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# Existence and Multiplicity of Nontrivial Solutions for a $(p,q)$-Laplacian System on Locally Finite Graphs

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Abstract. We generalize two embedding theorems and investigate the existence and multiplicity of nontrivial solutions for a $(p,q)$-Laplacian coupled system with perturbations and two parameters $\lambda_1$ and $\lambda_2$ on locally finite graph. By using the Ekeland's variational principle, we obtain that system has at least one nontrivial solution when the nonlinear term satisfies the sub-$(p,q)$ conditions. We also obtain a necessary condition for the existence of semi-trivial solutions to the system. Moreover, by using the mountain pass theorem and Ekeland's variational principle, we obtain that system has at least one solution of positive energy and one solution of negative energy when the nonlinear term satisfies the super-$(p,q)$ conditions which is weaker than the well-known Ambrosetti–Rabinowitz condition. Especially, in all of the results, we present the concrete ranges of the parameters $\lambda_1$ and $\lambda_2$.

## 1. Introduction

Some research results on the existence of solutions of partial differential equations on discrete graphs have been applied in machine learning, image processing and other fields. For example, in [7–9], Elmoataz et al. studied the existence and uniqueness of solutions of $p$-Laplacian equation subject to the Dirichlet boundary condition on a weighted connected graph, and showed that this operator can be applied to some inverse problems in image processing and machine learning, including filtering, segmentation, clustering, and inpainting. In [1], Bougleux et al. proposed a structure-preserving filtering framework based on $p$-Laplacian operator on directed graphs. They showed that this method can obtain better smoothing quality during imaging. In [10], Ennaji et al. discussed the relationship between some stochastic games named Tug-of-War games and a class of nonlocal partial differential equations on graphs and showed that it covers several nonlocal partial differential equations on graphs, such as $p$-Laplacian equation, $\infty$-Laplacian equation and Eikonal equation. Moreover, they also showed that it can be used to solve several inverse problems in imaging and data science.

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Next, we recall some basic knowledge of discrete graphs. Let $G = (V, E)$ be a locally finite and connected graph, where $V$ denotes the vertex set and $E$ denotes the edge set. We say that $(V, E)$ is a locally finite graph if for any $x \in V$ there are only finite edges $xy \in E$. Moreover, we say that $(V, E)$ is a connected graph if any two vertices $x$ and $y$ can be connected via finite edges. For any edge $xy \in E$, assume that its weight $\omega_{xy} > 0$ and $\omega_{xy} = \omega_{yx}$. For any $x \in V$, its degree is defined as $\deg(x) = \sum_{y\sim x} \omega_{xy}$, where we denote $y \sim x$ if there exists $y \in V$ such that edge $xy \in E$. The distance of two vertices $x$, $y$, denoted by $\operatorname{dist}(x, y)$, is defined as the minimal number of edges which connect $x$, $y$. Let $\mu\colon V \to \mathbb{R}^+$ be a finite measure, $\mu(x) \geq \mu_0 > 0$, and $C(V)$ be the set of all real functions on $V$. Define $\Delta\colon C(V) \to C(V)$ as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) - u(x)).$$

The associated gradient form is

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)).$$

Write $\Gamma(u) = \Gamma(u, u)$. We denote the length of the gradient is

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left( \frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x))^2 \right)^{1/2}.$$

We can obtain that the gradient has the following properties:

$$\Gamma(u_1 + u_2, v)(x) = \Gamma(u_1, v)(x) + \Gamma(u_2, v)(x),$$

$$\Gamma(u, v_1 + v_2)(x) = \Gamma(u, v_1)(x) + \Gamma(u, v_2)(x),$$

$$\Gamma(\theta u, v)(x) = \Gamma(u, \theta v)(x) = \theta\Gamma(u, v)(x) \quad \text{for all } \theta \in \mathbb{R},$$

$$\Gamma(u, v) \leq |\nabla u| \cdot |\nabla v|, \tag{1.1}$$

$$\big| |\nabla u_k| - |\nabla u| \big| \leq |\nabla(u_k - u)|. \tag{1.2}$$

For any $p > 1$, we define $\Delta_p\colon C(V) \to C(V)$ as follows:

$$\Delta_p(u)(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} \big( |\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x) \big) \omega_{xy}(u(y) - u(x)). \tag{1.3}$$

Let $C_c(V) := \{u\colon V \to \mathbb{R} \mid \operatorname{supp} u \subset V\}$. Then for any function $\phi \in C_c(V)$,

$$\int_V \Delta_p u \phi \, d\mu = -\int_V |\nabla u|^{p-2} \Gamma(u, \phi) \, d\mu. \tag{1.4}$$

For any function $u\colon V \to \mathbb{R}$, we denote

$$\int_V u(x) \, d\mu = \sum_{x\in V} u(x)\mu(x).$$

Define $L^r(V) = \{u\colon V \to \mathbb{R} \mid \int_V |u|^r \, d\mu < +\infty\}$ $(1 \le r < +\infty)$ with the norm defined by

$$\|u\|_{L^r(V)} = \left(\int_V |u(x)|^r \, d\mu\right)^{1/r}.$$

Then $(L^r(V), \|\cdot\|_{L^r(V)})$ is a reflexive Banach space. Define $L^\infty(V) = \{u\colon V \to \mathbb{R} \mid \sup_{x\in V} |u(x)| < +\infty\}$ with the norm defined by

$$\|u\|_\infty = \sup_{x\in V} |u(x)|.$$

For more details, one can see [11, 12].

Consider the following $p$-Laplacian equation on a locally finite graph $G = (V,E)$,

$$-\Delta_p u + h(x)|u|^{p-2}u = f(x,u), \quad x \in V, \tag{1.5}$$

where $p > 1$, $h\colon V \to \mathbb{R}$ and $f\colon V \times \mathbb{R} \to \mathbb{R}$.

In recent years, the existence and multiplicity of nontrivial solutions to (1.5) have attracted some attentions (for example, see [5, 11, 12, 15, 16, 19, 22, 23]). In [22], Zhang investigated (1.5) with $p = 2$ and $f(x,u) = |u|^{s-2}u$ for all $x \in V$, where $s > 2$. He obtained that equation (1.5) has a positive solution by using the mountain pass theorem. In [23], Zhang and Lin studied (1.5) with $f(x,u) = g(x)|u|^{r-2}u$ for all $x \in V$, where $g\colon V \to \mathbb{R}$ and $r > p > 2$. They obtained that equation (1.5) has a positive solution. In [5], by using the variational principles and Fatou's lemma, Chang and Zhang obtained the equation (1.5) has a solution when $f(x,u)$ is Lipschitz continuous in $u$. In [19], Shao investigated (1.5) with $f(x,u) = g(x,u) + e(x)$. When $\|e\|_{L^{\frac{p}{p-1}}(V)}$ is small enough, $g(x,u)$ satisfies sub-$(p-1)$-linear growth condition at origin and $|g(x,u)| < C(1 + |u|^{q-1})$ for all $x \in V$, where $q > p \ge 2$, Shao obtained the equation (1.5) has one nontrivial solution of positive energy and another nontrivial solution of negative energy by using the mountain pass theorem and Ekeland's variational principle. In [16], Man investigated (1.5) with $p = 2$ and $h$ replaced by a constant $\alpha$. When $\alpha$ is small enough and nonlinear term $f(x,u)$ satisfies super-$(r-1)$-linear growth condition at origin, where $r > 2$ and some additional assumptions, he obtained that equation (1.5) has a positive solution by using the mountain pass theorem. In [15], Liu investigated (1.5) with $p = 2$ and Dirichlet boundary condition, where $f(x,u) = |u|^{r-2}u + \epsilon e(x)$, where $r > 2$, $\epsilon > 0$ and $e(x) > 0$. When $\epsilon$ is small enough, he obtained that the equation has two positive solutions by using the mountain pass theorem and Ekeland's variational principle. Especially, in [12], Grigor'yan, Lin, Yang considered (1.5) with $p = 2$. They assumed that the measure $\mu(x) \ge \mu_{\min} > 0$ for all $x \in V$, where $\mu_{\min} = \min_{x\in V} \mu(x)$, and $h$ and $f$ satisfy the following conditions:

($\mathrm{K}_1$) there exists a constant $h_0 > 0$ such that $h(x) \ge h_0$ for all $x \in V$;

($K_2$) $\frac{1}{h} \in L^1(V)$;

($S_1$) $f(x,s)$ is continuous in $s$, $f(x,0) = 0$, and for any fixed $M > 0$, there exists a constant $A_M$ such that $\max_{s\in[0,M]} f(x,s) \le A_M$ for all $x \in V$;

($S_2$) $\limsup_{s\to 0^+} \frac{2F(x,s)}{s^2} < \lambda_1 = \inf_{\int_V u^2\,d\mu=1} \int_V (|\nabla u|^2 + hu^2)\,d\mu$;

($S_3$) there exists a constant $\theta > 2$ such that for all $x \in V$ and $s > 0$,

$$0 < \theta F(x,s) = \theta \int_0^s f(x,t)\,dt \le s f(x,s).$$

(The ($S_3$) condition is usually called as Ambrisetti–Rabinowitz condition ((AR)-condition for short).)

Then equation (1.5) with $p = 2$ has a strictly positive solution. Moreover, they also investigated the following equation with perturbation:

$$-\Delta u + hu = f(x,u) + \epsilon e(x), \quad x \in V, \tag{1.6}$$

where $e \ge 0$ for all $x \in V$ ($e \not\equiv 0$). They obtained that there exists a constant $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, (1.6) has at least two distinct strictly positive solutions under the above assumptions. When ($K_2$) is replaced by the following condition:

($K_2'$) $h(x) \to +\infty$ as $\operatorname{dist}(x,x_0) \to +\infty$ for some fixed $x_0 \in V$,

and ($S_1$) is replaced by the following condition:

($S_1'$) $f(x,0) = 0$, $f(x,s) > 0$ for all $x \in V$ and all $s > 0$, and there exists a constant $L > 0$ such that

$$|f(x,s) - f(x,t)| \le L|s-t| \quad \text{for all } x \in V \text{ and all } (s,t) \in \mathbb{R}^2.$$

They obtained that (1.6) has a strictly positive solution.

In this paper, inspired by [11, 12] we consider the following $(p,q)$-Laplacian coupled system with perturbation terms and two parameters on a locally finite graph $G = (V,E)$:

$$\begin{cases} -\Delta_p u + h_1(x)|u|^{p-2}u = F_u(x,u,v) + \lambda_1 e_1(x), & x \in V, \\ -\Delta_q v + h_2(x)|v|^{q-2}v = F_v(x,u,v) + \lambda_2 e_2(x), & x \in V, \end{cases} \tag{1.7}$$

where $\Delta_p$ and $\Delta_q$ are defined by (1.3) with $p \ge 2$ and $q \ge 2$, $F\colon V\times\mathbb{R}^2 \to \mathbb{R}$, $e_1 \in L^{\frac{p}{p-1}}(V)$, $e_2 \in L^{\frac{q}{q-1}}(V)$, $e_1(x), e_2(x) \not\equiv 0$ and $\lambda_1, \lambda_2 > 0$.

If $(u,v)$ is a solution of system (1.7) and $(u,v) \ne (0,0)$, then we call that $(u,v)$ is a nontrivial solution of system (1.7). Furthermore, if $(u,v)$ is a nontrivial solution of system (1.7), $(u,v) = (u,0)$ or $(u,v) = (0,v)$, then we call that $(u,v)$ is a semi-trivial solution of system (1.7). We obtain the following results.

**(I) The sub-$(p,q)$-linear case:**

**Theorem 1.1.** *Assume that the following conditions hold:*

($\mathrm{H}_1$) *there exists a constant* $h_0 > 0$ *such that* $h_i(x) \geq h_0 > 0$ *for all* $x \in V$, $i = 1, 2$;

($\mathrm{H}_2$) $h_i(x) \to \infty$ *as* $\operatorname{dist}(x, x_0) \to \infty$ *for some fixed* $x_0$, $i = 1, 2$;

($\mathrm{F}_0$) $F(x,s,t)$ *is continuously differentiable in* $(s,t) \in \mathbb{R}^2$ *for all* $x \in V$, *and there exists a function* $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and a function* $b \colon V \to \mathbb{R}^+$ *with* $b \in L^1(V)$ *such that*

$$|F_s(x,s,t)|, |F_t(x,s,t)|, |F(x,s,t)| \leq a(|(s,t)|)b(x)$$

*for all* $x \in V$ *and all* $(s,t) \in \mathbb{R}^2$;

($\mathrm{F}_1$) $F(x,0,0) = 0$, *and there exists* $f_i, g_i \colon V \to \mathbb{R}^+$, $i = 1, 2$, $g_1 \in L^{\frac{p}{p-1}}(V)$ *and* $g_2 \in L^{\frac{q}{q-1}}(V)$ *with* $\|f_1\|_\infty < \min\left\{\frac{h_0}{2}, \frac{ph_0}{q(p-1)}\right\}$ *and* $\|f_2\|_\infty < h_0 - \frac{q(p-1)}{p}\|f_1\|_\infty$ *such that*

$$|F_s(x,s,t)| \leq f_1(x)\big(|s|^{p-1}+|t|^{\frac{pq-q}{p}}\big)+g_1(x), \quad |F_t(x,s,t)| \leq f_2(x)\big(|s|^{p}+|t|^{q-1}\big)+g_2(x)$$

*for all* $x \in V$ *and all* $(s,t) \in \mathbb{R}^2$, *where* $p \geq 2$ *and* $q \geq 2$;

($\mathrm{F}_2$) *one of the following conditions holds:*

(i) *there exists* $\beta_1 > 1$ *and* $K_1 \colon V \to \mathbb{R}$ *such that* $K_1(x_1) > 0$ *for some* $x_1 \in V$ *with* $e_1(x_1) > 0$ *and*

$$F(x,s,0) \geq -K_1(x)|s|^{\beta_1} \quad \text{for all } s \in \mathbb{R} \text{ and all } x \in V;$$

(ii) *there exists* $\beta_2 > 1$ *and* $K_2 \colon V \to \mathbb{R}$ *such that* $K_2(x_2) > 0$ *for some* $x_2 \in V$ *with* $e_2(x_2) > 0$ *and*

$$F(x,0,t) \geq -K_2(x)|t|^{\beta_2} \quad \text{for all } t \in \mathbb{R} \text{ and all } x \in V.$$

*Then for each pair* $(\lambda_1, \lambda_2) \in (0, +\infty) \times (0, +\infty)$, *system* (1.7) *has at least one nontrivial solution* $(u_{\lambda\star}, v_{\lambda\star})$. *Furthermore, the necessary conditions for the existence of the semi-trivial solutions to the system* (1.7) *are obtained. If* $(u_{\lambda\star}, v_{\lambda\star}) = (u_{\lambda\star}, 0)$, *then*

$$\|u_{\lambda\star}\|_\infty \leq \mu_0^{-\frac{1}{p}} \left( \frac{\lambda_1 \|e_1\|_{L^{\frac{p}{p-1}}(V)} + \|g_1\|_{L^{\frac{p}{p-1}}(V)}}{h_0 - \|f_1\|_\infty} \right)^{\frac{1}{p-1}}.$$

*If* $(u_{\lambda\star}, v_{\lambda\star}) = (0, v_{\lambda\star})$, *then*

$$\|v_{\lambda\star}\|_\infty \leq \mu_0^{-\frac{1}{q}} \left( \frac{\lambda_2 \|e_2\|_{L^{\frac{q}{q-1}}(V)} + \|g_2\|_{L^{\frac{q}{q-1}}(V)}}{h_0 - \|f_2\|_\infty} \right)^{\frac{1}{q-1}}.$$

**Theorem 1.2.** *Assume that* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$, $(\mathrm{F}_0)$, $(\mathrm{F}_2)$ *and the following condition hold:*

($\mathrm{F}_1'$) $F(x,0,0)=0$, *and there exists* $f_i, g_i\colon V\to\mathbb{R}^+$, $i=1,2$, $g_1\in L^{\frac{p}{p-1}}(V)$ *and* $g_2\in L^{\frac{q}{q-1}}(V)$ *with* $\|f_1\|_\infty<\min\{\frac{h_0}{2},\frac{qh_0}{p(q-1)}\}$ *and* $\|f_2\|_\infty<h_0-\frac{p(q-1)}{q}\|f_2\|_\infty$ *such that*

$$|F_s(x,s,t)|\le f_2(x)(|t|^q+|s|^{p-1})+g_1(x),\quad |F_t(x,s,t)|\le f_1(x)(|t|^{q-1}+|s|^{\frac{qp-p}{q}})+g_2(x)$$

*for all* $x\in V$ *and all* $(s,t)\in\mathbb{R}^2$, *where* $p\ge 2$ *and* $q\ge 2$.

