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The Solutions of a Class of Sylvester-like Linear Matrix Equations and the Estimation of the Associated Measurements of Their Solutions

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Abstract. This paper studies solutions and relevant measures of a class of Sylvester-like linear matrix equations commonly encountered in control theory. Firstly, inequalities related to the singular values of solutions of a class of Sylvester-like linear matrix equations are obtained. These results improve upon existing relevant studies. Next, starting from the definition of singular values for any matrix, a lower bound for the product of solutions and their complex conjugate transpose matrices is directly obtained. Additionally, when a Hermite matrix is a solution to the matrix equation, a convergent matrix series is obtained, as the positive definite solution under certain conditions. Finally, we design two algorithms for solving the class of matrix equations, where each recursive iteration results in obtaining the upper and lower solution bounds. Numerical experiments demonstrate that our results outperform some existing studies.

1. Introduction

Let $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) denote the set of $n\times n$ real (complex) matrices. Suppose $A,B\in\mathbb{C}^{n\times n}$, the notation $A\succ 0$ ($A\succeq 0$) is used to denote that A is a Hermite positive (semi-)definite matrix. $A\succ B$ ($A\succeq B$) means $A-B$ is Hermite positive (semi-)definite. A^T, A^*, A^{-1} and $\det(A)$ denote the transpose, the complex conjugate transpose, the inverse and the determinant of A, respectively. Let $A\in\mathbb{C}^{n\times n}$, we assume the real parts of the eigenvalues of A are arranged such that $\operatorname{Re}\lambda_1(A)\geq\operatorname{Re}\lambda_2(A)\geq\cdots\geq\operatorname{Re}\lambda_n(A)$. The singular values of A are arranged such that $\sigma_1(A)\geq\sigma_2(A)\geq\cdots\geq\sigma_n(A)$. $\rho(A)$ is the set of all eigenvalues of A. $A\in\mathbb{C}^{n\times n}$ is said to be stable if the eigenvalues of A lie on the open left half-plane, i.e., $\operatorname{Re}\lambda_i(A)<0$, $i=1,2,\ldots,n$.

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In the analysis and design of a control system, such as optimal control and stability analysis, it is often reduced to solve the corresponding Sylvester-like linear matrix equation [12, 30]. For example, consider the following stability analysis problem of singularly perturbed system [31]:

$$E_\epsilon \dot{x} = A_c x + B_w w, \tag{1.1}$$

where x is the state vector, w is the external disturbance and $E_\epsilon \succ 0$. Regarding the stability problem of singularly perturbed system (1.1), when a Lyapunov function is defined as $V = x^T E_\varepsilon P_\varepsilon E_\varepsilon x$ with $P_\varepsilon \succ 0$, then its derivative is formulated as

$$\dot{V} = x^T A_c^T P_\varepsilon E_\varepsilon x + x^T E_\varepsilon P_\varepsilon A_c x + x^T E_\varepsilon P_\varepsilon B_w w + w^T B_w^T P_\varepsilon E_\varepsilon x,$$

from which we obtain that a sufficient and necessary condition for the asymptotic stability of system (1.1) with $w = 0$ is

$$A_c^T P_\varepsilon E_\varepsilon + E_\varepsilon P_\varepsilon A_c \prec 0.$$

It is obvious that, for any given positive definite symmetric matrix C, a sufficient and necessary condition for system (1.1) with $w = 0$ to be asymptotically stable is that the equation $A_c^T P_\epsilon E_\epsilon + E_\epsilon P_\epsilon A_c^T + C = 0$ has a unique symmetric positive definite solution P_ϵ. In summary, it is of great theoretical and practical significance for solving the corresponding Sylvester-like linear matrix equation.

In this paper, we consider a class of Sylvester-like linear matrix equations as follows:

$$A^* X B + B^* X^* A = C \tag{1.2}$$

and

$$A^* X B + B^* X A = C \quad \text{with } X = X^*, \tag{1.3}$$

where $A, B, C \in \mathbb{C}^{n\times n}$ are given, $X \in \mathbb{C}^{n\times n}$ is an unknown matrix. Equation (1.3) studies the case where a Hermite matrix is a solution of equation (1.2). Equations (1.2) and (1.3) contain equations

$$A^* X + X^* A = C \tag{1.4}$$

and

$$A^* X + X A = C \quad \text{with } X = X^*, \tag{1.5}$$

as their special forms, respectively. Specially, equations

$$AXE + DX^* B = C \tag{1.6}$$

and

$$AXE + DXB = C \quad \text{with } X = X^*, \tag{1.7}$$

are natural extension of equations (1.2) and (1.3), respectively. Equation (1.2) commonly occur in practical applications, such as structured generalized and quadratic inverse eigenvalue problems [1, 4, 34], an inverse problem of vibration theory [15], Hamiltonian systems [3, 6]. And in some applications, such as time-varying singular value decomposition [2], system balancing [19], Newton's method for solving continuous algebraic Riccati equations [13, 17, 26], model reduction [27] and complex network systems [10] etc., the Hemite solution is required. For more details, please refer to [28, 30]. Many significant research achievements had been made due to the important role of equations (1.2) and (1.3) in practical applications. The analytical expressions of the solution of equation (1.5) and its properties were explored in [18, 20], and the analytical expressions of the solution to equation (1.4) and its variant were presented in [3, 6]. [23, 24] studied the properties of the positive definite solution of a continuous time algebraic Riccati equation. Equation (1.7) was investigated in [7, 21, 35]. [11] explored the uniqueness condition of the solution of equation (1.6). The quaternion and operator equation form of equation (1.3) were explored in [5] and [32], respectively. In [8, 22, 33], the authors presented analytical expressions of the solutions and the solutions with the minimum norm of equations (1.2) and (1.3). However, these expressions seem complicated. In [9], the authors first proposed a lower bound for the determinants of the solutions of equations (1.2) and (1.3) in terms of real number domain. Later, Soares [29] further studied them, extending A and B that are stable to $\det(A)\det(B) > 0$. But for practical problems, these are not enough. Therefore, more accurate estimations related to the solutions of equations (1.2) and (1.3) are carried out in this paper.

Compared with existing relevant works on solving Sylvester-like linear matrix equations, the main contributions of this paper are as follows:

- The relationship between the singular values of a square matrix and the eigenvalues of its Hermitian part, as well as singular value inequalities, are used to derive inequalities related to the singular values of the solutions of equation (1.2). The results improve upon those of [9, 29].

- Starting from the definition of singular values of any matrix, a lower bound for the product of the solutions to the matrix equation and their complex conjugate transpose matrices is obtained. Numerical experiments reveal that the results obtained outperform those in [9, 29] when the result is reduced to the lower bound of the determinant.

- When a Hermite matrix is a solution to the matrix equation, a convergent matrix series is obtained as the positive definite solution to the matrix equation under certain conditions.

- Two algorithms are designed to solve the positive definite solution of the matrix equation, with each recursive iteration obtaining the upper and lower bounds of the positive definite solution. The superiority of our results over papers [9, 29] is demonstrated by a class of numerical experiments with random matrices.

