac{|u|^{p_{s_1}^*(b)-2}u}{|x|^b} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, and $p_{s_1}^*(b) = \frac{(N-b)p}{N-ps_1}$ denotes the fractional critical Hardy–Sobolev exponent. More precisely, when $\nu = 1$ and $\nu > 0$ is sufficiently small, using some asymptotic estimates and the Mountain Pass Theorem, we establish the existence results for the above fractional elliptic equation under some suitable hypotheses, respectively, which are gained over a wider range of parameters.

## 1. Introduction and main results

Consider the following fractional $p\&q$-Laplacian problems with two different Hardy potentials

$$(P_\nu(\lambda)) \qquad \begin{cases} (-\Delta)_p^{s_1} u + \nu(-\Delta)_q^{s_2} u = \lambda \frac{|u|^{r-2}u}{|x|^a} + \frac{|u|^{p_{s_1}^*(b)-2}u}{|x|^b} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, $N \geq 2$, $\lambda, \nu > 0$ are parameters. $0 < s_2 < s_1 < 1 < q < p < \frac{N}{s_1}$, $0 < a, b < ps_1 < N$, $r \in (q, p_{s_1}^*(b))$, $p_{s_1}^*(b) = (N-b)p/(N-ps_1)$ denotes the fractional critical Hardy–Sobolev exponent. Up to normalization, the nonlocal operator $(-\Delta)_p^s$ $(p \geq 1)$ is the fractional $p$-Laplacian defined by

$$(-\Delta)_p^s u(x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

along any function $u(x) \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x - y| < \varepsilon\}$.

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As the basic theories of fractional Sobolev space gradually mature, a good deal of study on nonlocal operators has achieved extensive popularity. Nonlocal operators are applied in an extremely natural way in lots of different contexts such as water waves, nonlocal phase transitions, finance, geophysics, and image recovery, see [3,12,24,25,33], one of these operators is the fractional $p$-Laplacian.

Research on the fractional $p$-Laplacian has many interesting results; see for instance [7, 17, 19, 23, 27, 29, 31] and references therein. Among them, we pay special attention to [27] where Mosconi et al. studied the well-known Brézis–Niréberg problem (1983) for the fractional $p$-Laplacian and obtained the nontrivial weak solution by working with certain asymptotic estimates for minimizers and an abstract link theorem in [32].

More recently, due to the research done for problems driven by the local and nonlocal operators $-\Delta_p u$, $-\Delta_q u$, and $(-\Delta)_p^s u$, the existence, multiplicity, regularity, and maximum principles of solutions for fractional $p\&q$-Laplacian problems have received plenty of attention, see [1,2,4–6,8–11,14,16,21,28,30]. In [11] Bhakta and Mukherjee got the existence of infinitely many nontrivial solutions of a class of fractional $p\&q$ elliptic equations involving concave-critical nonlinearities in bounded domains in $\mathbb{R}^N$. When the nonlinearity was of convex-critical type, they also established the multiplicity of nonnegative solutions by variational methods. In [4] Ambrosio proved a strong maximum principle in an open set $\Omega \subset \mathbb{R}^N$ for weak supersolutions of

$$(-\Delta)_p^s u + \beta(-\Delta)_q^s u + c(x)\big(|u|^{p-2}u + |u|^{q-2}u\big) = 0.$$

In [10] Behboudi et al. studied a quasi-linear problem in a bounded Lipschitz domain $\Omega$:

$$\begin{cases} (-\Delta)_p^r u + \gamma(-\Delta)_q^s u = \lambda|u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

They investigated the existence of a mountain pass solution via critical point theory and variational methods. On the other hand, Goel et al. [21] studied the following nonlinear doubly nonlocal equation

$$\begin{cases} (-\Delta)_p^{s_1} u + \beta(-\Delta)_q^{s_2} u = \lambda a(x)|u|^{\delta-2}u + b(x)|u|^{r-2}u & \text{on } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.1}$$

By analyzing the fibering maps and the energy functional over suitable subsets of the Nehari manifold, they got the multiplicity of weak solutions, and in the case of $\delta = q$, they obtained the existence of solutions.

When $\beta = a(x) = b(x) = 1$ in (1.1), Chen and Yang [14] studied the related Brézis–

Niréberg problem for the fractional $p\&q$-Laplacian

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \mu |u|^{q-2}u + \lambda |u|^{p-2}u + |u|^{p^*_{s_1}-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\mu, \lambda > 0$, $0 < s_2 < s_1 < 1 < q < p < \frac{N}{s_1}$. By the mountain pass theorem and certain asymptotic estimates for minimizers, the existence of a nonnegative nontrivial weak solution was obtained, which extended the results of $p\&q$-Laplacian in [13] to the fractional $p\&q$-Laplacian.

Moreover, we note that the elliptic equations with Hardy term have been studied recently in many papers such as [8, 18–20, 31]. In [8], Ambrosio and Isernia studied the following problem with critical Sobolev–Hardy exponents in an open bounded domain with smooth boundary $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} (-\Delta)_p^{s} u + (-\Delta)_q^{s} u = \frac{|u|^{p^*_s(\alpha)-2}u}{|x|^\alpha} + \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where $\lambda > 0$ is a parameter, $p^*_s(\alpha) = \frac{p(N-\alpha)}{N-ps}$ is the so-called Hardy–Sobolev critical exponent. Using concentration-compactness principle and the mountain pass lemma, they showed that there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, problem (1.2) has infinitely many nontrivial solutions.

In addition, Fan [18] studied the related problems

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = f(x)|u|^{m-2}u + \frac{|u|^{r-2}u}{|u|^\alpha} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

If $2 < q+1 < p < m < r = p^*_\alpha = \frac{(N-\alpha)p}{(N-ps_1)}$, via variational methods, at least one nontrivial solution was obtained when $N < p^2 s_1$ and $m > Np/(N-ps_1) - p/(p-1)$.

Motivated by the results mentioned above, in the present paper, we consider the fractional $p\&q$-Laplacian problem $(P_\nu(\lambda))$ with a subcritical and a critical Hardy term, which generalize the results of [22] to the fractional $p\&q$-Laplacian problem with critical Hardy nonlinearity. The purpose of this paper is using Lemmas 2.1 and 2.7 to study the nontrivial weak solutions of $(P_\nu(\lambda))$ under the cases $\nu = 1$ and $\nu > 0$ sufficiently small, respectively. To the best of our knowledge, our result is new.

Here are the results:

**Theorem 1.1.** *When $\nu = 1$, problem $(P_1(\lambda))$ has a nontrivial weak solution for all $\lambda > 0$ in each to following cases when $N < p^2 s_1$:*

(i) $1 < q < p - \frac{p(N-ps_1)}{N(p-1)}$, $\frac{p(N-a)}{N-ps_1} - \frac{p}{p-1} < r < p^*_{s_1}(b)$, *and* $\max\left\{0, b - \frac{N-ps_1}{p-1}\right\} < a \leq \frac{p^2 s_1 - N}{p-1}$;

(ii) $p - \frac{p(N-ps_1)}{N(p-1)} \le q < p$, $\frac{Nq-ap}{N-ps_1} < r < p^*_{s_1}(b)$, *and* $0 < b \le a \le \frac{p^2s_1-N}{p-1}$.

*Remark* 1.2. When $a = b = 0$, the result of Theorem 1.1 also holds.

We find that when $1 < p \le \frac{3N+\sqrt{5N^2-4Ns_1}}{2(N+s_1)}$, case (i) in Theorem 1.1 cannot hold and the first inequality in case (ii) holds for $q > 1$, so we have the following corollary.

**Corollary 1.3.** *When* $N < p^2s_1$, *if* $1 < q < p \le \frac{3N+\sqrt{5N^2-4Ns_1}}{2(N+s_1)}$, $\frac{Nq-ap}{N-ps_1} < r < p^*_{s_1}(b)$, $0 < b \le a \le \frac{p^2s_1-N}{p-1}$, *then the problem* $(P_1(\lambda))$ *has a nontrivial weak solution for all* $\lambda > 0$.

