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Chain Recurrence Rates and Topological Entropy of Free Semigroup Actions

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Abstract. In this paper, we first introduce the pseudo-entropy of free semigroup actions and show that it is equal to the topological entropy of free semigroup actions defined by Bufetov [9]. Second, for free semigroup actions, the concepts of chain recurrence and chain recurrence time, chain mixing and chain mixing time are introduced, and upper bounds for these recurrence times are calculated. Furthermore, the lower box dimension and the chain mixing time provide a lower bound on topological entropy of free semigroup actions. Third, the structure of chain transitive systems of free semigroup actions is discussed. Our analysis generalizes the results obtained by Misiurewicz [21], Richeson and Wiseman [23], and Bufetov [9] etc.

1. Introduction

Topological entropy was first introduced by Adler et al. [1]. Bowen [8] and Dinaburg [13] extended the topological entropy to a uniformly continuous map on metric space and proved that it coincides with that defined by Adler et al. for a compact metric space. The topological entropy turned out to be a surprisingly universal concept in dynamical systems since it appears in the study of different subjects such as fractal, Poincaré recurrence, and in the analysis of either local or global complexities. Pseudo-orbits, or chains, have always been one of the significant tools for studying the topological entropy. In the past 40 years, a large number of scholars have used pseudo-orbits or chains as tools to study topological entropy, and have obtained some excellent results, for examples $[4, 6, 17, 21, 23, 28]$. In particular, Misiurewicz [21] stated that the topological entropy can be estimated by the exponential growth rate of the number of pseudo-orbits. Barge and Swanson [4] further found that replacing pseudo-orbits with periodic pseudo-orbits, and the result proved by Misiurewicz [21] was still valid. In [17], Hurley established relations between topological entropy, preimage relation entropy, preimage branch entropy and point entropy. Taking the topological entropy defined by Misiurewicz [21] as a bridge, Richeson and Wiseman [23] related the chain mixing time and the lower box dimension to topological entropy and obtained a lower bound of topological entropy.

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Akin [2, Exercises 8.22 and 9.18] initially discussed the structure of the chain transitive maps, a map of chain transitive but not chain mixing factors a cyclic permutation on a finite set with at least two elements. Richeson and Wiseman [23] enriched the result of Akin [2] and filled in the gaps in the proofs sketched. They obtained the structural theorem of chain transitive dynamical systems, that is, if f is a chain transitive map on a compact metric space either then there is a period $k \geq 1$ such that f cyclically permutes k closed and open equivalence classes and f^k restricted to each equivalence class is chain mixing; or f factors onto an adding machine map.

People have become increasingly concerned with the research of free semigroup actions in recent years. On the one hand, it is required by some other disciplines, such as physics, that the system that describes what really happened be given time to readjust in order to take into account the inescapable experimental errors in [19]. Some dynamic system theories, on the other hand, are closely related to it, such as the case of a foliation on a manifold and a pseudo-group of holonomy maps. The geometric entropy of finitely generated pseudogroup has been introduced [15] and shown to be a useful tool for studying the topology and dynamics of foliated manifolds. Bufetov [9] introduced the topological entropy of free semigroup actions and a topological analog of the classical Abramov–Rokhlin formula for the entropy of a skew-product transformation was obtained. Many remarkable results have been obtained [10–12, 15, 18, 20].

Naturally, we wonder if the results of Misiurewicz [21] and Richeson and Wiseman [23] remain valid in the case of free semigroup actions. Notice that some concepts such as pseudo-entropy, chain recurrence rates etc. are vacant in dynamical systems of free semigroup actions. First of all, we need to introduce the pseudo-entropy of free semigroup actions and further consider whether it is equal to the topological entropy defined by Bufetov [9]. Secondly, we intend to introduce the chain recurrence time and chain mixing time of free semigroup actions, and then consider their basic properties and compute the lower bound of the topological entropy. Finally, we focus on whether the chain transitive theorem holds in dynamical systems of free semigroup actions. If the theorem holds, we gain a deeper understanding of chain transitive dynamical systems of free semigroup actions.

Now we start to state our main results. Let (X, G) be a dynamical system where X is a compact metric space and G is the free semigroup acting on the space X generated by $\{f_0, \ldots, f_{m-1}\}$, where f_i is a continuous self-map on X for all $i = 0, \ldots, m-1$. Firstly, we show that the topological entropy $h(G)$ is equal to the pseudo-entropy $h^*(G)$ of free semigroup actions.

Theorem 1.1. *Let (X, G) be a dynamical system. Then*

$$h(G) = h^*(G).$$

Secondly, we show that under special conditions, there is a quantitative relationship between the upper box dimension of the space and the chain recurrence (mixing) time. Let $r_\varepsilon(G)$ and $m_\varepsilon(\delta, G)$ be the chain recurrence time and the chain mixing time of the free semigroup action G, more precisely in Section 4.

Theorem 1.2. *Let $\overline{\dim}_B X$ be the upper box dimension of X. There exists a constant $C > 0$ such that for small enough $\varepsilon > 0$:*

(1) *if G is chain transitive, then $r_\varepsilon(G) \le C/\varepsilon^{\overline{\dim}_B X+1}$;*

(2) *if G is chain mixing, then $m_\varepsilon(\delta, G) \le C/\varepsilon^{2(\overline{\dim}_B X+1)}$.*

In addition, we obtain a lower bound on the topological entropy of free semigroup actions using Theorem 1.1.

Theorem 1.3. *Let (X, G) be chain mixing. Then the topological entropy $h(G)$ satisfies*

$$h(G) \ge \max\left\{0,\ \underline{\dim}_B X \cdot \limsup_{\delta\to 0} \frac{\log(1/\delta)}{\lim_{\varepsilon\to 0} m_\varepsilon(\delta)} - \log m\right\},$$

where $\underline{\dim}_B X$ is the lower box dimension of X.

Finally, we describe the structure of chain transitive dynamical systems.

Theorem 1.4. *Let (X, G) be chain transitive. Then either*

(1) *There is a period $k \ge 1$, such that G cyclically permutes k closed and open equivalence classes of X, and G^k restricted to each equivalence classes is chain mixing; or*

(2) *G factors onto an adding machine map.*

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we define the pseudo-entropy of free semigroup actions and prove Theorem 1.1. In Section 4, we introduce these concepts of the chain recurrence, the chain mixing, the chain recurrence time, and the chain mixing time of free semigroup actions, and prove Theorem 1.2. Furthermore, Theorem 1.3 is obtained by means of the definition of the pseudo-entropy of free semigroup actions. In Section 5, we discuss the structure of chain transitive systems and prove Theorem 1.4. Our analysis generalizes the results obtained by Misiurewicz [21], Bufetov [9] and Richeson et al. [23].

2. Preliminaries

Let (X, d) be a compact metric space and f be a continuous map on X. A *δ-pseudo-orbit* is an infinite sequence $\{x_i\}_{i=0}^{\infty}$ such that $d(f(x_i), x_{i+1}) \le \delta$ for $i \ge 0$. We say that f has the *pseudo-orbit tracing property* if for $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that each δ_ε-pseudo-orbit can be ε-shadowed, that is, if $\{x_i\}_{i=0}^{\infty}$ is a δ-pseudo-orbit, then there exists $z \in X$ such that $d(f^i(z), x_i) < \varepsilon$ for all $i \geq 0$.