The remainder of this paper is organized as follows. In Section 2, relevant measures for the solutions of equation (1.2) are explored. In Section 3, the solution of equation (1.3) and its related properties are explored under certain conditions.

2. Relevant measures for the solutions of the matrix equation

In [9], lower bounds for the product of the eigenvalues of the solutions of equations (1.2) and (1.3) were presented, in terms of real number domain.

Theorem 2.1. [9] *Assume equation* (1.2) *is consistent. If* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times n}$ *are stable matrices and* $C \in \mathbb{R}^{n\times n}$ *is a symmetric positive definite matrix, then*

$$\delta_1\delta_2\cdots\delta_n \geq \frac{\prod_{i=1}^n \gamma_i}{2^n \prod_{i=1}^n \alpha_i\beta_i} \triangleq d_{l2}, \tag{2.1}$$

where $\rho(A) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $\rho(B) = \{\beta_1, \beta_2, \ldots, \beta_n\}$, $\rho(C) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ *and* $\rho(X) = \{\delta_1, \delta_2, \ldots, \delta_n\}$.

Remark 2.2. The result in [29] generalized that of Theorem 2.1 and extended the conditions satisfying the inequality (2.1) to $\det(A)\det(B) > 0$ and C is a symmetric positive definite matrix.

Applying the same technique to equation (1.3), which studies the case when a symmetric matrix is a solution of equation (1.2), similar results as Theorem 2.1 and Remark 2.2 can be obtained. In this paper, we will improve them.

Lemma 2.3. [36] *Let* $A \in \mathbb{C}^{n\times n}$, *then*

$$\lambda_i\left(\frac{A+A^*}{2}\right) \leq \sigma_i(A).$$

Lemma 2.4. [36] *Let* $A, B \in \mathbb{C}^{n\times n}$ *and* $1 \leq i_1 \leq \cdots \leq i_k \leq n$, *then*

$$\prod_{t=1}^k \sigma_{i_t}(AB) \leq \prod_{i=1}^k \sigma_i(A) \prod_{t=1}^k \sigma_{i_t}(B).$$

Lemma 2.5. [14] *Let $A \in \mathbb{C}^{n\times n}$ be a Hermite matrix, then*

$$\lambda_1(A)I \succeq A \succeq \lambda_n(A)I.$$

Lemma 2.6. [36] *Let $A, B \in \mathbb{C}^{n\times n}$, $r+s \le n-1$, then*

$$\sigma_{r+s+1}(AB) \le \sigma_{r+1}(A)\sigma_{s+1}(B).$$

Lemma 2.7. [16] *For any square matrix $A \in \mathbb{C}^{n\times n}$, it can be decomposed into $A = E+F$ with $E^* = E \succ 0$ and $F^* = -F$, then $\det(E) \le |\det(A)|$. Equality holds if and only if A is Hermite.*

Lemma 2.8. [36] *Let $A, B, C \in \mathbb{C}^{n\times n}$ be Hermite matrices. If $A \succeq 0$, $B \succeq 0$, and $A \succeq B$, then $A^{1/2} \succeq B^{1/2}$.*

Lemma 2.9. *Assume that $A, B, C \in \mathbb{R}^{n\times n}$ such that $\det(A)\det(B) > 0$ and C is a symmetric positive definite matrix. Let $X \in R^{n\times n}$ be a solution of equation(1.2), then $\det(X) > 0$ holds.*

Proof. In fact, for any matrix $A \in \mathbb{R}^{n\times n}$, one can conclude

$$\lambda_n\left(\frac{A+A^T}{2}\right) \le \operatorname{Re}\lambda_i(A) \le \lambda_1\left(\frac{A+A^T}{2}\right).$$

According to equation (1.2) and C is a symmetric positive definite matrix, one can conclude

$$\operatorname{Re}\lambda_i(A^TXB) > 0.$$

Since $A, X, B \in \mathbb{R}^{n\times n}$ and the complex eigenvalues of a real matrix appear in pairs, one can conclude

$$\det(A)\det(X)\det(B) = \det(A^TXB) > 0.$$

Thus, when $\det(A)\det(B) > 0$, $\det(X) > 0$. □

Taking advantage of the relationship between the singular values of a square matrix and the eigenvalues of its Hermitian part and singular value inequalities, a lower bound of the product of k singular values of the solutions of matrix equation (1.2) can be obtained.

Theorem 2.10. *Assuming that A and B are non-singular and C is Hermite positive definite. Let X be a solution of equation (1.2), then*

$$\prod_{t=1}^{k}\sigma_{i_t}(X) \ge \frac{\prod_{t=1}^{k}\lambda_{i_t}(A^{-*}CA^{-1})}{2^k\prod_{i=1}^{k}\sigma_i(BA^{-1})}. \tag{2.2}$$

Proof. When A is non-singular, multiplying by A^{-*} to the left and by A^{-1} to the right, equation (1.2) can be converted to

$$\widetilde{A}^*X^* + X\widetilde{A} = \widetilde{C}, \tag{2.3}$$

where $\widetilde{A} = BA^{-1}$, $\widetilde{C} = A^{-*}CA^{-1}$. By the use of Lemmas 2.3 and 2.4 to (2.3), one can conclude

$$\prod_{t=1}^{k} \lambda_{i_t}\left(\frac{\widetilde{C}}{2}\right) = \prod_{t=1}^{k} \lambda_{i_t}\left(\frac{\widetilde{A}^*X^* + X\widetilde{A}}{2}\right) \leq \prod_{t=1}^{k} \sigma_{i_t}(X\widetilde{A}) \leq \prod_{t=1}^{k} \sigma_{i_t}(X) \prod_{i=1}^{k} \sigma_i(\widetilde{A}). \tag{2.4}$$

Since A and B are non-singular, then

$$\prod_{t=1}^{k} \sigma_{i_t}(X) \geq \frac{\prod_{t=1}^{k} \lambda_{i_t}(\widetilde{C})}{2^k \prod_{i=1}^{k} \sigma_i(\widetilde{A})}.$$ □

When $k = n$, we can also get some inequalities related to the determinants of the solutions of equation (1.2).

Remark 2.11. Assuming that A and B are non-singular and C is Hermite positive definite such that $C \neq 2A^*XB$. Let X be a solution of equation (1.2), then

$$|\det(X)| > \frac{\det(C)}{2^n |\det(A)\det(B)|}.$$

Moreover, when equation (1.2) is defined in real number domain and $\det(A)\det(B) > 0$, one can conclude

$$\det(X) > \frac{\det(C)}{2^n \det(A) \cdot \det(B)}. \tag{2.5}$$

It is the result of Theorem 2.4 in [29].