**Theorem 1.4.** *When* $\nu > 0$, *there exists* $\nu_0 > 0$ *such that problem* $(P_\nu(\lambda))$ *has a nontrivial weak solution for all* $\nu \in (0, \nu_0)$ *and* $\lambda > 0$ *in each of the following cases:*

(i) $a \in \big[\max\big\{0, \frac{p^2s_1-N}{p-1}\big\}, ps_1\big) \setminus \{0\}$ *and* $r \in (q, p^*_{s_1}(b))$;

(ii) $p^2s_1 > N$, *either* $0 < a < \frac{p^2s_1-N}{p-1}$, *and* $q < r < p$ *or* $\max\big\{0, b - \frac{N-ps_1}{p-1}\big\} < a < \frac{p^2s_1-N}{p-1}$, *and* $\frac{p(N-a)}{N-ps_1} - \frac{p}{p-1} < r < p^*_{s_1}(b)$.

*Remark* 1.5. When $a = b = 0$, the result of Theorem 1.4 also holds.

When $q < r < p$, we have the following corollary.

**Corollary 1.6.** *When* $0 < a < (p^2s_1 - N)/(p-1)$, *if* $q < r < p$, *then there exists* $\nu_0 > 0$ *such that problem* $(P_\nu(\lambda))$ *has a nontrivial weak solution for all* $\nu \in (0, \nu_0)$ *and* $\lambda > 0$.

The main novelty of this paper is that the form of $(P_\nu(\lambda))$ is new and the existence of solutions is gained over a wider range of parameter $\lambda$ and $r$, compared with the relevant study. Precisely, in contrast to [8], Theorem 1.1 gives a nontrivial solution of $(P_\nu(\lambda))$ for all $\lambda > 0$. On the other hand, we obtain a nontrivial solution, not only allows $p < r < p^*_{s_1}(b)$, but also allows $q < r \le p$, which is different from [18], where $p < r < p^*_{s_1}(b)$.

Our main difficulty in this paper is the lack of explicit formulas for minimizers, we will overcome this by working with certain estimates for minimizers recently obtained in [26, 31]. At the same time, the nodus of lacking compactness is also overcome. The outline of this paper is as follows. In Section 2, we analyze the behavior of the Palais–Smale sequence, the mountain pass geometry, and some useful asymptotic estimates that can be applied to prove our theorems. In Section 3 and Section 4, we give the proofs of Theorems 1.1 and 1.4, respectively.

## 2. Preliminaries

In order to precisely state our main theorems, we first introduce some notations. We denote the fractional Sobolev space by $W^{s,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|_p^p + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y\right)^{1/p},$$

where $\|u\|_t = \big(\int_\Omega |u|^t\,\mathrm{d}x\big)^{1/t}$ denotes the norm of the space $L^t(\Omega)$ $(1 \le t \le p_{s_1}^*(b))$.

For $p > 1$ and $s \in (0,1)$, we set $\mathcal{Q} := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$, where $\Omega^c = \mathbb{R}^N \setminus \Omega$ and $t \in \{p,q\}$, we define

$$X_{s,t} := \left\{ u\colon \mathbb{R}^N \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^t(\Omega), \text{ and } \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^t}{|x-y|^{N+st}}\,\mathrm{d}x\mathrm{d}y < +\infty \right\}.$$

We work in the closed linear subspace

$$X_{0,s_1,p} = \big\{ u \in X_{s_1,p} : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \big\},$$

which is a uniformly convex Banach space endowed with the norm as

$$\|u\|_{X_{0,s_1,p}} = \left( \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1 p}}\,\mathrm{d}x\mathrm{d}y \right)^{1/p}.$$

Since $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, the above integral can be extended to all of the $\mathbb{R}^N$.

From [11, Lemma 2.2], we know if $0 < s_2 < s_1 < 1$ and $1 < q \le p$, $\Omega$ is a smooth bounded in $\mathbb{R}^N$, when $N > ps_1$, then $X_{0,s_1,p} \subset X_{0,s_2,q}$, and there exists a constant $C = C(|\Omega|, N, p, q, s_1, s_2) > 0$ such that for all $u \in X_{0,s_1,p}$,

$$\|u\|_{X_{0,s_2,q}} \le C \|u\|_{X_{0,s_1,p}}. \tag{2.1}$$

We define

$$S := \inf_{u \in W^{s_1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{X_{0,s_1,p}}}{\big(\int_\Omega \frac{|u|^{p_{s_1}^*(b)}}{|x|^b}\,\mathrm{d}x\big)^{\frac{p}{p_{s_1}^*(b)}}}. \tag{2.2}$$

Let $L^j(\Omega, |x|^{-b}\,\mathrm{d}x)$ be the weighted $L^j$ space with the norm

$$\|u\|_{L^j(\Omega,|x|^{-b}\,\mathrm{d}x)} = \left( \int_\Omega \frac{|u|^j}{|x|^b}\,\mathrm{d}x \right)^{1/j}.$$

Then from [15], we know that the embedding $X_{0,s_1,p} \hookrightarrow L^j(\Omega, |x|^{-b}\,\mathrm{d}x)$ is continuous for $j \in [1, p_{s_1}^*(b)]$ and compact for $j \in [1, p_{s_1}^*(b))$. So we have the inequality as follows:

$$\int_\Omega \frac{|u|^r}{|x|^a}\,\mathrm{d}x \le C \|u\|^r_{X_{0,s_1,p}}$$

for all $0 < r \le p_{s_1}^*(b)$, where $C$ is a suitable constant.

## 2.1. A compactness result

For $\nu \geq 0$, a function $u \in X_{0,s_1,p}$ is a weak solution of $(P_\nu(\lambda))$ for all $h \in X_{0,s_1,p}$ if

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(h(x)-h(y))}{|x-y|^{N+ps_1}}\,\mathrm{d}x\mathrm{d}y\\&+\nu\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(h(x)-h(y))}{|x-y|^{N+qs_2}}\,\mathrm{d}x\mathrm{d}y\\&=\lambda\int_\Omega \frac{|u|^{r-2}uh}{|x|^a}\,\mathrm{d}x+\int_\Omega \frac{|u|^{p^*_{s_1}(b)-2}uh}{|x|^b}\,\mathrm{d}x.\end{aligned}$$

Weak solutions of $(P_\nu(\lambda))$ coincide with critical points of $C^1$-functional

$$E_\nu(u)=\frac{1}{p}\|u\|^p_{X_{0,s_1,p}}+\frac{\nu}{q}\|u\|^q_{X_{0,s_2,q}}-\frac{\lambda}{r}\int_\Omega\frac{|u|^r}{|x|^a}\,\mathrm{d}x-\frac{1}{p^*_{s_1}(b)}\int_\Omega\frac{|u|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x.$$

Our main results will be based on the following lemma which extends Ho et al. [22, Proposition 3.1] to the fractional $p\&q$-Laplacian.