*Then for each pair* $(\lambda_1,\lambda_2)\in(0,+\infty)\times(0,+\infty)$, *system* (1.7) *has at least one nontrivial solution* $(u_{\lambda\star},v_{\lambda\star})$. *Furthermore, the necessary conditions for the existence of the semi-trivial solutions to the system* (1.7) *are obtained. If* $(u_{\lambda\star},v_{\lambda\star})=(u_{\lambda\star},0)$, *then*

$$\|u_{\lambda\star}\|_\infty\le\mu_0^{-\frac{1}{p}}\left(\frac{\lambda_1\|e_1\|_{L^{\frac{p}{p-1}}(V)}+\|g_1\|_{L^{\frac{p}{p-1}}(V)}}{h_0-\|f_2\|_\infty}\right)^{\frac{1}{p-1}}.$$

*If* $(u_{\lambda\star},v_{\lambda\star})=(0,v_{\lambda\star})$, *then*

$$\|v_{\lambda\star}\|_\infty\le\mu_0^{-\frac{1}{q}}\left(\frac{\lambda_2\|e_2\|_{L^{\frac{q}{q-1}}(V)}+\|g_2\|_{L^{\frac{q}{q-1}}(V)}}{h_0-\|f_1\|_\infty}\right)^{\frac{1}{q-1}}.$$

**(II) The super-$(p,q)$-linear case:**

**Theorem 1.3.** *Let* $\lambda_1=\lambda_2=\lambda$. *Assume* $(\mathrm{H}_1)$, $(\mathrm{F}_0)$ *and the following conditions hold:*

($\mathrm{H}_2'$) *for any given constant* $B>0$, $\sum_{x\in A_i}\mu(x)<\infty$, *where* $A_i=\{x\in V\mid h_i(x)\le B\}$, $i=1,2$;

($\mathrm{C}_1$) $F(x,0,0)=0$ *for all* $x\in V$, *and there exists a constant* $l_0>0$ *such that*

$$|F_s(x,s,t)|\le\frac{h_0}{q+1}(|s|^{p-1}+|t|^{\frac{pq-q}{p}}),\quad |F_t(x,s,t)|\le\frac{h_0}{q+1}(|s|^p+|t|^{q-1})$$

*for all* $x\in V$ *and all* $(s,t)\in\mathbb{R}^2$ *with* $|(s,t)|<l_0$, *where* $p\ge 2$ *and* $q\ge 2$;

($\mathrm{C}_2$) *there exists* $l_1>0$ *such that* $F(x_3,s,s)\ge M(s^p+s^q)$ *for some* $x_3\in\{x\in V\mid e_1(x)+e_2(x)>0\}$ *and all* $s\in\mathbb{R}$ *with* $s>l_1$, *where*

$$M>\max\left\{\frac{D_1+\mu(x_3)h_1(x_3)}{p\mu(x_3)},\frac{D_2+\mu(x_3)h_2(x_3)}{q\mu(x_3)}\right\},$$
$$D_1=\left(\frac{\deg(x_3)}{2}\right)^{\frac{p}{2}}\left(\sum_{x\sim x_3}\left(\frac{1}{\mu(x)}\right)^{\frac{p}{2}-1}+\frac{1}{\mu(x_3)^{\frac{p}{2}-1}}\right),$$
$$D_2=\left(\frac{\deg(x_3)}{2}\right)^{\frac{q}{2}}\left(\sum_{x\sim x_3}\left(\frac{1}{\mu(x)}\right)^{\frac{q}{2}-1}+\frac{1}{\mu(x_3)^{\frac{q}{2}-1}}\right);$$

(C$_3$) *there exists a constant* $\nu > \max\{p,q\}$ *and* $0 \le A < \min\left\{\frac{\nu}{p} - 1, \frac{\nu}{q} - 1\right\}h_0$ *such that*

$$\nu F(x,s,t) - F_s(x,s,t)s - F_t(x,s,t)t \le A(|s|^p + |t|^q) \quad \text{for all } x \in V.$$

*Then for each* $\lambda$ *satisfying*

$$0 < \lambda < \lambda_0 = \frac{\min\{1, q-1\} \cdot \Lambda_0^{\max\{p,q\}-1}}{2^{\max\{p,q\}-1}(pq+p) \max\left\{h_0^{-\frac{1}{p}} \|e_1\|_{L^{\frac{p}{p-1}}(V)}, h_0^{-\frac{1}{q}} \|e_2\|_{L^{\frac{q}{q-1}}(V)}\right\}}, \tag{1.8}$$

*where*

$$\Lambda_0 = \min\left\{\frac{l_0}{2} \min\left\{h_0^{1/p}\mu_0^{1/p}, h_0^{1/q}\mu_0^{1/q}\right\}, 1\right\},$$

*system* (1.7) *has one nontrivial solution* $(u_{\star,1}, v_{\star,1})$ *of positive energy. Furthermore, if the following condition holds:*

(C$_4$) *there exists* $l_2 > 0$, $\beta_3 > 1$ *and* $K_3(x)\colon V \to \mathbb{R}$ *such that* $K_3(x_4) > 0$, *and* $F(x_4,s,s) \ge K_3(x_4)|t|^{\beta_3}$ *for some* $x_4 \in \{x \in V \mid e_1(x) + e_2(x) > 0\}$ *with* $\mu(x_4) > 0$ *and all* $s \in \mathbb{R}$ *with* $0 < s < l_2$,

*then system* (1.7) *has another nontrivial solution* $(u_{\star,2}, v_{\star,2})$ *of negative energy for each* $\lambda \in (0, \lambda_0)$.

By using similar proofs, we can also obtain some results similar to Theorems 1.1 and 1.3 to the following equation on locally finite graph $(V,E)$:

$$-\Delta_p u + h(x)|u|^{p-2}u = F_u(x,u) + \epsilon e(x), \quad x \in V. \tag{1.9}$$

**Theorem 1.4.** *Assume that the following conditions hold:*

(h$_1$) *there exists a constant* $h_0 > 0$ *such that* $h(x) \ge h_0$ *for all* $x \in V$;

(h$_2$) $h(x) \to \infty$ *as* $\operatorname{dist}(x,x_0) \to \infty$ *for some fixed* $x_0$;

(f$_0$) $F(x,s)$ *is continuously differentiable in* $s \in \mathbb{R}$ *for all* $x \in V$, *and there exists a function* $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and a function* $b\colon V \to \mathbb{R}^+$ *with* $b \in L^1(V)$ *such that*

$$|F_s(x,s)| \le a(|s|)b(x), \quad |F(x,s)| \le a(|s|)b(x)$$

*for all* $x \in V$ *and all* $s \in \mathbb{R}$;

(f$_1$) $F(x,0) = 0$, *and there exists* $f_1, g_1\colon V \to \mathbb{R}^+$ *with* $f_1 \in L^\infty(V)$ *and* $g_1 \in L^{\frac{p}{p-1}}(V)$ *satisfying* $\|f_1\|_\infty < h_{01}$ *such that*

$$|F_s(x,s)| \le f_1(x)|s|^{p-1} + g_1(x) \quad \text{for all } x \in V \text{ and all } s \in \mathbb{R};$$

($f_2$) *there exists* $\beta_1 > 1$ *and* $K_1 \colon V \to \mathbb{R}$ *such that* $K_1(x_1) > 0$ *for some* $x_1 \in V$ *with* $e(x_1) > 0$ *and*

$$F(x,s) \geq -K_1(x)|s|^{\beta_1} \quad \text{for all } x \in V \text{ and all } s \in \mathbb{R}.$$

*Then for each* $\epsilon \in (0, +\infty)$*, system* (1.9) *has at least one nontrivial solution.*

**Theorem 1.5.** *Assume* ($h_1$), ($f_0$) *and the following conditions hold:*

($h_2'$) *for any given constant* $B > 0$, $\sum_{x \in A} \mu(x) < \infty$*, where* $A = \{x \in V \mid h(x) \leq B\}$*;*

($c_1$) $F(x,0) = 0$ *for all* $x \in V$*, and there exists a constant* $l_0 > 0$ *such that*

$$|f(x,s)| \leq \frac{h_0}{p+1}|s|^{p-1}$$

*for all* $x \in V$ *and all* $s \in \mathbb{R}$ *with* $|s| < l_0$*;*

($c_2$) *there exists* $l_1 > 0$ *such that* $F(x_2, s) \geq Ms^p$ *for some* $x_2 \in V$ *with* $e(x_2) > 0$ *and all* $s \in \mathbb{R}$ *with* $s > l_1$*, where*

$$M > \frac{D_1 + \mu(x_2)h_1(x_2)}{p\mu(x_2)}, \quad D_1 = \left(\frac{\deg(x_2)}{2}\right)^{\frac{p}{2}} \left(\sum_{x \sim x_2} \left(\frac{1}{\mu(x)}\right)^{\frac{p}{2}-1} + \left(\frac{1}{\mu(x_2)}\right)^{\frac{p}{2}-1}\right);$$

($c_3$) *there exists a constant* $\nu > p$ *and* $0 \leq A < h_0\left(\frac{\nu}{p} - 1\right)$ *such that*

$$\nu F(x,s) - F_s(x,s)s \leq A|s|^p \quad \text{for all } x \in V.$$

*Then for each* $\epsilon$ *satisfying*

$$0 < \epsilon < \epsilon_0 = \frac{\left(\min\left\{l_0(h_0\mu_0)^{1/p}, 1\right\}\right)^{p-1}}{(p+1)h_0^{-\frac{1}{p}}\|e\|_{L^{\frac{p}{p-1}}(V)}},$$

*equation* (1.9) *has one nontrivial solution of positive energy. Furthermore, if the following condition holds:*

($c_4$) *there exists* $l_2 > 0$ *and* $\beta_3 > 1$ *such that* $F(x_3, s) \geq K_3(x_3)|s|^{\beta_3}$ *for some* $x_3 \in \{x \in V \mid e(x) > 0\}$ *and all* $s \in \mathbb{R}$ *with* $0 < s < l_2$*,*

*then system* (1.9) *has another nontrivial solution of negative energy for each* $\epsilon \in (0, \epsilon_0)$*.*

*Remark* 1.6. In Theorem 1.3, the condition ($C_2$) is interesting, which implies that the inequality $F(x,s,t) \geq M(s^p + t^q)$ holds only for a point $x_3$ rather than all $x \in V$ and only for a ray $s = t$ starting at the point $(l_1, l_1)$ in the plane $\mathbb{R}^2$ rather than for all $(s,t) \in \mathbb{R}^2$

with $|(s,t)| > l_1$ (see Figure 1.1), which is usually assumed in investigating the existence of solutions for the elliptic partial differential system with the nonlinear term satisfying the super-quadratic conditions (for example, see [14]).

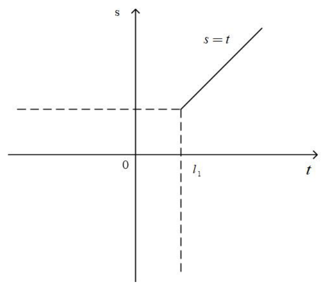


Figure 1.1: $F(x,s,t) \geq M(s^p + t^q)$ holds only for a ray $s = t$ starting at the point $(l_1, l_1)$ in the plane $\mathbb{R}^2$.

*Remark* 1.7. Theorem 1.4 is different from Theorem 1.4 in [12], where they consider (1.9) with $p = 2$ and they assume that $f(x,s) := F_s(x,s)$ satisfies the (AR)-condition ($S_3$) and ($S_1'$). It is easy to see that ($f_1$) in Theorem 1.4 is weaker than ($S_1'$) even if $p = 2$ and we do not need the (AR)-condition in Theorem 1.4. Theorem 1.5 is also different from Theorem 1.3 in [12] even if $p = 2$. It is easy to see that ($h_2$) is weaker than ($K_2$). Moreover, ($h_2'$) is weaker than ($h_2$). In fact, by ($h_2$) we have for any positive constant $B$ when $h(x) < B$, there exists positive constant $B_1$ such that $\operatorname{dist}(x, x_0) < B_1$. So, $A = \{x \in V \mid h(x) < B\}$ is a finite set. Moreover, ($c_2$) together with ($c_3$) is weaker than ($S_3$). There exists examples satisfying Theorem 1.5 but not satisfying Theorem 1.3 in [12], for example, let

$$F(x,s) = M \ln(1+s^2)|s|^3,$$

where $M$ is defined as Theorem 1.5.

## 2. Sobolev embedding

Let $W^{1,s}(V)$ be the completion of $C_c(V)$ under the norm

$$\|u\|_{W^{1,s}(V)} = \left( \int_V [|\nabla u(x)|^s + |u(x)|^s] \, d\mu \right)^{1/s},$$

where $s > 1$ and $W^{1,s}(V)$ is a reflexive Banach space (see [20, Theorem 1.1]). Let $h(x) \geq h_0 > 0$. Define the space

$$W_h^{1,s}(V) = \left\{ u \in W^{1,s}(V) \;\middle|\; \int_V h(x)|u(x)|^s \, d\mu < \infty \right\}$$

endowed with the norm

$$\|u\|_{W_h^{1,s}(V)} = \left( \int_V [|\nabla u(x)|^s + h(x)|u(x)|^s] \, d\mu \right)^{1/s}.$$

**Lemma 2.1.** *If $\mu(x) \geq \mu_0 > 0$ and $h$ satisfies* $(\mathrm{H}_1)$*, then $W_h^{1,s}(V)$ is continuously embedded into $L^r(V)$ for all $1 < s \leq r \leq \infty$, and the following inequalities hold:*

$$\|u\|_\infty \leq \frac{1}{h_0^{1/s} \mu_0^{1/s}} \|u\|_{W_h^{1,s}(V)} \tag{2.1}$$

*and*

$$\|u\|_{L^r(V)} \leq \mu_0^{\frac{s-r}{sr}} h_0^{-\frac{1}{s}} \|u\|_{W_h^{1,s}(V)} \quad \text{for all } s \leq r < \infty. \tag{2.2}$$

*Furthermore, if* $(\mathrm{H}_2)$ *also holds, then $W_h^{1,s}(V)$ is compactly embedded into $L^r(V)$ for all $1 < s \leq r \leq \infty$.*

*Proof.* For any $u \in W_h^{1,s}(V)$, we claim that

$$\sum_{x \in V} |u(x)|^s \geq \|u\|_\infty^s. \tag{2.3}$$

In fact, assume that

$$\left( \sum_{x \in V} |u(x)|^s \right)^{1/s} < \|u\|_\infty.$$

Then there exists a $\varepsilon > 0$ such that

$$\sum_{x \in V} |u(x)|^s < (\|u\|_\infty - \varepsilon)^s. \tag{2.4}$$

Note that $\|u\|_\infty = \sup_{x \in V} |u(x)|$. Then by the definition of supremum, there exists an $x_* \in V$ such that $|u(x_*)| > \|u\|_\infty - \varepsilon$. Then $|u(x_*)|^s > (\|u\|_\infty - \varepsilon)^s$, which together with (2.4) implies that

$$|u(x_*)|^s > (\|u\|_\infty - \varepsilon)^s > \sum_{x \in V} |u(x)|^s \geq |u(x_*)|^s,$$

a contradiction.