We recall the definition of pseudo-entropy of f. The first one is due to Misiurewicz [21]. Say a collection E of δ-pseudo-orbits of f is (n, ε, δ)-separated if, for each $\{x_i\}_{i=0}^{\infty}, \{y_i\}_{i=0}^{\infty} \in E$, $\{x_i\}_{i=0}^{\infty} \neq \{y_i\}_{i=0}^{\infty}$, there is a k, $0 \leq k \leq n-1$, for which $d(x_k, y_k) \geq \varepsilon$. Denote by $s(n, \varepsilon, \delta)$ the maximal cardinality of an (n, ε, δ)-separated set of f.

A collection K of δ-pseudo-orbits of f is (n, ε, δ)-spanning if for each δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$, there is $\{y_i\}_{i=0}^{\infty} \in K$ such that $d(x_i, y_i) < \varepsilon$ for all $0 \leq i \leq n-1$. The minimum cardinality of an (n, ε, δ)-spanning set of f is denoted by $r(n, \varepsilon, \delta)$.

Let
$$h^*(f, \varepsilon, \delta) = \limsup_{n\to\infty} \frac{1}{n} \log s(n, \varepsilon, \delta), \quad h^*(f, \varepsilon) = \lim_{\delta \to 0} h^*(f, \varepsilon, \delta),$$
and
$$h^*(f) := \lim_{\varepsilon\to 0} h^*(f, \varepsilon).$$
The number $h^*(f)$ is called the *pseudo-entropy* of f. Obviously,
$$r(n, \varepsilon/2, \delta) \geq s(n, \varepsilon, \delta) \geq r(n, \varepsilon, \delta).$$
Thus,
$$h^*(f) = \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r(n, \varepsilon, \delta).$$

Theorem 2.1. [21] *Let (X, f) be a dynamical system. Then*
$$h(f) = h^*(f).$$

Let F_m^+ be the set of all finite words of symbols $0, 1, \ldots, m-1$. For any $w \in F_m^+$, $|w|$ stands for the length of w, that is, the digits of symbols in w. Obviously, F_m^+ with respect to the law of composition is a free semigroup with m generators. We write $w' \leq w$ if there exists a word $w'' \in F_m^+$ such that $w = w''w'$. For $w = i_0 i_1 \ldots i_k \in F_m^+$, denote $\overline{w} = i_k \ldots i_1 i_0$.

Denote by Σ_m^+ the set of all one-side infinite sequences of symbols $0, 1, \ldots, m-1$, that is,
$$\Sigma_m^+ = \big\{\omega = (i_0 i_1 \ldots) \mid i_k = 0, 1, \ldots, m-1, k \in \mathbb{N}_0\big\}.$$
The metric on Σ_m^+ is given by
$$d'(\omega, \omega') = \frac{1}{m^k}, \quad \text{where } k = \inf\{n \mid i_n \neq i'_n\}.$$
Obviously, Σ_m^+ is compact with respect to this metric. The shift $\sigma\colon \Sigma_m^+ \to \Sigma_m^+$ is given by the formula, for each $\omega = (i_0 i_1 \ldots) \in \Sigma_m^+$,
$$\sigma(\omega) = (i_1 i_2 \ldots).$$

Suppose that $\omega \in \Sigma_m^+$, and $a, b \in \mathbb{N}$ with $a \le b$. We write $\omega|_{[a,b]} = w$ if $w = i_a i_{a+1} \dots i_b$.

Let G be a free semigroup generated by m generators $f_0, f_1, \dots, f_{m-1}$ which are continuous self-maps on X, denoted as $G := \{f_0, f_1, \dots, f_{m-1}\}$. To each $w \in F_m^+$, $w = i_0 i_1 \dots i_{k-1}$, let $f_w = f_{i_0} \circ f_{i_1} \circ \cdots \circ f_{i_{k-1}}$ if $k > 0$, and $f_w = \mathrm{Id}$ if $k = 0$, where Id is the identity map. Obviously, $f_{ww'} = f_w f_{w'}$ and $f_{\overline{w}} = f_{i_{k-1}} \circ \cdots \circ f_{i_1} \circ f_{i_0}$. We assign a metric d_w on X by setting

$$\begin{aligned} d_w(x_1, x_2) &:= \max\left\{d(f_{i_{j-1}\dots i_0}(x), f_{i_{j-1}\dots i_0}(y)) : j = 0, 1, \dots, k\right\} \\ &= \max_{w' \le \overline{w}} d(f_{w'}(x_1), f_{w'}(x_2)). \end{aligned}$$

A subset B of X is called a (w, ε, G)-*spanning* subset if, for any $x \in X$, there exists $y \in B$ with $d_w(x, y) < \varepsilon$. The minimum cardinality of a (w, ε, G)-spanning subset of X is denoted by $B(w, \varepsilon, G)$.

A subset K of X is called a (w, ε, G)-*separated* subset if, for any $x_1, x_2 \in K$ with $x_1 \neq x_2$, one has $d_w(x_1, x_2) \ge \varepsilon$. The maximum cardinality of a (w, ε, G)-separated subset of X is denoted by $N(w, \varepsilon, G)$. Let

$$B(n, \varepsilon, G) = \frac{1}{m^n} \sum_{|w|=n} B(w, \varepsilon, G), \quad N(n, \varepsilon, G) = \frac{1}{m^n} \sum_{|w|=n} N(w, \varepsilon, G).$$

In [9], the author introduced the topological entropy of free semigroup actions. The topological entropy of free semigroup actions is defined by the formula

$$\begin{aligned} h(G) &:= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log B(n, \varepsilon, G) \\ &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, G). \end{aligned}$$

Remark 2.2. If $m = 1$, this definition coincides with the topological entropy of a single map defined by [8]. For more information, see Chapter 7 of [25].

The dynamical systems of free semigroup actions have a strong connection with skew-products which has been analyzed to obtain properties of free semigroup actions through fiber associated with the skew-product (see for instance [29]). Recall that the skew-product transformation is given as follows:

$$F \colon \Sigma_m^+ \times X \to \Sigma_m^+ \times X, \quad (\omega, x) \mapsto (\sigma(\omega), f_{i_0}(x)),$$

where $\omega = (i_0 i_1 \dots)$ and σ is the shift map of Σ_m^+. The metric $d_{\Sigma_m^+ \times X}$ on $\Sigma_m^+ \times X$ is given by the formula

$$d_{\Sigma_m^+ \times X}((\omega, x), (\omega', x')) = \max\{d'(\omega, \omega'), d(x, x')\}.$$

Theorem 2.3. [9] *Topological entropy of the skew-product transformation F satisfies*

$$h(F) = \log m + h(G).$$

We recall the definitions of box dimension more precisely in [14]. Let E be a non-empty subset of X. Let $N_\delta(E)$ be the smallest number of sets of diameter at most δ which can cover E. The *lower* and *upper box dimensions* of E respectively are defined as

$$\underline{\dim}_B E := \liminf_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta} \quad \text{and} \quad \overline{\dim}_B E := \limsup_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta}.$$

3. The pseudo-entropy of free semigroup actions

In this section, we will introduce the concept of pseudo-entropy of free semigroup actions and prove Theorem 1.1.

According to Bahabadi [3], recall that for $w = i_0 \dots i_{n-1} \in F_m^+$, a (w,δ)-*chain* (or (w,δ)-pseudo-orbit) of G from x to y is a sequence $(x_0 = x, x_1, \dots, x_n = y)$ such that $d(f_{i_j}(x_j), x_{j+1}) \le \delta$ for $j = 0, \dots, n-1$. To simplify notation, we sometimes write $(x_j)_{j=0}^n$. For $w \in F_m^+$ with $|w| = n$, denote by $E(w,\delta)$ the set of all (w,δ)-chain of G.

