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Non-vanishing of L-functions of Vector-valued Modular Forms

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Abstract. Kohnen proved a non-vanishing result for L-functions associated to Hecke eigenforms of integral weights on the full group. In this paper, we show a non-vanishing result for the averages of L-functions associated with the orthogonal basis of the space of cusp forms of vector-valued modular forms of weight $k \in \frac{1}{2}\mathbb{Z}$ on the full group. We also show the existence of at least one basis element whose L-function does not vanish under certain conditions. As an application, we generalize the result of Kohnen to $\Gamma_0(N)$ and prove the analogous result for Jacobi forms.

1. Introduction

Vector-valued modular forms have played a crucial role in the theory of modular forms. In particular, Selberg used these forms to give an estimation for the Fourier coefficients of the classical modular forms [13]. Moreover, vector-valued modular forms arise naturally in the theory of Jacobi forms, Siegel modular forms, and Moonshine. Some applications of vector-valued modular forms stand out in high-energy physics by mainly providing a method of differential equations in order to construct the modular multiplets, and also revealing the simple structure of the modular invariant mass models [11]. Other applications concerning vector-valued modular forms of half-integer weight seem to provide a simple solution to the Riemann–Hilbert problem for representations of the modular group [2].

In [7,8], Knopp and/or Mason gave a systematic development of the theory of vector-valued modular forms where they introduced the foundation of the space of these forms mainly through the introduction of vector-valued Poincaré series and vector-valued Eisenstein series leading to a better understanding of the space of vector-valued modular forms. More recently, several algorithms for computing Fourier coefficients of vector-valued modular forms were determined in connection to Weil representations due to their importance in the Moonshine applications [12].

On the other hand, L-functions of vector-valued modular forms play important role in the above-mentioned computations as well so it is natural to study them. In this paper, we show a non-vanishing result for averages of L-functions associated with vector-valued

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modular forms. To illustrate, we let $\{f_{k,1}, \ldots, f_{k,d_k}\}$ be an orthogonal basis of $S_{k,\chi,\rho}$ with Fourier expansions

$$f_{k,l}(\tau) = \sum_{j=1}^{m} \sum_{n+\kappa_j>0} b_{k,l,j}(n) e^{2\pi i(n+\kappa_j)\tau} \mathbf{e}_j, \quad 1 \le l \le d_k,$$

where χ is a multiplier system of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\mathrm{SL}_2(\mathbb{Z})$ and $\rho\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{C})$ is an m-dimensional unitary complex representation. Here and throughout the paper, κ_j is a certain positive number with $0 \le \kappa_j < 1$. We let $t_0 \in \mathbb{R}$, $\epsilon > 0$, and $1 \le i \le m$. Then, there exists a constant $C(t_0, \epsilon, i) > 0$ such that for $k > C(t_0, \epsilon, i)$ the function

$$\sum_{l=1}^{d_k} \frac{\langle L^*(f_{k,l}, s), \mathbf{e}_i \rangle}{(f_{k,l}, f_{k,l})} b_{k,l,i}(n_{i,0})$$

does not vanish at any point $s = \sigma + it_0$ with $(k-1)/2 < \sigma < k/2 - \epsilon$, where $\langle L^*(f_{k,l}, s), \mathbf{e}_i \rangle$ denotes the i-th component of $L^*(f_{k,l}, s)$ and $n_{i,0}$ is defined by

$$n_{i,0} := \begin{cases} 1 & \text{if } \kappa_i = 0, \\ 0 & \text{if } \kappa_i \neq 0. \end{cases}$$

By using the integral weight case, we generalize a result of Kohnen in [9] to $\Gamma_0(N)$ in Section 5. On the other hand, by using the half-integral weight case, we prove the analogous result for Jacobi forms in Section 6.

2. Preliminaries

Let $k \in \frac{1}{2}\mathbb{Z}$ and χ be a unitary multiplier system of weight k on Γ, i.e., $\chi\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ satisfies the following conditions:

(1) $|\chi(\gamma)| = 1$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

(2) χ satisfies the consistency condition

$$\chi(\gamma_3)(c_3\tau + d_3)^k = \chi(\gamma_1)\chi(\gamma_2)(c_1\gamma_2\tau + d_1)^k (c_2\tau + d_2)^k,$$

where $\gamma_3 = \gamma_1\gamma_2$ and $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ for $i = 1, 2, 3$.

Throughout this paper, we use the convention that $\sqrt{\tau}$ is chosen so that $\arg(\sqrt{\tau}) \in (-\pi/2, \pi/2]$. Let m be a positive integer and $\rho\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{C})$ an m-dimensional unitary complex representation. We assume that $\rho(-I)$ is the identity matrix, where I denotes the identity matrix. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ denote the standard basis of $\mathbb{C}^m$. For a vector-valued function $f = \sum_{j=1}^{m} f_j \mathbf{e}_j$ on $\mathbb{H}$ and $\gamma \in \Gamma$, define a slash operator by

$$(f|_{k,\chi,\rho}\gamma)(\tau) := \chi^{-1}(\gamma)(c\tau + d)^{-k} \rho^{-1}(\gamma) f(\gamma\tau).$$

Definition 2.1. A vector-valued modular form of weight k and multiplier system χ with respect to ρ on $\mathrm{SL}_2(\mathbb{Z})$ is a sum $f = \sum_{j=1}^{m} f_j \mathbf{e}_j$ of functions holomorphic in $\mathbb{H}$ satisfying the following conditions:

(1) $f|_{k,\chi,\rho}\gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

(2) For each $1 \le j \le m$, each function f_j has a Fourier expansion of the form

$$f_j(\tau) = \sum_{n+\kappa_j \ge 0} a_j(n) e^{2\pi i (n+\kappa_j)\tau}.$$

We write $M_{k,\chi,\rho}$ for the space of vector-valued modular forms of weight k and multiplier system χ with respect to ρ on $\mathrm{SL}_2(\mathbb{Z})$. There is a subspace $S_{k,\chi,\rho}$ of vector-valued cusp forms for which we require that each $a_j(n) = 0$ when $n + \kappa_j$ is non-positive.