Proof. Note that when $k = n$, for any matrix $X \in \mathbb{C}^{n\times n}$, $|\det(X)| = \prod_{i=1}^{n} \sigma_i(X)$. When $k = n$, (2.2) can be transformed into

$$|\det(X)| \geq \frac{\det(C)}{2^n |\det(A)| \cdot |\det(B)|}. \tag{2.6}$$

If $C \neq 2A^*XB$, then

$$\widetilde{C} \neq 2X\widetilde{A}. \tag{2.7}$$

Combining (2.7) with (2.3), one can conclude $X\widetilde{A} \neq \widetilde{A}^*X^*$. By the use of Lemma 2.7, the first inequality sign in (2.4) strictly holds. Thus (2.6) strictly holds. Moreover, when equation (1.2) is defined in real number domain and $\det(A)\det(B) > 0$, applying Lemma 2.9, one can conclude $\det(X) > 0$, and then (2.5) holds. □

The above results are related with the product of certain singular values of the solutions of equation (1.2), and we can also get the results about individual singular value.

Theorem 2.12. *Assuming that A and B are non-singular and C is Hermite positive definite. Let X be a solution of equation* (1.2)*, then for any integer $i \in \{1, 2, \ldots, n\}$,*

$$\sigma_i(X) \geq \max_{j=i,i+1,\ldots,n} \left\{ \frac{\lambda_j(A^{-*}CA^{-1})}{2\sigma_{j-i+1}(BA^{-1})} \right\}. \tag{2.8}$$

Proof. For any integer $i \in \{1, 2, \ldots, n\}$, $j = i, i+1, \ldots, n$, by the use of Lemmas 2.3 and 2.6 to (2.3), one can conclude

$$\lambda_j \left(\frac{\widetilde{A}^* X^* + X\widetilde{A}}{2} \right) \leq \sigma_j(X\widetilde{A}) \leq \sigma_i(X)\sigma_{j-i+1}(\widetilde{A}).$$

When i takes every value of the set $\{1, 2, \ldots, n\}$, and A and B are non-singular, one can conclude (2.8) holds. □

Applying Theorem 2.10, when $k = 1$, (2.2) is equal to

$$\sigma_i(X) \geq \frac{\sigma_i(A^{-*}CA^{-1})}{2\sigma_1(BA^{-1})}.$$

Compared with (2.8), due to $\frac{\sigma_i(A^{-*}CA^{-1})}{2\sigma_1(BA^{-1})} \in \left\{ \frac{\sigma_j(A^{-*}CA^{-1})}{2\sigma_{j-i+1}(BA^{-1})} \mid j = i, i+1, \ldots, n \right\}$, then we have

$$\frac{\sigma_i(A^{-*}CA^{-1})}{2\sigma_1(BA^{-1})} \leq \max_{j=i,i+1,\ldots,n} \left\{ \frac{\sigma_j(A^{-*}CA^{-1})}{2\sigma_{j-i+1}(BA^{-1})} \right\}.$$

Thus, we can obtain the remark as follows.

Remark 2.13. The result of Theorem 2.12 is better than that of Theorem 2.10, when using them to estimate individual singular values of the solutions of equation (1.2).

Example 2.14 illustrates the superiority of our results, compared with that of [9] and [29].

Example 2.14. Consider the following equation (1.2) with

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 0 & 0.5 & 0 \\ 0 & 6 & 0 & 0 \\ 1 & 0 & -5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad and \quad C = \begin{bmatrix} 10 & 0 & 0 & 1 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 1 & 0 & 0 & 7 \end{bmatrix}.$$

By computation, $\det(A) = 288$, $\det(B) = -915$, and C is a symmetric positive definite matrix. Due to $\det(A)\det(B) < 0$, a lower bound of the determinant of the solution X

of matrix equation (1.2) cannot be obtained with Theorem 2.1 and Remark 2.2, since the conditions of Theorem 2.1 and Remark 2.2 can not be met. While here a lower bound of the absolute value of the determinant of the solution can be obtained. We can also obtain lower bounds of singular values of the solution (see Tables 2.1 and 2.2). Thus, the results obtained in this section improved that of Theorem 2.1 and Remark 2.2.

Table 2.1: Lower bounds for the product of singular values in Theorem 2.10.

$\sigma_1(X) \geq 0.9419$	$\sigma_2(X) \geq 0.0330$	$\sigma_3(X) \geq 0.0100$
$\sigma_4(X) \geq 0.0049$	$\sigma_1(X)\sigma_2(X) \geq 0.1663$	$\sigma_1(X)\sigma_3(X) \geq 0.0505$
$\sigma_1(X)\sigma_4(X) \geq 0.0245$	$\sigma_2(X)\sigma_3(X) \geq 0.0018$	$\sigma_2(X)\sigma_4(X) \geq 8.5854 \times 10^{-4}$
$\sigma_3(X)\sigma_4(X) \geq 2.6052 \times 10^{-4}$	$\sigma_1(X)\sigma_2(X)\sigma_3(X)\sigma_4(X) \geq 0.0010$	$\sigma_1(X)\sigma_2(X)\sigma_3(X) \geq 0.0154$
$\sigma_1(X)\sigma_2(X)\sigma_4(X) \geq 0.0075$	$\sigma_1(X)\sigma_3(X)\sigma_4(X) \geq 0.0023$	$\sigma_2(X)\sigma_3(X)\sigma_4(X) \geq 7.9530 \times 10^{-5}$

Table 2.2: Lower bounds of individual singular value in Theorem 2.12.

$\sigma_1(X) \geq 0.9419$	$\sigma_2(X) \geq 0.0536$	$\sigma_3(X) \geq 0.0260$	$\sigma_4(X) \geq 0.0049$

It can be seen from Tables 2.1 and 2.2 that the result of Theorem 2.12 is superior to that of Theorem 2.10 in terms of individual singular value.

The above results are related with the singular values of the solutions of equation (1.2). Next, starting from the definition of singular values of any matrix $X \in C^{n\times n}$ i.e., $\sigma_i(X) = \sqrt{\lambda_i(XX^*)}$, by ingeniously constructing a positive semi-definite matrix and combining with matrix inequality for special matrices, we can directly obtain a lower bound of the matrix XX^*.

Theorem 2.15. *Assuming that A and B are non-singular and C is Hermite positive definite. Let X be a solution of equation* (1.2), *then for any positive constant α such that*

$$\alpha \leq \lambda_n\big[(A^{-*}B^*BA^{-1})^{-1/2}A^{-*}CA^{-1}(A^{-*}B^*BA^{-1})^{-1/2}\big], \tag{2.9}$$

we have

$$XX^* \succeq \alpha A^{-*}CA^{-1} - \alpha^2 A^{-*}B^*BA^{-1} \triangleq Y(\alpha) \succeq 0.$$

Proof. According to (2.3), for any positive constant α, one can conclude

$$\begin{aligned}0 &\preceq (\alpha\widetilde{A} - X^*)^*(\alpha\widetilde{A} - X^*) = \alpha^2\widetilde{A}^*\widetilde{A} - \alpha(\widetilde{A}^*X^* + X\widetilde{A}) + XX^* \\ &= \alpha^2\widetilde{A}^*\widetilde{A} - \alpha\widetilde{C} + XX^*.\end{aligned}$$

Applying Lemma 2.5 to (2.9), one can conclude

$$\alpha I \preceq (\widetilde{A}^*\widetilde{A})^{-1/2}\widetilde{C}(\widetilde{A}^*\widetilde{A})^{-1/2},$$

i.e.,

$$\alpha^2\widetilde{A}^*\widetilde{A} - \alpha\widetilde{C} \preceq 0.$$

Thus,

$$XX^* \succeq \alpha\widetilde{C} - \alpha^2\widetilde{A}^*\widetilde{A}. \tag{2.10}$$

□

When $X = X^* \succeq 0$, using Lemma 2.8 to (2.10), we can easily get the corollary as follows.