Similar to Misiurewicz [21], we mimic this definition of Bufetov [9] by pseudo-orbits to introduce the following definitions of free semigroup actions. A collection K of (w,δ)-chain of G is called a (w,ε,δ,G)-*pseudo-separated set* of X if, for any $(x_0,\dots,x_n),(y_0,\dots,y_n) \in K$, $(x_0,\dots,x_n) \ne (y_0,\dots,y_n)$, there exists $0 \le i < n$, such that $d(x_i,y_i) > \varepsilon$. The maximum cardinality of a (w,ε,δ,G)-pseudo-separated set of X is denoted by $N^*(w,\varepsilon,\delta,G)$.

A collection B of (w,δ)-chain of G is called a (w,ε,δ,G)-*pseudo-spanning set* of X if, for any (w,δ)-chain $(x_0,\dots,x_n)$, there is $(y_0,\dots,y_n) \in B$, such that $d(x_i,y_i) \le \varepsilon$ for every $0 \le i < n$. The minimum cardinality of a (w,ε,δ,G)-pseudo-spanning set of X is denoted by $B^*(w,\varepsilon,\delta,G)$.

Let

$$B^*(n,\varepsilon,\delta,G) = \frac{1}{m^n}\sum_{|w|=n} B^*(w,\varepsilon,\delta,G), \quad N^*(n,\varepsilon,\delta,G) = \frac{1}{m^n}\sum_{|w|=n} N^*(w,\varepsilon,\delta,G).$$

Obviously,

$$B^*(w,\varepsilon/2,\delta,G) \ge N^*(w,\varepsilon,\delta,G) \ge B^*(w,\varepsilon,\delta,G),$$

whence

$$B^*(n,\varepsilon/2,\delta,G) \ge N^*(n,\varepsilon,\delta,G) \ge B^*(n,\varepsilon,\delta,G).$$

Now let,

$$h^*(\varepsilon,\delta,G) := \limsup_{n\to\infty} \frac{1}{n}\log N^*(n,\varepsilon,\delta,G), \quad h^*(\varepsilon,G) := \lim_{\delta\to 0} h^*(\varepsilon,\delta,G),$$

and

$$h^*(G) := \lim_{\varepsilon\to 0} h^*(\varepsilon,G).$$

Definition 3.1. The number $h^*(G)$ is called pseudo-entropy of the free semigroup action G.

It easily follows that

$$h^*(G) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log B^*(n,\varepsilon,\delta,G).$$

Remark 3.2. (1) If $G=\{f\}$, it is clear that $h^*(G)=h^*(f)$, where $h^*(f)$ is the pseudo-entropy of f defined by Misiurewicz [21].

(2) Recall that the topological entropy of free semigroup actions introduced by Biś [5]. Two points $x,y\in X$ are called (n,ε)-separated by G if there exists $w\in F_m^+$ with $|w|\le n$ such that $d(f_w(x),f_w(y))\ge\varepsilon$. Denote by $s(n,\epsilon,X)$ the maximum cardinality of (n,ε)-separated set by G. Biś [5] called the quantity $h^B(G)$

$$h^B(G) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s(n,\epsilon,X),$$

the topological entropy of free semigroup action G. It is easy to check that $h(G)\le h^B(G)$, where $h(G)$ is topological entropy of G introduced by Bufetov [9]. Example 5.7 of [26] showed that the preceding inequality may be strictly true. In [6], the pseudo-entropy was generalized to a case of finitely generated pseudo-group, and it corresponds to the pseudo-entropy of the free semigroup action G generated by $f_0,\dots,f_{m-1}$ described in [5], as follows. A map $x\colon G\to X$ is called a δ-pseudo-orbit of G if, for any $h\in\{f_0,\dots,f_{m-1}\}$ and any $g\in G$, $d(h(x(g)),x(h\circ g))\le\delta$. Two δ-pseudo-orbits x and y are (n,ε)-separated if $d(x(f_w),y(f_w))\ge\varepsilon$ for some $w\in F_m^+$ with $|w|\le n$. Denote by $N_\delta(n,\varepsilon)$ the maximal cardinality of (n,ε)-separated set of the set of all δ-pseudo-orbit of G. Let

$$h_{\mathrm{ps}}(G) := \lim_{\varepsilon\to 0}\inf_{\delta}\limsup_{n\to\infty}\frac{1}{n}\log N_{\delta_n}(n,\varepsilon),$$

where the infimum is taken over all sequences $\delta=\{\delta_n\}$ such that $0<\delta_n\searrow 0$. The quantity $h_{\mathrm{ps}}(G)$ is called the pseudo-entropy of free semigroup actions in [5, 6]. It follows from [5, 6] that

$$h^B(G) = h_{\mathrm{ps}}(G).$$

Hence, we get that the pseudo-entropy of free semigroup actions introduced in [5,6] differs from Definition 3.1. It follows from Theorem 1.1 that

$$h^*(G)\le h_{\mathrm{ps}}(G).$$

Next, we will prove Theorem 1.1. In fact, it is enough to show that

$$h^*(F) = \log m + h^*(G)$$

by Theorems 2.1 and 2.3. To this end, we adopt the method of Bufetov [9]. In fact, the dynamical system of free semigroup actions is more complex than the classical dynamical system of a single map. The obstacles in the proof of Theorem 1.1 are mainly, when proving that the set H we construct in Lemma 3.4 is an (n,ε,δ,F)-pseudo-spanning set of $\Sigma_m^+\times X$, we use the fact that (Σ_m^+,σ) has pseudo-orbit tracing property to constrain the gap of pseudo-orbit, which give us the result we desired. We obtain the following two lemmas.

Lemma 3.3. *For any natural number $n\in\mathbb{N}$ and $0<\varepsilon,\delta<1/2$,*

$$N^*(n,\varepsilon,\delta,F)\geq\sum_{|w|=n}N^*(w,\varepsilon,\delta,G).$$

Proof. Let $N=m^n$, there are N distinct words of length n in F_m^+. Denote these words by $w^{(1)},\ldots,w^{(N)}$. For any $i=1,\ldots,N$, let $\omega(i)\in\Sigma_m^+$ be an arbitrary sequence such that $\omega(i)|_{[0,n-1]}=w^{(i)}$. Suppose that B_i is a $(w^{(i)},\varepsilon,\delta,G)$-pseudo-separated set of maximum cardinality of X for all $1\leq i\leq N$. For any $(x_0^{(i)},\ldots,x_n^{(i)})\in B_i$, consider that

$$\big((\omega(i),x_0^{(i)}),(\sigma(\omega(i)),x_1^{(i)}),\ldots,(\sigma^{n-1}(\omega(i)),x_{n-1}^{(i)}),(\sigma^n(\omega(i)),x_n^{(i)})\big).$$

It clear that it is an (n,δ)-chain of F as $(x_0^{(i)},\ldots,x_n^{(i)})$ is a $(w^{(i)},\delta)$-chain of G.