From the condition (2) in Definition 2.1, we see that $\chi\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right) \rho\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)$ is a diagonal matrix whose (j,j) entry is $e^{2\pi i \kappa_j}$. If $f \in S_{k,\chi,\rho}$ is a vector-valued cusp form, then $a_j(n) = O(n^{k/2})$ for every $1 \le j \le m$, as $n \to \infty$ by the same argument for classical modular forms (for example, see [7, Section 1]). For a vector-valued cusp form $f(z) = \sum_{j=1}^{m} \sum_{n+\kappa_j > 0} a_j(n) e^{2\pi i (n+\kappa_j) z} \mathbf{e}_j$ we define the L-series

$$L(f,s) = \sum_{j=1}^{m} \sum_{n+\kappa_j > 0} \frac{a_j(n)}{(n+\kappa_j)^s} \mathbf{e}_j.$$

This series converges absolutely for $\mathrm{Re}(s) \gg 0$.

The following theorem for vector-valued modular forms follows from the same argument used for classical modular forms.

Theorem 2.2. *Let $k \in \frac{1}{2}\mathbb{Z}$. If $f \in S_{k,\chi,\rho}$ is a vector-valued cusp form, then*

$$\frac{\Gamma(s)}{(2\pi)^s} L(f,s) = \int_0^\infty f(iy) y^s \, \frac{dy}{y}.$$

Furthermore, $L(f,s)$ has an analytic continuation to the whole complex plane and the functional equation

$$L^*(f,s) = i^k \chi(S) \rho(S) L^*(f, k-s),$$

where

$$L^*(f,s) = \frac{\Gamma(s)}{(2\pi)^s} L(f,s)$$

and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

3. The construction of the kernel function

In what follows, we define the kernel function $R_{k,s,l}$ which will play a crucial role in determining the Fourier coefficients of the orthogonal basis of the space of vector-valued cusp forms using Petersson's scalar product. Moreover, we determine the Fourier coefficients of this kernel function using the Lipshitz summation formula.

Let l be an integer with $1 \le l \le m$. Define

$$p_{s,l}(\tau) := \tau^{-s}\mathbf{e}_l.$$

For $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$, we define

$$R_{k,s,l} := \gamma_k(s) \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} p_{s,l}|_{k,\chi,\rho}\gamma,$$

where $\gamma_k(s) := \frac{1}{2}e^{\pi i s/2}\Gamma(s)\Gamma(k-s)$.

We write $\langle \cdot, \cdot \rangle$ for the standard scalar product on $\mathbb{C}^m$, i.e.,

$$\left\langle \sum_{j=1}^{m} \lambda_j \mathbf{e}_j, \sum_{j=1}^{m} \mu_j \mathbf{e}_j \right\rangle = \sum_{j=1}^{m} \lambda_j \overline{\mu_j}.$$

Then, for $f, g \in M_{k,\chi,\rho}$, we define the Petersson scalar product of f and g by

$$(f, g) := \int_{\mathcal{F}} \langle f(\tau), g(\tau) \rangle v^k \, \frac{dudv}{v^2}$$

if the integral converges, where $\mathcal{F}$ is the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ and $\tau = u + iv$.

Lemma 3.1. *Let $k \in \frac{1}{2}\mathbb{Z}$ with $k > 2$, and let $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$.*

(1) *The series $R_{k,s,l}$ converges absolutely uniformly whenever $\tau = u + iv$ satisfies $v \ge \epsilon$, $u \le 1/\epsilon$ for a given $\epsilon > 0$, and s varies over a compact set.*

(2) *The series $R_{k,s,l}$ is a vector-valued cusp form in $S_{k,\chi,\rho}$.*

(3) *For $f \in S_{k,\chi,\rho}$, we have*

$$(f, R_{k,\bar{s},l}) = c_k \langle L^*(f,s), \mathbf{e}_l \rangle,$$

where $c_k := \frac{(-1)^{k/2}\pi(k-2)!}{2^{k-2}}$.

Proof. For the first part, note that for each $1 \le j \le m$, we have

$$(R_{k,s,l})_j(\tau) = \gamma_k(s) \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l} (c\tau + d)^{-k} \left(\frac{a\tau + b}{c\tau + d}\right)^{-s},$$

where $\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l}$ denotes the (j,l)-th entry of the matrix $\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)$ and $(R_{k,s,l})_j$ denotes the j-th component of $R_{k,s,l}$. Then, we have

$$\begin{aligned}
&\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})} \left| \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l} (c\tau+d)^{-k} \left(\frac{a\tau+b}{c\tau+d}\right)^{-s} \right| \\
&\leq \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})} \left| (c\tau+d)^{-k} \left(\frac{a\tau+b}{c\tau+d}\right)^{-s} \right|
\end{aligned}$$

since ρ is a unitary representation. It is known that the series

$$\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})} (c\tau+d)^{-k} \left(\frac{a\tau+b}{c\tau+d}\right)^{-s}$$

converges absolutely uniformly whenever $\tau = u+iv$ satisfies $v \geq \epsilon$, $u \leq 1/\epsilon$ for a given $\epsilon > 0$, and s varies over a compact set of $1 < \mathrm{Re}(s) < k-1$ (see [9, Section 4]). Therefore, the series $R_{k,s,l}$ converges absolutely uniformly whenever $\tau = u+iv$ satisfies $v \geq \epsilon$, $u \leq 1/\epsilon$ for a given $\epsilon > 0$, and s varies over a compact set of $1 < \mathrm{Re}(s) < k-1$. The second part follows from the Fourier expansion of $R_{k,s,l}$ given in Theorem 4.1.