Corollary 2.16. *Assuming that there exists a positive semi-definite solution to matrix equation* (1.3)*. If A and B are non-singular and C is Hermite positive definite, then for any positive constant α such that*

$$\alpha \leq \lambda_n\big[(A^{-*}B^*BA^{-1})^{-1/2}A^{-*}CA^{-1}(A^{-*}B^*BA^{-1})^{-1/2}\big],$$

we have

$$X \succeq (\alpha A^{-*}CA^{-1} - \alpha^2A^{-*}B^*BA^{-1})^{1/2} \succeq 0.$$

In Example 2.14, choose $\alpha = \frac{1}{2}\lambda_4\big[(\widetilde{A}^T\widetilde{A})^{-1/2}\widetilde{C}(\widetilde{A}^T\widetilde{A})^{-1/2}\big] \approx 0.1191$, by the use of Theorem 2.15, one can directly obtain a lower bound of the product for the solution and its complex conjugate transpose matrix, i.e.,

$$XX^T \succeq \begin{bmatrix} 0.8577 & 0 & -0.0079 & 0.1043 \\ 0 & 0.0088 & -0.0022 & 0 \\ -0.0079 & -0.0022 & 0.0303 & -0.0008 \\ 0.1043 & 0 & -0.0008 & 0.0171 \end{bmatrix}.$$

3. The solution of matrix equation (1.3) and its related properties

When A and B are non-singular and $X \neq X^*$, by calculation, $X = \frac{1}{2}A^{-*}CB^{-1}$ is a solution of matrix equation (1.2), and the solution is not unique. In [9, 29], the authors proposed an lower bound of the determinants of the solutions of matrix equation (1.2). And the lower bound is the determinant of the solution $X = \frac{1}{2}A^{-*}CB^{-1}$. In this section, we obtain an analytical expression for the positive definite solution of equation (1.3) and design two algorithms for the solution. Each recursive iteration is its upper bound or lower bound of the positive definite solution of equation (1.3).

Lemma 3.1. [14] *Let $A, B \in \mathbb{C}^{n\times n}$ be Hermite matrices, if $A \succeq B$, then $\lambda_i(A) \geq \lambda_i(B)$, $i = 1, 2, \ldots, n$.*

Lemma 3.2. [14] *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be Hermite matrices, then*

$$\lambda_i(A) + \lambda_n(B) \leq \lambda_i(A + B) \leq \lambda_i(A) + \lambda_1(B), \quad i = 1, 2, \ldots, n.$$

Lemma 3.3. [25] *For given $A \in \mathbb{C}^{n\times n}$, $\rho(A) < 1$ if and only if there exists a nonsingular matrix $D \in \mathbb{C}^{n\times n}$ such that $\sigma_1(DAD^{-1}) < 1$.*

Remark 3.4. An algorithm for obtaining a nonsingular matrix D was given in Algorithm II of [25].

Lemma 3.5. [14] *Let $A^{(k)} \in \mathbb{C}^{m\times n}$, then the series $\sum_{k=1}^{\infty} A^{(k)}$ absolutely converges if and only if the series $\sum_{k=1}^{\infty} \|A^{(k)}\|$ converges, where $\|\cdot\|$ denotes any matrix norm on $\mathbb{C}^{m\times n}$.*

Lemma 3.6. [14] *Let $A \in \mathbb{C}^{n\times n}$, then $\lim_{k\to\infty} A^k = 0$ if and only if $\rho(A) < 1$.*

By utilizing the characteristics of the matrix equation, and by combining them with the eigenvalue inequalities of positive semi-definite matrices and matrix inequalities, new inequalities are obtained. We then construct a convergent matrix series under certain conditions and prove that this series is the positive definite solution to equation (1.3).

Theorem 3.7. *Assuming that the coefficient matrices A, B and C satisfy*

$$\operatorname{Re}\lambda_i(BA^{-1}) > 0, \quad i = 1, 2, \ldots, n, \tag{3.1}$$

and C is a Hermite positive definite matrix. Let X be the positive definite solution of matrix equation (1.3), then we have the following results:

(i) *For arbitrary positive constant q, there exists a non-singular matrix D such that*

$$\sigma_1[D(\widetilde{A} - qI)(\widetilde{A} + qI)^{-1}D^{-1}] < 1. \tag{3.2}$$

(ii) *For arbitrary integer $k = 0, 1, 2, \ldots,$*

$$\begin{aligned} X \succeq X_l^{(k)} \triangleq\ & \eta_1\big[(\widetilde{A} + qI)^{-*}(\widetilde{A} - qI)^*\big]^{k+1} D^*D\big[(\widetilde{A} - qI)(\widetilde{A} + qI)^{-1}\big]^{k+1} \\ & + \sum_{i=0}^{k} \big[(\widetilde{A} + qI)^{-*}(\widetilde{A} - qI)^*\big]^i (\widetilde{A} + qI)^{-*} 2q\widetilde{C}(\widetilde{A} + qI)^{-1} \\ & \times \big[(\widetilde{A} - qI)(\widetilde{A} + qI)^{-1}\big]^i, \end{aligned} \tag{3.3}$$

where $\widetilde{A} = BA^{-1}$, $\widetilde{C} = A^{-*}CA^{-1}$ and

$$\eta_1 \triangleq \frac{\lambda_n\big[D^{-*}(\widetilde{A} + qI)^{-*}2q\widetilde{C}(\widetilde{A} + qI)^{-1}D^{-1}\big]}{1 - \lambda_n\big[D^{-*}(\widetilde{A} + qI)^{-*}(\widetilde{A} - qI)^*D^*D(\widetilde{A} - qI)(\widetilde{A} + qI)^{-1}D^{-1}\big]}.$$

(iii) *For arbitrary integer* $k = 1, 2, \ldots,$

$$X_l^{(k)} \succeq X_l^{(k-1)}. \tag{3.4}$$

Moreover, the monotonically increasing lower bound sequence $X_l^{(k)}$ *converges, and there exists a Hermite positive definite matrix* X_l *such that*

$$X_l = \lim_{k\to\infty} X_l^{(k)}. \tag{3.5}$$

(iv) *The limit of monotonically increasing lower bound sequence is equal to*

$$X_l = \sum_{i=0}^{\infty} \big[(\widetilde{A}+qI)^{-*}(\widetilde{A}-qI)^*\big]^i (\widetilde{A}+qI)^{-*} 2q\widetilde{C}(\widetilde{A}+qI)^{-1}\big[(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}\big]^i. \tag{3.6}$$

It is the positive definite solution of equation (1.3).