Put

$$K:=\big\{\big((\sigma^k(\omega(i)),x_k^{(i)})\big)_{k=0}^n\mid(x_0^{(i)},\ldots,x_n^{(i)})\in B_i,1\leq i\leq N\big\}.$$

We claim that K forms an (n,ε,δ,F)-pseudo-separated set of $\Sigma_m^+\times X$. Indeed, it suffices to check that these (n,δ)-chains of F determined by B_i and B_j are ε-separated where $i\neq j$ and $1\leq i,j\leq N$. For any $(x_0^{(i)},\ldots,x_n^{(i)})\in B_i$ and $(x_0^{(j)},\ldots,x_n^{(j)})\in B_j$, we have

$$\big((\sigma^k(\omega(i)),x_k^{(i)})\big)_{k=0}^n,\big((\sigma^k(\omega(j)),x_k^{(j)})\big)_{k=0}^n\in K.$$

Since $\omega(i)|_{[0,n-1]}=w^{(i)}$, $\omega(j)|_{[0,n-1]}=w^{(j)}$ and $w^{(i)}\neq w^{(j)}$, then $w_k^{(i)}\neq w_k^{(j)}$ for some $0\leq k\leq n-1$, this gives us $d'(\sigma^k(\omega(i)),\sigma^k(\omega(j)))=1>\varepsilon$. Therefore K is an (n,ε,δ,F)-pseudo-separated set of $\Sigma_m^+\times X$. The lemma is proved. □

Lemma 3.4. *For any $\varepsilon>0$, there is some $\delta_\varepsilon>0$, for any $0<\delta<\delta_\varepsilon$ and $n\in\mathbb{N}$, we have*

$$B^*(n,\varepsilon,\delta,F)\leq K(\varepsilon)\left(\sum_{|w|=n}B^*(w,\varepsilon,\delta,G)\right),$$

where $K(\varepsilon)$ is a positive constant that depends only on ε.

Proof. For any $\varepsilon>0$, let $C(\varepsilon)$ be a minimum positive integer such that $m^{-C(\varepsilon)}<\varepsilon$. Let $N=m^{n+C(\varepsilon)}$, there are N distinct words of length $n+C(\varepsilon)$ in F_m^+. Denote these words by $w^{(1)},\dots,w^{(N)}$. For any $i=1,\dots,N$, let $\omega(i)\in\Sigma_m^+$ be an arbitrary sequence such that $\omega(i)|_{[0,n+C(\varepsilon)-1]}=w^{(i)}$. Since (Σ_m^+,σ) has pseudo-orbit tracing property, there is $\delta_\varepsilon>0$, for any $0<\delta<\delta_\varepsilon$, such that each δ-pseudo-orbit of σ can be ε-shadowed. Suppose that E_i is $(\omega(i)|_{[0,n-1]},\varepsilon,\delta,G)$-pseudo-spanning set of minimum cardinality of X for all $i=1,\dots,N$. For any $(\omega(i)|_{[0,n-1]},\delta)$-chain $(y_0^{(i)},\dots,y_n^{(i)})$ in E_i, we can construct an (n,δ)-chain of F similar to Lemma 3.3, that is,

$$\big((\omega(i),y_0^{(i)}),(\sigma(\omega(i)),y_1^{(i)}),\dots,(\sigma^{n-1}(\omega(i)),y_{n-1}^{(i)}),(\sigma^n(\omega(i)),y_n^{(i)})\big).$$

Put

$$H:=\big\{\big((\sigma^k(\omega(i)),y_k^{(i)})\big)_{k=0}^n \mid (y_0^{(i)},\dots,y_n^{(i)})\in E_i, 1\le i\le N\big\}.$$

We claim that H forms an (n,ε,δ,F)-pseudo-spanning set of $\Sigma_m^+\times X$. Indeed, suppose now that $((\omega^{(0)},x^{(0)}),\dots,(\omega^{(n)},x^{(n)}))$ is an (n,δ)-chain of F where $\omega^{(k)}=(i_0^{(k)}i_1^{(k)}\dots)\in\Sigma_m^+$ for every $k=0,\dots,n$. Clearly, $(\omega^{(0)},\omega^{(1)},\dots,\omega^{(n)})$ is an (n,δ)-chain of σ, by the pseudo-orbit tracing property of (Σ_m^+,σ), this implies that there is an $\omega\in\Sigma_m^+$ such that $d'(\sigma^k(\omega),\omega^{(k)})<\varepsilon$ for all $0\le k\le n$. This yields that $\sigma^k(\omega)|_{[0,C(\varepsilon)-1]}=\omega^{(k)}|_{[0,C(\varepsilon)-1]}$. Moreover, we have $\omega|_{[0,n-1]}=i_0^{(0)}i_0^{(1)}\dots i_0^{(n-1)}$. It is clear to see that $\omega|_{[0,C(\varepsilon)+n-1]}=\omega(i)|_{[0,C(\varepsilon)+n-1]}=w^{(i)}$ for some $1\le i\le N$, this implies

$$\sigma^k(\omega)|_{[0,C(\varepsilon)-1]}=\sigma^k(\omega(i))|_{[0,C(\varepsilon)-1]},$$

and hence $\sigma^k(\omega(i))|_{[0,C(\varepsilon)-1]}=\omega^{(k)}|_{[0,C(\varepsilon)-1]}$ for all $0\le k\le n$. Since $(x^{(0)},x^{(1)},\dots,x^{(n)})$ is an $(i_0^{(0)}i_0^{(1)}\dots i_0^{(n-1)},\delta)$-chain of G and

$$\omega(i)|_{[0,n-1]}=\omega|_{[0,n-1]}=i_0^{(0)}i_0^{(1)}\dots i_0^{(n-1)},$$

this gives us $(x^{(0)},x^{(1)},\dots,x^{(n)})$ is an $(\omega(i)|_{[0,n-1]},\delta)$-chain of G. Therefore, there is $(y_0^{(i)},y_1^{(i)},\dots,y_n^{(i)})$ in E_i, such that $d(y_k^{(i)},x^{(k)})<\varepsilon$ for each $k=0,\dots,n-1$. As $(y_0^{(i)},y_1^{(i)},\dots,y_n^{(i)})$ is an $(\omega(i)|_{[0,n-1]},\delta)$-chain of G, we deduce that there exists an (n,δ)-chain of F, that is,

$$\big((\omega(i),y_0^{(i)}),(\sigma(\omega(i)),y_1^{(i)}),\dots,(\sigma^{n-1}(\omega(i)),y_{n-1}^{(i)}),(\sigma^n(\omega(i)),y_n^{(i)})\big)\in H$$

such that

$$d_{\Sigma_m^+\times X}\big((\sigma^k(\omega(i)),y_k^{(i)}),(\omega^{(k)},x^{(k)})\big)<\varepsilon$$

for all $0\le k\le n$. Consequently, H is an (n,ε,δ,F)-pseudo-spanning set of $\Sigma_m^+\times X$. The number of H is not greater than $K(\varepsilon)\big(\sum_{|w|=n}B^*(w,\varepsilon,\delta,G)\big)$, where $K(\varepsilon)$ is a positive constant that depends only on ε. The lemma is proved. □

Now, we can obtain immediately Theorem 1.1.

Proof of Theorem 1.1. From Lemma 3.3 we have

$$N^*(n, \varepsilon, \delta, F) \geq \sum_{|w|=n} N^*(w, \varepsilon, \delta, G),$$

whence, taking logarithms and limits, we obtain that

$$h^*(F) \geq \log m + h^*(G).$$

In this way, from Lemma 3.4, we have

$$B^*(n, \varepsilon, \delta, F) \leq K(\varepsilon) m^n B^*(n, \varepsilon, \delta, G),$$

whence

$$h^*(F) \leq \log m + h^*(G).$$

Combining these two inequalities we find that

$$h^*(F) = \log m + h^*(G).$$

We conclude by Theorems 2.1 and 2.3 that

$$h(G) = h^*(G).$$

This finishes the proof of the theorem. □

4. Chain recurrence rates and topological entropy

In this section, we mainly introduce these concepts of the chain recurrence and the chain mixing, the chain recurrence time and the chain mixing time of free semigroup actions and discuss some propositions of these notions, and prove Theorems 1.2 and 1.3.