For the last part, we follow the argument in [9, Lemma 1]. It is enough to consider the case when $1 < \mathrm{Re}(s) < (k-1)/2$. Note that for each $(c,d) \in \mathbb{Z}^2$ with $(c,d) = 1$, we can find $(a,b) \in \mathbb{Z}^2$ such that $ad - bc = 1$. Then, we see that $R_{k,s,l}(\tau)$ is equal to

$$\begin{aligned}
\gamma_k(s) &\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \sum_{n\in\mathbb{Z}} (c\tau+d)^{-k} \left(\frac{a\tau+b}{c\tau+d}+n\right)^{-s} \\
&\times \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \chi^{-1}\left(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\right) \mathbf{e}_l,
\end{aligned}$$

where for each coprime pair $(c,d) \in \mathbb{Z}^2$, one chooses a fixed pair $(a,b) \in \mathbb{Z}^2$ such that $ad - bc = 1$. Therefore, we have

$$\begin{aligned}
&(R_{k,s,l})_j(\tau) \\
&= \gamma_k(s) \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \sum_{n\in\mathbb{Z}} (c\tau+d)^{-k} \left(\frac{a\tau+b}{c\tau+d}+n\right)^{-s} e^{-2\pi i n\kappa_l} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l}.
\end{aligned}$$

Next, we will use the Lipschitz summation formula [10]: For $0 \leq a < 1$, $\mathrm{Re}(s) > 1$ and $\tau \in \mathbb{H}$, we have

$$\frac{\Gamma(s)}{(-2\pi i)^s} \sum_{k\in\mathbb{Z}} \frac{e^{2\pi i a k}}{(k+\tau)^s} = \sum_{n=1}^{\infty} \frac{e^{2\pi i \tau(n-a)}}{(n-a)^{1-s}}. \tag{3.1}$$

Therefore, we have

$$(R_{k,s,l})_j(\tau) = \frac{1}{2}(2\pi)^s\Gamma(k-s)\sum_{n+\kappa_l>0}(n+\kappa_l)^{s-1} \times \sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)=1}} \chi^{-1}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)_{j,l}(c\tau+d)^{-k}e^{2\pi i(n+\kappa_l)(a\tau+b)/(c\tau+d)}.$$

From this, we have

$$R_{k,s,l}(\tau) = (2\pi)^s\Gamma(k-s)\sum_{n+\kappa_l>0}(n+\kappa_l)^{s-1}P_{k,n,l}(\tau),$$

where

$$P_{k,n,l}(\tau) = \frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)=1}} \chi^{-1}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)(c\tau+d)^{-k}e^{2\pi i(n+\kappa_l)(a\tau+b)/(c\tau+d)}\mathbf{e}_l$$

is a vector-valued Poincaré series. Suppose that $f\in S_{k,\chi,\rho}$ has a Fourier expansion of the form

$$f(\tau) = \sum_{j=1}^{m}\sum_{n+\kappa_j>0} a_j(n)e^{2\pi i(n+\kappa_j)\tau}\mathbf{e}_j.$$

By following the argument as in [7, Theorem 5.3], we have

$$(f, P_{k,n,l}) = a_l(n)\frac{\Gamma(k-1)}{(4\pi(n+\kappa_l))^{k-1}}.$$

Therefore, we see that

$$(f, R_{k,\bar{s},l}) = c_k(-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)\langle L(f,k-s),\mathbf{e}_l\rangle = c_k\langle L^*(f,s),\mathbf{e}_l\rangle. \qquad \square$$

We now compute the Fourier expansion of $R_{k,s,l}$ by following a similar argument as in [9, Lemma 2].

Lemma 3.2. *Let $k\in\frac{1}{2}\mathbb{Z}$ with $k>2$. The function $R_{k,s,l}$ has the Fourier expansion*

$$R_{k,s,l}(\tau) = \sum_{j=1}^{m}\sum_{n+\kappa_j>0} r_{k,s,l,j}(n)e^{2\pi i(n+\kappa_j)\tau}\mathbf{e}_j,$$

where $r_{k,s,l,j}(n)$ is given by

$$\begin{aligned} &r_{k,s,l,j}(n)\\ &= \delta_{l,j}(2\pi)^s\Gamma(k-s)(n+\kappa_l)^{s-1} + \chi^{-1}(S)\rho^{-1}(S)_{j,l}(-1)^{k/2}(2\pi)^{k-s}\Gamma(s)(n+\kappa_j)^{k-s-1}\\ &\quad + \frac{(-1)^{k/2}}{2}(2\pi)^k(n+\kappa_j)^{k-1}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)=1,\ ac>0}} c^{-k}\left(\frac{c}{a}\right)^s\\ &\quad\times\Big(e^{2\pi i(n+\kappa_j)d/c}e^{\pi i s}\chi^{-1}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\right)_{j,l}\ {}_1F_1(s,k;-2\pi i n/(ac))\\ &\qquad + e^{-2\pi i(n+\kappa_j)d/c}e^{-\pi i s}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b\\ c & -d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b\\ c & -d\end{smallmatrix}\right)\right)_{j,l}\ {}_1F_1(s,k;2\pi i n/(ac))\Big),\end{aligned}$$

where ${}_1F_1(\alpha, \beta; z)$ is Kummer's degenerate hypergeometric function.