**Lemma 2.1.** *Let $1<q<p<\frac{N}{s_1}$, $0<s_2<s_1<1$, $0<a,b<ps_1<N$, $r\in(q,p^*_{s_1}(b))$. If $0<c<c^*=\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(b)}\right)S^{\frac{N-b}{ps_1-b}}$, then every $(\mathrm{PS})_c$ sequence has a subsequence that converges weakly to a nontrivial critical point of $E_\nu(u)$.*

*Proof.* Recall that a sequence $(u_j)\subset X_{0,s_1,p}$ such that $E_\nu(u_j)\to c$ and $E'_\nu(u_j)\to 0$ is called a $(\mathrm{PS})_c$ sequence, i.e.,

$$\begin{aligned}E_\nu(u_j)&=\frac{1}{p}\|u_j\|^p_{X_{0,s_1,p}}+\frac{\nu}{q}\|u_j\|^q_{X_{0,s_2,q}}-\frac{\lambda}{r}\int_\Omega\frac{|u_j|^r}{|x|^a}\,\mathrm{d}x-\frac{1}{p^*_{s_1}(b)}\int_\Omega\frac{|u_j|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x\\&=c+o(1),\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\langle E'_\nu(u_j),h\rangle&=\int_{\mathbb{R}^{2N}}\frac{|u_j(x)-u_j(y)|^{p-2}(u_j(x)-u_j(y))(h(x)-h(y))}{|x-y|^{N+ps_1}}\,\mathrm{d}x\mathrm{d}y\\&\quad+\nu\int_{\mathbb{R}^{2N}}\frac{|u_j(x)-u_j(y)|^{q-2}(u_j(x)-u_j(y))(h(x)-h(y))}{|x-y|^{N+qs_2}}\,\mathrm{d}x\mathrm{d}y\\&\quad-\lambda\int_\Omega\frac{|u_j|^{r-2}uh}{|x|^a}\,\mathrm{d}x-\int_\Omega\frac{|u_j|^{p^*_{s_1}(b)-2}uh}{|x|^b}\,\mathrm{d}x\\&=o(1)\|h\|_{X_{0,s_1,p}}.\end{aligned}\tag{2.4}$$

Taking $h=u_j$ in (2.4) gives

$$\begin{aligned}\langle E'_\nu(u_j),u_j\rangle&=\|u_j\|^p_{X_{0,s_1,p}}+\nu\|u_j\|^q_{X_{0,s_2,q}}-\lambda\int_\Omega\frac{|u_j|^r}{|x|^a}\,\mathrm{d}x-\int_\Omega\frac{|u_j|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x\\&=o(1)\|u_j\|_{X_{0,s_1,p}}.\end{aligned}\tag{2.5}$$

Fix $m \in (p, p_{s_1}^*(b))$, dividing (2.5) by $m$ and subtracting from (2.3) gives

$$
\begin{aligned}
c + o(1) + o(1)\|u_j\|_{X_{0,s_1,p}} &= \left(\frac{1}{p} - \frac{1}{m}\right)\|u_j\|_{X_{0,s_1,p}}^p + \left(\frac{1}{m} - \frac{1}{p_{s_1}^*(b)}\right)\int_\Omega \frac{|u_j|^{p_{s_1}^*(b)}}{|x|^b}\,\mathrm{d}x \\
&\quad + \nu\left(\frac{1}{q} - \frac{1}{m}\right)\|u_j\|_{X_{0,s_2,q}}^q - \lambda\left(\frac{1}{r} - \frac{1}{m}\right)\int_\Omega \frac{|u_j|^r}{|x|^a}\,\mathrm{d}x.
\end{aligned} \tag{2.6}
$$

From this and Hölder inequality, we conclude that

$$
\left(\frac{1}{m} - \frac{1}{p_{s_1}^*(b)}\right)\int_\Omega \frac{|u_j|^{p_{s_1}^*(b)}}{|x|^b}\,\mathrm{d}x \le c + o(1) + o(1)\|u_j\|_{X_{0,s_1,p}} + C\left(\int_\Omega \frac{|u_j|^{p_{s_1}^*(b)}}{|x|^b}\,\mathrm{d}x\right)^{\frac{r}{p_{s_1}^*(b)}},
$$

combining with (2.6), we conclude that $(u_j)$ is bounded. So there exists a subsequence (still denoted by $(u_j)$) and $u \in X_{0,s_1,p}$ such that

$$
\begin{aligned}
&u_j \rightharpoonup u \quad \text{weakly in } X_{0,s_1,p},\\
&u_j \to u \quad \text{a.e. on } \Omega,\\
&u_j \rightharpoonup u \quad \text{weakly in } L^{p_{s_1}^*(b)}(\Omega, |x|^{-b}\,\mathrm{d}x),\\
&u_j \to u \quad \text{strongly in } L^r(\Omega, |x|^{-b}\,\mathrm{d}x),\ r \in [1, p_{s_1}^*(b)).
\end{aligned}
$$

Denoting by $p' = \frac{p}{p-1}$ the Hölder conjugate of $p$, $q' = \frac{q}{q-1}$ the Hölder conjugate of $q$.

Similar as in [14], let us observe that the sequence

$$
\left\{\frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))}{|x-y|^{\frac{N+ps_1}{p'}}}\right\}_{j\in\mathbb{N}} \quad \text{is bounded in } L^{p'}(\mathbb{R}^{2N}),
$$

and

$$
\frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))}{|x-y|^{\frac{N+ps_1}{p'}}} \to \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{\frac{N+ps_1}{p'}}} \quad \text{a.e. in } \mathbb{R}^{2N},
$$

and

$$
\frac{h(x) - h(y)}{|x-y|^{\frac{N+ps_1}{p}}} \in L^p(\mathbb{R}^{2N}).
$$

Hence, up to a subsequence, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))(h(x) - h(y))}{|x-y|^{N+ps_1}}\,\mathrm{d}x\mathrm{d}y \\
&\to \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(h(x) - h(y))}{|x-y|^{N+ps_1}}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Similarly for $q$,

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|u_j(x) - u_j(y)|^{q-2}(u_j(x) - u_j(y))(h(x) - h(y))}{|x-y|^{N+qs_2}}\,\mathrm{d}x\mathrm{d}y \\
&\to \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))(h(x) - h(y))}{|x-y|^{N+qs_2}}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

On the other hand, for any $h \in X_{0,s_1,p}$, we have

$$\int_\Omega \frac{|u_j|^{r-1}h}{|x|^a}\,\mathrm{d}x \to \int_\Omega \frac{|u|^{r-1}h}{|x|^a}\,\mathrm{d}x, \quad \int_\Omega \frac{|u_j|^{p^*_{s_1}(b)-1}h}{|x|^b}\,\mathrm{d}x \to \int_\Omega \frac{|u|^{p^*_{s_1}(b)-1}h}{|x|^b}\,\mathrm{d}x.$$

Passing to the limit in (2.4) shows that $u \in X_{0,s_1,p}$ is a weak solution of $(P_\nu(\lambda))$, that is, $E'_\nu(u) = 0$.

Suppose $u = 0$, then (2.3) and (2.5) reduce to

$$E_\nu(u_j) = \frac{1}{p}\|u_j\|^p_{X_{0,s_1,p}} + \frac{\nu}{q}\|u_j\|^q_{X_{0,s_2,q}} - \frac{1}{p^*_{s_1}(b)}\int_\Omega \frac{|u_j|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x = c + o(1), \tag{2.7}$$

and

$$\langle E'_\nu(u_j), u_j\rangle = \|u_j\|^p_{X_{0,s_1,p}} + \nu\|u_j\|^q_{X_{0,s_2,q}} - \int_\Omega \frac{|u_j|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x = o(1)\|u_j\|_{X_{0,s_1,p}}. \tag{2.8}$$

Equation (2.8) together with (2.2) gives

$$\|u_j\|^p_{X_{0,s_1,p}} \le \int_\Omega \frac{|u_j|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x + o(1) \le S^{-\frac{p^*_{s_1}(b)}{p}}\|u_j\|^{p^*_{s_1}(b)}_{X_{0,s_1,p}} + o(1). \tag{2.9}$$

If $\|u_j\|_{X_{0,s_1,p}} \to 0$ for a renamed subsequence, then (2.7) gives $c = 0$, contrary to our assumption that $c > 0$. So $\|u_j\|_{X_{0,s_1,p}}$ is bounded away from 0, and hence (2.9) implies that

$$\|u_j\|^p_{X_{0,s_1,p}} \ge S^{\frac{p^*_{s_1}(b)}{p^*_{s_1}(b)-p}} + o(1) = S^{\frac{N-b}{ps_1-b}} + o(1).$$