Let (X, d) be a compact metric space and G be a free semigroup generated by m generators $f_0, f_1, \ldots, f_{m-1}$ which are continuous maps on X. We define the chain recurrence of free semigroup actions as follows:

Definition 4.1. One says that $x \in X$ is the chain recurrence point of G if for every $\varepsilon > 0$, there is a (w, ε)-chain from x to itself for some $w \in F_m^+$. We say that G is chain recurrence if every point of X is chain recurrent.

Chain transitive and totally chain transitive of free semigroup actions were introduced by [3] and [16], respectively. Recall that G is *chain transitive* if for every $\varepsilon > 0$ and any $x, y \in X$, there is a (w, ε)-chain from x to y for some $w \in F_m^+$. G is *totally chain transitive* if G^k is chain transitive for all $k \geq 1$. According to [23], we now may define the chain mixing of free semigroup actions as follows:

Definition 4.2. G is said to be ε-chain mixing if there is an $N > 0$ such that for any $x, y \in X$ and any $n \geq N$, there is a (w, ε)-chain from x to y for some $w \in F_m^+$ with $|w| = n$. G is called chain mixing if it admits ε-chain mixing for every $\varepsilon > 0$.

Remark 4.3. Since X is compact, we can obtain an equivalent statement that G is chain mixing as for any $\varepsilon > 0$ and $x, y \in X$, there is an $N > 0$ such that for any $n \geq N$, there is a (w, ε)-chain from x to y for some $w \in F_m^+$ with $|w| = n$.

If x is a chain recurrence point, define the *ε-chain recurrence time* $r_\varepsilon(x, G)$ to be the smallest n such that there is a (w, ε)-chain from x to itself for some $w \in F_m^+$ with $|w| = n$. If G is chain recurrent, define $r_\varepsilon(G)$ to be the maximum over all x of $r_\varepsilon(x, G)$. To see that this maximum exists, observe that if there is a (w, ε)-chain from x to itself for some $w \in F_m^+$, there is a neighborhood U of x such that for all $y \in U$, there is (w, ε)-chain from y to itself for above w. Then the compactness of X gives an upper bound on $r_\varepsilon(G)$.

If G is chain mixing, for $0 < \varepsilon < \delta$ and $x \in X$, define *chain mixing time* $m_\varepsilon(x, \delta, G)$ to be the smallest N such that for any $n \geq N$ and $y \in X$, there exists a (w, ε)-chain from some point in $B(x, \delta)$ to y for some $w \in F_m^+$ with $|w| = n$. We define $m_\varepsilon(\delta, G)$ to be the maximum over all x of $m_\varepsilon(x, \delta, G)$. The maximum exists by compactness.

Inspired by Example 2 in [23], we give the following Example 4.4 to illustrate the existence of a system of free semigroup actions that is chain transitive but not chain mixing.

Example 4.4. Let X be the disjoint union of two circles and $G = \{f_0, f_1\}$ where f_0 is the map sending a point x to the point $2x$ in the other circle and f_1 is the map sending a point x to the point $3x$ in the other circle. Obviously, G is chain transitive but not chain mixing, because it is not ε-chain mixing for any ε smaller than the distance between the two circles. However, G^2 restricted to one circle is chain mixing.

Next, we provide an example of chain mixing of free semigroup actions as follows.

Example 4.5. We define two continuous maps f_0, f_1 on Σ_2^+ as follows:

$$f_0(s_0 s_1 \ldots) = 0 s_0 s_1 \ldots, \quad f_1(s_0 s_1 \ldots) = 1 s_0 s_1 \ldots.$$

Put $G = \{f_0, f_1\}$. In [3], the author proved G has the shadowing property. We claim that (Σ_2^+, G) is chain mixing. Indeed, suppose that $\varepsilon > 0$ and $\omega' = (i_0 i_1 \ldots), \omega'' = (j_0 j_1 \ldots) \in \Sigma_2^+$ are given. Note that there is an $N \in \mathbb{N}$ such that $\frac{1}{2^{N-1}} < \varepsilon$. Next we have to construct an ε-chain $(\omega_i)_{i=0}^n$ from ω' to ω'' of length exactly n with $n \geq N$. Firstly, put $\omega_0 = \omega'$ and $\omega_n = \omega''$. Then, for $1 \leq i \leq n-2$, let $\omega_i = (j_{n-i} \ldots j_{n-2} j_{n-1} i_0 i_1 \ldots i_N \ldots)$ such that $\omega_i\big|_{[i,i+N]} = \omega'\big|_{[0,N]} = i_0 \ldots i_N$. Then it is easy to see that $(\omega_i)_{i=0}^n$ is a (w, ε)-chain from ω' to ω'' of length n where $w = j_{n-1} j_{n-2} \cdots j_0$.

Let (X,d) and (Y,d_Y) be compact metric spaces. Let $G := \{f_0, f_1, \ldots, f_{m-1}\}$ where $f_0, f_1, \ldots, f_{m-1}$ are continuous maps on X, and $H := \{g_0, g_1, \ldots, g_{n-1}\}$ where $g_0, g_1, \ldots, g_{n-1}$ are continuous maps on Y. Let

$$G \times H := \big\{(f \times g)_0, \ldots, (f \times g)_{mn-1}\big\},$$

where $(f \times g)_i \in \{f_j \times g_k \mid 0 \le j \le m-1, 0 \le k \le n-1\}$, and $(f \times g)(x,y) = (f(x), g(y))$ for any $f \times g \in G \times H$ and $x \in X$, $y \in Y$. A metric $d_{X\times Y}$ on the product space $X \times Y$ is given by

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) = \max\big\{d(x_1,x_2), d_Y(y_1,y_2)\big\}.$$

For any $v = v_0 \ldots v_{r-1} \in F_{mn}^+$, there exist unique $w^{(1)} = i_0 \ldots i_{r-1} \in F_m^+$ and unique $w^{(2)} = j_0 \ldots j_{r-1} \in F_n^+$ such that $(f \times g)_{v_l} = f_{i_l} \times g_{j_l}$ for any $0 \le l \le r-1$ and thus $(f \times g)_v = f_{w^{(1)}} \times g_{w^{(2)}}$. On the other hand, if $w^{(1)} = i_0 \ldots i_{r-1} \in F_m^+$, $w^{(2)} = j_0 \ldots j_{r-1} \in F_n^+$, there exists unique $v = v_0 \ldots v_{r-1} \in F_{mn}^+$ such that $f_{i_l} \times g_{j_l} = (f \times g)_{v_l}$ for any $0 \le l \le r-1$ and thus $f_{w^{(1)}} \times g_{w^{(2)}} = (f \times g)_v$. Thus, the map $v \to (w^{(1)}, w^{(2)})$ is a one-to-one correspondence.