Proof. First, we consider the contribution of the terms where $ac = 0$. The contribution of the terms $\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$

$$2\gamma_k(s) \sum_{n\in\mathbb{Z}} (z+n)^{-s} e^{-2\pi n\kappa_l} \mathbf{e}_l$$

can be written, by the Lipschitz summation formula in (3.1), as follows:

$$(2\pi)^s \Gamma(k-s) \sum_{n+\kappa_l>0} (n+\kappa_l)^{s-1} e^{2\pi i(n+\kappa_l)z} \mathbf{e}_l.$$

Note that $\left(\begin{smallmatrix} 0 & -1 \\ 1 & n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, the contribution of the terms $\pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & n \end{smallmatrix}\right)$ is equal to

$$\begin{aligned} &2\gamma_k(s) \sum_{n\in\mathbb{Z}} (-z)^{-s}(z+n)^{s-k} \chi^{-1}\left(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\right) \chi^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \mathbf{e}_l \\ &= 2\gamma_k(s)(-z)^{-s} \sum_{j=1}^{m} \chi^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)_{j,l} \sum_{n\in\mathbb{Z}} e^{-2\pi i n\kappa_j}(z+n)^{s-k} \mathbf{e}_j. \end{aligned} \tag{3.2}$$

By the similar computation as in the case of the terms $\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, we see that (3.2) is equal to

$$(-1)^{k/2}(2\pi)^{k-s}\Gamma(s) \sum_{j=1}^{m} \chi^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)_{j,l} \sum_{n+\kappa_j>0} (n+\kappa_j)^{k-s-1} e^{2\pi i(n+\kappa_j)z} \mathbf{e}_j.$$

The contribution of the terms with $ac \neq 0$ at the j-th component is given by

$$\begin{aligned} &\gamma_k(s) \int_{iC}^{iC+1} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac\neq 0}} (cz+d)^{-k} \left(\frac{az+d}{cz+d}\right)^{-s} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l} e^{-2\pi i(n+\kappa_j)z}\, dz \\ &= \gamma_k(s) \int_{iC}^{iC+1} \sum_{m\in\mathbb{Z}} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac\neq 0}} (c(z+m)+d)^{-k} \left(\frac{a(z+m)+b}{c(z+m)+d}\right)^{-s} \\ &\qquad \times e^{-2\pi i m\kappa_j} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l} e^{-2\pi i(n+\kappa_j)z}\, dz \end{aligned} \tag{3.3}$$

for any fixed positive real number C. By the change of variables $z \mapsto z+m$ $(m \in \mathbb{Z})$, we see that (3.3) is equal to

$$\gamma_k(s) \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac\neq 0}} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l} \int_{iC-\infty}^{iC+\infty} (cz+d)^{-k} \left(\frac{az+b}{cz+d}\right)^{-s} e^{-2\pi i(n+\kappa_j)z}\, dz. \tag{3.4}$$

By the change of variables $z \mapsto z - d/c$, we see that (3.4) is equal to

$$\gamma_k(s) \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac\neq 0}} c^{-k} e^{2\pi i(n+\kappa_j)d/c} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l} \times \int_{iC-\infty}^{iC+\infty} z^{-k}\left(-\frac{1}{c^2 z} + \frac{a}{c}\right)^{-s} e^{-2\pi i(n+\kappa_j)z}\, dz. \tag{3.5}$$

If $ac > 0$, then we have

$$z^{-s}\left(-\frac{1}{c^2 z} + \frac{a}{c}\right)^{-s} = \left(-\frac{1}{c^2} + \frac{a}{c}z\right)^{-s}.$$

Therefore, by the change of variable $z \mapsto (c/a)it$, we see that the integral in (3.5) is equal to

$$(-1)^{k/2} 2\pi \left(\frac{c}{a}\right)^{-k+s+1} \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} t^{-k+s}\left(t + \frac{i}{c^2}\right)^{-s} e^{2\pi(n+\kappa_j)(c/a)t}\, dt. \tag{3.6}$$

Note that for $\mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0$, $p \in \mathbb{C}$, we have

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} (t+\alpha)^{-\mu}(t+\beta)^{-\nu} e^{pt}\, dt = \frac{1}{\Gamma(\mu+\nu)} p^{\mu+\nu-1} e^{-\beta p}\, {}_1F_1(\mu, \mu+\nu; (\beta-\alpha)p)$$

(see [6]). Therefore, (3.6) can be written as

$$(-1)^{k/2}\frac{(2\pi)^k}{\Gamma(k)}(n+\kappa_j)^{k-1}\left(\frac{c}{a}\right)^{s} {}_1F_1(s, k; -2\pi i n/(ac)).$$

From this, we see that the contribution of the terms with $ac > 0$ at the j-th component is equal to

$$\frac{(-1)^{k/2}}{2}(2\pi)^k (n+\kappa_j)^{k-1}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} \times \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac>0}} c^{-k}\left(\frac{c}{a}\right)^{s} e^{2\pi i(n+\kappa_j)d/c} e^{\pi i s} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l} {}_1F_1(s, k; -2\pi i n/(ac)).$$

The contribution of the terms with $ac < 0$ at the j-th component is obtained by the same argument if we replace (a, c) by $(-a, c)$. $\square$

4. The main result

In this section, we give the main result where we determine the non-vanishing of the averages of L-functions associated with the orthogonal basis of the space of cusp forms. We also show the existence of at least one basis element whose L-function does not vanish under certain conditions. Let

$$n_{j,0} := \begin{cases} 1 & \text{if } \kappa_j = 0, \\ 0 & \text{if } \kappa_j \neq 0. \end{cases}$$