For any $u \in W_h^{1,s}(V)$, we have

$$\|u\|_{W_h^{1,s}(V)}^s \geq \int_V h(x)|u(x)|^s \, d\mu \geq h_0 \int_V |u(x)|^s \, d\mu \quad \text{for all } s > 1, \tag{2.5}$$

and by (2.3), we have

$$\|u\|_{W_h^{1,s}(V)}^s \geq \int_V h(x)|u(x)|^s \, d\mu = \sum_{x \in V} \mu(x) h(x) |u(x)|^s \geq h_0 \mu_0 \|u\|_\infty^s \quad \text{for all } s > 1, \tag{2.6}$$

which implies that

$$\|u\|_\infty \leq \frac{1}{h_0^{1/s}\mu_0^{1/s}}\|u\|_{W_h^{1,s}(V)} \quad \text{for all } s > 1.$$

When $s < r < \infty$, it follows from (2.5) and (2.6) that

$$\int_V |u(x)|^r \, d\mu \leq \|u\|_\infty^{r-s} \int_V |u(x)|^s \, d\mu \leq \mu_0^{\frac{s-r}{s}} h_0^{-\frac{r}{s}} \|u\|_{W_h^{1,s}(V)}^r.$$

So,

$$\|u\|_{L^r(V)} \leq \mu_0^{\frac{s-r}{sr}} h_0^{-\frac{1}{s}} \|u\|_{W_h^{1,s}(V)} \quad \text{for all } s \leq r < \infty.$$

Suppose that $\{u_k\}$ is a bounded sequence in $W_h^{1,s}(V)$. Note that $W_h^{1,s}(V)$ is reflexive. Then there exists a subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u$ weakly in $W_h^{1,s}(V)$ for some $u \in W_h^{1,s}(V)$. In particular,

$$\lim_{k\to\infty} \int_V u_k \varphi \, d\mu = \int_V u\varphi \, d\mu, \quad \forall\, \varphi \in C_c(V),$$

which implies that

$$\lim_{k\to\infty} u_k(x) = u(x) \quad \text{for any fixed } x \in V, \tag{2.7}$$

if we choose $\varphi \in C_c(V)$ defined by

$$\varphi(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

We now prove $u_k \to u$ in $L^r(V)$ for all $s \leq r \leq \infty$, if $(\mathrm{H}_2)$ holds. Since $\{u_k\}$ is bounded in $W_h^{1,s}(V)$ and $u \in W_h^{1,s}(V)$, by the definition of norm $\|\cdot\|_{W_h^{1,s}(V)}$, there exists a constant $c_0 > 0$ such that

$$\int_V h|u_k - u|^s \, d\mu \leq c_0.$$

For any given $\epsilon > 0$, in view of $(\mathrm{H}_2)$, there exists a constant $R(\epsilon) > 0$ such that

$$\frac{1}{h(x)} < \epsilon \quad \text{as } \operatorname{dist}(x, x_0) \geq R(\epsilon).$$

Hence,

$$\int_{\operatorname{dist}(x,x_0)\geq R(\epsilon)} |u_k - u|^s \, d\mu = \int_{\operatorname{dist}(x,x_0)\geq R(\epsilon)} \frac{1}{h} h|u_k - u|^s \, d\mu \leq c_0\epsilon. \tag{2.8}$$

Note that $\{x \mid \operatorname{dist}(x, x_0) \leq R(\epsilon)\}$ is a finite set. Then (2.7) implies that

$$\lim_{k\to\infty} \int_{\operatorname{dist}(x,x_0)\leq R(\epsilon)} |u_k - u|^s \, d\mu = 0. \tag{2.9}$$

So, by the arbitrary of $\epsilon$, (2.8) and (2.9) imply that

$$\lim_{k\to\infty}\int_V |u_k-u|^s\,d\mu = 0. \tag{2.10}$$

Then by (2.3) and (2.10), we have

$$\begin{aligned}\|u_k-u\|_\infty^s &\le \sum_{x\in V}|u_k(x)-u(x)|^s = \sum_{x\in V}\frac{1}{\mu(x)}\mu(x)|u_k(x)-u(x)|^s\\ &\le \frac{1}{\mu_0}\sum_{x\in V}\mu(x)|u_k(x)-u(x)|^s = \frac{1}{\mu_0}\int_V |u_k-u|^s\,d\mu \to 0 \quad \text{as } k\to\infty,\end{aligned} \tag{2.11}$$

and when $s<r<\infty$, we have

$$\int_V |u_k-u|^r\,d\mu \le \|u_k-u\|_\infty^{r-s}\int_V |u_k-u|^s\,d\mu \to 0 \quad \text{as } k\to\infty. \tag{2.12}$$

Hence, (2.10), (2.11) and (2.12) imply that $u_k\to u$ in $L^r(V)$ for all $s\le r\le\infty$. □

**Lemma 2.2.** *If $\mu(x)\ge\mu_0>0$ and $h$ satisfies* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2')$, *then $W_h^{1,s}(V)$ is compactly embedded into $L^r(V)$ for all $1<s\le r\le\infty$.*

*Proof.* Suppose that $\{u_k\}$ is a bounded sequence in $W_h^{1,s}(V)$ and there exists a positive constant $C_0$ such that

$$\|u_k\|_{W_h^{1,s}(V)} \le C_0. \tag{2.13}$$

Since $\|u_k\|_{L^s(V)}^s \le \frac{1}{h_0}\int_V h(x)|u_k|^s\,d\mu \le \frac{1}{h_0}\|u_k\|_{W_h^{1,s}(V)}^s$, we also have that $\left\{\|u_k\|_{L^s(V)}\right\}$ is bounded in $\mathbb{R}$. Noting that $W_h^{1,s}(V)$ is reflexive, we have, up to a subsequence, $u_k\rightharpoonup u$ weakly in $W_h^{1,s}(V)$ for some $u\in W_h^{1,s}(V)$ and $\delta_k=\|u_k\|_{L^s(V)}\to\delta$ for some $\delta\in\mathbb{R}$ as $k\to\infty$. Similar to the argument of (2.7), we have $\lim_{k\to\infty}u_k(x)=u(x)$ for all $x\in V$. Then for any bounded domain $\Omega\subset V$, we have

$$\int_\Omega |u_k|^s\,d\mu \to \int_\Omega |u|^s\,d\mu \quad\text{and}\quad \int_\Omega |u_k|^s\,d\mu \le \int_V |u_k|^s\,d\mu \to \delta^s \quad \text{as } k\to\infty.$$

Then

$$\delta^s \ge \|u\|_{L^s(\Omega)}^s. \tag{2.14}$$

For any given constant $B>0$, define $\Omega=\{x\in V \mid \mathrm{dist}(x,x_0)\le B, h(x)\le B\}$ for some fixed $x_0\in V$. Let $A(\Omega)=\{x\in V/\Omega, h(x)\le B\}$. Then $A=\Omega\cup A(\Omega)$, where $A=\{x\in V\mid h(x)\le B\}$. By $(\mathrm{H}_2')$, we have $\sum_{x\in A}\mu(x)<\infty$ and then by the definition of convergent series, for any sufficient small $\epsilon>0$, there exists a sufficient large $B>\frac{1}{\epsilon}$ such that

$$\sum_{x\in A(\Omega)}\mu(x)<\epsilon. \tag{2.15}$$

Moreover, since $h$ satisfies ($\mathrm{H}_1$), by Lemma 2.1 we know that $W_h^{1,s}(V)$ is continuously embedded into $L^r(V)$, $s \leq r \leq \infty$. So, by (2.2), (2.13) and (2.15), we have

$$\begin{aligned}\int_{A(\Omega)} |u_k|^s \, d\mu &= \int_{A(\Omega)} 1 \cdot |u_k|^s \, d\mu \\ &\leq \left( \int_{A(\Omega)} |u_k|^{2s} \, d\mu \right)^{1/2} \left( \sum_{x \in A(\Omega)} \mu(x) \right)^{1/2} \\ &\leq \mu_0^{-1/2} h_0^{-1} \|u_k\|_{W_h^{1,s}(V)}^s \left( \sum_{x \in A(\Omega)} \mu(x) \right)^{1/2} \\ &\leq \mu_0^{-1/2} h_0^{-1} C_0^s \epsilon \quad \text{for all } k \in \mathbb{N}.\end{aligned}$$

Define $B(\Omega) = \{x \in V/\Omega, h(x) > B\}$. Then

$$\int_{B(\Omega)} |u_k|^s \, d\mu \leq \int_{B(\Omega)} \frac{h(x)}{B} |u_k|^s \, d\mu \leq \frac{1}{B} \|u_k\|_{W_h^{1,s}(V)}^s \leq \frac{C_0^s}{B} \leq C_0^s \epsilon \quad \text{for all } k \in \mathbb{N}.$$

Then

$$\int_{V/\Omega} |u_k|^s \, d\mu = \int_{B(\Omega)} |u_k|^s \, d\mu + \int_{A(\Omega)} |u_k|^s \, d\mu < \big(\mu_0^{-1/2} h_0^{-1} + 1\big) C_0^s \epsilon \quad \text{for all } k \in \mathbb{N}.$$

Similarly, we also have

$$\int_{V/\Omega} |u|^s \, d\mu = \int_{B(\Omega)} |u|^s \, d\mu + \int_{A(\Omega)} |u|^s \, d\mu < \epsilon \big(\mu_0^{-1/2} h_0^{-1} + 1\big) \|u\|_{W_h^{1,s}(V)}^s. \tag{2.16}$$

Let $C_1 = \max\big\{C_0, \|u\|_{W_h^{1,s}(V)}\big\}$. So, by (2.14) and (2.16), we have

$$\|u\|_{L^s(V)}^s = \|u\|_{L^s(\Omega)}^s + \|u\|_{L^s(V/\Omega)}^s \leq \delta^s + \big(\mu_0^{-1/2} h_0^{-1} + 1\big) C_1^s \epsilon.$$

On the other hand,

$$\begin{aligned}\|u\|_{L^s(V)}^s = \|u\|_{L^s(\Omega)}^s + \|u\|_{L^s(V/\Omega)}^s &\geq \lim_{k \to \infty} \|u_k\|_{L^s(\Omega)}^s \\ &= \lim_{k \to \infty} \|u_k\|_{L^s(V)}^s - \lim_{k \to \infty} \|u_k\|_{L^s(V/\Omega)}^s \geq \delta^s - \big(\mu_0^{-1/2} h_0^{-1} + 1\big) C_1^s \epsilon.\end{aligned}$$

Hence, by the arbitrary of $\epsilon$, we obtain that $\delta^s = \|u\|_{L^s(V)}^s$. Thus we have proved that $\|u_k\|_{L^s(V)} \to \|u\|_{L^s(V)}$ as $k \to \infty$. By the uniform convexity of $L^s(V)$ (see [13, Lemma 2.2]) and that $u_k \rightharpoonup u$ weakly in $W_h^{1,s}(V)$, it follows from the Kadec–Klee property that $\|u_k - u\|_{L^s(V)} \to 0$ as $k \to \infty$. Then similar to the argument of (2.11) and (2.12), we have $\|u_k - u\|_\infty \to 0$ and $\|u_k - u\|_{L^r(V)} \to 0$ for all $s < r < \infty$. $\square$

*Remark* 2.3. Lemma 2.1 generalizes [12, Lemma 2.2] and [13, Lemma 2.6], and Lemma 2.2 generalizes [4, Lemma 3]. To be precise, when $s = 2$, Lemmas 2.1 and 2.2 reduce to [12, Lemma 2.2] and [4, Lemma 3], respectively. In [4, Lemma 3], the potential $h(x)$ is allowed to be sigh-changing, which satisfies $(H_1')$: $\inf_{x\in V} h(x) \geq h_0$ for some $h_0 \in (-1, 0)$. One can prove that Lemma 2.2 still holds under $(H_1')$ and $(\mathrm{H}_2')$. Moreover, if $h(x) = \lambda a(x) + 1$, where $a\colon V \to \mathbb{R}$ with $a(x) \geq 0$ for all $x \in V$, then Lemma 2.1 reduces to [13, Lemma 2.6]. The proofs of Lemmas 2.1 and 2.2 are based on those in [4, 12, 13] and we make some appropriate modifications.

Assume that $\varphi \in C^1(X, \mathbb{R})$. An sequence $\{u_n\}$ is called as the Palais–Smale sequence of $\varphi$ if $\varphi(u_n)$ is bounded for all $n \in \mathbb{N}$ and $\varphi'(u_n) \to 0$ as $n \to \infty$. If any Palais–Smale sequence $\{u_n\}$ of $\varphi$ has a convergent subsequence, we call that $\varphi$ satisfies the Palais–Smale condition ((PS)-condition for short).

**Lemma 2.4.** (Ekeland's variational principle [17]) *Let $M$ be a complete metric space with metric $d$, and $\varphi\colon M \to \mathbb{R}$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ be given and $U \in M$ such that*

$$\varphi(U) \leq \inf_M \varphi + \varepsilon.$$

*Then there exists $V \in M$ such that*

$$\varphi(V) \leq \varphi(U), \quad d(U, V) \leq 1,$$

*and for each $W \in M$, one has*

$$\varphi(V) \leq \varphi(W) + \varepsilon d(V, W).$$

By the Ekeland's variational principle, it is easy to obtain the following corollary.

**Lemma 2.5.** [17] *Suppose that $X$ is a Banach space, $M \subset X$ is closed, $\varphi \in C^1(X, \mathbb{R})$ is bounded from below on $M$ and satisfies the (PS)-condition. Then $\varphi$ attains its infimum on $M$.*

**Lemma 2.6.** (Mountain pass theorem [18]) *Let $X$ be a real Banach space and $\varphi \in C^1(X, \mathbb{R})$, $\varphi(0) = 0$ satisfy (PS)-condition. Suppose that $\varphi$ satisfies the following conditions:*

(i) *there exists a constant $\rho > 0$ and $\alpha > 0$ such that $\varphi|_{\partial B_\rho(0)} \geq \alpha$, where $B_\rho = \{w \in X : \|w\|_X < \rho\}$;*

(ii) *there exists $w \in X \setminus \overline{B}_\rho(0)$ such that $\varphi(w) \leq 0$.*

*Then $\varphi$ has a critical value $c_* \geq \alpha$ with*

$$c_* := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)),$$

*where $\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = w\}$.*

## 3. Proofs for the sub-$(p,q)$-linear case

Define the space $W := W_h^{1,p}(V) \times W_h^{1,q}(V)$ with the norm

$$\|(u,v)\|_W = \|u\|_{W_h^{1,p}(V)} + \|v\|_{W_h^{1,q}(V)}.$$

Then $W$ is a Banach space. Consider the functional $\varphi\colon W \to \mathbb{R}$ defined as

$$\begin{aligned} \varphi_\lambda(u,v) &= \frac{1}{p}\int_V (|\nabla u|^p + h_1|u|^p)\,d\mu + \frac{1}{q}\int_V (|\nabla v|^q + h_2|v|^q)\,d\mu \\ &\quad - \int_V F(x,u,v)\,d\mu - \lambda_1\int_V e_1 u\,d\mu - \lambda_2\int_V e_2 v\,d\mu. \end{aligned} \tag{3.1}$$

Then $\varphi_\lambda(u,v) \in C^1(W,\mathbb{R})$, and

$$\begin{aligned} &\langle \varphi_\lambda'(u,v), (\phi_1,\phi_2)\rangle \\ &= \int_V \left[|\nabla u|^{p-2}\Gamma(u,\phi_1) + h_1|u|^{p-2}u\phi_1 - F_u(x,u,v)\phi_1 - \lambda_1 e_1\phi_1\right] d\mu \\ &\quad + \int_V \left[|\nabla v|^{q-2}\Gamma(v,\phi_2) + h_2|v|^{q-2}v\phi_2 - F_v(x,u,v)\phi_2 - \lambda_2 e_2\phi_2\right] d\mu \end{aligned} \tag{3.2}$$

for all $(\phi_1,\phi_2) \in W$ (see Lemma A.2).

**Definition 3.1.** $(u,v) \in W$ is called as a weak solution of system (1.7) if

$$\int_V \left[|\nabla u|^{p-2}\Gamma(u,\phi_1) + h_1|u|^{p-2}u\phi_1\right] d\mu = \int_V [F_u(x,u,v)\phi_1 + \lambda_1 e_1\phi_1]\,d\mu, \tag{3.3}$$

$$\int_V \left[|\nabla v|^{q-2}\Gamma(v,\phi_2) + h_2|v|^{q-2}v\phi_2\right] d\mu = \int_V [F_v(x,u,v)\phi_1 + \lambda_2 e_2\phi_2]\,d\mu \tag{3.4}$$

for all $(\phi_1,\phi_2) \in W$.

Obviously, $(u,v) \in W$ is a weak solution of system (1.7) if and only if $(u,v)$ is a critical point of $\varphi$ and similar to the arguments in [13], we have the following proposition.

**Proposition 3.2.** *If $(u,v) \in W$ is a weak solution of system* (1.7)*, then $(u,v) \in W$ is also a point-wise solution of* (1.7).

*Proof.* For any fixed $y \in V$, we take a test function $\phi_1\colon V \to \mathbb{R}$ in (3.3) with

$$\phi_1(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$$

and a test function $\phi_2\colon V \to \mathbb{R}$ in (3.4) with

$$\phi_2(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Thus, by (1.4), we have

$$\begin{aligned}-\Delta_p u(y)+h_1(y)|u(y)|^{p-2}u(y)&=F_u(y,u(y),v(y))+\lambda_1 e_1(y),\\-\Delta_q v(y)+h_2(y)|v(y)|^{q-2}v(y)&=F_v(y,u(y),v(y))+\lambda_2 e_2(y).\end{aligned}$$

By the arbitrary of $y$, we complete the proof. □

**Lemma 3.3.** *Assume that* ($\mathrm{H}_1$) *and* ($\mathrm{F}_1$) *hold. Then* $\varphi$ *is coercive, that is,* $\varphi(u,v)\to+\infty$ *as* $\|(u,v)\|_W\to\infty$.