For $k \in \mathbb{N}$, we denote

$$G^k = \big\{f_w \mid w \in F_m^+, |w| = k\big\} := \big\{(f)_0, (f)_1, \ldots, (f)_{m^k-1}\big\},$$

that is, G^k denotes the free semigroup generated by $\{f_w \mid w \in F_m^+, |w| = k\}$. For any $u = u_0 \ldots u_{r-1} \in F_{m^k}^+$, there exists unique $w^{(0)}, \ldots, w^{(r-1)} \in F_m^+$ with $|w^{(i)}| = k$ for all $0 \le i \le r-1$ such that $(f)_{u_i} = f_{w^{(i)}}$ for all $0 \le i \le r-1$ and thus $(f)_u = f_{w^{(0)} \ldots w^{(r-1)}}$. On the other hand, if $w^{(0)}, \ldots, w^{(r-1)} \in F_m^+$ with $|w^{(i)}| = k$ for all $0 \le i \le r-1$, there exists unique $u = u_0 \ldots u_{r-1} \in F_{m^k}^+$ such that $f_{w^{(i)}} = (f)_{u_i}$ for all $0 \le i \le r-1$, this implies that $f_{w^{(0)} \ldots w^{(r-1)}} = (f)_u$. Consequently, the map $u \to w^{(0)} \ldots w^{(r-1)}$ is a one-to-one correspondence.

Proposition 4.6. *(X,G) and (Y,H) are chain recurrence if and only if $(X \times Y, G \times H)$ is chain recurrence. If (X,G) and (Y,H) both are chain recurrence. Then for all $\varepsilon > 0$, $x \in X$ and $y \in Y$,*

(1) $r_\varepsilon((x,y), G \times H) \ge \max\{r_\varepsilon(x,G), r_\varepsilon(y,H)\}$;

(2) $r_\varepsilon((x,y), G \times H) \le \operatorname{lcm}(r_\varepsilon(x,G), r_\varepsilon(y,H))$,

where $\operatorname{lcm}(\cdot)$ *denotes the least common multiple.*

Proof. Observe that if $((x_0,y_0), \ldots, (x_k,y_k))$ is a (v,ε)-chain with $v = v_0 \ldots v_{k-1} \in F_{mn}^+$ for $G \times H$, we have $(x_0, \ldots, x_k)$ is a $(w^{(1)}, \varepsilon)$-chain with some $w^{(1)} \in F_m^+$ for G, and $(y_0, \ldots, y_k)$ is a $(w^{(2)}, \varepsilon)$-chain with some $w^{(2)} \in F_n^+$ for H. This shows statement (1), and (X,G) and (Y,H) both are chain recurrence if $(X \times Y, G \times H)$ is.

If (X, G) and (Y, H) both are chain recurrence, let $(x_0 = x, \dots, x_k = x)$ be a $(w^{(1)}, \varepsilon)$-chain with some $w^{(1)} \in F_m^+$ for G, and $(y_0 = y, \dots, y_s = y)$ be a $(w^{(2)}, \varepsilon)$-chain with some $w^{(2)} \in F_n^+$ for H. Then the $(w^{(1)} \dots w^{(1)}, \varepsilon)$-chain

$$(x_0, \dots, x_k = x_0, x_1, \dots, x_k = x_0, \dots, x_k = x_0)$$

formed by concatenating $(x_0 = x, \dots, x_k = x)$ with itself $s/\gcd(k, s)$ times has length $\operatorname{lcm}(k, s)$, where $\gcd(\cdot)$ denotes the greatest common divisor. As does the $(w^{(2)} \dots w^{(2)}, \varepsilon)$-chain

$$(y_0, \dots, y_s = y_0, y_1, \dots, y_s = y_0, \dots, y_s = y_0)$$

formed by concatenating $(y_0 = y, \dots, y_s = y)$ with itself $k/\gcd(k, s)$ times has length $\operatorname{lcm}(k, s)$. Combining the two gives a (v, ε)-chain from (x, y) to itself of $G \times H$ for some $v \in F_{mn}^+$. This shows (2) and $(X \times Y, G \times H)$ is chain recurrence. □

Proposition 4.7. *Let $k \in \mathbb{N}$. (X, G) is chain recurrence if and only if (X, G^k) is chain recurrence. If (X, G) is chain recurrence. Then for all $\varepsilon > 0$, $x \in X$,*

(1) $r_\varepsilon(x, G^k) \geq \frac{1}{k} r_\varepsilon(x, G)$;

(2) *there exists an $\varepsilon' \leq \varepsilon$ such that $r_\varepsilon(x, G^k) \leq r_{\varepsilon'}(x, G)$.*

Proof. Suppose that $(x_0, \dots, x_l)$ is a (u, ε)-chain for G^k with $u = u_0 \dots u_{l-1} \in F_{m^k}^+$. There are $w^{(0)}, \dots, w^{(l-1)} \in F_m^+$ such that $(f)_{u_j} = f_{w^{(j)}}$ for all $j = 0, \dots, l-1$. Let $w^{(j)} = i_{k-1}^{(j)} \dots i_0^{(j)} \in F_m^+$ for $j = 0, \dots, l-1$. Then we insert the orbit $(f_{i_0^{(j)}}(x_j), \dots, f_{i_{k-2}^{(j)} \dots i_0^{(j)}}(x_j))$ between x_j and x_{j+1}, for all $j = 0, \dots, l-1$. That is,

$$\big(x_0, f_{i_0^{(0)}}(x_0), \dots, f_{i_{k-2}^{(0)} \dots i_0^{(0)}}(x_0), x_1, f_{i_0^{(1)}}(x_1), \dots, f_{i_{k-2}^{(l-1)} \dots i_0^{(l-1)}}(x_{l-1}), x_l\big).$$

It is clear that this is a (w, ε)-chain of length lk for G, where $w = \overline{w^{(0)}} \dots \overline{w^{(l-1)}} \in F_m^+$. This last fact shows statement (1) and proves that G is chain recurrence if G^k is.

Next, we show to prove that G^k is chain recurrence if G is. By uniform continuity of $f_0, \dots, f_{m-1}$, there is an $\varepsilon' < \varepsilon/k$ such that any (w, ε')-chain of length $|w| = k$ for G, $(x_0, \dots, x_k)$, we have $d(f_{\overline{w}}(x_0), x_k) < \varepsilon$. For $x \in X$, we suppose that $(x_0 = x, \dots, x_{lk} = x)$ is a (w', ε')-chain of length $|w'| = lk$ of G where $w' = i'_0 \dots i'_{lk-1} \in F_m^+$. We deduce that

$$d\big(f_{\overline{i'_{jk} \dots i'_{(j+1)k-1}}}(x_{jk}), x_{(j+1)k}\big) < \varepsilon$$

for all $0 \leq j \leq l-1$. Moreover, there is $u = u_0 \dots u_{l-1} \in F_{m^k}^+$ such that $(f)_{u_j} = f_{\overline{i'_{jk} \dots i'_{(j+1)k-1}}}$ for all $0 \leq j \leq l-1$. Consequently, $(x_0, x_k, x_{2k}, \dots, x_{lk})$ is a (u, ε)-chain of length $|u| = l$ of G^k.

Analogously, observe that by taking a (w', ε')-chain of length $|w'| = l$ for G from x to itself, and concatenating it with itself k times, we can get a (v, ε)-chain of length $|v| = l$ for G^k from x to x where $v \in F_{m^k}^+$. This shows (2). □

Proposition 4.8. *Let (X,G) and (Y,H) be chain mixing, and $k \in \mathbb{N}$. Then (X,G^k) and $(X \times Y, G \times H)$ are also chain mixing, and for all $\varepsilon > 0$,*

(1) $m_\varepsilon(\delta, G \times H) = \max\{m_\varepsilon(\delta,G), m_\varepsilon(\delta,H)\}$;

(2) $m_\varepsilon(\delta, G^k) \geq \frac{1}{k} m_\varepsilon(\delta,G)$;

(3) *there exists an $\varepsilon' \leq \varepsilon$ such that $m_\varepsilon(\delta,G^k) \leq m_{\varepsilon'}(\delta,G)$.*

Proof. The fact that $G \times H$ is chain mixing if and only if both G and H follow from the definition of chain mixing, thus we have a statement (1). The proofs of (2), (3), and the fact that G^k is chain mixing are analogous to those of the corresponding statements for chain recurrence in Proposition 4.7. □

Remark 4.9. If $m = 1$, then Propositions 24 and 26 of [23] are obtained by Propositions 4.6, 4.7 and 4.8.