(v) *For arbitrary integer* $k = 0, 1, 2, \ldots,$

$$\begin{aligned} X \preceq X_u^{(k)} \triangleq\ & \xi_1 \big[(\widetilde{A}+qI)^{-*}(\widetilde{A}-qI)^*\big]^{k+1} D^* D \big[(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}\big]^{k+1} \\ & + \sum_{i=0}^{k} \big[(\widetilde{A}+qI)^{-*}(\widetilde{A}-qI)^*\big]^i (\widetilde{A}+qI)^{-*} 2q\widetilde{C}(\widetilde{A}+qI)^{-1} \\ & \quad \times \big[(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}\big]^i, \end{aligned}$$

where

$$\xi_1 \triangleq \frac{\lambda_1\big[D^{-*}(\widetilde{A}+qI)^{-*}2q\widetilde{C}(\widetilde{A}+qI)^{-1}D^{-1}\big]}{1-\lambda_1\big[D^{-*}(\widetilde{A}+qI)^{-*}(\widetilde{A}-qI)^*D^*D(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}D^{-1}\big]}.$$

(vi) *For arbitrary integer* $k = 1, 2, \ldots,$

$$X_u^{(k-1)} \succeq X_u^{(k)}, \quad k = 1, 2, \ldots.$$

The monotonically decreasing upper bound sequence $X_u^{(k)}$ *converges, and the limit of the sequence is equal to*

$$\lim_{k\to\infty} X_u^{(k)} = \sum_{i=0}^{\infty} \big[(\widetilde{A}+qI)^{-*}(\widetilde{A}-qI)^*\big]^i (\widetilde{A}+qI)^{-*} 2q\widetilde{C}(\widetilde{A}+qI)^{-1}\big[(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}\big]^i.$$

Proof. (i) let

$$V = \widetilde{A} + qI,$$

then

$$(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1} = (V-2qI)V^{-1} = I - 2qV^{-1}.$$

Thereupon,

$$\lambda_i\big[(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}\big]=1-2q\lambda_{n-i+1}(V^{-1})=1-2q\frac{1}{\lambda_i(\widetilde{A})+q}=\frac{\lambda_i(\widetilde{A})-q}{\lambda_i(\widetilde{A})+q}. \tag{3.7}$$

According to (3.1) and (3.7), one can conclude

$$\big|\lambda_i\big[(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}\big]\big|^2=\frac{\lambda_i(\widetilde{A})-q}{\lambda_i(\widetilde{A})+q}\cdot\frac{\overline{\lambda}_i(\widetilde{A})-q}{\overline{\lambda}_i(\widetilde{A})+q}=\frac{q^2-2q\operatorname{Re}\lambda_i(\widetilde{A})+|\lambda_i(\widetilde{A})|^2}{q^2+2q\operatorname{Re}\lambda_i(\widetilde{A})+|\lambda_i(\widetilde{A})|^2}<1,$$

and then

$$\rho\big[(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}\big]<1. \tag{3.8}$$

Applying Lemma 3.3 to (3.8), one can conclude there exists a nonsingular matrix D such that

$$\sigma_1\big[D(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}D^{-1}\big]<1. \tag{3.9}$$

(ii) Multiplying by A^{-*} on the left and by A^{-1} on the right, (1.3) can be converted to

$$\widetilde{A}^*X+X\widetilde{A}=\widetilde{C}. \tag{3.10}$$

Multiplying by D^{-*} on the left and by D^{-1} on the right, (3.10) can be converted to

$$\widehat{A}^*\widehat{X}+\widehat{X}\widehat{A}=\widehat{C},$$

where $\widehat{A}=D\widetilde{A}D^{-1}$, $\widehat{X}=D^{-*}XD^{-1}$, $\widehat{C}=D^{-*}\widetilde{C}D^{-1}$. According to the equality

$$(\widehat{A}+qI)^*\widehat{X}(\widehat{A}+qI)-(\widehat{A}-qI)^*\widehat{X}(\widehat{A}-qI)=2q\big(\widehat{A}^*\widehat{X}+\widehat{X}\widehat{A}\big)=2q\widehat{C},$$

one can conclude

$$\widehat{X}=(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\widehat{X}(\widehat{A}-qI)(\widehat{A}+qI)^{-1}+(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}. \tag{3.11}$$

By the use of Lemma 2.5, one can conclude

$$\widehat{X}\succeq(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\lambda_n(\widehat{X})I(\widehat{A}-qI)(\widehat{A}+qI)^{-1}+(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}.$$

By the use of Lemmas 3.1 and 3.2, one can conclude

$$\begin{aligned}&\lambda_n(\widehat{X})\\ &\ge\lambda_n\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\lambda_n(\widehat{X})(\widehat{A}-qI)(\widehat{A}+qI)^{-1}+(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big]\\ &\ge\lambda_n(\widehat{X})\lambda_n\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]+\lambda_n\big[(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big].\end{aligned} \tag{3.12}$$

By the use of (3.2), we have

$$\begin{aligned}&\sigma_1^2\big[D(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}D^{-1}\big]\\ &=\sigma_1^2\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]=\lambda_1\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]<1,\end{aligned}\tag{3.13}$$

and then

$$\lambda_n\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]<1.\tag{3.14}$$

According to (3.14), (3.12) can be converted into

$$\lambda_n(\widehat{X})\geq\frac{\lambda_n\big((\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big)}{1-\lambda_n\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]}\triangleq\eta_1.\tag{3.15}$$

According to (3.11) and $\widehat{X}\succeq\eta_1 I$, it is easy to known that

$$\begin{aligned}\widehat{X}&\succeq\eta_1\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\big]^{k+1}\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]^{k+1}\\ &\quad+\sum_{i=0}^{k}\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\big]^i(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]^i\\ &\triangleq\widehat{X}_l^{(k)}.\end{aligned}\tag{3.16}$$

(3.16) is multiplied by D^* on the left and by D on the right, then $X\succeq X_l^{(k)}$.

(iii) For arbitrary integer $k=1,2,\ldots,$

$$\begin{aligned}&\widehat{X}_l^{(k)}-\widehat{X}_l^{(k-1)}\\ &=\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\big]^k\\ &\quad\times\big\{\eta_1(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*(\widehat{A}-qI)(\widehat{A}+qI)^{-1}-\eta_1 I+(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big\}\\ &\quad\times\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]^k\\ &=\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\big]^k(\widehat{A}+qI)^{-*}\\ &\quad\times\big\{\eta_1(\widehat{A}-qI)^*(\widehat{A}-qI)-\eta_1(\widehat{A}+qI)^*(\widehat{A}+qI)+2q\widehat{C}\big\}\\ &\quad\times(\widehat{A}+qI)^{-1}\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]^k\\ &=\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\big]^k(\widehat{A}+qI)^{-*}\big\{-2q\eta_1\widehat{A}^*-2q\eta_1\widehat{A}+2q\widehat{C}\big\}(\widehat{A}+qI)^{-1}\\ &\quad\times\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]^k.\end{aligned}\tag{3.17}$$