*Proof.* By ($\mathrm{F}_1$) and Lemma 2.1, we have

$$\begin{aligned}&\int_V |F(x,u,v)|\,d\mu\\&=\int_V |F(x,u,v)-F(x,0,0)|\,d\mu\\&\le\int_V |F(x,u,v)-F(x,0,v)|+|F(x,0,v)-F(x,0,0)|\,d\mu\\&\le\int_V\int_0^{|u|}|F_s(x,s,v)|\,dsd\mu+\int_V\int_0^{|v|}|F_t(x,0,t)|\,dtd\mu\\&\le\int_V\int_0^{|u|}[f_1(x)(|s|^{p-1}+|v|^{\frac{pq-q}{p}})+g_1(x)]\,dsd\mu+\int_0^{|v|}[f_2(x)|t|^{q-1}+g_2(x)]\,dtd\mu\\&\le\int_V\left[\frac{|u|^p}{p}f_1(x)+f_1(x)|v|^{\frac{pq-q}{p}}|u|+\frac{|v|^q}{q}f_2(x)+g_1(x)|u|+g_2(x)|v|\right]d\mu\\&\le\frac{2\|f_1\|_\infty}{p}\int_V|u|^p\,d\mu+\frac{(p-1)\|f_1\|_\infty}{p}\int_V|v|^q\,d\mu\\&\quad+\frac{\|f_2\|_\infty}{q}\int_V|v|^q\,d\mu+\|g_1\|_{L^{\frac{p}{p-1}}(V)}\|u\|_{L^p(V)}+\|g_2\|_{L^{\frac{q}{q-1}}(V)}\|v\|_{L^q(V)}\\&\le\frac{2\|f_1\|_\infty}{ph_0}\|u\|^p_{W_h^{1,p}(V)}+\left(\frac{(p-1)\|f_1\|_\infty}{ph_0}+\frac{\|f_2\|_\infty}{qh_0}\right)\|v\|^q_{W_h^{1,q}(V)}\\&\quad+\frac{\|g_1\|_{L^{\frac{p}{p-1}}(V)}}{h_0^{1/p}}\|u\|_{W_h^{1,p}(V)}+\frac{\|g_2\|_{L^{\frac{q}{q-1}}(V)}}{h_0^{1/q}}\|v\|_{W_h^{1,q}(V)}.\end{aligned}\tag{3.5}$$

Then, by (3.1) and (3.5), we have

$$\begin{aligned}\varphi_\lambda(u,v)\ge&\left(\frac{1}{p}-\frac{2\|f_1\|_\infty}{ph_0}\right)\|u\|^p_{W_h^{1,p}(V)}+\left(\frac{1}{q}-\frac{(p-1)\|f_1\|_\infty}{ph_0}-\frac{\|f_2\|_\infty}{qh_0}\right)\|v\|^q_{W_h^{1,q}(V)}\\&-\frac{1}{h_0^{1/p}}\big(\lambda_1\|e_1\|_{L^{\frac{p}{p-1}}(V)}+\|g_1\|_{L^{\frac{p}{p-1}}(V)}\big)\|u\|_{W_h^{1,p}(V)}\\&-\frac{1}{h_0^{1/q}}\big(\lambda_2\|e_2\|_{L^{\frac{q}{q-1}}(V)}+\|g_2\|_{L^{\frac{q}{q-1}}(V)}\big)\|v\|_{W_h^{1,q}(V)}.\end{aligned}$$

So $\varphi$ is coercive in $W$. □

**Lemma 3.4.** *Assume that* $(\mathrm{H}_1)$ *and* $(\mathrm{F}_1)$ *hold. Then* $\varphi_\lambda$ *satisfies the (PS)-condition.*

*Proof.* The proof is motivated by [13, 24]. Assume that $\{(u_k, v_k)\}$ is a Palais–Smale sequence, then $\varphi'_\lambda(u_k, v_k) \to 0$ as $k \to \infty$ and $\varphi_\lambda(u_k, v_k)$ is bounded. By Lemma 3.3, we obtain that $\{(u_k, v_k)\}$ is bounded in $W$. Then $\{u_k\}$ is bounded in $W_h^{1,p}(V)$ and $\{v_k\}$ is bounded in $W_h^{1,q}(V)$. Hence we can find a subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u_{\lambda\star}$ for some $u_{\lambda\star} \in W_h^{1,p}(V)$ as $k \to \infty$, and a subsequence of $\{v_k\}$, which has the same subscript as the subsequence of $\{u_k\}$, still denoted by $\{v_k\}$, such that $v_k \rightharpoonup v_{\lambda\star}$ for some $v_{\lambda\star} \in W_h^{1,q}(V)$ as $k \to \infty$. By Lemma 2.1, we know that

$$u_k \to u_{\lambda\star} \ \text{ in } L^p(V), \quad v_k \to v_{\lambda\star} \ \text{ in } L^q(V) \quad \text{as } k \to \infty. \tag{3.6}$$

Then by (3.2), we have

$$\begin{aligned}
&\langle \varphi'_\lambda(u_k, v_k) - \varphi'_\lambda(u_{\lambda\star}, v_{\lambda\star}), (u_k - u_{\lambda\star}, 0)\rangle \\
&= \int_V \big[|\nabla u_k|^{p-2}\Gamma(u_k, u_k - u_{\lambda\star}) + (h_1(x)|u_k|^{p-2}u_k - F_u(x, u_k, v_k))(u_k - u_{\lambda\star})\big]\, d\mu \\
&\quad - \int_V \big[|\nabla u_{\lambda\star}|^{p-2}\Gamma(u_{\lambda\star}, u_k - u_{\lambda\star}) + (h_1(x)|u_{\lambda\star}|^{p-2}u_{\lambda\star} - F_u(x, u_{\lambda\star}, v_{\lambda\star}))(u_k - u_{\lambda\star})\big]\, d\mu \\
&= \|u_k\|^p_{W_h^{1,p}(V)} + \|u_{\lambda\star}\|^p_{W_h^{1,p}(V)} - \int_V \big[|\nabla u|^{p-2}\Gamma(u_k, u_{\lambda\star}) + h_1(x)|u_k|^{p-2}u_k u_{\lambda\star}\big]\, d\mu \\
&\quad - \int_V \big[|\nabla u|^{p-2}\Gamma(u_{\lambda\star}, u_k) + h_1(x)|u_{\lambda\star}|^{p-2}u_{\lambda\star}u_k\big]\, d\mu \\
&\quad + \int_V [F_u(x, u_{\lambda\star}, v_{\lambda\star}) - F_u(x, u_k, v_k)](u_k - u_{\lambda\star})\, d\mu.
\end{aligned} \tag{3.7}$$

By $(\mathrm{F}_1)$ and (3.6), we have

$$\begin{aligned}
&\int_V [F_u(x, u_{\lambda\star}, v_{\lambda\star}) - F_u(x, u_k, v_k)](u_k - u_{\lambda\star})\, d\mu \\
&\le \int_V |F_u(x, u_{\lambda\star}, v_{\lambda\star}) - F_u(x, u_k, v_k)||u_k - u_{\lambda\star}|\, d\mu \\
&\le \int_V [|F_u(x, u_k, v_k)| + |F_u(x, u_{\lambda\star}, v_{\lambda\star})|]|u_k - u_{\lambda\star}|\, d\mu \\
&\le \|f_1\|_\infty \int_V \big(|u_k|^{p-1} + |v_k|^{\frac{pq-q}{p}} + |u_{\lambda\star}|^{p-1} + |v_{\lambda\star}|^{\frac{pq-q}{p}}\big)|u_k - u_{\lambda\star}|\, d\mu \\
&\quad + \int_V g_1(x)|u_k - u_{\lambda\star}|\, d\mu \\
&\le \|f_1\|_\infty \big(\|u_k\|^{p-1}_{L^p(V)} + \|v_k\|^{\frac{pq-q}{p}}_{L^q(V)} + \|u_{\lambda\star}\|^{p-1}_{L^p(V)} + \|v_{\lambda\star}\|^{\frac{pq-q}{p}}_{L^q(V)}\big)\|u_k - u_{\lambda\star}\|_{L^p(V)} \\
&\quad + \|g_1\|_{L^{\frac{p}{p-1}}(V)}\|u_k - u_{\lambda\star}\|_{L^p(V)} \\
&\to 0.
\end{aligned} \tag{3.8}$$

Moreover, by (1.1), we have

$$\begin{aligned}
&\int_V \left[|\nabla u_k|^{p-2}\Gamma(u_k,u_{\lambda\star})+h_1|u_k|^{p-2}u_k u_{\lambda\star}\right]d\mu\\
&\le \int_V |\nabla u_k|^{p-2}|\nabla u_k||\nabla u_{\lambda\star}|\,d\mu+\int_V \big(h_1^{\frac{p-1}{p}}|u_k|^{p-2}u_k\big)\big(h_1^{1/p}u_{\lambda\star}\big)\,d\mu\\
&\le \|\nabla u_k\|_{L^p(V)}^{p-1}\|\nabla u_{\lambda\star}\|_{L^p(V)}+\left(\int_V h|u_k|^p\,d\mu\right)^{\frac{p-1}{p}}\left(\int_V h|u_{\lambda\star}|^p\,d\mu\right)^{1/p}\\
&\le \|u_k\|_{W_h^{1,p}(V)}^{p-1}\|u_{\lambda\star}\|_{W_h^{1,p}(V)}.
\end{aligned}\tag{3.9}$$

Similarly, we also have

$$\int_V \left[|\nabla u_{\lambda\star}|^{p-2}\Gamma(u_{\lambda\star},u_k)+h_1(x)|u_{\lambda\star}|^{p-2}u_{\lambda\star}u_k\right]d\mu\le \|u_{\lambda\star}\|_{W_h^{1,p}(V)}^{p-1}\|u_k\|_{W^{m_1,p}(V)}.$$

So, by (3.7), (3.8) and (3.9), we have

$$\begin{aligned}
&\langle \varphi_\lambda'(u_k,v_k)-\varphi_\lambda'(u_{\lambda\star},v_{\lambda\star}),(u_k-u_{\lambda\star},0)\rangle\\
&\ge \|u_k\|_{W_h^{1,p}(V)}^p+\|u_{\lambda\star}\|_{W_h^{1,p}(V)}^p-\|u_k\|_{W^{m_1,p}(V)}^{p-1}\|u_{\lambda\star}\|_{W_h^{1,p}(V)}\\
&\quad-\|u_{\lambda\star}\|_{W_h^{1,p}(V)}^{p-1}\|u_k\|_{W_h^{1,p}(V)}+o_k(1)\\
&=\big(\|u_k\|_{W_h^{1,p}(V)}^{p-1}-\|u_{\lambda\star}\|_{W_h^{1,p}(V)}^{p-1}\big)\big(\|u_k\|_{W_h^{1,p}(V)}-\|u_{\lambda\star}\|_{W_h^{1,p}(V)}\big)+o_k(1).
\end{aligned}$$

Hence, $\|u_k\|_{W_h^{1,p}(V)}\to\|u_{\lambda\star}\|_{W_h^{1,p}(V)}$ as $k\to\infty$. Then it follows from the uniformly convexity of $W_h^{1,p}(V)$ (see Lemma A.1) and the Kadec–Klee property that

$$u_k\to u_{\lambda\star}\ \text{ strongly in }\ W_h^{1,p}(V)\ \text{ as } k\to\infty.$$

Similarly, we can also prove

$$v_k\to v_{\lambda\star}\ \text{ strongly in }\ W_h^{1,q}(V)\ \text{ as } k\to\infty.$$

Therefore,

$$(u_k,v_k)\to(u_{\lambda\star},v_{\lambda\star})\ \text{ strongly in }\ W\ \text{ as } k\to\infty. \qquad \square$$

*Proof of Theorem* 1.1. By Lemma 3.3 and the continuity of $\varphi_\lambda$, we know that $\varphi_\lambda$ is bounded from below. Then by Lemmas 3.4 and 2.5, we obtain that $\varphi_\lambda$ attains its infimum on $W$. Hence, there exists a $(u_{\lambda\star},v_{\lambda\star})\in W$ such that $\varphi(u_{\lambda\star},v_{\lambda\star})=\inf_{(u,v)\in W}\varphi(u,v)$.

Next, we prove $(u_{\lambda\star},v_{\lambda\star})\ne(0,0)$. Assume that $(u_{\lambda\star},v_{\lambda\star})=(0,0)$. Then $\varphi(0,0)=0=\inf_{(u,v)\in W}\varphi(u,v)$. Let

$$u_*(x)=\begin{cases}1 & \text{if } x=x_1,\\ 0 & \text{if } x\ne x_1,\end{cases}$$

where $x_1 \in V$ with $e_1(x_1) > 0$. If (i) of ($F_2$) holds, then

$$\begin{aligned}\inf_{(u,v)\in W} \varphi_\lambda(u,v) &\le \inf_{\theta\in(0,+\infty)} \varphi_\lambda(\theta u_*,0)\\ &= \inf_{\theta\in(0,+\infty)} \left(\frac{1}{p}\theta^p\|u_*\|^p_{W_h^{1,p}(V)} - \int_V F(x,\theta u_*,0)\,d\mu - \lambda_1\theta\int_V e_1u_*\,d\mu\right)\\ &\le \inf_{\theta\in(0,+\infty)} \left(\frac{1}{p}\theta^p\|u_*\|^p_{W_h^{1,p}(V)} + \int_V K_1(x)|\theta u_*|^{\beta_1}\,d\mu - \lambda_1\theta\int_V e_1u_*\,d\mu\right)\\ &= \inf_{\theta\in(0,+\infty)} \left(\frac{1}{p}\theta^p\|u_*\|^p_{W_h^{1,p}(V)} + \mu(x_1)\theta^{\beta_1}K_1(x_1) - \lambda_1\theta\mu(x_1)e_1(x_1)\right).\end{aligned}$$

Note that $\beta_1 > 1$, $p > 1$, $\mu(x_1) > 0$, $e_1(x_1) > 0$, $K_1(x_1) > 0$ and $\lambda_1 > 0$. Then for each $\lambda_1 > 0$, there exists sufficiently small $\theta > 0$ such that $\inf_{(u,v)\in W}\varphi(u,v) < 0$, which is a contradiction. Similarly, if (ii) of ($F_2$) holds, we also can obtain the same contradiction.

Moreover, if $(u_{\lambda\star}, v_{\lambda\star}) = (u_{\lambda\star}, 0)$, then by (3.3), we have

$$\int_V (|\nabla u_{\lambda\star}|^p + h_1|u_{\lambda\star}|^p)\,d\mu = \int_V F_u(x,u_{\lambda\star},0)u_{\lambda\star}\,d\mu + \lambda_1\int_V e_1u_{\lambda\star}\,d\mu.$$

Hence, combining with ($F_1$), we have

$$\|u_{\lambda\star}\|_{W_h^{1,p}(V)} \le h_0^{1/p}\left(\frac{\lambda_1\|e_1\|_{L^{\frac{p}{p-1}}(V)} + \|g_1\|_{L^{\frac{p}{p-1}}(V)}}{h_0 - \|f_1\|_\infty}\right)^{\frac{1}{p-1}},$$

then by (2.1), we have

$$\|u_{\lambda\star}\|_\infty \le \mu_0^{-\frac{1}{p}}\left(\frac{\lambda_1\|e_1\|_{L^{\frac{p}{p-1}}(V)} + \|g_1\|_{L^{\frac{p}{p-1}}(V)}}{h_0 - \|f_1\|_\infty}\right)^{\frac{1}{p-1}}.$$

Similarly, when $(u_{\lambda\star}, v_{\lambda\star}) = (0, v_{\lambda\star})$, we have

$$\|v_{\lambda\star}\|_\infty \le \mu_0^{-\frac{1}{q}}\left(\frac{\lambda_2\|e_2\|_{L^{\frac{q}{q-1}}(V)} + \|g_2\|_{L^{\frac{q}{q-1}}(V)}}{h_0 - \|f_2\|_\infty}\right)^{\frac{1}{q-1}}. \qquad \square$$

*Proof of Theorem* 1.2. The proof is similar to that of Theorem 1.1, in which we only need to slightly modify the proof of Lemma 3.3 with replacing ($F_1$) by ($F_1'$). We omit the details. $\square$

## 4. Proofs for the super-$(p,q)$-linear case

**Lemma 4.1.** *Assume that* ($H_1$) *and* ($C_1$) *hold. Then for each* $\lambda \in (0, \lambda_0)$, *there exists a positive constant* $\rho_\lambda$ *such that* $\varphi(u,v) > 0$ *whenever* $\|(u,v)\|_W = \rho_\lambda$.