Now, multiplying (2.8) by $\frac{1}{p^*_{s_1}(b)}$ and subtracting from (2.7) gives

$$\begin{aligned} c &= \left(\frac{1}{p} - \frac{1}{p^*_{s_1}(b)}\right)\|u_j\|^p_{X_{0,s_1,p}} + \nu\left(\frac{1}{q} - \frac{1}{p^*_{s_1}(b)}\right)\|u_j\|^q_{X_{0,s_2,q}} + o(1) \\ &\ge \left(\frac{1}{p} - \frac{1}{p^*_{s_1}(b)}\right) S^{\frac{N-b}{ps_1-b}} + o(1), \end{aligned}$$

which is contrary to our assumption. □

### 2.2. Mountain pass geometry

Weak solutions of $(P_1(\lambda))$ coincide with the critical points of the $C^1$-functional

$$E_1(u) = \frac{1}{p}\|u\|^p_{X_{0,s_1,p}} + \frac{1}{q}\|u\|^q_{X_{0,s_2,q}} - \frac{\lambda}{r}\int_\Omega \frac{|u|^r}{|x|^a}\,\mathrm{d}x - \frac{1}{p^*_{s_1}(b)}\int_\Omega \frac{|u|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x.$$

Let

$$\eta_1 = \inf_{u \in X_{0,s_1,p}\setminus\{0\}} \frac{\|u\|^p_{X_{0,s_1,p}}}{\int_\Omega \frac{|u|^p}{|x|^a}\,\mathrm{d}x} \tag{2.10}$$

be the first eigenvalue of the eigenvalue problem

$$\begin{cases} (-\Delta)_p^{s_1} u = \eta \frac{|u|^{p-2}u}{|x|^a} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Let

$$\mu_1 = \inf_{u \in X_{0,s_2,q} \setminus \{0\}} \frac{\|u\|^q_{X_{0,s_2,q}}}{\int_\Omega \frac{|u|^q}{|x|^a} \, \mathrm{d}x} \tag{2.11}$$

be the first eigenvalue of the eigenvalue problem

$$\begin{cases} (-\Delta)_q^{s_2} u = \mu \frac{|u|^{q-2}u}{|x|^a} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We note that when $r \in (q, p^*_{s_1}(b))$, for some $\eta \in (0, \eta_1)$ and some constant $b_1 > 0$, we have

$$\frac{\lambda}{r} \frac{|u|^r}{|x|^a} \leq \frac{\eta}{p} \frac{|u|^p}{|x|^a} + \frac{\mu_1}{q} \frac{|u|^q}{|x|^a} + b_1 \frac{|u|^s}{|x|^a} \tag{2.12}$$

holds for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$, where $s \in (p, p^*_{s_1}(b))$.

**Lemma 2.2.** (i) *There exist constants $\rho > 0$ and $\alpha > 0$ such that $E_1(u) \geq \alpha$ for all $u \in X_{0,s_1,p}$ with $\|u\|_{X_{0,s_1,p}} = \rho$.*

(ii) *There exists $u_1 \in X_{0,s_1,p}$ with $\|u_1\|_{X_{0,s_1,p}} > \rho$ such that $E_1(u_1) < \alpha$.*

*Proof.* (i) By (2.2), (2.10)–(2.12), and Sobolev inequality, we have

$$\begin{aligned} E_1(u) \geq{}& \frac{1}{p}\|u\|^p_{X_{0,s_1,p}} + \frac{1}{q}\|u\|^q_{X_{0,s_2,q}} - \frac{\eta}{p}\int_\Omega \frac{|u|^p}{|x|^a} \, \mathrm{d}x \\ & - \frac{\mu_1}{q}\int_\Omega \frac{|u|^q}{|x|^a} \, \mathrm{d}x - b_1 \int_\Omega \frac{|u|^s}{|x|^a} \, \mathrm{d}x - \frac{1}{p^*_{s_1}(b)} \int_\Omega \frac{|u|^{p^*_{s_1}(b)}}{|x|^b} \, \mathrm{d}x \\ \geq{}& \frac{1}{p}\left(1 - \frac{\eta}{\eta_1}\right)\|u\|^p_{X_{0,s_1,p}} - b_1 \int_\Omega \frac{|u|^s}{|x|^a} \, \mathrm{d}x - \frac{1}{p^*_{s_1}(b)} \int_\Omega \frac{|u|^{p^*_{s_1}(b)}}{|x|^b} \, \mathrm{d}x \\ \geq{}& \frac{1}{p}\left(1 - \frac{\eta}{\eta_1}\right)\|u\|^p_{X_{0,s_1,p}} - b_2\|u\|^s_{X_{0,s_1,p}} - \frac{1}{p^*_{s_1}(b)} S^{-\frac{p^*_{s_1}(b)}{p}} \|u\|^{p^*_{s_1}(b)}_{X_{0,s_1,p}} \end{aligned}$$

for some constant $b_2 > 0$. Since $p < s < p^*_{s_1}(b)$, it follows that the origin is a strict local minimizer of $E_1(u)$. Thus we can choose $\|u\|_{X_{0,s_1,p}} = \rho$ sufficiently small and Lemma 2.2(i) holds.

(ii) Let $h \in X_{0,s_1,p}$ with $h > 0$, then we get for $t \to +\infty$,

$$\begin{aligned} E_1(th) ={}& \frac{t^p}{p}\|h\|^p_{X_{0,s_1,p}} + \frac{t^q}{q}\|h\|^q_{X_{0,s_2,q}} - \lambda\frac{t^r}{r}\int_\Omega \frac{|h|^r}{|x|^a} \, \mathrm{d}x - \frac{t^{p^*_{s_1}(b)}}{p^*_{s_1}(b)} \int_\Omega \frac{|h|^{p^*_{s_1}(b)}}{|x|^b} \, \mathrm{d}x \\ & \to -\infty. \end{aligned}$$

By taking $u_1 = th$, we can conclude the proof of Lemma 2.2(ii). $\square$

Let $\Gamma_1 = \{\gamma \in C([0,1], X_{0,s_1,p}) : \gamma(0) = 0, E_1(\gamma(1)) < 0\}$ be the class of paths in $X_{0,s_1,p}$ joining the origin to the set $\{u \in X_{0,s_1,p} : E_1(u) < 0\}$ and set

$$c := \inf_{\gamma\in\Gamma_1} \max_{u\in\gamma([0,1])} E_1(u). \tag{2.13}$$

Since the origin is a strict local minimizer of $E_1(u)$, $c > 0$.

## 2.3. Some estimates

In the following, we shall fix a radially symmetric nonnegative decreasing minimizer $U = U(r)$ for the Sobolev constant $S$. If necessary multiplying $U$ by a positive constant, we may assume that

$$(-\Delta)_p^{s_1} u = \frac{U^{p^*_{s_1}(b)-1}}{|x|^b}, \quad x \in \mathbb{R}^N. \tag{2.14}$$

Testing this equation with $U$ and using (2.2) shows that

$$\|U\|^p_{X_{0,s_1,p}} = |U|^{p^*_{s_1}(b)}_{p^*_{s_1}(b)} = S^{\frac{N-b}{ps_1-b}}. \tag{2.15}$$

For any $\varepsilon > 0$, the function

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-ps_1}{p}} U\left(\frac{|x|}{\varepsilon}\right) \tag{2.16}$$

is also a minimizer for $S$ satisfying (2.14) and (2.15), so after a rescaling we may assume that $U(0) = 1$. Henceforth $U$ will denote such a normalized (with respect to constant multiples and rescaling) minimizer and $U_\varepsilon$ will denote the associated family of minimizer given by (2.16). In the absence of an explicit formula for $U$, we will use the following estimates.