According to (3.13), we have

$$\begin{aligned}0&\succ(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*(\widehat{A}-qI)(\widehat{A}+qI)^{-1}-I\\ &=(\widehat{A}+qI)^{-*}\big[(\widehat{A}-qI)^*(\widehat{A}-qI)-(\widehat{A}+qI)^*(\widehat{A}+qI)\big](\widehat{A}+qI)^{-1}\\ &=-(\widehat{A}+qI)^{-*}\big[2q(\widehat{A}+\widehat{A}^*)\big](\widehat{A}+qI)^{-1}.\end{aligned}$$

Thereupon,

$$\widehat{A}+\widehat{A}^{*}\succ 0. \tag{3.18}$$

According to the definition of η_1 in (3.15) and (3.18), one can conclude

(3.19)

$$\begin{aligned}
\eta_1\widehat{A}+\eta_1\widehat{A}^{*} &= \frac{\lambda_n\big[(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big]}{1-\lambda_n\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^{*}(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]}(\widehat{A}+\widehat{A}^{*})\\
&= \frac{\lambda_n\big[(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big]}{\lambda_1\big[I-(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^{*}(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]}(\widehat{A}+\widehat{A}^{*})\\
&= \frac{\lambda_n\big[(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big]}{\lambda_1\big[(\widehat{A}+qI)^{-*}2q(\widehat{A}+\widehat{A}^{*})(\widehat{A}+qI)^{-1}\big]}(\widehat{A}+\widehat{A}^{*})\\
&= (\widehat{A}+\widehat{A}^{*})^{1/2}\lambda_n\big[(\widehat{A}+qI)^{-*}\widehat{C}(\widehat{A}+qI)^{-1}\big]\\
&\quad\times\lambda_n\big[(\widehat{A}+qI)(\widehat{A}+\widehat{A}^{*})^{-1}(\widehat{A}+qI)^{*}\big](\widehat{A}+\widehat{A}^{*})^{1/2}\\
&\preceq (\widehat{A}+\widehat{A}^{*})^{1/2}\lambda_n\big[(\widehat{A}+qI)^{-*}\widehat{C}(\widehat{A}+qI)^{-1}(\widehat{A}+qI)(\widehat{A}+\widehat{A}^{*})^{-1}(\widehat{A}+qI)^{*}\big]\\
&\quad\times(\widehat{A}+\widehat{A}^{*})^{1/2}\\
&= (\widehat{A}+\widehat{A}^{*})^{1/2}\lambda_n\big[(\widehat{A}+\widehat{A}^{*})^{-1/2}\widehat{C}(\widehat{A}+\widehat{A}^{*})^{-1/2}\big](\widehat{A}+\widehat{A}^{*})^{1/2}\\
&\preceq \widehat{C}.
\end{aligned}$$

Substituting (3.19) into (3.17), one can conclude $\widehat{X}_l^{(k)}\succeq\widehat{X}_l^{(k-1)}$. The above equality is multiplied by D^* on the left and by D on the right, (3.4) holds. Due to (3.3) and (3.4), there exists a Hermite positive definite matrix X_l such that

$$X_l=\lim_{k\to\infty}X_l^{(k)}.$$

(iv) By the use of (3.9), we have

$$\begin{aligned}
&\sum_{i=0}^{n}\|A^{(i)}\|\\
&\triangleq\sum_{i=0}^{n}\Big\|\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^{*}\big]^{i}(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]^{i}\Big\|\\
&\le\sum_{i=0}^{n}\sigma_1^{i}\big[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^{*}\big]\sigma_1\big[(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big]\sigma_1^{i}\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]\\
&\le\frac{\sigma_1\big[(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\big]}{1-\sigma_1^{2}\big[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\big]},
\end{aligned}$$

and then the series $\sum_{i=0}^{\infty}\|A^{(i)}\|$ converges. Applying Lemma 3.5, the series $\sum_{i=0}^{\infty}A^{(i)}$ converges. Due to $\rho(A)\le\sigma_1(A)$, for given $A\in\mathbb{C}^{n\times n}$, one can conclude $\rho((\widehat{A}-qI)(qI+$

$\widehat{A})^{-1}) < 1$ from (3.9). Applying Lemma 3.6, we have

$$\lim_{k\to\infty} \eta_1 \left[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\right]^{k+1}\left[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\right]^{k+1} = 0. \tag{3.20}$$

Since the series $\sum_{i=0}^{\infty} A^{(i)}$ converges and (3.20) holds,

$$\lim_{k\to\infty} \widehat{X}_l^k = \sum_{i=0}^{\infty} \left[(\widehat{A}+qI)^{-*}(\widehat{A}-qI)^*\right]^i(\widehat{A}+qI)^{-*}2q\widehat{C}(\widehat{A}+qI)^{-1}\left[(\widehat{A}-qI)(\widehat{A}+qI)^{-1}\right]^i. \tag{3.21}$$

(3.21) is multiplied by D^* on the left and by D on the right, then

$$\lim_{k\to\infty} X_l^{(k)} = \sum_{i=0}^{\infty} \left[(\widetilde{A}+qI)^{-*}(\widetilde{A}-qI)^{-*}\right]^i(\widetilde{A}+qI)^{-*}2q\widetilde{C}(\widetilde{A}+qI)^{-1}\left[(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}\right]^i. \tag{3.22}$$

According to (3.5) and (3.22), (3.6) holds. (3.6) can be rewritten as

$$X_l = (\widetilde{A}+qI)^{-*}2q\widetilde{C}(\widetilde{A}+qI)^{-1} + (\widetilde{A}+qI)^{-*}(\widetilde{A}-qI)^{-*}X_l(\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}. \tag{3.23}$$

(3.23) is multiplied by $(\widetilde{A}+qI)^*$ on the left and by $(\widetilde{A}+qI)$ on the right, then (3.23) can be converted to

$$\widetilde{A}^* X_l + X_l\widetilde{A} + \widetilde{C} = 0.$$

Thereupon, X_l in (3.6) is the positive definite solution of equation (1.3).

(v) The proof is similar to (ii).

(vi) The proof is similar to (iii) to (iv). □

Moreover, we designed two algorithms to solve the positive definite solution of equation (1.3). Each recursive iteration of these algorithms obtains the upper or lower bounds of the positive definite solution of equation (1.3). The results obtained from these algorithms improve upon the results of existing relevant studies.

Based on the monotonically increasing lower bound sequence $X_l^{(k)}$, $k = 0, 1, 2, \ldots$, we design the following algorithm for the positive definite solution of equation (1.3).

Algorithm 3.1

1: For arbitrary given positive constant q, let $W = (\widetilde{A}-qI)(\widetilde{A}+qI)^{-1}$, applying Algorithm II in [25] can obtain a non-singular matrix D such that $\sigma_1(DWD^{-1}) < 1$.
2: For arbitrary given error precision ε, by calculating $\|X_l^{(k)} - X_l^{(k-1)}\| < \varepsilon$, we can determine n.
3: Compute $X_l^{(n)}$, then $X_l^{(n)}$ is an approximate solution of equation (1.3) that satisfies the precision ε.

Based on the monotonically decreasing upper bound sequence $X_u^{(k)}$, $k = 0, 1, 2, \ldots$, we design the following algorithm for the positive definite solution of equation (1.3).