*Proof.* Note that $F(x,0,0)=0$. By ($C_1$), for all $(s,t)\in\mathbb{R}^2$ with $|(s,t)|<l_0$, we have

$$\begin{aligned}|F(x,s,t)| &= |F(x,s,t)-F(x,0,0)|\\ &\le |F(x,s,t)-F(x,0,t)|+|F(x,0,t)-F(x,0,0)|\\ &\le \int_0^{|s|}|F_s(x,s,t)|\,ds+\int_0^{|t|}|F_t(x,0,t)|\,dt\\ &\le \int_0^{|s|}\frac{h_0}{q+1}\left(|s|^{p-1}+|t|^{\frac{pq-q}{p}}\right)ds+\int_0^{|t|}\frac{h_0}{q+1}|t|^{q-1}\,dt\\ &\le \frac{h_0}{p(q+1)}|s|^p+\frac{h_0}{q+1}|t|^{\frac{pq-q}{p}}|s|+\frac{h_0}{q(q+1)}|t|^q\\ &\le \frac{2h_0}{p(q+1)}|s|^p+\frac{(pq-q+p)h_0}{pq(q+1)}|t|^q.\end{aligned}$$

It is easy to obtain that for each $\lambda$ satisfying (1.8), there exists a $\varepsilon_\lambda>0$ such that

$$0<\lambda<\lambda_\varepsilon:=\frac{\min\{1,q-1\}\cdot(\Lambda_0-\varepsilon_\lambda)^{\max\{p,q\}-1}}{2^{\max\{p,q\}-1}(pq+p)\max\left\{h_0^{-\frac{1}{p}}\|e_1\|_{L^{\frac{p}{p-1}}(V)},h_0^{-\frac{1}{q}}\|e_2\|_{L^{\frac{q}{q-1}}(V)}\right\}}.$$

For any $(u,v)\in W$ with $\|(u,v)\|_W=\Lambda_0-\varepsilon_\lambda$, by (2.1), we have $\|u\|_\infty<\frac{l_0}{2}$ and $\|v\|_\infty<\frac{l_0}{2}$, and so $|(u(x),v(x))|\le\|u\|_\infty+\|v\|_\infty<l_0$ for all $x\in V$. Then

$$\begin{aligned}\varphi_\lambda(u,v) &\ge \frac{1}{p}\|u\|^p_{W_h^{1,p}(V)}+\frac{1}{q}\|v\|^q_{W_h^{1,q}(V)}-\frac{2h_0}{p(q+1)}\int_V|u|^p\,d\mu\\ &\quad-\frac{(pq-q+p)h_0}{pq(q+1)}\int_V|v|^q\,d\mu-\lambda\int_V(e_1u+e_2v)\,d\mu\\ &\ge\left(\frac{1}{p}-\frac{2}{p(q+1)}\right)\|u\|^p_{W_h^{1,p}(V)}+\left(\frac{1}{q}-\frac{pq-q+p}{pq(q+1)}\right)\|v\|^q_{W_h^{1,q}(V)}\\ &\quad-\lambda\int_V(e_1u+e_2v)\,d\mu\\ &\ge\frac{\min\{1,q-1\}}{pq+p}\left(\|u\|^p_{W_h^{1,p}(V)}+\|v\|^q_{W_h^{1,q}(V)}\right)\\ &\quad-\lambda\max\left\{h_0^{-\frac{1}{p}}\|e_1\|_{L^{\frac{p}{p-1}}(V)},h_0^{-\frac{1}{q}}\|e_2\|_{L^{\frac{q}{q-1}}(V)}\right\}\|(u,v)\|_W\\ &\ge\frac{\min\{1,q-1\}}{2^{\max\{p,q\}-1}(pq+p)}\|(u,v)\|_W^{\max\{p,q\}}\\ &\quad-\lambda\max\left\{h_0^{-\frac{1}{p}}\|e_1\|_{L^{\frac{p}{p-1}}(V)},h_0^{-\frac{1}{q}}\|e_2\|_{L^{\frac{q}{q-1}}(V)}\right\}\|(u,v)\|_W\end{aligned}\tag{4.1}$$

for any $(u,v)\in W$ with $\|(u,v)\|_W=\Lambda_0-\varepsilon_\lambda$. Let $\rho_\lambda=\Lambda_0-\varepsilon_\lambda$. Hence, for each $\lambda\in(0,\lambda_0)$, there exists a $\rho_\lambda$ such that $\varphi(u,v)\ge\alpha_\lambda>0$ whenever $\|(u,v)\|_W=\rho_\lambda$, where

$$\alpha_\lambda=\frac{\min\{1,q-1\}}{2^{\max\{p,q\}-1}(pq+p)}\rho_\lambda^{\max\{p,q\}}-\lambda\max\left\{h_0^{-\frac{1}{p}}\|e_1\|_{L^{\frac{p}{p-1}}(V)},h_0^{-\frac{1}{q}}\|e_2\|_{L^{\frac{q}{q-1}}(V)}\right\}\rho_\lambda.\tag{4.2}$$

□

**Lemma 4.2.** *Assume that* $(\mathrm{C}_2)$ *holds. Then for each* $\lambda \in (0, \lambda_0)$, *there exists a* $(u_{**\lambda}, v_{**\lambda}) \in W$ *with* $\|(u_{**\lambda}, v_{**\lambda})\|_W > \rho_\lambda$ *such that* $\varphi(u_{**\lambda}, v_{**\lambda}) < 0$.

*Proof.* Let

$$u^*(x) = v^*(x) = \begin{cases} 1 & \text{if } x = x_3, \\ 0 & \text{if } x \neq x_3, \end{cases}$$

where $x_3 \in V$ with $\mu(x_3) > 0$ and $e_1(x_3) + e_2(x_3) > 0$. Then

$$\begin{aligned} \int_V |\nabla u^*|^p \, d\mu &= \sum_{x \in V} |\nabla u^*|^p(x)\mu(x) \\ &= \sum_{x \in V} \left( \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u^*(y) - u^*(x))^2 \right)^{p/2} \mu(x) \\ &= \left( \frac{1}{2\mu(x_3)} \sum_{y \sim x_3} w_{x_3 y} \right)^{p/2} \mu(x_3) + \sum_{x \sim x_3} \left( \frac{1}{2\mu(x)} \sum_{x_3 \sim x} w_{x x_3} \right)^{p/2} \mu(x) \\ &= \left( \frac{\deg(x_3)}{2\mu(x_3)} \right)^{p/2} \mu(x_3) + \sum_{x \sim x_3} \left( \frac{\deg(x_3)}{2\mu(x)} \right)^{p/2} \mu(x) \\ &= \left( \frac{\deg(x_3)}{2} \right)^{p/2} \left( \sum_{x \sim x_3} \left( \frac{1}{\mu(x)} \right)^{p/2-1} + \frac{1}{\mu(x_3)^{p/2-1}} \right) \\ &:= D_1. \end{aligned} \tag{4.3}$$

Similarly, we have

$$\int_V |\nabla v^*|^q \, d\mu = \left( \frac{\deg(x_3)}{2} \right)^{q/2} \left( \sum_{x \sim x_3} \left( \frac{1}{\mu(x)} \right)^{q/2-1} + \frac{1}{\mu(x_3)^{q/2-1}} \right) := D_2.$$

Thus, by $(\mathrm{C}_2)$, for all $s \in \mathbb{R}$ with $s > l_1$, we have

$$\begin{aligned} \varphi_\lambda(su^*, sv^*) = {} & \frac{s^p}{p} \|u^*\|^p_{W_h^{1,p}(V)} + \frac{s^q}{q} \|v^*\|^q_{W_h^{1,q}(V)} - \int_V F(x, su^*(x), sv^*(x)) \, d\mu \\ & - \lambda \int_V (se_1 u^* + se_2 v^*) \, d\mu \\ = {} & \frac{s^p}{p} (D_1 + \mu(x_3)h_1(x_3)) + \frac{s^q}{q} (D_2 + \mu(x_3)h_2(x_3)) \\ & - \mu(x_3)F(x_3, s, s) - \lambda s\mu(x_3)(e_1(x_3) + e_2(x_3)) \\ \leq {} & \frac{s^p}{p} (D_1 + \mu(x_3)h_1(x_3)) + \frac{s^q}{q} (D_2 + \mu(x_3)h_2(x_3)) \\ & - M\mu(x_3)(s^p + s^q) - \lambda s\mu(x_3)(e_1(x_3) + e_2(x_3)) \\ = {} & s^p \left( \frac{D_1 + \mu(x_3)h_1(x_3)}{p} - M\mu(x_3) \right) + s^q \left( \frac{D_2 + \mu(x_3)h_2(x_3)}{q} - M\mu(x_3) \right) \\ & - \lambda s\mu(x_3)(e_1(x_3) + e_2(x_3)), \end{aligned}$$

which implies $\varphi(su^*, sv^*) \to -\infty$ as $s \to +\infty$. Hence, for each $\lambda \in (0, \lambda_0)$, there exists $s_\lambda$ large enough such that $\|(s_\lambda u^*, s_\lambda v^*)\|_W > \rho_\lambda$ and $\varphi(s_\lambda u^*, s_\lambda v^*) < 0$. Let $u_{**\lambda} = s_\lambda u^*$ and $v_{**\lambda} = s_\lambda v^*$. Then the proof is completed. □

**Lemma 4.3.** *Assume that* $(\mathrm{F}_0)$, $(\mathrm{C}_3)$, $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2')$ *hold. Then for each* $\lambda \in (0, \lambda_0)$, $\varphi_\lambda$ *satisfies the (PS)-condition.*

*Proof.* Let $\{(u_k, v_k)\} \subset W$ be a Palais–Smale sequence of $\varphi_\lambda$. Then there exists a positive constant $c$ such that

$$|\varphi_\lambda(u_k, v_k)| \leq c \text{ for all } k \in \mathbb{N} \quad \text{and} \quad \varphi_\lambda'(u_k, v_k) \to 0 \quad \text{as } k \to \infty.$$

Then, by $(\mathrm{C}_3)$, we have

$$\begin{aligned}
&c + \|u_k\|_{W_h^{1,p}(V)} + \|v_k\|_{W_h^{1,q}(V)} \\
&= c + \|(u_k, v_k)\|_W \\
&\geq \varphi_\lambda(u_k, v_k) - \frac{1}{\nu}\langle \varphi_\lambda'(u_k, v_k), (u_k, v_k)\rangle \\
&= \left(\frac{1}{p} - \frac{1}{\nu}\right) \|u_k\|^p_{W_h^{1,p}(V)} + \left(\frac{1}{q} - \frac{1}{\nu}\right) \|v_k\|^q_{W_h^{1,q}(V)} \\
&\quad - \frac{1}{\nu}\int_V [\nu F(x, u_k, v_k) - F_u(x, u_k, v_k)u_k - F_v(x, u_k, v_k)v_k] \, d\mu \\
&\quad - \frac{\nu - 1}{\nu}\lambda \int_V (e_1 u_k + e_2 v_k) \, d\mu \\
&\geq \left(\frac{1}{p} - \frac{1}{\nu}\right) \|u_k\|^p_{W_h^{1,p}(V)} + \left(\frac{1}{q} - \frac{1}{\nu}\right) \|v_k\|^q_{W_h^{1,q}(V)} \\
&\quad - \frac{A}{\nu}\int_V (|u|^p + |v|^q) \, d\mu - \frac{\nu - 1}{\nu}\lambda \int_V (e_1 u_k + e_2 v_k) \, d\mu \\
&\geq \left(\frac{1}{p} - \frac{1}{\nu} - \frac{A}{\nu h_0}\right) \|u_k\|^p_{W_h^{1,p}(V)} + \left(\frac{1}{q} - \frac{1}{\nu} - \frac{A}{\nu h_0}\right) \|v_k\|^q_{W_h^{1,q}(V)} \\
&\quad - \frac{(\nu - 1)\lambda}{\nu}\big(h_0^{-\frac{1}{p}} \|e_1\|_{L^{\frac{p}{p-1}}(V)} \|u_k\|_{W_h^{1,p}(V)} + h_0^{-\frac{1}{q}} \|e_2\|_{L^{\frac{q}{q-1}}(V)} \|v_k\|_{W_h^{1,q}(V)}\big).
\end{aligned} \tag{4.4}$$

We claim that $\|(u_k, v_k)\|_W$ is bounded. In fact, if

$$\|u_k\|_{W_h^{1,p}(V)} \to \infty \quad \text{and} \quad \|v_k\|_{W_h^{1,q}(V)} \to \infty \quad \text{as } k \to \infty, \tag{4.5}$$

then (4.4) implies that

$$\begin{aligned}
&c + \|(u_k, v_k)\|_W + \frac{(\nu - 1)\lambda}{\nu}\max\big\{h_0^{-\frac{1}{p}} \|e_1\|_{L^{\frac{p}{p-1}}(V)}, h_0^{-\frac{1}{q}} \|e_2\|_{L^{\frac{q}{q-1}}(V)}\big\} \|(u_k, v_k)\|_W \\
&\geq \min\left\{\left(\frac{1}{p} - \frac{1}{\nu} - \frac{A}{\nu h_0}\right), \left(\frac{1}{q} - \frac{1}{\nu} - \frac{A}{\nu h_0}\right)\right\} \big(\|u_k\|^p_{W_h^{1,p}(V)} + \|v_k\|^q_{W_h^{1,q}(V)}\big) \\
&\geq \min\left\{\left(\frac{1}{p} - \frac{1}{\nu} - \frac{A}{\nu h_0}\right), \left(\frac{1}{q} - \frac{1}{\nu} - \frac{A}{\nu h_0}\right)\right\} \frac{1}{2^{\min\{p,q\}-1}} \|(u_k, v_k)\|_W^{\min\{p,q\}}
\end{aligned}$$

for all large $k$, which contradicts with (4.5). If

$$\|u_k\|_{W_h^{1,p}(V)} \to \infty \quad \text{as } k \to \infty \tag{4.6}$$

and $\|v_k\|_{W_h^{1,q}(V)}$ is bounded for all $k \in \mathbb{N}$, then by (4.4), there exists two positive constants $c_0$ and $c_1$ such that

$$c_0 + c_1\|u_k\|_{W_h^{1,p}(V)} \geq \left(\frac{1}{p} - \frac{1}{\nu} - \frac{A}{\nu h_0}\right) \|u_k\|^p_{W_h^{1,p}(V)}$$

which contradicts with (4.6). Similarly, if $\|v_k\|_{W_h^{1,q}(V)} \to \infty$ as $k \to \infty$ and $\|u_k\|_{W_h^{1,p}(V)}$ is bounded for all $k \in \mathbb{N}$, we can also obtain the same contradiction. Hence, the above arguments imply that both $\|u_k\|_{W_h^{1,p}(V)}$ and $\|v_k\|_{W_h^{1,q}(V)}$ are bounded. So there exists a positive constant $c_2$ such that $\|u_k\|_{W_h^{1,p}(V)} \leq c_2$ and $\|v_k\|_{W_h^{1,q}(V)} \leq c_2$. Then we can find a subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u_\lambda^\star$ for some $u_\lambda^\star \in W_h^{1,p}(V)$ as $k \to \infty$, and a subsequence of $\{v_k\}$, which has the same subscript as the subsequence of $\{u_k\}$, still denoted by $\{v_k\}$, such that $v_k \rightharpoonup v_\lambda^\star$ for some $v_\lambda^\star \in W_h^{1,q}(V)$ as $k \to \infty$. By Lemma 2.2, we know that

$$u_k \to u_\lambda^\star \text{ and } v_k \to v_\lambda^\star \text{ in } L^\infty(V) \quad \text{as } k \to \infty. \tag{4.7}$$

Then by (3.2), we have

$$\begin{aligned}
&\langle \varphi_\lambda'(u_k,v_k) - \varphi_\lambda'(u_\lambda^\star, v_\lambda^\star), (u_k - u_\lambda^\star, 0)\rangle \\
&= \|u_k\|^p_{W_h^{1,p}(V)} + \|u_\lambda^\star\|^p_{W_h^{1,p}(V)} - \int_V \left[|\nabla u_k|^{p-2}\Gamma(u_k,u_\lambda^\star) + h_1(x)|u_k|^{p-2}u_k u_\lambda^\star\right] d\mu \\
&\quad - \int_V \left[|\nabla u_\lambda^\star|^{p-2}\Gamma(u_\lambda^\star,u_k) + h_1(x)|u_\lambda^\star|^{p-2}u_\lambda^\star u_k\right] d\mu \\
&\quad + \int_V [F_u(x,u_\lambda^\star,v_\lambda^\star) - F_u(x,u_k,v_k)](u_k - u_\lambda^\star)\, d\mu.
\end{aligned}$$

Let $A_1 = c_2\frac{1}{h_0^{1/p}\mu_0^{1/p}} + c_2\frac{1}{h_0^{1/q}\mu_0^{1/q}}$ and $A_2 = \|u_\lambda^\star\|_\infty + \|v_\lambda^\star\|_\infty$. By $(\mathrm{F}_0)$ and (4.7), we have

$$\begin{aligned}
&\int_V [F_u(x,u_\lambda^\star,v_\lambda^\star) - F_u(x,u_k,v_k)](u_k - u_\lambda^\star)\, d\mu \\
&\leq \int_V |F_u(x,u_\lambda^\star,v_\lambda^\star) - F_u(x,u_k,v_k)||u_k - u_\lambda^\star|\, d\mu \\
&\leq \int_V [|F_u(x,u_k,v_k)| + |F_u(x,u_\lambda^\star,v_\lambda^\star)|]|u_k - u_\lambda^\star|\, d\mu \\
&\leq \left[\max_{|(s,t)|\leq A_1} a(|(s,t)|)\int_V b(x)\, d\mu + \max_{|(s,t)|\leq A_2} a(|(s,t)|)\int_V b(x)\, d\mu\right] \|u_k - u_\lambda^\star\|_\infty \\
&\to 0.
\end{aligned}$$

The rest of arguments are the same as Lemma 3.4. □

**Lemma 4.4.** *Assume that* $(C_1)$ *and* $(C_4)$ *holds. Then for each* $\lambda \in (0, \lambda_0)$, $-\infty < \inf\left\{\varphi(u,v) : (u,v) \in \overline{B}_{\rho_\lambda}\right\} < 0$, *where* $\rho_\lambda$ *is given in Lemma* 4.1 *and* $\overline{B}_{\rho_\lambda} = \left\{(u,v) \in W \mid \|(u,v)\|_W \leq \rho_\lambda\right\}$.