**Lemma 2.3.** [26] *There exist constants $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r > 1$,*

$$\frac{c_1}{r^{\frac{N-ps_1}{p-1}}} \le U(r) \le \frac{c_2}{r^{\frac{N-ps_1}{p-1}}} \quad \textit{and} \quad \frac{U(\theta r)}{U(r)} \le \frac{1}{2}.$$

We construct some auxiliary functional and estimate their norms. In what follows $\theta$ is a universal constant in Lemma 2.3 that depends only on $N$, $p$, and $s_1$. We may assume without loss of generality that $0 \in \Omega$, for $\varepsilon, \delta > 0$, let

$$m_{\varepsilon,\delta} = \frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\delta)},$$

and let

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0, & 0 \le t \le U_\varepsilon(\theta\delta), \\ m^p_{\varepsilon,\delta}(t - U_\varepsilon(\theta\delta)), & U_\varepsilon(\theta\delta) \le t \le U_\varepsilon(\delta), \\ t + U_\varepsilon(\delta)\big(m^{p-1}_{\varepsilon,\delta} - 1\big), & t \ge U_\varepsilon(\delta), \end{cases}$$

and let

$$G_{\varepsilon,\delta}(t)=\int_0^t g'_{\varepsilon,\delta}(\tau)^{1/p}\,\mathrm{d}\tau=\begin{cases}0, & 0\le t\le U_\varepsilon(\theta\delta),\\ m_{\varepsilon,\delta}(t-U_\varepsilon(\theta\delta)), & U_\varepsilon(\theta\delta)\le t\le U_\varepsilon(\delta),\\ t, & t\ge U_\varepsilon(\delta).\end{cases}$$

The functions $g_{\varepsilon,\delta}(t)$ and $G_{\varepsilon,\delta}(t)$ are absolutely continuous and nondecreasing. By definition, we obtain

$$G'_{\varepsilon,\delta}(t)=\big(g'_{\varepsilon,\delta}(t)\big)^{1/p}=\begin{cases}0, & 0\le t\le U_\varepsilon(\theta\delta),\\ m_{\varepsilon,\delta}, & U_\varepsilon(\theta\delta)\le t\le U_\varepsilon(\delta),\\ 1, & t\ge U_\varepsilon(\delta),\end{cases}$$

therefore,

$$G'_{\varepsilon,\delta}(t)\le \max\{m_{\varepsilon,\delta},1\}\le m_{\varepsilon,\delta}+1. \tag{2.17}$$

Next we estimate $m_{\varepsilon,\delta}$ as follows. Choosing $\varepsilon>0$ small enough such that $\delta/\varepsilon>1$ and thus $U(\theta\delta/\varepsilon)/U(\delta/\varepsilon)\le 1/2$, we get

$$m_{\varepsilon,\delta}=\frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta)-U_\varepsilon(\theta\delta)}=\frac{U(\delta/\varepsilon)}{U(\delta/\varepsilon)-U(\theta\delta/\varepsilon)}\le \frac{c_2}{c_1}\theta^{\frac{N-ps_1}{p-1}}. \tag{2.18}$$

Consider the radially symmetric nonincreasing function $u_{\varepsilon,\delta}=G_{\varepsilon,\delta}(U_\varepsilon(r))$, which satisfies

$$U_{\varepsilon,\delta}(r)=\begin{cases}U_\varepsilon(r) & \text{if } r\le\delta,\\ 0 & \text{if } r\ge\theta\delta.\end{cases}$$

Recall that $h(x)=\Theta(g(x))$ as $x\to 0$ if there exist constants $c,C>0$ such that

$$c\cdot|g(x)|\le|h(x)|\le C\cdot|g(x)|$$

for all sufficiently small $\varepsilon>0$. We have the following estimates.

**Lemma 2.4.** [31, Lemma 2.7] *For any $0<2\varepsilon\le\delta<\theta^{-1}\operatorname{dist}(0,\partial\Omega)$, we have the following estimates*

$$\|u_{\varepsilon,\delta}\|^p_{X_{0,s_1,p}}=S^{\frac{N-b}{ps_1-b}}+\Theta\big((\varepsilon/\delta)^{\frac{N-ps_1}{p-1}}\big), \tag{2.19}$$

*and*

$$|u_{\varepsilon,\delta}|^{p^*_{s_1}(b)}_{p^*_{s_1}(b)}=S^{\frac{N-b}{ps_1-b}}-\Theta\big((\varepsilon/\delta)^{\frac{N-b}{p-1}}\big). \tag{2.20}$$

**Lemma 2.5.** *Let $u_{\varepsilon,\delta}$ be defined as above, inspired by* [11], *we have*

$$\|u_{\varepsilon,\delta}\|^t_{X_{0,s_2,t}} = \Theta\big(\varepsilon^{\frac{N(p-t)}{p}}\big).$$

*Proof.* By (2.1) we have

$$\begin{aligned}\|u_{\varepsilon,\delta}\|^t_{X_{0,s_2,t}} &\le C\|u_{\varepsilon,\delta}\|^t_{X_{0,s_1,t}}\\ &= C\int_{\mathbb{R}^{2N}} \frac{\big|G_{\varepsilon,\delta}(U_\varepsilon(x)) - G_{\varepsilon,\delta}(U_\varepsilon(y))\big|^t}{|x-y|^{N+ts_1}}\,\mathrm{d}x\mathrm{d}y\\ &\le C\int_{\mathbb{R}^{2N}} \frac{\big|G'_{\varepsilon,\delta}\big(U_\varepsilon(x)+\tau(U_\varepsilon(y)-U_\varepsilon(x))\big)\big|^t|U_\varepsilon(x)-U_\varepsilon(y)|^t}{|x-y|^{N+ts_1}}\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.21}$$

In the last line, we have used the mean value theorem for some $\tau \in (0,1)$. Thus from (2.17) and (2.18) we get

$$G'_{\varepsilon,\delta}\big(U_\varepsilon(x)+\tau(U_\varepsilon(y)-U_\varepsilon(x))\big) \le 1 + \frac{c_2}{c_1}\theta^{\frac{N-ps_1}{p-1}} = C. \tag{2.22}$$

Putting (2.22) into (2.21) yields

$$\begin{aligned}\|u_{\varepsilon,\delta}\|^t_{X_{0,s_2,t}} &\le C\int_{\mathbb{R}^{2N}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^t}{|x-y|^{N+ts_1}}\,\mathrm{d}x\mathrm{d}y\\ &= \varepsilon^{N-ts_1-\frac{(N-ps_1)t}{p}}\int_{\mathbb{R}^{2N}} \frac{|U(x)-U(y)|^t}{|x-y|^{N+ts_1}}\,\mathrm{d}x\mathrm{d}y\\ &= \varepsilon^{\frac{N(p-t)}{p}}\|U\|^t_{X_{0,s_1,t}}.\end{aligned}$$

In particular,

$$\|u_{\varepsilon,\delta}\|^q_{X_{0,s_2,q}} = \Theta\big(\varepsilon^{\frac{N(p-q)}{p}}\big). \tag{2.23}$$

□

**Lemma 2.6.** *Let $u_{\varepsilon,\delta}$ be defined as above, we have*

$$\int_\Omega \frac{|u_{\varepsilon,\delta}|^r}{|x|^a}\,\mathrm{d}x = \begin{cases}\Theta\big(\varepsilon^{(N-ps_1)\left(\frac{r}{p-1}-\frac{r}{p}\right)}\delta^{N-a-\frac{(N-ps_1)r}{p-1}}\big) & \text{if } r < \frac{(N-a)(p-1)}{N-ps_1},\\ \Theta\big(\varepsilon^{\frac{N-a}{p}}|\ln(\delta/\varepsilon)|\big) & \text{if } r = \frac{(N-a)(p-1)}{N-ps_1},\\ \Theta\big(\varepsilon^{N-a-\frac{r(N-ps_1)}{p}}\big) & \text{if } r > \frac{(N-a)(p-1)}{N-ps_1}.\end{cases}$$

*Proof.*

$$\begin{aligned}\int_\Omega \frac{|u_{\varepsilon,\delta}|^r}{|x|^a}\,\mathrm{d}x &\ge \int_{B_\delta(0)} \frac{|U_\varepsilon(x)|^r}{|x|^a}\,\mathrm{d}x\\ &= \varepsilon^{-\frac{r(N-ps_1)}{p}}\int_{B_\delta(0)} \frac{|U(\frac{x}{\varepsilon})|^r}{|x|^a}\,\mathrm{d}x\\ &= \varepsilon^{N-a-\frac{r(N-ps_1)}{p}}\int_{B_{\delta/\varepsilon}(0)} \frac{|U(x)|^r}{|x|^a}\,\mathrm{d}x\\ &\ge C\varepsilon^{N-a-\frac{r(N-ps_1)}{p}}\int_1^{\delta/\varepsilon} \rho^{-\frac{r(N-ps_1)}{p-1}+N-a-1}\,\mathrm{d}\rho.\end{aligned}$$