Theorem 4.1. *Let $k \in \frac{1}{2}\mathbb{Z}$ with $k > 2$. Let $\{f_{k,1}, \ldots, f_{k,d_k}\}$ be an orthogonal basis of $S_{k,\chi,\rho}$ with Fourier expansions*

$$f_{k,l}(\tau) = \sum_{j=1}^{m} \sum_{n+\kappa_j>0} b_{k,l,j}(n) e^{2\pi i(n+\kappa_j)\tau} \mathbf{e}_j, \quad 1 \le l \le d_k.$$

Let $t_0 \in \mathbb{R}$, $\epsilon > 0$, and $1 \le j \le m$. Then, there exists a constant $C(t_0, \epsilon, j) > 0$ such that for $k > C(t_0, \epsilon, j)$ the function

$$\sum_{l=1}^{d_k} \frac{\langle L^*(f_{k,l}, s), \mathbf{e}_j\rangle}{(f_{k,l}, f_{k,l})} b_{k,l,j}(n_{j,0})$$

does not vanish at any point $s = \sigma + it_0$ with $(k-1)/2 < \sigma < k/2 - \epsilon$.

Remark 4.2. Note that

$$\sum_{l=1}^{d_k} \frac{\langle L^*(f_{k,l}, s), \mathbf{e}_j\rangle}{(f_{k,l}, f_{k,l})} b_{k,l,j}(n_{j,0})$$

is the $n_{j,0}$-th Fourier coefficient of $R_{k,\bar{s},j}$.Therefore, it is equal to a nonzero constant multiple of

$$(R_{k,\bar{s},j}, P_{k,n_{j,0},j}).$$

Therefore, Theorem 4.1 implies the nonvanishing of $\langle L^*(P_{k,n_{j,0},j}, s), \mathbf{e}_j\rangle$.

Proof of Theorem 4.1. By Lemma 3.1, we have

$$R_{k,\bar{s},j} = \sum_{l=1}^{d_k} \frac{(f_{k,l}, R_{k,\bar{s},j})}{(f_{k,l}, f_{k,l})} f_{k,l} = c_k \sum_{l=1}^{d_k} \frac{\langle L^*(f_{k,l}, s), \mathbf{e}_j\rangle}{(f_{k,l}, f_{k,l})} f_{k,l}.$$

If we take the first Fourier coefficients of both sides at the j-th component, then by Lemma 3.2 we have

$$\begin{aligned} &c_k \sum_{l=1}^{d_k} \frac{\langle L^*(f_{k,l}, s), \mathbf{e}_j\rangle}{(f_{k,l}, f_{k,l})} b_{k,l,j}(n_{j,0}) \\ &= (2\pi)^s \Gamma(k-s) \widetilde{\kappa}_j^{s-1} + \chi^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)_{j,j} (-1)^{k/2} (2\pi)^{k-s} \Gamma(s) \widetilde{\kappa}_j^{k-s-1} \\ &\quad + \frac{(-1)^{k/2}}{2} (2\pi)^k \widetilde{\kappa}_j^{k-1} \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\, ac>0}} c^{-k} \left(\frac{c}{a}\right)^s \\ &\quad \times \Big(e^{2\pi i(n+\kappa_j)d/c} e^{\pi i s} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,j} {}_1F_1(s, k; -2\pi i n/(ac)) \\ &\qquad + e^{-2\pi i(n+\kappa_j)d/c} e^{-\pi i s} \chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right)_{j,j} {}_1F_1(s, k; 2\pi i n/(ac)) \Big), \end{aligned} \tag{4.1}$$

where

$$\widetilde{\kappa}_j := \begin{cases} 1 & \text{if } \kappa_j = 0, \\ \kappa_i & \text{if } \kappa_j \neq 0. \end{cases}$$

Suppose that

$$c_k \sum_{l=1}^{d_k} \frac{\langle L^*(f_{k,l}, s), \mathbf{e}_j\rangle}{(f_{k,l}, f_{k,l})} b_{k,l,j}(n_{j,0}) = 0.$$

If we divide (4.1) by $(2\pi)^s\Gamma(k-s)\kappa_j^{s-1}$, we have

$$\begin{aligned}
-1 = {} & \rho^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)_{j,j} (-1)^{k/2}(2\pi)^{k-2s}\frac{\Gamma(s)}{\Gamma(k-s)}\widetilde{\kappa}_j^{k-2s} \\
& + \frac{(-1)^{k/2}(2\pi)^{k-s}\widetilde{\kappa}_j^{k-s}}{2\Gamma(k-s)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac>0}} c^{-k}\left(\frac{c}{a}\right)^s \\
& \quad \times \Big(e^{2\pi i(n+\kappa_j)d/c} e^{\pi i s}\chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,j}\ {}_1f_1(s,k;-2\pi i n/(ac)) \\
& \qquad + e^{-2\pi i(n+\kappa_j)d/c} e^{-\pi i s}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right)_{j,j}\ {}_1f_1(s,k;2\pi i n/(ac))\Big),
\end{aligned}$$

where

$${}_1f_1(\alpha,\beta;z) := \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\,{}_1F_1(\alpha,\beta;z).$$