*Proof.* Let

$$u^{**}(x) = v^{**}(x) = \begin{cases} 1 & \text{if } x = x_4, \\ 0 & \text{if } x \neq x_4, \end{cases}$$

where $x_4 \in V$ with $\mu(x_4) > 0$ and $e_1(x_4) + e_2(x_4) > 0$. Hence, by (4.3), we obtain that

$$D_3 := \int_V |\nabla u^{**}|^p \, d\mu = \left(\frac{\deg(x_4)}{2}\right)^{p/2} \left(\sum_{x \sim x_4} \left(\frac{1}{\mu(x)}\right)^{p/2-1} + \frac{1}{\mu(x_4)^{p/2-1}}\right),$$

$$D_4 := \int_V |\nabla v^{**}|^q \, d\mu = \left(\frac{\deg(x_4)}{2}\right)^{q/2} \left(\sum_{x \sim x_4} \left(\frac{1}{\mu(x)}\right)^{q/2-1} + \frac{1}{\mu(x_4)^{q/2-1}}\right).$$

Then for each $\lambda \in (0, \lambda_0)$, by $(C_4)$, for all $t \in \mathbb{R}$ with $0 < t < l_2$, we have

$$\begin{aligned} \varphi_\lambda(tu^{**}, tv^{**}) &= \frac{t^p}{p}\|u^{**}\|^p_{W_h^{1,p}(V)} + \frac{t^q}{q}\|v^{**}\|^q_{W_h^{1,q}(V)} - \int_V F(x, tu^{**}(x), tv^{**}(x)) \, d\mu \\ &\quad - \lambda \int_V (te_1 u^{**} + te_2 v^{**}) \, d\mu \\ &= \frac{t^p}{p}(D_3 + \mu(x_4)h_1(x_4)) + \frac{t^q}{q}(D_4 + \mu(x_4)h_2(x_4)) \\ &\quad - \mu(x_4)F(x_4, t, t) - \lambda t\mu(x_4)(e_1(x_4) + e_2(x_4)) \\ &\leq \frac{t^p}{p}(D_3 + \mu(x_4)h_1(x_4)) + \frac{t^q}{q}(D_4 + \mu(x_4)h_2(x_4)) \\ &\quad + K_3(x_4)\mu(x_4)|t|^{\beta_3} - \lambda t\mu(x_4)(e_1(x_4) + e_2(x_4)). \end{aligned} \tag{4.8}$$

Note that $p > 1$, $q > 1$, $\beta_3 > 1$ and $K_3(x_4) > 0$. By (4.8), there exists a sufficiently small $t_{1,\lambda}$ satisfying

$$0 < t_{1,\lambda} < \min\left\{\frac{\rho_\lambda}{2\|u^{**}\|_{W_h^{1,p}(V)}}, \frac{\rho_\lambda}{2\|v^{**}\|_{W_h^{1,q}(V)}}\right\}$$

such that $\varphi(t_{1,\lambda}u^{**}, t_{1,\lambda}v^{**}) < 0$. Clearly, $\|(t_{1,\lambda}u^{**}, t_{1,\lambda}v^{**})\|_W < \rho_\lambda$. Hence, $\inf\left\{\varphi(u,v) : (u,v) \in \overline{B}_{\rho_\lambda}\right\} \leq \varphi(t_{1,\lambda}u^{**}, t_{1,\lambda}v^{**}) < 0$. Moreover, it is easy to see that (4.1) still holds for all $(u,v) \in \overline{B}_{\rho_\lambda}$. Then

$$\varphi_\lambda(u,v) \geq -\lambda \max\left\{h_0^{-\frac{1}{p}}\|e_1\|_{L^{\frac{p}{p-1}}(V)}, h_0^{-\frac{1}{q}}\|e_2\|_{L^{\frac{q}{q-1}}(V)}\right\}\rho_\lambda,$$

which shows that $\varphi_\lambda$ is bounded from below in $\overline{B}_{\rho_\lambda}$ for each $\lambda \in (0, \lambda_0)$. So $\inf\left\{\varphi(u,v) : (u,v) \in \overline{B}_{\rho_\lambda}\right\} > -\infty$. □

*Proof of Theorem* 1.3. By Lemmas 2.6, 4.1, 4.2 and 4.3, we obtain that for each $\lambda \in (0, \lambda_0)$, $\varphi_\lambda$ has a critical value $c_* \geq \alpha_\lambda > 0$ with

$$c_* := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi_\lambda(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = (0,0), \gamma(1) = (u_{*,\lambda}, v_{*,\lambda})\}$$

and $\alpha_\lambda$ is defined by (4.2). Hence, by Proposition 3.2, system (1.7) has one solution $(u^{\lambda\star}, v^{\lambda\star})$ of positive energy. Obviously, $(u^{\lambda\star}, v^{\lambda\star}) \neq (0,0)$. Otherwise, by the fact that $F(x,0,0) = 0$ for all $x \in V$, we have $\varphi(u^{\lambda\star}, v^{\lambda\star}) = 0$, which contradicts with $c_* > 0$.

Next, we prove that system (1.7) has one solution of negative energy if $(\mathrm{C}_4)$ also holds. The proof is motivated by [6, Theorem 3.3]. In fact, by Lemmas 4.1 and 4.4, we know that

$$-\infty < \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda < 0 < \inf_{\partial B_{\rho_\lambda}} \varphi_\lambda$$

for each $\lambda \in (0, \lambda_0)$. Set

$$\frac{1}{n} \in \left(0, \inf_{\partial B_{\rho_\lambda}} \varphi_\lambda - \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda\right), \quad n \in \mathbb{Z}^+.$$

Then there exists a $(u_n, v_n) \in \overline{B}_{\rho_\lambda}$ such that

$$\varphi_\lambda(u_n, v_n) \leq \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda + \frac{1}{n}. \tag{4.9}$$

As $\varphi_\lambda(u,v) \in C^1(W, \mathbb{R})$, we know $\varphi_\lambda(u,v)$ is lower semicontinuous. Thus, by Lemma 2.4 we have

$$\varphi_\lambda(u_n, v_n) \leq \varphi_\lambda(u,v) + \frac{1}{n}\|(u,v) - (u_n, v_n)\|_W, \quad \forall\,(u,v) \in \overline{B}_{\rho_\lambda}.$$

Note that

$$\varphi_\lambda(u_n, v_n) \leq \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda + \frac{1}{n} < \inf_{\partial B_{\rho_\lambda}} \varphi_\lambda.$$

Thus, $(u_n, v_n) \in B_{\rho_\lambda}$. Defining $M_n \colon W \to R$ by

$$M_n(u,v) = \varphi_\lambda(u,v) + \frac{1}{n}\|(u,v) - (u_n, v_n)\|_W,$$

we have $(u_n, v_n) \in B_{\rho_\lambda}$ minimizes $M_n$ on $\overline{B}_{\rho_\lambda}$. Therefore, for all $(u,v) \in W$ with $\|(u,v)\|_W = 1$, taking $t > 0$ small enough such that $(u_n + tu, v_n + tv) \in \overline{B}_{\rho_\lambda}$, then

$$\frac{M_n(u_n + tu, v_n + tv) - M_n(u_n, v_n)}{t} \geq 0,$$

which implies that

$$\langle \varphi_\lambda'(u_n, v_n), (u,v) \rangle \geq -\frac{1}{n}.$$

Similarly, when $t < 0$ and $|t|$ small enough, we have

$$\langle \varphi'_\lambda(u_n, v_n), (u, v)\rangle \le \frac{1}{n}.$$

Hence,

$$\|\varphi'_\lambda(u_n, v_n)\| \le \frac{1}{n}. \tag{4.10}$$

Passing to the limit in (4.9) and (4.10), we conclude that

$$\varphi_\lambda(u_n, v_n) \to \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda \quad \text{and} \quad \|\varphi'_\lambda(u_n, v_n)\| \to 0 \quad \text{as } n \to \infty.$$

Hence, $\{(u_n, v_n)\} \subset \overline{B}_{\rho_\lambda}$ is a Palais–Smale sequence of $\varphi_\lambda$. By Lemma 4.3, $\{(u_n, v_n)\}$ has a strongly convergent subsequence $\{(u_{nk}, v_{nk})\} \subset \overline{B}_{\rho_\lambda}$, and $(u_{nk}, v_{nk}) \to (u^{\star\star}, v^{\star\star}) \in \overline{B}_{\rho_\lambda}$ as $n_k \to \infty$. Consequently,

$$\varphi_\lambda(u^{\star\star}, v^{\star\star}) = \inf_{\overline{B}_{\rho_\lambda}} \varphi_\lambda < 0 \quad \text{and} \quad \varphi'_\lambda(u^{\star\star}, v^{\star\star}) = 0,$$

which implies that system (1.7) has a solution $(u^{\star\star}, v^{\star\star}) \neq (0,0)$ of negative energy. □

## 5. Examples

**Example 5.1.** Let $p = 2$ and $q = 3$. Consider the following system

$$\begin{cases} -\Delta u + h_1(x)u = F_u(x,u,v) + \lambda_1 e_1(x), & x \in V, \\ -\Delta_3 v + h_2(x)v = F_v(x,u,v) + \lambda_2 e_2(x), & x \in V, \end{cases} \tag{5.1}$$

where $G = (V, E)$ is locally finite graph, the measure $\mu(x) \ge \mu_0 = 1$ for all $x \in V$, $h_i \colon V \to \mathbb{R}^+$, $i = 1, 2$, $h_1(x) = 3 + \operatorname{dist}(x, x_1)$, $h_2(x) = 3 + \operatorname{dist}(x, x_2)$, where $x_1$ and $x_2$ are two fixed points in $V$ and $\mu(x_1) = \mu(x_2) = 1$,

$$F(x,s,t) = \begin{cases} \frac{3}{5}\left(s^{5/3} + t^{5/3}\right) & \text{if } x = x_1, x_2, \\ 0 & \text{if } x \neq x_1, x_2, \end{cases} \qquad e_1(x) = e_2(x) = \begin{cases} 1 & \text{if } x = x_1, x_2, \\ 0 & \text{if } x \neq x_1, x_2, \end{cases}$$

and $\lambda_1, \lambda_2 > 0$. Next, we verify that $h_1$, $h_2$ and $F$ satisfy the conditions in Theorem 1.1:

- Obviously, when $\operatorname{dist}(x, x_i) \to +\infty$, $h_i(x) \to +\infty$ and $h_i \ge h_0 = 3$, $i = 1, 2$. Hence, $h_i$ satisfies $(\mathrm{H}_1)$, $(\mathrm{H}_2)$, $i = 1, 2$.

- Let

$$f_1(x) \equiv 1, \quad g_1(x) = \begin{cases} 1 & \text{if } x = x_1, x_2, \\ 0 & \text{if } x \neq x_1, x_2. \end{cases}$$

Then

$$\|f_1\|_\infty = 1 < \min\left\{\frac{h_0}{2}, \frac{ph_0}{q(p-1)}\right\} = \frac{3}{2}, \quad \|g_1\|_{L^2(V)} = \sqrt{2}.$$

Moreover,

$$|F_s(x,s,t)| = |s|^{2/3} \le |s| + 1 \le f_1(x)(|s| + |t|^{3/2}) + g_1(x).$$

Similarly, let

$$f_2(x) = f_1(x), \quad g_2(x) = g_1(x).$$

We also have

$$|F_t(x,s,t)| = |t|^{2/3} \le |t|^2 + 1 \le f_2(x)(|s|^2 + |t|^2) + g_2(x).$$

Then $F(x,s,t)$ satisfies $(\mathrm{F}_1)$. Hence, $F(x,s,t)$ also satisfies $(\mathrm{F}_0)$.

- Let

$$\beta_1 = \frac{5}{3}, \quad K_1(x) \equiv \frac{3}{5}.$$

Then

$$F(x,s,0) \ge -\frac{3}{5}|s|^{5/3}.$$

Hence, $F(x,s,t)$ satisfies (i) of $(\mathrm{F}_2)$.

Hence, by Theorem 1.1, for each pair $(\lambda_1, \lambda_2) \in (0,+\infty) \times (0,+\infty)$, system (5.1) has one nontrivial solution $(u_{\lambda\star}, v_{\lambda\star})$. Furthermore, if $(u_{\lambda\star}, v_{\lambda\star}) = (u_{\lambda\star}, 0)$, then $\|u_{\lambda\star}\|_\infty \le \frac{\sqrt{2}}{2}(\lambda_1 + 1)$. If $(u_{\lambda\star}, v_{\lambda\star}) = (0, v_{\lambda\star})$, then $\|v_{\lambda\star}\|_\infty \le 2^{1/3}\left(\frac{\lambda_2+1}{2}\right)^{1/2}$.

**Example 5.2.** Let $p = 2$ and $q = 3$. Consider the following system

$$\begin{cases} -\Delta u + h_1(x)u = F_u(x,u,v) + \lambda e_1(x), & x \in V, \\ -\Delta_3 v + h_2(x)v = F_v(x,u,v) + \lambda e_2(x), & x \in V, \end{cases} \tag{5.2}$$

where $G = (V,E)$ is locally finite graph, the measure $\mu(x) \ge \mu_0 > 0$ for all $x \in V$, $e_1, e_2 \in L^2(V)$, $e_1(x), e_2(x) \not\equiv 0$ and $\lambda > 0$. $h_1(x) = h_2(x) = c_1 \operatorname{dist}(x,x_1) - \frac{1}{\operatorname{dist}(x,x_2)+1} + 2$, where $c_1$ is positive constant and $x_1$ and $x_2$ are two fixed points in $V$ with $e_1(x_1) + e_2(x) > 0$, $F(x,s,t) = M\ln(1 + s^4 + t^4)(s^4 + t^4)$, $M = \max\left\{\frac{D_1 + \mu(x_1)h_1(x_1)}{2\mu(x_1)}, \frac{D_2 + \mu(x_1)h_2(x_1)}{3\mu(x_1)}\right\} + 1 > 1$, $D_1 = \frac{\deg(x_1)}{2}(\sharp A + 1)$, and

$$D_2 = \left(\frac{\deg(x_1)}{2}\right)^{3/2}\left(\sum_{x\sim x_1}\frac{1}{\sqrt{\mu(x)}} + \frac{1}{\sqrt{\mu(x_1)}}\right),$$

where $\sharp A$ is the number of elements in the set $A = \{x \in V \mid x \sim x_1\}$.

We verify that $h_1$, $h_2$ and $F$ satisfy the conditions in Theorem 1.3:

- Obviously, $h_1$, $h_2$ satisfy $(\mathrm{H}_1)$ and $h_0 = 1$.
- For any given constant $B$, when $h_1 = h_2 = c_1 \operatorname{dist}(x, x_1) - \frac{1}{\operatorname{dist}(x,x_2)+1} + 2 < B$, we have
$$c_1 \operatorname{dist}(x, x_1) < B - 2 + \frac{1}{\operatorname{dist}(x, x_2) + 1} < B - 1.$$
Moreover, since $V$ is a locally finite graph, the set $A_i = \{x \in V \mid h_i \leq B\} \subseteq \{x \in V \mid \operatorname{dist}(x, x_1) < B - 1\}$ is finite. So, $\sum_{x \in A_i} \mu(x)$ is finite, $(\mathrm{H}_2')$ holds.
- By $F(x, s, t)$, when $|s|$ and $|t| < 1$, we have
$$\begin{aligned} |F_s(x, s, t)| &= M \left| \frac{4s^3(s^4 + t^4)}{1 + s^4 + t^4} + 4s^3 \ln(1 + s^4 + t^4) \right| \\ &\leq 4M \left( \frac{|s|^3(s^4 + t^4)}{1 + s^4 + t^4} + |s|^3(1 + s^4 + t^4) \right) \\ &\leq 4M(4|s|^3 + t^4). \end{aligned}$$
Moreover, when $|s| < \frac{1}{16\sqrt{M}}$ and $|t| < \frac{1}{(16M)^{2/5}}$, we have
$$16M|s|^3 \leq \frac{1}{4}|s|, \quad 4Mt^4 \leq \frac{1}{4}|t|^{3/2}.$$
So, when $|(s, t)| \leq \frac{1}{16\sqrt{M}}$, $F_s(x, s, t)| \leq \frac{1}{4}(|s| + |t|^{3/2})$. Similarly, when $|s| < \frac{1}{4\sqrt{M}}$ and $|t| < \frac{1}{64M}$, we can prove that
$$|F_t(x, s, t)| \leq 4M(4|t|^3 + s^4) \leq \frac{1}{4}(s^2 + t^2).$$
Hence, when $|(s, t)| \leq \frac{1}{64M}$,
$$|F_s(x, s, t)| \leq \frac{1}{4}(|s| + |t|^{3/2}) \quad \text{and} \quad |F_t(x, s, t)| \leq \frac{1}{4}(s^2 + t^2).$$
It is that $(\mathrm{C}_1)$ holds.
- When $s > 1$, $F(x, s, s) = 4s^4 \ln(1 + 2s^4) \geq 4(s^2 + s^3)$. So, $F$ satisfies $(\mathrm{C}_2)$.
- Let $\nu = 4$ and $A = \frac{1}{4}$. For all $x \in V$, we have
$$4F(x, s, t) - F_s(x, s, t)s - F_t(x, s, t)t = -4M \frac{(s^4 + t^4)^2}{(1 + s^4 + t^4)} \leq \frac{1}{4}(s^2 + |t|^3).$$
So, $(\mathrm{C}_3)$ holds.
- Let $\beta_3 = 2$ and $K_3(x) \equiv 1$. For all $s \in \mathbb{R}$, $F(x, s, s) = 2Ms^4 \ln(1 + 2s^4) \geq -s^2$. So, $(\mathrm{C}_4)$ holds.