Algorithm 3.2

1: For arbitrary given positive constant q, let $W = (\widetilde{A} - qI)(\widetilde{A} + qI)^{-1}$, applying Algorithm II in [25] can obtain a non-singular matrix D such that $\sigma_1(DWD^{-1}) < 1$.
2: For arbitrary given error precision ε, by calculating $\|X_u^{(k)} - X_u^{(k-1)}\| < \varepsilon$, we can determine n.
3: Compute $X_u^{(n)}$, then $X_u^{(n)}$ is an approximate solution of equation (1.3) that satisfies the precision ε.

For Algorithms 3.1 and 3.2, it is difficult to determine the corresponding n from Step 2. We give the following two simple criteria to determine n.

Theorem 3.8. *For equation* (1.3)*, assuming that the coefficient matrices A, B and C satisfy*

$$\operatorname{Re}\lambda_i(BA^{-1}) > 0, \quad i = 1, 2, \ldots, n,$$

and C is a Hermite positive definite matrix. For arbitrary given error precision ε, take an natural number N_1 such that

$$\begin{aligned} &\frac{\ln\left(\frac{\varepsilon}{\lambda_1\left(\eta_1 W^* D^* DW - \eta_1 D^* D + (\widetilde{A}+qI)^{-*}2q\widetilde{C}(\widetilde{A}+qI)^{-1}\right)}\right)}{\ln(\lambda_1(W^*W))} \\ &< N_1 \le \frac{\ln\left(\frac{\varepsilon}{\lambda_1\left(\eta_1 W^* D^* DW - \eta_1 D^* D + (\widetilde{A}+qI)^{-*}2q\widetilde{C}(\widetilde{A}+qI)^{-1}\right)}\right)}{\ln(\lambda_1(W^*W))} + 1, \end{aligned} \tag{3.24}$$

then $X_l^{(N_1)}$ obtained by Algorithm 3.1 *can be regarded as an approximate solution of equation* (1.3) *that meets the precision ε.*

Proof. Take the natural number N_1 defined as (3.24), then

$$\begin{aligned} &\|X_l^{(N_1)} - X_l^{(N_1-1)}\| \\ &= \|(W^*)^{N_1}\big(\eta_1 W^* D^* DW - \eta_1 D^* D + (\widetilde{A} + qI)^{-*}2q\widetilde{C}(\widetilde{A} + qI)^{-1}\big)W^{N_1}\| \\ &\le \lambda_1^{N_1}(W^*W)\lambda_1\big(\eta_1 W^* D^* DW - \eta_1 D^* D + (\widetilde{A} + qI)^{-*}2q\widetilde{C}(\widetilde{A} + qI)^{-1}\big) < \varepsilon. \qquad \square \end{aligned}$$

Similar to the proof of Theorem 3.8, we obtain the following theorem.

Theorem 3.9. *Under the condition of Theorem* 3.8*, for arbitrary given error precision ε, take an natural number N_2 such that*

$$\begin{aligned} &\frac{\ln\left(\frac{\varepsilon}{\sigma_1\left(\xi_1 W^* D^* DW - \xi_1 D^* D + (\widetilde{A}+qI)^{-*}2q\widetilde{C}(\widetilde{A}+qI)^{-1}\right)}\right)}{\ln(\lambda_1(W^*W))} \\ &< N_2 \le \frac{\ln\left(\frac{\varepsilon}{\sigma_1\left(\xi_1 W^* D^* DW - \xi_1 D^* D + (\widetilde{A}+qI)^{-*}2q\widetilde{C}(\widetilde{A}+qI)^{-1}\right)}\right)}{\ln(\lambda_1(W^*W))} + 1, \end{aligned}$$

then $X_u^{(N_2)}$ obtained by Algorithm 3.2 *can be regarded as an approximate solution of equation* (1.3) *that meets the precision ε.*

Example 3.10 illustrates that the obtained results in this section improve that of Theorems 2.10, 2.12 and 2.15 and Corollary 2.16.

Example 3.10. Consider the following matrix equation (1.3) with

$$A=\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{bmatrix},\quad B=\begin{bmatrix}1&1&0&0\\1&2&1&0\\0&0&1&2\\0&0&0&1\end{bmatrix}\quad\text{and}\quad C=\begin{bmatrix}28&37&9&3\\37&64&27&5\\9&27&34&26\\3&5&26&92\end{bmatrix}.$$

By computation, the elements of the set of the spectrum of the matrix BA^{-1} are $\{1,1,1,1\}$, and C is a symmetric positive definite matrix. Thus the conditions of Theorem 3.7 are met.

Let X be the positive definite solution of matrix equation (1.3), choose $q=1$, by the use of Algorithm II in [25], one can obtain

$$D=\begin{bmatrix}0&2.5&0&0\\1.5625&0&0&0\\0&0&2.5&0\\0&0&0&1.5625\end{bmatrix}.$$

By the use of Theorem 3.7, one can conclude

$$X\succeq P_l^{(0)}=\begin{bmatrix}13.75&0&0&1\\0&9&0&1\\0&0&8&0\\1&1&0&30\end{bmatrix}\quad\text{and}\quad X\preceq X_u^{(0)}=\begin{bmatrix}61.6649&0&0&1\\0&9&0&1\\0&0&8&0\\1&1&0&77.9149\end{bmatrix}.$$

By the use of Corollary 2.16, choose $\alpha=\frac{1}{2}\lambda_6\big[(\widetilde{A}^T\widetilde{A})^{-1/2}\widetilde{C}(\widetilde{A}^T\widetilde{A})^{-1/2}\big]=6.5$, one can get

$$X\succeq Y^{1/2}(\alpha)=\begin{bmatrix}9.8113&0.8650&-0.0148&0.7003\\0.8650&8.5892&-0.0104&0.4761\\-0.0148&-0.0104&7.8486&0.3859\\0.7003&0.4761&0.3859&17.4538\end{bmatrix}.$$

By computation, the elements of the set of the spectrum of the matrix $P_l^{(0)}-Y^{1/2}(\alpha)$ are $\{12.5887,4.1359,0.1995,0.1229\}$, then $P_l^{(0)}\succeq Y^{1/2}(\alpha)$.

Due to $X=X^T\succ 0$, one can conclude $\lambda_i(X)=\sigma_i(X)>0$. Applying Theorems 2.10, 2.12 and 3.7 to equation (1.3), we can obtain upper and lower bounds of the singular values of the solution of matrix equation (1.3), see Tables 3.1 and 3.2.