Let $s = k/2 - \delta - it_0$, where $\epsilon < \delta < 1/2$. Then, we have

$$\begin{aligned}
-1 = {} & \chi^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)_{j,j} (-1)^{k/2}(2\pi\widetilde{\kappa}_j)^{2\delta+2it_0}\frac{\Gamma(k/2-\delta-it_0)}{\Gamma(k/2+\delta+it_0)} \\
& + \frac{(-1)^{k/2}(2\pi\widetilde{\kappa}_j)^{k/2+\delta+it_0}}{2\Gamma(k/2+\delta+it_0)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac>0}} c^{-k/2-\delta-it_0}a^{-k/2+\delta+it_0} \\
& \quad \times \Big(e^{2\pi i(n+\kappa_j)d/c} e^{\pi i(k/2-\delta-it_0)}\chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,j} \\
& \qquad \times {}_1f_1(k/2-\delta-it_0,k;-2\pi i n/(ac)) \\
& \qquad + e^{-2\pi i(n+\kappa_j)d/c} e^{-\pi i(k/2-\delta-it_0)}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right)_{j,j} \\
& \qquad \times {}_1f_1(k/2-\delta-it_0,k;2\pi i n/(ac))\Big).
\end{aligned} \tag{4.2}$$

For $\mathrm{Re}(\beta) > \mathrm{Re}(\alpha) > 0$, we have

$${}_1f_1(\alpha,\beta;z) = \int_0^1 e^{zu}u^{\alpha-1}(1-u)^{\beta-\alpha-1}\,du.$$

By [1, 13.21], for $\mathrm{Re}(\alpha) > 1$, $\mathrm{Re}(\beta-\alpha) > 1$, and $|z| = 1$, we have

$$|{}_1f_1(\alpha,\beta;z)| \le 1.$$

If we take absolute values in (4.2), then we have

$$\begin{aligned}
1 \le {} & (2\pi\widetilde{\kappa}_j)^{2\delta}\frac{|\Gamma(k/2-\delta-it_0)|}{|\Gamma(k/2+\delta+it_0)|} \\
& + \frac{(2\pi\widetilde{\kappa}_j)^{k/2+\delta}}{2|\Gamma(k/2+\delta+it_0)|} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac>0}} |c|^{-k/2-\delta}|a|^{-k/2+\delta}(e^{\pi t_0}+e^{-\pi t_0}).
\end{aligned} \tag{4.3}$$

By [1, 6.147], we have

$$\frac{|\Gamma(k/2-\delta-it_0)|}{|\Gamma(k/2+\delta+it_0)|} \sim \left(\frac{k}{2}\right)^{-2\delta}$$

as $k \to \infty$. Therefore, (4.3) becomes $1 \le 0$ as $k \to \infty$, which is a contradiction. $\square$

We now give a corollary that is a direct consequence of Theorem 4.1 which basically demonstrates the existence of a basis element of the space of vector-valued cusp forms whose L-function does not vanish.

Corollary 4.3. *Let $k \in \frac{1}{2}\mathbb{Z}$ with $k > 2$. Let $\{f_{k,1}, \dots, f_{k,d_k}\}$ be an orthogonal basis of $S_{k,\chi,\rho}$ with Fourier expansions*

$$f_{k,l}(\tau) = \sum_{j=1}^{m} \sum_{n+\kappa_j>0} b_{k,l,j}(n) e^{2\pi i(n+\kappa_j)\tau} \mathbf{e}_j, \quad 1 \le l \le d_k.$$

Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$.

(1) *For any $k > C(t_0, \epsilon, j)$, any $1 \le j \le m$, and any $s = \sigma + it_0$ with*

$$\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon,$$

there exists a basis element $f_{k,l} \in S_{k,\chi,\rho}$ such that

$$\langle L^*(f_{k,l}, s), \mathbf{e}_j \rangle \ne 0 \quad \textit{and} \quad b_{k,l,j}(n_{j,0}) \ne 0.$$

(2) *There exists a constant $C(t_0, \epsilon) > 0$ such that for any $k > C(t_0, \epsilon)$, and any $s = \sigma + it_0$ with*

$$\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon \quad \textit{and} \quad \frac{k}{2} + \epsilon < \sigma < \frac{k+1}{2},$$

there exists a basis element $f_{k,l} \in S_{k,\chi,\rho}$ such that

$$L(f_{k,l}, s) \ne 0.$$

5. The case of $\Gamma_0(N)$

In what follows, we consider the case of a scalar-valued modular form on the congruence subgroup $\Gamma_0(N)$. By using Theorem 4.1, we can extend Kohnen's result in [9] to the case of $\Gamma_0(N)$. To illustrate, let N be a positive integer and let $\Gamma = \Gamma_0(N)$. Let $S_k(\Gamma)$ be the space of cusp forms of weight k on Γ. Let $\{\gamma_1, \dots, \gamma_m\}$ be the set of representatives of $\Gamma \setminus \mathrm{SL}_2(\mathbb{Z})$ with $\gamma_1 = I$. For $f \in S_k(\Gamma)$, we define a vector-valued function $\widetilde{f}\colon \mathbb{H} \to \mathbb{C}^m$ by $\widetilde{f} = \sum_{j=1}^{m} f_j \mathbf{e}_j$ and

$$f_j = f|_k \gamma_j, \quad 1 \le j \le m,$$

where $(f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(z) := (cz+d)^{-k} f(\gamma z)$. Then, $\widetilde{f}$ is a vector-valued modular form of weight k and the trivial multiplier system with respect to ρ on $\mathrm{SL}_2(\mathbb{Z})$, where ρ is a certain m-dimensional unitary complex representation such that $\rho(\gamma)$ is a permutation matrix for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and is an identity matrix if $\gamma \in \Gamma$. Then, the map $f \mapsto \widetilde{f}$ induces an isomorphism between $S_k(\Gamma)$ and $S_{k,\rho}$, where $S_{k,\rho}$ denotes the space of vector-valued cusp forms of weight k and trivial multiplier system with respect to ρ on $\mathrm{SL}_2(\mathbb{Z})$.