Hence, by Theorem 1.3, when $0 < \lambda < \frac{\min\left\{\frac{1}{128M} \min\left\{\mu_0^{1/2}, \mu_0^{1/3}\right\}, 1\right\}^2}{32 \max\left\{\|e_1\|_{L^2(V)}, \|e_2\|_{L^{3/2}(V)}\right\}}$, system (5.2) has one nontrivial solution of positive energy and another nontrivial solution of negative energy.

## 6. Conclusion

The existence of nontrivial solutions for system (1.7) is investigated when the nonlinear term $F$ satisfies the sub-$(p, q)$-linear condition or super-$(p, q)$-linear condition, which generalize some results in [12] in some sense. We present the concrete ranges of the parameter $\lambda_1$ and $\lambda_2$. For the sub-$(p, q)$-linear case, we furthermore obtain a necessary condition for the existence of the semi-trivial solutions, and for the super-$(p, q)$-linear case, we present a weaker assumption of $F$ than the well-known (AR)-condition. However, we do not investigate the existence of the non-semi-trivial solutions. A possible method to solve the problem can be referred to [2, 3] and we shall try to do it in future works.

## A. Appendix

In this section, we present some conclusions about $W_h^{1,s}(V)$ and $\varphi_\lambda$.

**Lemma A.1.** *$W_h^{1,s}(V)$ is uniformly convex for all $s > 1$.*

*Proof.* Since $L^s(V)$ is uniformly convex for all $s > 1$, by using Theorem 8 in [21], we have $E$ is uniformly convex, where $E = L^s(V) \times L^s(V)$ with $\|(u, v)\|_E = \big(\|u\|_{L^s(V)}^s + \|v\|_{L^s(V)}^s\big)^{1/s}$. Define $T\colon W_h^{1,s}(V) \to E$ by

$$T(u(x)) = \big(|\nabla u|(x), h(x)^{1/s}u(x)\big),$$

where $h(x) \geq h_0 > 0$. Then

$$\|T(u)\|_E = \big(\|\nabla u\|_{L^s(V)}^s + \|h^{1/s}u\|_{L^s(V)}^s\big)^{1/s} = \left(\int_V |\nabla u|^s + h|u|^s \, d\mu\right)^{1/s} = \|u\|_{W_h^{1,s}(V)}.$$

So, $T$ is an isometry. Hence, $W_h^{1,s}(V)$ is uniformly convex. □

**Lemma A.2.** *If $F(x, s, t)$ satisfies* ($\mathrm{F}_0$), *then $\varphi_\lambda \in C^1(W, \mathbb{R})$, and*

$$\begin{aligned}\langle \varphi'(u, v), (\phi_1, \phi_2)\rangle &= \int_V \big[|\nabla u|^{p-2}\Gamma(u, \phi_1) + h_1|u|^{p-2}u\phi_1 - F_u(x, u, v)\phi_1 - \lambda_1 e_1\phi_1\big] \, d\mu \\ &\quad + \int_V \big[|\nabla v|^{q-2}\Gamma(v, \phi_2) + h_2|v|^{q-2}v\phi_2 - F_v(x, u, v)\phi_2 - \lambda_2 e_2\phi_2\big] \, d\mu.\end{aligned}$$

*Proof.* Let

$$\begin{aligned}G(x, u, v) = {}& \frac{1}{p}(|\nabla u|^p + h_1|u|^p)\mu(x) + \frac{1}{q}(|\nabla v|^q + h_2|v|^q)\mu(x) \\ & - F(x, u, v)\mu(x) - \lambda_1 e_1 u\mu(x) - \lambda_2 e_2 v\mu(x).\end{aligned}$$

Then $\sum_{x\in V} G(x,u,v)=\varphi_\lambda(u,v)$. For any given $(\phi_1,\phi_2)\in W$ and $\theta\in[-1,1]$, we have

$$\begin{aligned}
G_x(\theta) &\triangleq G(x,u+\theta\phi_1,v+\theta\phi_2)\\
&=\frac{1}{p}(|\nabla(u+\theta\phi_1)|^p+h_1|u+\theta\phi_1|^p)\mu(x)\\
&\quad+\frac{1}{q}(|\nabla(v+\theta\phi_2)|^q+h_2|v+\theta\phi_2|^q)\mu(x)\\
&\quad-F(x,u+\theta\phi_1,v+\theta\phi_2)\mu(x)-\lambda_1e_1(u+\theta\phi_1)\mu(x)-\lambda_2e_2(v+\theta\phi_2)\mu(x).
\end{aligned}$$

Hence, by $(\mathrm{F}_0)$ and (1.2), we have

(A.1)
$$\begin{aligned}
G_x(\theta)\le{}& \frac{2^{p-1}}{p}(|\nabla u|^p+h_1|u|^p+|\nabla\phi_1|^p+h_1|\phi_1|^p)\mu(x)\\
&+\frac{2^{q-1}}{q}(|\nabla v|^q+h_2|v|^q+|\nabla\phi_2|^q+h_2|\phi_2|^q)\mu(x)\\
&+a(|(u+\theta\phi_1,v+\theta\phi_2)|)b(x)\mu(x)+\lambda_1|e_1(u+\theta\phi_1)|\mu(x)+\lambda_2|e_2(v+\theta\phi_2)|\mu(x)\\
\le{}& \frac{2^{p-1}}{p}(|\nabla u|^p+h_1|u|^p+|\nabla\phi_1|^p+h_1|\phi_1|^p)\mu(x)\\
&+\frac{2^{q-1}}{q}(|\nabla v|^q+h_2|v|^q+|\nabla\phi_2|^q+h_2|\phi_2|^q)\mu(x)\\
&+\max_{|(s,t)|\le\|u\|_\infty+\|v\|_\infty+\|\phi_1\|_\infty+\|\phi_2\|_\infty} a(|(s,t)|)b(x)\mu(x)\\
&+\lambda_1|e_1(u+\phi_1)|\mu(x)+\lambda_2|e_2(v+\phi_2)|\mu(x).
\end{aligned}$$

Since $u,\phi_1\in W_h^{1,p}(V)$, $v,\phi_2\in W_h^{1,q}(V)$, $a\in L^\infty(V)$, $b\in L^1(V)$, $e_1\in L^{\frac{p}{p-1}}(V)$, $e_2\in L^{\frac{q}{q-1}}(V)$, then $\sum_{x\in V}G_x(\theta)$ is convergence for all $\theta\in[-1,1]$. Moreover,

$$\begin{aligned}
&\frac{1}{p}\frac{\partial}{\partial\theta}|\nabla(u+\theta\phi_1)|^p\\
&=\frac{1}{p}\frac{\partial}{\partial\theta}(|\nabla(u+\theta\phi_1)|^2)^{p/2}\\
&=\frac{1}{2}|\nabla(u+\theta\phi_1)|^{p-2}\frac{\partial}{\partial\theta}\Gamma(u+\theta\phi_1,u+\theta\phi_1)\\
&=\frac{1}{2}|\nabla(u+\theta\phi_1)|^{p-2}\frac{\partial}{\partial\theta}\left(\frac{1}{2\mu(x)}\sum_{y\sim x}w_{xy}[(u(y)+\theta\phi_1(y))-(u(x)+\theta\phi_1(x))]^2\right)\\
&=|\nabla(u+\theta\phi_1)|^{p-2}\frac{1}{2\mu(x)}\sum_{y\sim x}w_{xy}[(u(y)+\theta\phi_1(y))-(u(x)+\theta\phi_1(x))](\phi_1(y)-\phi_1(x))\\
&=|\nabla(u+\theta\phi_1)|^{p-2}\frac{1}{2\mu(x)}\sum_{y\sim x}w_{xy}\big[(u(y)-u(x))(\phi_1(y)-\phi_1(x))+\theta(\phi_1(y)-\phi_1(x))^2\big]\\
&=|\nabla(u+\theta\phi_1)|^{p-2}(\Gamma(u,\phi_1)+\Gamma(\theta\phi_1,\phi_1))\\
&=|\nabla(u+\theta\phi_1)|^{p-2}\Gamma(u+\theta\phi_1,\phi_1).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{\partial G_x(\theta)}{\partial\theta}\\
&= |\nabla(u+\theta\phi_1)|^{p-2}\Gamma(u+\theta\phi_1,\phi_1)\mu(x) + |\nabla(v+\theta\phi_2)|^{q-2}\Gamma(v+\theta\phi_2,\phi_2)\mu(x)\\
&\quad + h_1|u+\theta\phi_1|^{p-2}(u+\theta\phi_1)\phi_1\mu(x) + h_2|v+\theta\phi_2|^{q-2}(v+\theta\phi_2)\phi_2\mu(x)\\
&\quad - F_{u+\theta\phi_1}(x,u+\theta\phi_1,v+\theta\phi_2)\phi_1\mu(x) - F_{v+\theta\phi_2}(x,u+\theta\phi_1,v+\theta\phi_2)\phi_2\mu(x)\\
&\quad - \lambda_1 e_1\phi_1\mu(x) - \lambda_2 e_2\phi_2\mu(x).
\end{aligned} \tag{A.2}$$

Since $F(x,s,t)$ is continuously differentiable in $(s,t)\in\mathbb{R}^2$ for all $x\in V$, it is easy to obtain that $\frac{\partial G_x(\theta)}{\partial\theta}$ is continuous in $[-1,1]$. By $(\mathrm{F}_0)$, we have

$$\begin{aligned}
&\frac{\partial G_x(\theta)}{\partial\theta}\\
&\le |\nabla(u+\theta\phi_1)|^{p-1}|\nabla\phi_1|\mu(x) + \big(h_1^{1/p}|u+\theta\phi_1|\big)^{p-1}h_1^{1/p}|\phi_1|\mu(x)\\
&\quad + |\nabla(v+\theta\phi_2)|^{q-1}|\nabla\phi_2|\mu(x) + \big(h_2^{1/q}|v+\theta\phi_2|\big)^{q-1}h_2^{1/q}|\phi_2|\mu(x)\\
&\quad + \big(|F_u|\phi_1 + |F_v|\phi_2 + \theta\lambda_1 e_1\phi_1 + \theta\lambda_2 e_2\phi_2\big)\mu(x)\\
&\le \big(|\nabla(u+\theta\phi_1)|^p + h_1|u+\theta\phi_1|^p\big)^{\frac{p-1}{p}}\big(|\nabla\phi_1|^p + h_1|\phi_1|^p\big)^{1/p}\mu(x)\\
&\quad + \big(|\nabla(v+\theta\phi_2)|^q + h_2|v+\theta\phi_2|^q\big)^{\frac{q-1}{q}}\big(|\nabla\phi_2|^q + h_2|\phi_2|^q\big)^{1/q}\mu(x)\\
&\quad + \max_{|(s,t)|\le\|u\|_\infty+\|v\|_\infty} a|(s,t)|b(x)(\phi_1+\phi_2)\mu(x) + \lambda_1|e_1\phi_1|\mu(x) + \lambda_2|e_2\phi_2|\mu(x)\\
&\le 2^{\frac{(p-1)^2}{p}}\big(|\nabla u|^p + |\nabla\phi_1|^p + h_1|u|^p + h_1|\phi_1|^p\big)^{\frac{p-1}{p}}\big(|\nabla\phi_1|^p + h_1|\phi_1|^p\big)^{1/p}\mu(x)\\
&\quad + 2^{\frac{(q-1)^2}{q}}\big(|\nabla v|^q + |\nabla\phi_2|^q + h_2|v|^q + h_2|\phi_2|^q\big)^{\frac{q-1}{q}}\big(|\nabla\phi_2|^q + h_2|\phi_2|^q\big)^{1/q}\mu(x)\\
&\quad + \max_{|(s,t)|\le\|u\|_\infty+\|v\|_\infty} a|(s,t)|b(x)(\phi_1+\phi_2)\mu(x) + \lambda_1|e_1\phi_1|\mu(x) + \lambda_2|e_2\phi_2|\mu(x)\\
&\le 2^{\frac{(p-1)^2}{p}}\Big(\big(|\nabla u|^p + h_1|u|^p\big)^{\frac{p-1}{p}} + \big(|\nabla\phi_1|^p + h_1|\phi_1|^p\big)^{\frac{p-1}{p}}\Big)\big(|\nabla\phi_1|^p + h_1|\phi_1|^p\big)^{1/p}\mu(x)\\
&\quad + 2^{\frac{(q-1)^2}{q}}\Big(\big(|\nabla v|^q + h_2|v|^q\big)^{\frac{q-1}{q}} + \big(|\nabla\phi_2|^q + h_2|\phi_2|^q\big)^{\frac{q-1}{q}}\Big)\big(|\nabla\phi_2|^q + h_2|\phi_2|^q\big)^{1/q}\mu(x)\\
&\quad + \max_{|(s,t)|\le\|u\|_\infty+\|v\|_\infty} a|(s,t)|b(x)(\phi_1+\phi_2)\mu(x) + \lambda_1|e_1\phi_1|\mu(x) + \lambda_2|e_2\phi_2|\mu(x).
\end{aligned} \tag{A.3}$$

Moreover, we have

$$\begin{aligned}
&\sum_{x\in V}\frac{\partial G_x(\theta)}{\partial\theta}\\
&\le 2^{\frac{(p-1)^2}{p}}\left(\sum_{x\in V}\Big(\big(|\nabla u|^p + h_1|u|^p\big)^{\frac{p-1}{p}} + \big(|\nabla\phi_1|^p + h_1|\phi_1|^p\big)^{\frac{p-1}{p}}\Big)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\\
&\qquad\times\left(\sum_{x\in V}|\nabla\phi_1|^p + h_1|\phi_1|^p\right)^{1/p}\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&+ 2^{\frac{(q-1)^2}{q}} \left( \sum_{x\in V} \left( (|\nabla v|^q + h_2|v|^q)^{\frac{q-1}{q}} + (|\nabla \phi_2|^q + h_2|\phi_2|^q)^{\frac{q-1}{q}} \right)^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \\
&\quad \times \left( \sum_{x\in V} |\nabla \phi_2|^q + h_2|\phi_2|^q \right)^{1/q} \mu(x) \\
&+ \max_{|(s,t)|\le \|u\|_\infty + \|v\|_\infty} a|(s,t)| \|b(x)\|_{L^1(V)} \|\phi_1 + \phi_2\|_\infty \\
&+ \lambda_1 \|e_1\|_{L^{\frac{p}{p-1}}(V)} \|\phi_1\|_{L^p(V)} + \lambda_2 \|e_2\|_{L^{\frac{q}{q-1}}(V)} \|\phi_2\|_{L^q(V)} \\
\le\ & 2^{\frac{(p-1)^2+1}{p}} \left( \sum_{x\in V} \left( |\nabla u|^p + h_1|u|^p + |\nabla \phi_1|^p + h_1|\phi_1|^p \right) \mu(x) \right)^{\frac{p-1}{p}} \\
&\quad \times \left( \sum_{x\in V} (|\nabla \phi_1|^p + h_1|\phi_1|^p) \mu(x) \right)^{1/p} \\
&+ 2^{\frac{(q-1)^2+1}{q}} \left( \sum_{x\in V} \left( |\nabla v|^q + h_2|v|^q + |\nabla \phi_2|^q + h_2|\phi_2|^q \right) \mu(x) \right)^{\frac{q-1}{q}} \\
&\quad \times \left( \sum_{x\in V} (|\nabla \phi_2|^q + h_2|\phi_2|^q) \mu(x) \right)^{1/q} \\
&+ \max_{|(s,t)|\le \|u\|_\infty + \|v\|_\infty} a|(s,t)| \|b(x)\|_{L^1(V)} \|\phi_1 + \phi_2\|_\infty \\
&+ \lambda_1 \|e_1\|_{L^{\frac{p}{p-1}}(V)} \|\phi_1\|_{L^p(V)} + \lambda_2 \|e_2\|_{L^{\frac{q}{q-1}}(V)} \|\phi_2\|_{L^q(V)} \\
=\ & 2^{\frac{(p-1)^2+1}{p}} \left( \|u\|^p_{W^{1,p}(V)} + \|\phi_1\|^p_{W^{1,p}(V)} \right)^{\frac{p-1}{p}} \|\phi_1\|_{W^{1,p}(V)} + \lambda_1 \|e_1\|_{L^{\frac{p}{p-1}}(V)} \|\phi_1\|_{L^p(V)} \\
&2^{\frac{(q-1)^2+1}{q}} \left( \|v\|^q_{W^{1,q}(V)} + \|\phi_2\|^q_{W^{1,q}(V)} \right)^{\frac{q-1}{q}} \|\phi_2\|_{W^{1,q}(V)} + \lambda_2 \|e_2\|_{L^{\frac{q}{q-1}}(V)} \|\phi_2\|_{L^q(V)} \\
&+ \max_{|(s,t)|\le \|u\|_\infty + \|v\|_\infty} a|(s,t)| \|b(x)\|_{L^1(V)} \|\phi_1 + \phi_2\|_\infty \\
\le\ & 2^{\frac{(p-1)^2+1}{p}} \left( \|u\|^{p-1}_{W^{1,p}(V)} + \|\phi_1\|^{p-1}_{W^{1,p}(V)} \right) \|\phi_1\|_{W^{1,p}(V)} + \lambda_1 \|e_1\|_{L^{\frac{p}{p-1}}(V)} \|\phi_1\|_{L^p(V)} \\
&+ 2^{\frac{(q-1)^2+1}{q}} \left( \|v\|^{q-1}_{W^{1,q}(V)} + \|\phi_2\|^{q-1}_{W^{1,q}(V)} \right) \|\phi_2\|_{W^{1,q}(V)} + \lambda_2 \|e_2\|_{L^{\frac{q}{q-1}}(V)} \|\phi_2\|_{L^q(V)} \\
&+ \max_{|(s,t)|\le \|u\|_\infty + \|v\|_\infty} a|(s,t)| \|b(x)\|_{L^1(V)} \|\phi_1 + \phi_2\|_\infty .
\end{aligned} \tag{A.4}
$$