(i) If $r < \frac{(N-a)(p-1)}{N-ps_1}$, then

$$\int_1^{\delta/\varepsilon} \rho^{-\frac{r(N-ps_1)}{p-1}+N-a-1}\, \mathrm{d}\rho = C(\delta/\varepsilon)^{-\frac{N(p-s_1)r}{p-1}+N-a},$$

so

$$\int_\Omega \frac{|u_{\varepsilon,\delta}|^r}{|x|^a}\, \mathrm{d}x = \Theta\big(\varepsilon^{(N-ps_1)\left(\frac{r}{p-1}-\frac{r}{p}\right)} \delta^{N-a-\frac{(N-ps_1)r}{p-1}}\big).$$

(ii) If $r = \frac{(N-a)(p-1)}{N-ps_1}$, then

$$\int_1^{\delta/\varepsilon} \rho^{-\frac{r(N-ps_1)}{p-1}+N-a-1}\, \mathrm{d}\rho = \int_1^{\delta/\varepsilon} \frac{1}{\rho}\, \mathrm{d}\rho = C|\ln(\delta/\varepsilon)|,$$

so

$$\int_\Omega \frac{|u_{\varepsilon,\delta}|^r}{|x|^a}\, \mathrm{d}x = \Theta\big(\varepsilon^{\frac{N-a}{p}} |\ln(\delta/\varepsilon)|\big).$$

(iii) If $r > \frac{(N-a)(p-1)}{N-ps_1}$, then there exists $C$ such that

$$\left|\int_1^{\delta/\varepsilon} \rho^{-\frac{r(N-ps_1)}{p-1}+N-a-1}\, \mathrm{d}\rho\right| \leq C,$$

so

$$\int_\Omega \frac{|u_{\varepsilon,\delta}|^r}{|x|^a}\, \mathrm{d}x = \Theta\big(\varepsilon^{N-a-\frac{r(N-ps_1)}{p}}\big). \qquad \square$$

**Lemma 2.7.** *If* $(\varepsilon_j)$, $(\delta_j)$ *are sequences such that* $\varepsilon_j \to 0$, $0 < \delta_j \leq 1$, $\varepsilon_j/\delta_j \to 0$,

$$\frac{\nu \|u_{\varepsilon_j,\delta_j}\|^q_{X_{0,s_2,q}}}{\int_\Omega \frac{|u_{\varepsilon_j,\delta_j}|^r}{|x|^a}\, \mathrm{d}x} \to 0, \quad \frac{(\varepsilon_j/\delta_j)^{\frac{N-ps_1}{p-1}}}{\int_\Omega \frac{|u_{\varepsilon_j,\delta_j}|^r}{|x|^a}\, \mathrm{d}x} \to 0, \tag{2.24}$$

*then*

$$\max_{t\geq 0} E_\nu\big(tu_{\varepsilon_j,\delta_j}(x)\big) < c^* = \left(\frac{1}{p} - \frac{1}{p^*_{s_1}(b)}\right) S^{\frac{N-b}{ps_1-b}}$$

*for all sufficiently large* $j$.

*Proof.* Write $\widetilde{u_j} = u_{\varepsilon_j,\delta_j}(x)$. We know that

$$\begin{aligned} E_\nu(t\widetilde{u_j}) &= \frac{t^p}{p}\|\widetilde{u_j}\|^p_{X_{0,s_1,p}} + \nu\frac{t^q}{q}\|\widetilde{u_j}\|^q_{X_{0,s_2,q}} - \lambda\frac{t^r}{r}\int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\, \mathrm{d}x - \frac{t^{p^*_{s_1}(b)}}{p^*_{s_1}(b)}\int_\Omega \frac{|\widetilde{u_j}|^{p^*_{s_1}(b)}}{|x|^b}\, \mathrm{d}x \\ &=: \varphi(t). \end{aligned}$$

Suppose that the conclusion of the Lemma 2.7 is false. Then there are renamed subsequences $(\varepsilon_j)$, $(\delta_j)$ and $t_j > 0$ such that

$$\begin{aligned} \varphi(t_j) &= \frac{t_j^p}{p}\|\widetilde{u_j}\|^p_{X_{0,s_1,p}} + \nu\frac{t_j^q}{q}\|\widetilde{u_j}\|^q_{X_{0,s_2,q}} - \lambda\frac{t_j^r}{r}\int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\, \mathrm{d}x - \frac{t_j^{p^*_{s_1}(b)}}{p^*_{s_1}(b)}\int_\Omega \frac{|\widetilde{u_j}|^{p^*_{s_1}(b)}}{|x|^b}\, \mathrm{d}x \\ &\geq c^*, \end{aligned} \tag{2.25}$$

and

$$t_j\varphi'(t_j) = t_j^p\|\widetilde{u_j}\|^p_{X_{0,s_1,p}} + \nu t_j^q\|\widetilde{u_j}\|^q_{X_{0,s_2,q}} - \lambda t_j^r\int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\,\mathrm{d}x - t_j^{p^*_{s_1}(b)}\int_\Omega \frac{|\widetilde{u_j}|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x = 0. \tag{2.26}$$

Noting that by (2.23), Lemmas 2.4 and 2.6, we have

$$\|\widetilde{u_j}\|^p_{X_{0,s_1,p}} \to S^{\frac{N-b}{ps_1-b}}, \quad \|\widetilde{u_j}\|^q_{X_{0,s_2,q}} \to 0, \quad \int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\,\mathrm{d}x \to 0, \quad \int_\Omega \frac{|\widetilde{u_j}|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x \to S^{\frac{N-b}{ps_1-b}}.$$

So (2.25) implies that the sequence $(t_j)$ is bounded and hence converges to some $t_0 > 0$ for a subsequence. Passing to the limit in (2.26) gives

$$t_0^p S^{\frac{N-b}{ps_1-b}} - t_0^{p^*_{s_1}(b)} S^{\frac{N-b}{ps_1-b}} = 0, \tag{2.27}$$

so $t_0 = 1$. Subtracting (2.27) from (2.26) and using (2.19) and (2.20) gives

$$\begin{aligned} &t_j^p\left(S^{\frac{N-b}{ps_1-b}} + \Theta\big((\varepsilon_j/\delta_j)^{\frac{N-ps_1}{p-1}}\big)\right) + \nu t_j^q\|\widetilde{u_j}\|^q_{X_{0,s_2,q}} - \lambda t_j^r\int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\,\mathrm{d}x \\ &- t_j^{p^*_{s_1}(b)}\left(S^{\frac{N-b}{ps_1-b}} - \Theta\big((\varepsilon_j/\delta_j)^{\frac{N-b}{p-1}}\big)\right) - \left(t_0^p S^{\frac{N-b}{ps_1-b}} - t_0^{p^*_{s_1}(b)} S^{\frac{N-b}{ps_1-b}}\right) = 0. \end{aligned}$$

Simplifying this gives

$$\begin{aligned} &S^{\frac{N-b}{ps_1-b}}\big(t_j^p - t_0^p\big) - S^{\frac{N-b}{ps_1-b}}\big(t_j^{p^*_{s_1}(b)} - t_0^{p^*_{s_1}(b)}\big) \\ &= \lambda t_j^r\int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\,\mathrm{d}x - \nu t_j^q\|\widetilde{u_j}\|^q_{X_{0,s_2,q}} + \Theta\big((\varepsilon_j/\delta_j)^{\frac{N-ps_1}{p-1}}\big) + \Theta\big((\varepsilon_j/\delta_j)^{\frac{N-b}{p-1}}\big). \end{aligned}$$