Suppose that $f, g \in S_k(\Gamma)$. Then, we have

$$(\widetilde{f}, \widetilde{g}) = \int_{\mathcal{F}} \langle \widetilde{f}, \widetilde{g} \rangle y^k \frac{dxdy}{y^2} = \sum_{j=1}^{m} \int_{\mathcal{F}} (f|_k \gamma_j)(z) \overline{(g|_k \gamma)(z)} y^k \frac{dxdy}{y^2} = (f, g)$$

where (f, g) denotes the Petersson inner product. Therefore, if $f, g \in S_k(\Gamma)$ such that f and g are orthogonal, then $\widetilde{f}$ and $\widetilde{g}$ is also orthogonal.

Corollary 5.1. *Let k be a positive even integer with $k > 2$. Let N be a positive integer and $\Gamma = \Gamma_0(N)$. Let $\{f_{k,1}, \ldots, f_{k,e_k}\}$ be an orthogonal basis of $S_k(\Gamma)$. Let $t_0 \in \mathbb{R}$, $\epsilon > 0$. Then, there exists a constant $C(t_0, \epsilon) > 0$ such that for $k > C(t_0, \epsilon)$ there exists a basis element $f_{k,l} \in S_k(\Gamma)$ satisfying*

$$L(\widetilde{f_{k,l}}, s) \neq 0$$

at any point $s = \sigma + it_0$ with

$$\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon \quad \text{and} \quad \frac{k}{2} + \epsilon < \sigma < \frac{k+1}{2}.$$

6. The case of Jacobi forms

Let k be a positive even integer and m be a positive integer. Let $J_{k,m}$ be the space of Jacobi forms of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$. From now, we use the notation $\tau = u + iv \in \mathbb{H}$ and $z = x + iy \in \mathbb{C}$. We review basic notions of Jacobi forms (for more details, see [5]). Let F be a complex-valued function on $\mathbb{H} \times \mathbb{C}$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $X = (\lambda, \mu) \in \mathbb{Z}^2$, we define

$$(F|_{k,m}\gamma)(\tau, z) := (c\tau + d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau + d}} F(\gamma(\tau, z))$$

and

$$(F|_m X)(\tau, z) := e^{2\pi i m(\lambda^2 \tau + 2\lambda z)} F(\tau, z + \lambda\tau + \mu),$$

where $\gamma(\tau, z) = \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$.

With these notations, we introduce the definition of a Jacobi form.

Definition 6.1. A *Jacobi form* of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function F on $\mathbb{H} \times \mathbb{C}$ satisfying

(1) $F|_{k,m}\gamma = F$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

(2) $F|_m X = F$ for every $X \in \mathbb{Z}^2$,

(3) F has the Fourier expansion of the form

$$F(\tau, z) = \sum_{\substack{l,r\in\mathbb{Z} \\ 4ml-r^2\geq 0}} a(l,r)e^{2\pi i l\tau}e^{2\pi i rz}. \tag{6.1}$$

We denote by $J_{k,m}$ the vector space of all Jacobi forms of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$. If a Jacobi form satisfies the condition $a(l,r) = 0$ if $4ml - r^2 = 0$, then it is called a Jacobi cusp form. We denote by $S_{k,m}$ the vector space of all Jacobi cusp forms of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$.

For $1 \leq j \leq 2m$, we consider the theta series

$$\theta_{m,j}(\tau, z) := \sum_{\substack{r\in\mathbb{Z} \\ r\equiv j \pmod{2m}}} e^{2\pi i r^2\tau/(4m)}e^{2\pi i rz}.$$

Suppose that $F(\tau, z)$ is a holomorphic function of z and satisfies

$$F|_m X = F \quad \text{for every } X \in \mathbb{Z}^2.$$

Then we have

$$F(\tau, z) = \sum_{1\leq j\leq 2m} F_j(\tau)\theta_{m,j}(\tau, z) \tag{6.2}$$

with uniquely determined holomorphic functions $F_a \colon \mathbb{H} \to \mathbb{C}$. Furthermore, if F is a Jacobi form in $J_{k,m}$ with the Fourier expansion (6.1), then functions in $\{F_j \mid 1 \leq j \leq 2m\}$ have the Fourier expansions

$$F_j(\tau) = \sum_{\substack{n\geq 0 \\ n+j^2\equiv 0 \pmod{4m}}} a\left(\frac{n+j^2}{4m}, j\right) e^{2\pi i n\tau/(4m)}.$$

In [3], it is proved that the Petersson inner product of skew-holomorphic Jacobi cusp forms can be expressed as the sum of partial L-values of skew-holomorphic Jacobi cusp forms. Similarly, for a Jacobi cusp form $F \in J_{k,m}$ with its Fourier expansion (6.1), we define partial L-functions of F by

$$L(F, j, s) := \sum_{\substack{n\in\mathbb{Z},\, n>0 \\ n+j^2\equiv 0 \pmod{4m}}} \frac{a\left(\frac{n+j^2}{4m}, j\right)}{\left(\frac{n}{4m}\right)^s}$$

for $1 \leq j \leq 2m$.

We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic group. The elements of $\mathrm{Mp}_2(\mathbb{R})$ are pairs $(\gamma, \phi(\tau))$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, and ϕ denotes a holomorphic function on $\mathbb{H}$ with $\phi(\tau)^2 = c\tau + d$. The product of $(\gamma_1, \phi_1(\tau)), (\gamma_2, \phi_2(\tau)) \in \mathrm{Mp}_2(\mathbb{R})$ is given by

$$(\gamma_1, \phi_1(\tau))(\gamma_2, \phi_2(\tau)) = (\gamma_1\gamma_2, \phi_1(\gamma_2\tau)\phi_2(\tau)).$$

The map

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau + d}\right)$$

defines a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$. Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by $\widetilde{T}$ and $\widetilde{S}$.