Proof of Theorem 1.2. By Theorem 7.5 of [14],

$$\overline{\dim}_B(\Sigma_m^+ \times X) \leq \overline{\dim}_B(\Sigma_m^+) + \overline{\dim}_B(X).$$

Since $\overline{\dim}_B(\Sigma_m^+) = 1$, we have that $\overline{\dim}_B(\Sigma_m^+ \times X) \leq \overline{\dim}_B X + 1$. It follows from [27] that F is chain transitive if and only if G is chain transitive. Note that if $((\omega_0,x_0),\dots,(\omega_n,x_n))$ is an (n,ε)-chain of F, then $(x_0,\dots,x_n)$ is a (w,ε)-chain of G for some $w \in F_m^+$ with $|w| = n$. Thus, we have

$$r_\varepsilon(x,G) \leq \max_{\omega \in \Sigma_m^+} r_\varepsilon((\omega,x),F).$$

This gives us $r_\varepsilon(G) \leq r_\varepsilon(F)$. By [23], there is a $C > 0$ such that $r_\varepsilon(F) \leq C/\varepsilon^{\overline{\dim}_B(\Sigma_m^+ \times X)}$, we deduce that

$$r_\varepsilon(G) \leq C/\varepsilon^{\overline{\dim}_B X + 1}.$$

This shows (1).

Let $(\omega,x) \in \Sigma_m^+ \times X$ and $m_\varepsilon((\omega,x),\delta,F) = N$. Since F is chain mixing if and only if G is chain mixing by [27], this shows that for any $(\xi,y) \in \Sigma_m^+ \times X$, there exist $(\zeta,x') \in B((\omega,x),\delta)$ and ε-chain of F of length exactly N from (ζ,x') to (ξ,y), denoted by

$$((\zeta,x'),(\zeta_1,x_1),\dots,(\zeta_N,x_N) = (\xi,y)).$$

It follows that $(x',x_1,\dots,x_N = y)$ is a (w,ε)-chain for some $w \in F_m^+$ with $|w| = N$. We deduce that

$$m_\varepsilon(x,\delta,G) \leq \max_{\omega \in \Sigma_m^+} m_\varepsilon((\omega,x),\delta,F).$$

This implies that $m_\varepsilon(\delta, G) \le m_\varepsilon(\delta, F)$. By [23], there is a $C > 0$ independent of δ such that $m_\varepsilon(\delta, F) \le C/\varepsilon^{2\overline{\dim}_B(\Sigma_m^+ \times X)}$, we conclude that

$$m_\varepsilon(\delta, G) \le C/\varepsilon^{2(\overline{\dim}_B X+1)}.$$

This shows (2). □

Proof of Theorem 1.3. By Theorem 1.1 we know

$$h(G) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N^*(n,\varepsilon,\delta,G).$$

For $\alpha > 0$, let $N_\alpha := N^*(0,\alpha,0,G)$. Let $\{x_1,\ldots,x_{N_{3\delta}}\}$ be a 3δ-separated set of points. Each sequence $(i_0,\ldots,i_k) \in \{1,\ldots,N_{3\delta}\}^{k+1}$ corresponds to $\{x_{i_0},\ldots,x_{i_k}\}$. Taking $x'_{i_k} = x_{i_k}$, by the definition of $m_\varepsilon(\delta)$, there are $x'_{i_{k-1}} \in B(x_{i_{k-1}},\delta)$ and $(w^{(k-1)},\varepsilon)$-chain of length $m_\varepsilon(\delta)$ for some $w^{(k-1)} \in F_m^+$ from $x'_{i_{k-1}}$ to x'_{i_k}. Now, suppose that this process continues until, there are $x'_{i_0} \in B(x_{i_0},\delta)$ and $(w^{(0)},\varepsilon)$-chain of length $m_\varepsilon(\delta)$ for some $w^{(0)} \in F_m^+$ from x'_{i_0} to x'_{i_1}. We can derive that $(x'_{i_0},\ldots,x'_{x_k})$ is a $(w^{(0)}\ldots w^{(k-1)},\varepsilon)$-chain of length $km_\varepsilon(\delta)$. Since the set $\{x_1,\ldots,x_{N_{3\delta}}\}$ is 3δ-separated, the set

$$E := \left\{(x'_{i_1},\ldots,x'_{i_k}) \mid (i_0,\ldots,i_k) \in \{1,\ldots,N_{3\delta}\}^{k+1}\right\}$$

is a set of δ-separated ε-chains of length $km_\varepsilon(\delta)$. Obviously, $\sharp E = (N_{3\delta})^{k+1}$, then

$$N^*(km_\varepsilon(\delta),\varepsilon,\delta,G) \ge \frac{(N_{3\delta})^{k+1}}{m^{km_\varepsilon(\delta)}}.$$

Next, let $B_\alpha := B^*(0,\alpha,0,G)$ be the minimum number of balls of radius α necessary to cover X. By the definition of lower box dimension, for small enough α, $B_\alpha \ge C(1/\alpha)^{\underline{\dim}_B X}$ for some positive constant C. Clearly $N_\alpha \ge B_\alpha$, then $N_\alpha \ge C(1/\alpha)^{\underline{\dim}_B X}$ as well.

Finally, we have that

$$\begin{aligned}
h(G) &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N^*(n,\delta,\varepsilon,G)\\
&\ge \limsup_{\delta\to 0}\lim_{\varepsilon\to 0}\limsup_{k\to\infty}\frac{1}{km_\varepsilon(\delta)}\log N^*(km_\varepsilon(\delta),\delta,\varepsilon,G)\\
&\ge \limsup_{\delta\to 0}\lim_{\varepsilon\to 0}\limsup_{k\to\infty}\frac{1}{km_\varepsilon(\delta)}\log\frac{(N_{3\delta})^{k+1}}{m^{km_\varepsilon(\delta)}}\\
&= \limsup_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\log N_{3\delta}}{m_\varepsilon(\delta)} - \log m\\
&\ge \limsup_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\log C(1/3\delta)^{\underline{\dim}_B X}}{m_\varepsilon(\delta)} - \log m\\
&= \underline{\dim}_B X \cdot \limsup_{\delta\to 0}\frac{\log(1/\delta)}{\lim_{\varepsilon\to 0} m_\varepsilon(\delta)} - \log m.
\end{aligned}$$

Since X is not a single point, and it cannot be a finite collection of points as (X,G) is chain mixing, then $\lim_{\delta\to 0} m_\varepsilon(\delta) = \infty$. Thus this conclusion is proved. □

Remark 4.10. In fact, by virtue of Theorem 2.3 and Theorem 28 of [23], we immediately deduce that

$$h(G) \geq \max\left\{0, \underline{\dim}_B X \cdot \limsup_{\delta\to 0} \frac{\log(1/\delta)}{\lim_{\varepsilon\to 0} m_\varepsilon(\delta, F)} - \log m\right\}.$$

As $m_\varepsilon(\delta, G) \leq m_\varepsilon(\delta, F)$ by Theorem 1.2, we conclude that the estimation of the topological entropy of free semigroup actions is more accurate in Theorem 1.3.