Table 3.1: The eigenvalues values (singular values) of $X = X^T \succ 0$ (1).

eigenvalues (singular values)	lower bounds (Theorem 2.10)	lower bounds (Theorem 3.7)	upper bounds (Theorem 2.10)	upper bounds (Theorem 3.7)
$\lambda_1(X)\lambda_2(X)$	194.1750	412.1698	–	4804.5147
$\lambda_1(X)\lambda_3(X)$	86.0541	269.5318	–	700.7844
$\lambda_1(X)\lambda_4(X)$	75.0256	240.8680	–	623.9255
$\lambda_2(X)\lambda_3(X)$	45.7419	122.5484	–	553.5393
$\lambda_2(X)\lambda_4(X)$	39.8797	109.5158	–	492.8296
$\lambda_3(X)\lambda_4(X)$	17.6738	71.6161	–	71.8839
$\lambda_1(X)\lambda_2(X)\lambda_3(X)$	2288.8368	3689.7500	–	43170.9112
$\lambda_1(X)\lambda_2(X)\lambda_4(X)$	1995.5047	3297.3584	–	38436.1176
$\lambda_1(X)\lambda_3(X)\lambda_4(X)$	884.3643	2156.2541	–	5606.2751
$\lambda_2(X)\lambda_3(X)\lambda_4(X)$	470.0817	980.3874	–	4428.3147
$\lambda_1(X)\lambda_2(X)\lambda_3(X)\lambda_4(X)$	23522	29518	–	345367.2895

It can be seen that from Table 3.1 the results of Theorem 3.7 improve that of Theorem 2.10 in terms of the product of the singular values.

Table 3.2: The eigenvalues values (singular values) of $X = X^T \succ 0$ (2).

eigenvalues	lower bounds (Theorem 2.10)	lower bounds (Theorem 2.12)	lower bounds (Theorem 3.7)	upper bounds (Theorem 2.10)	upper bounds (Theorem 2.12)	upper bounds (Theorem 3.7)
$\lambda_1(X)$	19.1129	19.1129	30.1085	–	–	77.9907
$\lambda_2(X)$	10.1594	10.2768	13.6895	–	–	61.6037
$\lambda_3(X)$	4.5024	4.5024	8.9520	–	–	8.9855
$\lambda_4(X)$	3.9254	3.9254	8	–	–	8

It can be seen from Table 3.2 that the results of Theorem 3.7 improve that of Theorems 2.10 and 2.12 in terms of individual singular values.

Let $\varepsilon = 10^{-7}$, then Algorithms 3.1 and 3.2 iterate 15 and 16 respectively to obtain the matrix

$$X = \begin{bmatrix} 14 & 0 & 0 & 1 \\ 0 & 9 & 0 & 1 \\ 0 & 0 & 8 & 0 \\ 1 & 1 & 0 & 30 \end{bmatrix}.$$

It can be regarded as the symmetric positive definite solution of equation (1.3) that meets the precision ε.

It can be seen that when equation (1.3) has a symmetric positive definite solution, Remark 2.2 only gives the lower bound of the determinant of the solution, while we give the estimates of the solution, its upper and lower bounds and its singular values including the determinant of the solution. Thus, the results obtained improve the result of Remark 2.2.

The following example, consisting of random matrices, illustrates that through a certain number of iterations, our result improve the result of Remark 2.2 in most cases.

Example 3.11. We randomly construct the following 50 matrix equations in the form of (1.3) with

$$A=\begin{bmatrix}-2-2a_1 & a_{11} & a_{12} & a_{13}\\ a_5 & -3-3a_2 & a_{14} & a_{15}\\ a_6 & a_7 & -3-3a_3 & a_{16}\\ a_8 & a_9 & a_{10} & -3-3a_4\end{bmatrix},\ B=\begin{bmatrix}-2-2b_1 & b_{11} & b_{12} & b_{13}\\ b_5 & -2-2b_2 & b_{14} & b_{15}\\ b_6 & b_7 & -2-2b_3 & b_{16}\\ b_8 & b_9 & b_{10} & -2-2b_4\end{bmatrix},$$

$$C=\begin{bmatrix}20+10c_1 & c_1 & 0 & 0\\ c_1 & 20+10c_2 & c_2 & 0\\ 0 & c_2 & 20+10c_3 & c_3\\ 0 & 0 & c_3 & 20+10c_4\end{bmatrix},$$

under the condition that A and B are stable and C is symmetric positive definite.

Applying Remark 2.2, lower bounds of the determinants of symmetric positive definite solutions of 50 matrix equations in the form (1.3) can be obtained.

Taking $q_j = 1$, D_j is obtained by Algorithm II in [25]. Applying Theorem 3.7, upper and lower bounds of the determinants of symmetric positive definite solutions of 50 matrix equations in the form (1.3) can be obtained.

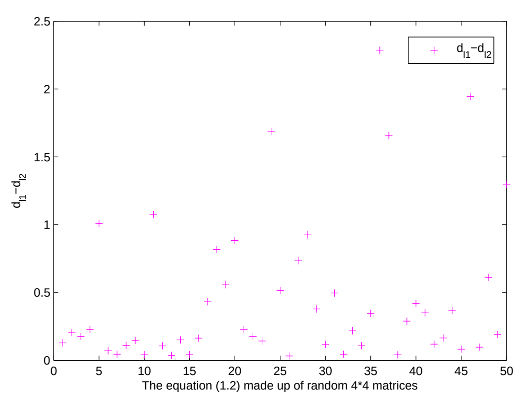

Figure 3.1: Comparison of lower bounds of the determinant between Theorem 3.7 and Remark 2.2.

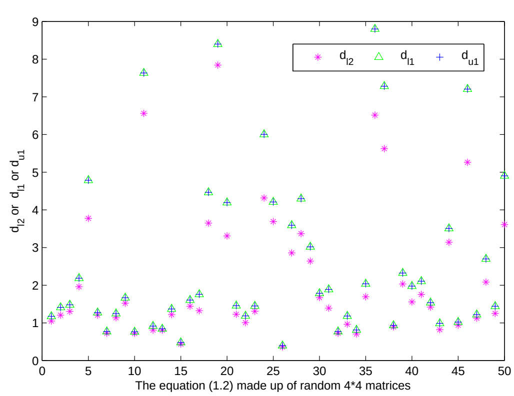

Figure 3.2: Comparison of upper and lower bounds of the determinant between Theorem 3.7 and Remark 2.2.

Tighter upper and lower bounds of Theorem 3.7 can be obtained, and the lower bound of Theorem 3.7 is superior to Remark 2.2, iterating 6 times, in most cases (see Figures 3.1 and 3.2). After 6 iterations, the result of Theorem 3.7 is better.

In Figures 3.1 and 3.2, $d_{l1} = \det(X_l^{(6)})$ and $d_{u1} = \det(X_u^{(6)})$ denote the lower and upper bounds of the determinant obtained by applying Theorem 3.7. d_{l2} denotes the lower bound of the determinant obtained by applying Theorem 2.1.

4. Conclusion

In this paper, inequalities related to the singular values of the solutions of a class of linear Sylvester-like matrix equations are presented. In addition, a lower matrix bound of the product for the solutions and their complex conjugate transpose matrices is presented directly. When a Hermite matrix is a solution of the matrix equation, we obtain a convergent matrix series under certain conditions and it is proved to be the positive definite solution of the matrix equation. Finally, two algorithms for solving the matrix equation are presented and each recursive iteration is its upper bound or lower bound of the positive definite solution of the matrix equation. Considering the importance of the matrix equation in practical applications, it is our future work to explore numerical algorithms for the solution of the matrix equation that are more accurate and easier to calculate and explore more accurate estimates of the associated measurements of the solutions.

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