By the mean value theorem, we can get

$$\begin{aligned} &S^{\frac{N-b}{ps_1-b}}\big(p\sigma_j^{p-1} - p^*_{s_1}(b)\tau_j^{p^*_{s_1}(b)-1}\big)(t_j - t_0) \\ &= \lambda t_j^r\int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\,\mathrm{d}x - \nu t_j^q\|\widetilde{u_j}\|^q_{X_{0,s_2,q}} + \Theta\big((\varepsilon_j/\delta_j)^{\frac{N-ps_1}{p-1}}\big), \end{aligned} \tag{2.28}$$

where $\sigma_j$ and $\tau_j$ are between $t_0$ and $t_j$. Since $t_j \to t_0$, $\sigma_j, \tau_j \to t_0$, hence

$$p\sigma_j^{p-1} - p^*_{s_1}(b)\tau_j^{p^*_{s_1}(b)-1} \to -(p^*_{s_1}(b) - p).$$

Thus (2.28) together with (2.24) gives

$$t_j - t_0 = \frac{\lambda t_j^r\int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\,\mathrm{d}x - \nu t_j^q\|\widetilde{u_j}\|^q_{X_{0,s_2,q}} + \Theta\big((\varepsilon_j/\delta_j)^{\frac{N-ps_1}{p-1}}\big)}{S^{\frac{N-b}{ps_1-b}}\big(p\sigma_j^{p-1} - p^*_{s_1}(b)\tau_j^{p^*_{s_1}(b)-1}\big)},$$

that is,

$$t_j = t_0 - \left(\frac{\lambda t_j^r}{(p^*_{s_1}(b) - p)S^{\frac{N-b}{ps_1-b}}} + o(1)\right)\int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\,\mathrm{d}x < t_0$$

for all sufficiently large $j$.

Dividing (2.26) by $p^*_{s_1}(b)$ and subtracting from (2.25) gives

$$\left(\frac{1}{p} - \frac{1}{p^*_{s_1}(b)}\right) t_j^p \|\widetilde{u_j}\|^p_{X_{0,s_1,p}} + \nu \left(\frac{1}{q} - \frac{1}{p^*_{s_1}(b)}\right) t_j^q \|\widetilde{u_j}\|^q_{X_{0,s_2,q}} - \lambda \left(\frac{1}{r} - \frac{1}{p^*_{s_1}(b)}\right) t_j^r \int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\, \mathrm{d}x \geq c^*,$$

then using (2.19) gives

$$\left(\frac{1}{p} - \frac{1}{p^*_{s_1}(b)}\right) t_j^p S^{\frac{N-b}{ps_1-b}} + \nu \left(\frac{1}{q} - \frac{1}{p^*_{s_1}(b)}\right) t_j^q \|\widetilde{u_j}\|^q_{X_{0,s_2,q}} - \lambda \left(\frac{1}{r} - \frac{1}{p^*_{s_1}(b)}\right) t_j^r \int_\Omega \frac{|\widetilde{u_j}|^r}{|x|^a}\, \mathrm{d}x$$
$$\geq \left(\frac{1}{p} - \frac{1}{p^*_{s_1}(b)}\right) S^{\frac{N-b}{ps_1-b}} + \Theta\big((\varepsilon_j/\delta_j)^{\frac{N-ps_1}{p-1}}\big).$$

This together with $t_j < t_0 = 1$ and (2.24) gives

$$\lambda \left(\frac{1}{r} - \frac{1}{p^*_{s_1}(b)}\right) \leq 0,$$

a contradiction since $r \in (q, p^*_{s_1}(b))$ and $\lambda > 0$. □

## 3. Proof of Theorem 1.1

As mentioned above, it suffices to show that the mountain pass level $c$ defined in (2.13) is below the threshold level $c^*$ defined in Lemma 2.1.

For any $u \in X_{0,s_1,p} \setminus \{0\}$, $E_1(tu) \to -\infty$ as $t \to +\infty$ and hence $\exists\, t_0 > 0$ such that $E_1(t_0 u) < 0$, then the line segment $\{tu : 0 \leq t \leq t_0\}$ belongs to $\Gamma_1$ and hence

$$c \leq \max_{0 \leq t \leq t_0} E_1(tu) \leq \max_{t \geq 0} E_1(tu). \tag{3.1}$$

If $r > (N-a)(p-1)/(N-ps_1)$, we will construct sequences $(\varepsilon_j)$, $(\delta_j)$ such that $\varepsilon_j \to 0$, $0 < \delta_j \leq 1$, $\varepsilon_j/\delta_j \to 0$ and (2.24) with $\nu = 1$.

(i) When $N < p^2 s_1$, we take a sequence $\varepsilon_j \to 0$ and set $\delta_j = \varepsilon_j^\kappa$, where $\kappa \in [0,1)$ is to be determined. Let

$$q < p - \frac{p(N-ps_1)}{N(p-1)}, \quad r > \frac{p(N-a)}{N-ps_1} - \frac{p}{p-1}, \quad \max\left\{0, b - \frac{N-ps_1}{p-1}\right\} < a \leq \frac{p^2 s_1 - N}{p-1}.$$

Since $a \leq (p^2 s_1 - N)/(p-1)$,

$$r > \frac{p(N-a)}{N-ps_1} - \frac{p}{p-1} \geq \frac{(N-a)(p-1)}{N-ps_1}.$$

By (2.23) and Lemma 2.6, we have

$$\frac{\|u_{\varepsilon_j,\delta_j}\|^q_{X_{0,s_2,q}}}{\int_\Omega \frac{|u_{\varepsilon_j,\delta_j}|^r}{|x|^a}\, \mathrm{d}x} = \Theta\big(\varepsilon_j^{\frac{N(p-q)}{p} - N + a + \frac{r(N-ps_1)}{p}}\big),$$

where the exponent of $\varepsilon_j$ is positive obviously, after that the first limit in (2.24) holds. For the second limit in (2.24), we have

$$\begin{aligned}\frac{(\varepsilon_j/\delta_j)^{\frac{N-ps_1}{p-1}}}{\int_\Omega \frac{|u_{\varepsilon_j,\delta_j}|^r}{|x|^a}\,\mathrm{d}x} &= \Theta\big(\varepsilon_j^{\frac{N-ps_1}{p-1}-N+a+\frac{r(N-ps_1)}{p}}\delta_j^{-\frac{N-ps_1}{p-1}}\big)\\ &= \Theta\big(\varepsilon_j^{\frac{N-ps_1}{p-1}-N+a+\frac{r(N-ps_1)}{p}-\kappa\frac{N-ps_1}{p-1}}\big)\\ &= \Theta\big(\varepsilon_j^{\frac{(\overline{\kappa}-\kappa)(N-ps_1)}{p-1}}\big),\end{aligned}$$

where

$$\overline{\kappa} = \frac{(N-ps_1)p + (a-N)(p-1)p + r(N-ps_1)(p-1)}{(N-ps_1)p}.$$

We want to choose $\kappa \in [0,1)$ such that $\overline{\kappa} > \kappa$, this is possible if and only if $\overline{\kappa} > 0$. Calculations show that inequality is equivalent to

$$r > -\frac{(N-ps_1)p + (a-N)(p-1)p}{(N-ps_1)(p-1)} = \frac{p(N-a)}{N-ps_1} - \frac{p}{p-1},$$

under our assumptions on $q$, $r$, $N$, and $a$, this is clearly true, then the second limit (2.24) holds.