We define a $2m$-dimensional unitary complex representation $\widetilde{\rho}_m$ of $\mathrm{Mp}_2(\mathbb{Z})$ by

$$\widetilde{\rho}_m(\widetilde{T})\mathbf{e}_j = e^{-2\pi i j^2/(4m)}\mathbf{e}_j \quad \text{and} \quad \widetilde{\rho}_m(\widetilde{S})\mathbf{e}_j = \frac{i^{1/2}}{\sqrt{2m}} \sum_{j'=1}^{2m} e^{2\pi i j j'/(2m)}\mathbf{e}_{j'},$$

Let χ be a multiplier system of weight $1/2$ on $\mathrm{SL}_2(\mathbb{Z})$. We define a map $\rho_m \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{2m}(\mathbb{C})$ by

$$\rho_m(\gamma) = \chi(\gamma)\widetilde{\rho}_m(\widetilde{\gamma})$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The map ρ_m gives a $2m$-dimensional unitary representation of $\mathrm{SL}_2(\mathbb{Z})$.

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{2m}\}$ denote the standard basis of $\mathbb{C}^{2m}$. For $F \in S_{k,m}$, we define a vector-valued function $\widetilde{F} \colon \mathbb{H} \to \mathbb{C}^{2m}$ by $\widetilde{F} = \sum_{j=1}^{2m} F_j \mathbf{e}_j$, where F_j is defined by the theta expansion in (6.2). Then, the map $F \mapsto \widetilde{F}$ induces an isomorphism between $S_{k,m}$ and $S_{k-1/2,\overline{\chi},\rho_m}$ (for more details, see [5, Section 5] and [4, Section 3.1]).

Suppose that $F, G \in S_{k,m}$. The Petersson inner product of F and G by

$$(F, G) := \int_{\mathrm{SL}_2(\mathbb{Z})^J \backslash \mathbb{H} \times \mathbb{C}} v^k e^{-4\pi m y^2/v} F(\tau, z)\overline{G(\tau, z)} \, \frac{dx dy du dv}{v^3},$$

where $\mathrm{SL}_2(\mathbb{Z})^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Then, by Theorem 5.3 in [5], we have

$$(F, G) = \frac{1}{\sqrt{2m}}(\widetilde{F}, \widetilde{G}).$$

Note that $\rho_m(-I)$ is not equal to the identity matrix in $\mathrm{GL}_{2m}(\mathbb{C})$. Instead, we have

$$\rho_m(-I)\mathbf{e}_j = i\mathbf{e}_{2m-j}.$$

Then, the corresponding kernel function $R_{k,s,l}$ has the Fourier expansion

$$R_{k,s,l}(\tau) = \sum_{j=1}^{2m} \sum_{n+\kappa_j > 0} r_{k,s,l,j}(n) e^{2\pi i (n+\kappa_j)\tau} \mathbf{e}_j,$$

where $r_{k,s,l,j}(n)$ is given by

$$
\begin{aligned}
r_{k,s,l,j}(n) = &\frac{1}{2}\delta_{l,j}(2\pi)^s\Gamma(k-s)(n+\kappa_i)^{s-1} + \frac{i}{2}\delta_{2m-l,j}(2\pi)^s\Gamma(k-s)(n+\kappa_{2m-l})^{s-1} \\
&+ \frac{1}{2}\chi^{-1}(S)\rho^{-1}(S)_{j,l}(-1)^{k/2}(2\pi)^{k-s}\Gamma(s)(n+\kappa_j)^{k-s-1} \\
&+ \frac{i}{2}\chi^{-1}(S)\rho^{-1}(S)_{j,2m-l}(-1)^{k/2}(2\pi)^{k-s}\Gamma(s)(n+\kappa_j)^{k-s-1} \\
&+ \frac{(-1)^{k/2}}{2}(2\pi)^k(n+\kappa_j)^{k-1}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1,\ ac>0}} c^{-k}\left(\frac{c}{a}\right)^s \\
&\times \Big(e^{2\pi i(n+\kappa_j)d/c}e^{\pi i s}\chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{j,l}\, {}_1F_1(s,k;-2\pi in/(ac)) \\
&\quad + e^{-2\pi i(n+\kappa_j)d/c}e^{-\pi i s}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right)_{j,l}\, {}_1F_1(s,k;2\pi in/(ac))\Big).
\end{aligned}
$$

By the similar argument, we prove the same result as in Corollary 4.3 for the representation ρ_m. From this, we have the following corollary.

Corollary 6.2. *Let k be a positive even integer with $k > 2$. Let $\{F_{k,m,1}, \ldots, F_{k,m,d}\}$ be an orthogonal basis of $S_{k,m}$. Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$.*

(1) *For any $k > C(t_0, \epsilon, j)$, any $1 \le j \le 2m$, and any $s = \sigma + it_0$ with*

$$\frac{2k-3}{4} < \sigma < \frac{2k-1}{4} - \epsilon,$$

there exists a basis element $F_{k,m,l} \in S_{k,m}$ such that

$$L(F_{k,m,l}, j, s) \neq 0.$$

(2) *There exists a constant $C(t_0, \epsilon) > 0$ such that for any $k > C(t_0, \epsilon)$, and any $s = \sigma + it_0$ with*

$$\frac{2k-3}{4} < \sigma < \frac{2k-1}{4} - \epsilon \quad \text{and} \quad \frac{2k-1}{4} + \epsilon < \sigma < \frac{2k+1}{4},$$

there exist a basis element $F_{k,m,l} \in S_{k,m}$ and $j \in \{1, \ldots, 2m\}$ such that

$$L(F_{k,m,l}, j, s) \neq 0.$$

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