So, we obtain that $\sum_{x\in V} \frac{\partial G_x(\theta)}{\partial \theta}$ is uniform convergence. Let $H(\theta) = \sum_{x\in V} \frac{\partial G_x(\theta)}{\partial \theta} = \varphi_\lambda(u + \theta\phi_1, v + \theta\phi_2)$. Then by (A.1)–(A.4), we have

$$
H'(0) = \left( \sum_{x\in V} \frac{\partial G_x(\theta)}{\partial \theta} \right) \Bigg|_{\theta=0}
$$

$$
\begin{aligned}
\text{(A.5)} \qquad &= \sum_{x\in V} \left[|\nabla u|^{p-2}\Gamma(u,\phi_1) + h_1|u|^{p-2}u\phi_1 - F_u(x,u,v)\phi_1 - \lambda_1 e_1\phi_1\right]\mu(x) \\
&\quad + \sum_{x\in V} \left[|\nabla v|^{q-2}\Gamma(v,\phi_2) + h_2|v|^{q-2}v\phi_2 - F_v(x,u,v)\phi_2 - \lambda_2 e_2\phi_2\right]\mu(x) \\
&= \langle \varphi'_\lambda(u,v),(\phi_1,\phi_2)\rangle.
\end{aligned}
$$

So, for any given $(\phi_1,\phi_2) \in W$, by (A.5) and (1.1), we have

$$
\begin{aligned}
&\langle \varphi'_\lambda(u,v),(\phi_1,\phi_2)\rangle \\
&\le \int_V \left[|\nabla u|^{p-1}|\nabla\phi_1| + h_1|u|^{p-2}u\phi_1 - F_u(x,u,v)\phi_1 - \lambda_1 e_1\phi_1\right] d\mu \\
&\quad + \int_V \left[|\nabla v|^{q-1}|\nabla\phi_2| + h_2|v|^{q-2}v\phi_2 - F_v(x,u,v)\phi_2 - \lambda_2 e_2\phi_2\right] d\mu \\
&\le \left(\int_V |\nabla u|^p\,d\mu\right)^{\frac{p-1}{p}} \left(\int_V |\nabla\phi_1|^p\,d\mu\right)^{1/p} + \left(\int_V h_1|u|^p\,d\mu\right)^{\frac{p-1}{p}} \left(\int_V h_1|\phi_1|^p\,d\mu\right)^{1/p} \\
&\quad + \left(\int_V |\nabla v|^q\,d\mu\right)^{\frac{q-1}{q}} \left(\int_V |\nabla\phi_2|^q\,d\mu\right)^{1/q} + \left(\int_V h_2|v|^q\,d\mu\right)^{\frac{q-1}{q}} \left(\int_V h_2|\phi_2|^q\,d\mu\right)^{1/q} \\
&\quad + \int_V \big(|F_u|\phi_1 + |F_v|\phi_2 + \lambda_1 e_1\theta\phi_1 + \lambda_2 e_2\theta\phi_2\big)\,d\mu \\
&\le \|u\|_{W^{1,p}(V)}^{p-1}\|\phi_1\|_{W^{1,p}(V)} + \|v\|_{W^{1,q}(V)}^{q-1}\|\phi_2\|_{W^{1,q}(V)} \\
&\quad + \int_V a(|(u,v)|)b(x)(\phi_1+\phi_2)\,d\mu + \int_V (\lambda_1 e_1\phi_1 + \lambda_2 e_2\phi_2)\,d\mu \\
&\le \|u\|_{W^{1,p}(V)}^{p-1}\|\phi_1\|_{W^{1,p}(V)} + \|v\|_{W^{1,q}(V)}^{q-1}\|\phi_2\|_{W^{1,q}(V)} \\
&\quad + \max_{|(s,t)|\le \|u\|_\infty + \|v\|_\infty} a|(s,t)|\|b\|_{L^1(V)}(\|\phi_1\|_\infty + \|\phi_2\|_\infty) \\
&\quad + \lambda_1 h_0^{-\frac{1}{p}}\|e_1\|_{L^{\frac{p}{p-1}}(V)}\|\phi_1\|_{W^{1,p}(V)} + \lambda_2 h_0^{-\frac{1}{q}}\|e_2\|_{L^{\frac{q}{q-1}}(V)}\|\phi_2\|_{W^{1,q}(V)} \\
&\le \max\left\{\|u\|_{W^{1,p}(V)}^{p-1}, \|v\|_{W^{1,q}(V)}^{q-1}\right\}\|(\phi_1,\phi_2)\|_W \\
&\quad + \max\left\{\frac{1}{(h_0\mu_0)^{1/p}}, \frac{1}{(h_0\mu_0)^{1/q}}\right\} \max_{|(s,t)|\le \|u\|_\infty + \|v\|_\infty} a|(s,t)|\|b\|_{L^1(V)}\|(\phi_1,\phi_2)\|_W \\
&\quad + \max\left\{\lambda_1 h_0^{-\frac{1}{p}}\|e_1\|_{L^{\frac{p}{p-1}}(V)}, \lambda_2 h_0^{-\frac{1}{q}}\|e_2\|_{L^{\frac{q}{q-1}}(V)}\right\}\|(\phi_1,\phi_2)\|_W.
\end{aligned}
$$

Thus, $\varphi'_\lambda(u,v)\colon W \to \mathbb{R}$ is bounded and linear operator, that is, $\varphi'_\lambda(u,v) \in W^*$ which is the dual space of $W$. Define the mapping $\varphi'_\lambda\colon W \to W^*$ by

$$\varphi'_\lambda\colon (u,v) \mapsto \varphi'_\lambda(u,v).$$

Next, we prove that $\varphi'_\lambda$ is continuous in $W$. For any sequence $\{(u_k,v_k)\} \subset W$ with

$(u_k, v_k) \to (u, v)$ in $W$ as $k \to \infty$, we have

$$\begin{aligned} &\int_V |\nabla(u_k - u)|^p \, d\mu \to 0, \qquad \int_V |(u_k - u)|^p \, d\mu \to 0, \\ &\int_V |\nabla(v_k - v)|^q \, d\mu \to 0, \qquad \int_V |(v_k - v)|^q \, d\mu \to 0 \end{aligned} \tag{A.6}$$

and by (2.1), we have

$$u_k(x) \to u(x), \quad v_k(x) \to v(x) \quad \text{for all } x \in V \text{ as } k \to \infty. \tag{A.7}$$

Note that

$$\begin{aligned} &\langle \varphi'_\lambda(u, v) - \varphi'_\lambda(u_k, v_k), (\phi_1, \phi_2) \rangle \\ &= \int_V \left[ |\nabla u|^{p-2}\Gamma(u, \phi_1) - |\nabla u_k|^{p-2}\Gamma(u_k, \phi_1) + h_1(|u|^{p-2}u - |u_k|^{p-2}u_k)\phi_1 \right] d\mu \\ &\quad + \int_V \left[ |\nabla v|^{q-2}\Gamma(v, \phi_2) - |\nabla v_k|^{q-2}\Gamma(v_k, \phi_2) + h_2(|v|^{q-2}v - |v_k|^{q-2}v_k)\phi_2 \right] d\mu \\ &\quad - \int_V (F_u(x, u, v) - F_{u_k}(x, u_k, v_k))\phi_1 \, d\mu - \int_V (F_v(x, u, v) - F_{v_k}(x, u_k, v_k))\phi_2 \, d\mu \\ &= \int_V |\nabla u|^{p-2}\Gamma(u - u_k, \phi_1) \, d\mu + \int_V \left( |\nabla u|^{p-2} - |\nabla u_k|^{p-2} \right)\Gamma(u_k, \phi_1) \, d\mu \\ &\quad + \int_V h_1(|u|^{p-2}u - |u_k|^{p-2}u_k)\phi_1 \, d\mu - \int_V (F_u(x, u, v) - F_{u_k}(x, u_k, v_k))\phi_1 \, d\mu \\ &\quad + \int_V |\nabla v|^{q-2}\Gamma(v - v_k, \phi_2) \, d\mu + \int_V (|\nabla v|^{q-2} - |\nabla v_k|^{q-2})\Gamma(v_k, \phi_2) \, d\mu \\ &\quad + \int_V h_2(|v|^{q-2}v - |v_k|^{q-2}v_k)\phi_2 \, d\mu - \int_V (F_v(x, u, v) - F_{v_k}(x, u_k, v_k))\phi_2 \, d\mu \\ &:= I + II. \end{aligned}$$

Firstly, we prove that

$$\begin{aligned} I &= \int_V |\nabla u|^{p-2}\Gamma(u - u_k, \phi_1) \, d\mu + \int_V (|\nabla u|^{p-2} - |\nabla u_k|^{p-2})\Gamma(u_k, \phi_1) \, d\mu \\ &\quad + \int_V h_1(|u|^{p-2}u - |u_k|^{p-2}u_k)\phi_1 \, d\mu - \int_V (F_u(x, u, v) - F_{u_k}(x, u_k, v_k))\phi_1 \, d\mu \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

By using Lemma 5.12 in [13], we have

$$\begin{aligned} &\int_V h_1(|u|^{p-2}u - |u_k|^{p-2}u_k)\phi_1 \, d\mu \\ &\le \left( \int_V h_1(|u|^{p-2}u - |u_k|^{p-2}u_k)^{\frac{p}{p-1}} \, d\mu \right)^{\frac{p-1}{p}} \left( \int_V h_1 |\phi_1|^p \, d\mu \right)^{1/p} \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned} \tag{A.8}$$

Similarly, by $(\mathrm{F}_0)$, Lebesgue dominated convergence theorem, (A.7) and the continuity of $F_u$, we also have

$$\begin{aligned}
&\int_V (F_u(x,u,v) - F_{u_k}(x,u_k,v_k))\phi_1\,d\mu \\
&\le \left(\int_V (F_u(x,u,v) - F_{u_k}(x,u_k,v_k))^{\frac{p}{p-1}}\,d\mu\right)^{\frac{p-1}{p}} \left(\int_V |\phi_1|^p\,d\mu\right)^{1/p} \\
&\to 0 \quad \text{as } k\to\infty.
\end{aligned} \tag{A.9}$$

Moreover, by Hölder inequality and (A.6), we get

$$\begin{aligned}
&\int_V |\nabla u|^{p-2}\Gamma(u-u_k,\phi_1)\,d\mu \\
&\le \int_V |\nabla u|^{p-2}|\nabla(u-u_k)|\cdot|\nabla\phi_1|\,d\mu \\
&\le \left(\int_V |\nabla(u-u_k)|^p\,d\mu\right)^{1/p}\left(\int_V |\nabla u|^{\frac{(p-2)p}{p-1}}\cdot|\nabla\phi_1|^{\frac{p}{p-1}}\,d\mu\right)^{\frac{p-1}{p}} \\
&\le \left(\int_V |\nabla(u-u_k)|^p\,d\mu\right)^{1/p}\left(\int_V |\nabla u|^p\,d\mu\right)^{\frac{p-2}{p}}\left(\int_V |\nabla\phi_1|^p\,d\mu\right)^{1/p} \\
&\to 0 \quad \text{as } k\to\infty,
\end{aligned} \tag{A.10}$$

and

$$\begin{aligned}
&\int_V (|\nabla u|^{p-2} - |\nabla u_k|^{p-2})\Gamma(u_k,\phi_1)\,d\mu \\
&\le \int_V \big||\nabla u|^{p-2} - |\nabla u_k|^{p-2}\big||\nabla u_k|\cdot|\nabla\phi_1|\,d\mu \\
&= \int_V \big||\nabla u|^{p-2}|\nabla u_k| - |\nabla u_k|^{p-1}\big||\nabla\phi_1|\,d\mu \\
&= \int_V \big||\nabla u|^{p-2}\big[|\nabla u| + (|\nabla u_k| - |\nabla u|)\big] - |\nabla u_k|^{p-1}\big||\nabla\phi_1|\,d\mu \\
&= \int_V \big||\nabla u|^{p-1} - |\nabla u_k|^{p-1}\big||\nabla\phi_1|\,d\mu + \int_V |\nabla u|^{p-2}\big||\nabla u_k| - |\nabla u|\big||\nabla\phi_1|\,d\mu \\
&\le \left(\int_V \big||\nabla u|^{p-1} - |\nabla u_k|^{p-1}\big|^{\frac{p}{p-1}}\,d\mu\right)^{\frac{p-1}{p}}\left(\int_V |\nabla\phi_1|^p\,d\mu\right)^{1/p} \\
&\quad + \left(\int_V |\nabla u|^{\frac{(p-2)p}{p-1}}\big||\nabla u_k| - |\nabla u|\big|^{\frac{p}{p-1}}\,d\mu\right)^{\frac{p-1}{p}}\left(\int_V |\nabla\phi_1|^p\,d\mu\right)^{1/p} \\
&\le (p-1)\left(\int_V |\nabla\phi_1|^p\,d\mu\right)^{1/p} \\
&\qquad \times \left(\int_V \big||\nabla u| - |\nabla u_k|\big|^{\frac{p}{p-1}}\big(|\nabla u_k|^{p-2} + |\nabla u|^{p-2}\big)^{\frac{p}{p-1}}\,d\mu\right)^{\frac{p-1}{p}}
\end{aligned} \tag{A.11}$$

$$
\begin{aligned}
&\quad + \left( \int_V |\nabla \phi_1|^p \, d\mu \right)^{1/p} \left( \int_V |\nabla u|^p \, d\mu \right)^{\frac{p-2}{p}} \left( \int_V \big| |\nabla u_k| - |\nabla u| \big|^p \, d\mu \right)^{1/p} \\
&\le (p-1) \left( \int_V |\nabla \phi_1|^p \, d\mu \right)^{1/p} \left( \int_V |\nabla (u - u_k)|^p \, d\mu \right)^{1/p} \\
&\quad \times \left( \int_V (|\nabla u_k|^{p-2} + |\nabla u|^{p-2})^{\frac{p}{p-2}} \, d\mu \right)^{\frac{p-2}{p}} \\
&\quad + \left( \int_V |\nabla \phi_1|^p \, d\mu \right)^{1/p} \left( \int_V |\nabla u|^p \, d\mu \right)^{\frac{p-2}{p}} \left( \int_V |\nabla (u_k - u)|^p \, d\mu \right)^{1/p} \\
&\le 2^{\frac{2}{p}} (p-1) \left( \int_V |\nabla \phi_1|^p \, d\mu \right)^{1/p} \left( \int_V |\nabla (u - u_k)|^p \, d\mu \right)^{1/p} \\
&\quad \times \left( \int_V (|\nabla u_k|^p + |\nabla u|^p) \, d\mu \right)^{\frac{p-2}{p}} \\
&\quad + \left( \int_V |\nabla \phi_1|^p \, d\mu \right)^{1/p} \left( \int_V |\nabla u|^p \, d\mu \right)^{\frac{p-2}{p}} \left( \int_V |\nabla (u_k - u)|^p \, d\mu \right)^{1/p} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}
$$

So, by (A.8)–(A.11), we obtain that

$$I \to 0 \quad \text{as } k \to \infty.$$

Similarly, we can prove that

$$II \to 0 \quad \text{as } k \to \infty.$$

Hence,

$$\langle \varphi_\lambda'(u,v) - \varphi_\lambda'(u_k, v_k), (\phi_1, \phi_2) \rangle \to 0 \quad \text{as } k \to \infty.$$

Then $\varphi_\lambda'$ is continuous. □

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