(ii) When $N < p^2 s_1$, we take a sequence $\varepsilon_j \to 0$ and set $\delta_j = 1$. Let

$$p - \frac{p(N-ps_1)}{N(p-1)} \le q < p, \quad r > \frac{Nq-ap}{N-ps_1}, \quad 0 < b \le a \le \frac{p^2 s_1 - N}{p-1}.$$

Since $q \ge p - p(N-ps_1)/N(p-1)$ and $a \le (p^2 s_1 - N)/(p-1)$,

$$r > \frac{Nq-ap}{N-ps_1} \ge \frac{p(N-a)}{N-ps_1} - \frac{p}{p-1} \ge \frac{(N-a)(p-1)}{N-ps_1}.$$

In this case, we have the following estimates of quotient in (2.24):

$$\frac{\|u_{\varepsilon_j,\delta_j}\|^q_{X_{0,s_2,q}}}{\int_\Omega \frac{|u_{\varepsilon_j,\delta_j}|^r}{|x|^a}\,\mathrm{d}x} = \Theta\big(\varepsilon_j^{\frac{N(p-q)}{p}-N+a+\frac{r(N-ps_1)}{p}}\big),$$

and

$$\frac{(\varepsilon_j/\delta_j)^{\frac{N-ps_1}{p-1}}}{\int_\Omega \frac{|u_{\varepsilon_j,\delta_j}|^r}{|x|^a}\,\mathrm{d}x} = \Theta\big(\varepsilon_j^{\frac{N-ps_1}{p-1}-N+a+\frac{r(N-ps_1)}{p}}\big).$$

Since $r > (Nq-ap)/(N-ps_1)$, the first limit in (2.24) holds, the second limit also holds since

$$\frac{Nq-ap}{N-ps_1} \ge \frac{p(N-a)}{N-ps_1} - \frac{p}{p-1}.$$

Whether in cases (i) or (ii), it follows from Lemma 2.7 and (3.1) that $c < c^*$.

## 4. Proof of Theorem 1.4

Weak solutions of $(P_\nu(\lambda))$ coincide with critical points of $C^1$-functional

$$E_\nu(u) = E_0(u) + \frac{\nu}{q}\|u\|^q_{X_{0,s_2,q}},$$

where

$$E_0(u) = \frac{1}{p}\|u\|^p_{X_{0,s_1,p}} - \frac{\lambda}{r}\int_\Omega \frac{|u|^r}{|x|^a}\,\mathrm{d}x - \frac{1}{p^*_{s_1}(b)}\int_\Omega \frac{|u|^{p^*_{s_1}(b)}}{|x|^b}\,\mathrm{d}x.$$

Let $\Gamma_\nu = \{\gamma \in C([0,1], X_{0,s_1,p}) : \gamma(0) = 0, E_\nu(\gamma(1)) < 0\}$, set

$$\widetilde{c} := \inf_{\gamma\in\Gamma_\nu} \max_{u\in\gamma([0,1])} E_\nu(u),$$

and note that $\tilde{c} > 0$ when $\nu > 0$. We will show that $\widetilde{c} < c^*$ for sufficiently small $\nu$.

Taking $\nu = 0$ and $\delta_j = 1$ in Lemma 2.7, for all sufficiently small $\varepsilon > 0$, we can conclude that $\max_{t\geq 0} E_0(tu_{\varepsilon,1}(x)) < c^*$ holds, provided

$$\frac{\varepsilon^{\frac{N-ps_1}{p-1}}}{\int_\Omega \frac{|u_{\varepsilon,1}|^r}{|x|^a}\,\mathrm{d}x} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.1}$$

As the proof of Theorem 1.1, in each of the two cases of Theorem 1.4, we will show that (4.1) holds for $u_0 = u_{\varepsilon,1}(x)$ with $\varepsilon > 0$ sufficiently small.

(i) Let $(p^2 s_1 - N)/(p-1) \leq a < ps_1$, we note that if $r < (N-a)(p-1)/(N-ps_1)$, then

$$\frac{\varepsilon^{\frac{N-ps_1}{p-1}}}{\int_\Omega \frac{|u_{\varepsilon,1}|^r}{|x|^a}\,\mathrm{d}x} = \Theta\big(\varepsilon^{\frac{N-ps_1}{p-1} - (N-ps_1)\left(\frac{r}{p-1} - \frac{r}{p}\right)}\big) \to 0 \quad \text{as } \varepsilon \to 0;$$

and if $r = (N-a)(p-1)/(N-ps_1)$, then

$$\frac{\varepsilon^{\frac{N-ps_1}{p-1}}}{\int_\Omega \frac{|u_{\varepsilon,1}|^r}{|x|^a}\,\mathrm{d}x} = \Theta\big(\varepsilon^{\left(\frac{N-ps_1}{p-1} - \frac{N-a}{p}\right)}/|\ln\varepsilon|\big) \to 0 \quad \text{as } \varepsilon \to 0;$$

and if $r > (N-a)(p-1)/(N-ps_1)$, then

$$\frac{\varepsilon^{\frac{N-ps_1}{p-1}}}{\int_\Omega \frac{|u_{\varepsilon,1}|^r}{|x|^a}\,\mathrm{d}x} = \Theta\big(\varepsilon^{\frac{N-ps_1}{p-1} - N + a + \frac{r(N-ps_1)}{p}}\big) \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus when $r \in (q, p^*_{s_1}(b))$ and $a \in \big[\max\big\{0, \frac{p^2 s_1 - N}{p-1}\big\}, ps_1\big) \setminus \{0\}$, the (4.1) holds obviously.

(ii) Let $a < (p^2 s_1 - N)/(p-1)$, we have

$$p < \frac{(N-a)(p-1)}{N-ps_1} < \frac{p(N-a)}{N-ps_1} - \frac{p}{p-1}.$$

If $q < r < p$, then

$$r < \frac{(N-a)(p-1)}{N-ps_1}.$$

On the other hand, when $a > \max\left\{0, b - \frac{N-ps_1}{p-1}\right\}$, we have

$$\frac{p(N-a)}{N-ps_1} - \frac{p}{p-1} < p^*_{s_1}(b).$$

Then if $p(N-a)/(N-ps_1) - p/(p-1) < r < p^*_{s_1}(b)$, then

$$r > \frac{(N-a)(p-1)}{N-ps_1}.$$

In this case, from Lemma 2.6, we have the following estimates for the quotient in (4.1):

$$\frac{\varepsilon^{\frac{N-ps_1}{p-1}}}{\int_\Omega \frac{|u_{\varepsilon,1}|^r}{|x|^a}\,\mathrm{d}x} = \begin{cases} \Theta\left(\varepsilon^{\frac{N-ps_1}{p-1} - (N-ps_1)\left(\frac{r}{p-1} - \frac{r}{p}\right)}\right) & \text{if } r < \frac{(N-a)(p-1)}{N-ps_1}, \\ \Theta\left(\varepsilon^{\frac{N-ps_1}{p-1} - N + a + \frac{r(N-ps_1)}{p}}\right) & \text{if } r > \frac{(N-a)(p-1)}{N-ps_1}, \end{cases}$$

where the exponents of $\varepsilon$ are both positive.

Whether in cases (i) or (ii), it follows from (4.1) that for some $u_0 \in X_{0,s_1,p} \setminus \{0\}$,

$$\max_{t \geq 0} E_0(tu_0) < c^*.$$

Therefore, we have

$$c_0 := \inf_{\gamma \in \Gamma_0} \max_{u \in \gamma([0,1])} E_0(u) < c^*.$$

Then there is a path $\gamma_0 \in \Gamma_0$ such that

$$\max_{u \in \gamma_0([0,1])} E_0(u) < c^*.$$

For all sufficiently small $\nu > 0$,

$$E_\nu(\gamma_0(1)) = E_0(\gamma_0(1)) + \frac{\nu}{q} \|\gamma_0(1)\|^q_{X_{0,s_2,q}} < 0,$$

and

$$\max_{u \in \gamma_0([0,1])} E_\nu(u) \leq \max_{u \in \gamma_0([0,1])} E_0(u) + \frac{\nu}{q} \left( \max_{u \in \gamma_0([0,1])} \|u\|^q_{X_{0,s_2,q}} \right) < c^*.$$

So $\gamma_0 \in \Gamma_\nu$ and

$$\widetilde{c} \leq \max_{u \in \gamma_0([0,1])} E_\nu(u) < c^*.$$

This completes the proof of Theorem 1.4.

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