On the other hand, if $m = 1$, this means that $G = \{f\}$, then

$$h(f) \geq \underline{\dim}_B X \cdot \limsup_{\delta\to 0} \frac{\log(1/\delta)}{\lim_{\varepsilon\to 0} m_\varepsilon(\delta, f)},$$

this yields that Theorem 28 in [23].

We provide an example that satisfies $\lim_{\varepsilon\to 0} m_\varepsilon(\delta, G) < \infty$ to show that Theorem 1.3 is non-trivial, because if this limit is infinite then Theorem 1.3 becomes trivial.

Example 4.11. Let X be a compact manifold, $f_0, \dots, f_{m-1} \colon X \to X$ be C^1-expanding maps, G be the free semigroup action generated by $f_0, \dots, f_{m-1}$. For any $\delta > 0$, by Lemma 18 of [24], there exists $N(\delta) \in \mathbb{N}$ so that $f_w(B(x, \delta)) = X$ for every $x \in X$ and every $w \in F_m^+$ with $|w| = N(\delta)$. Then, $m_0(x, \delta, G) \leq N(\delta)$ for every $x \in X$. Hence, $m_0(\delta, G) \leq N(\delta)$. Notice that $m_\varepsilon(\delta, G)$ is a nondecreasing function as $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon\to 0} m_\varepsilon(\delta, G) \leq N(\delta) < \infty.$$

5. The structure of chain transitive systems

In this section our main purpose is to prove Theorem 1.4. The proof is rather long and technical, so several auxiliary results and definitions are needed.

Let (X, d) be a compact metric space and G be a free semigroup generated by m generators $f_0, f_1, \dots, f_{m-1}$ which are continuous maps on X. The proof of Theorem 1.4 is technical and difficult, and below we provide an outline of the proof to help the readers understand our thoughts:

(1) We define an equivalence relation $\sim_\varepsilon$ which is related to ε. From Lemma 5.1 and Remark 5.2, we can obtain that equivalence relation $\sim_\varepsilon$ corresponds to k_ε equivalence classes.

(2) We define another equivalence relation on X by saying $x \sim y$ if $x \sim_\varepsilon y$ for all $\varepsilon > 0$. Next, we divide into three cases: (a) $k_\varepsilon = 1$ as $\varepsilon \to 0$; (b) $k_\varepsilon = k$ for sufficiently small ε, which can be transformed into Case (a) by Lemmas 5.4 and 5.6; (c) $k_\varepsilon \to \infty$ as $\varepsilon \to 0$, we deduce that G factors onto an adding machine map.

We assume G is chain transitive in this section. For $x \in X$, denote by $T_\varepsilon(x)$ the set of the lengths of all (w, ε)-chain from x to itself with some $w \in F_m^+$ for G. Recall that $\gcd(\cdot)$ denotes the greatest common divisor.

Lemma 5.1. *Let G be chain transitive and $\varepsilon > 0$. There exists $k_\varepsilon \geq 1$ such that $\gcd(T_\varepsilon(x)) = k_\varepsilon$ for any $x \in X$, in the sense that k_ε does not depend on the choice of x.*

Proof. This follows from the proof of Lemma 7 in [23]. For $x \in X$, define $k_\varepsilon := \gcd(T_\varepsilon(x))$. Consider that $y \in X$ and $(y_0 = y, y_1, \ldots, y_n = y)$ is a (w, ε)-chain from y to itself of length $|w| = n$. We claim that k_ε divides n. Indeed, as G is chain transitive, there are (w', ε)-chain $(x_0 = x, x_1, \ldots, x_{m_1} = y)$ from x to y of length $|w'| = m_1$, and (w'', ε)-chain $(z_0 = y, z_1, \ldots, z_{m_2} = x)$ from y to x of length $|w''| = m_2$. We have that

$$(x_0 = x, x_1, \ldots, x_{m_1} = y, z_1, \ldots, z_{m_2} = x)$$

is a $(w'w'', \varepsilon)$-chain from x to itself of length $m_1 + m_2$, and

$$(x_0 = x, x_1, \ldots, x_{m_1} = y, y_1, \ldots, y_n = y, z_1, \ldots, z_{m_2} = x)$$

is a $(w'ww'', \varepsilon)$-chain from x to itself of length $m_1 + n + m_2$. Note that both $m_1 + m_2$ and $m_1 + n + m_2$ are multiples of k_ε, which yields that k_ε divides n. □

Define a relation on X by setting $x \sim_\varepsilon y$ if there is a (w, ε)-chain from x to y of length a multiple of k_ε for some $w \in F_m^+$. It is clear that $\sim_\varepsilon$ is an equivalence relation. In fact, observe that reflexivity and transitivity are established by Lemma 5.1 and G is chain transitive. On the other hand, we claim that $\sim_\varepsilon$ is symmetry, that is, $y \sim_\varepsilon x$ if $x \sim_\varepsilon y$. Indeed, consider that $(y_0 = x, \ldots, y_n = y)$ is a (w', ε)-chain with $|w| = n$ and $k_\varepsilon \mid n$ from x to y. Let we suppose that $(x_0 = y, \ldots, x_l = x)$ is a (w'', ε)-chain with $|w''| = l$ from y to x as G is chain transitive. Then $(y_0 = x, \ldots, y_n = y, \ldots, x_l = x)$ is a $(w'w'', \varepsilon)$-chain from x to x of length $n + l$. This implies that l is a multiple of k_ε. Consequently, $\sim_\varepsilon$ is an equivalence relation.

Remark 5.2. In fact, by the definition of k_ε, if $x \sim_\varepsilon y$, then any ε-chain from x to y must have length a multiple of k_ε.

We now define another equivalence relation on X by saying $x \sim y$ if $x \sim_\varepsilon y$ for all $\varepsilon > 0$.

Lemma 5.3. *Let G be chain transitive. For any $\varepsilon > 0$, the equivalence relation $\sim_\varepsilon$ is both open and closed. The equivalence relation $\sim$ is closed.*

Proof. Similar to Lemma 9 of [23], so we won't repeat it here. □

The following result is essentially contained in Exercise 8.22 of [2], but we provide proof here for the reader's convenience.

Let $T_1, T_2 \subset \mathbb{N}$, and $T_1 + T_2 := \{t_1 + t_2 \mid t_1 \in T_1, t_2 \in T_2\}$. $T \subset \mathbb{N}$ is closed under addition if $T + T \subset T$.

Lemma 5.4. *Let $T \subset \mathbb{N}$ and T be close under addition, then there exists N so that $nd \in T$ for all $n \geq N$ where $\gcd(T) = d$. Furthermore, there exist $r, n \in T$ such that $\gcd(r, n) = d$.*

Proof. We only prove the case $d = 1$, others are similar. We claim that $\gcd(t_1, \ldots, t_p) = 1$ for some $p \in \mathbb{N}$. Indeed, for any $t_1, t_2 \in T$, $\gcd(t_1, t_2) = c$. If $c > 1$, there is $t_3 \in T$ such that $\gcd(c, t_3) = c_1$ with $c_1 < c$. If $c_1 = 1$, then $\gcd(t_1, t_2, t_3) = 1$, otherwise, repeat the above steps.

To prove the lemma, it is to show that there is N such that $n = n_1t_1 + \cdots + n_pt_p$ for all $n \geq N$ where $n_1, \ldots, n_p \in \mathbb{N}_0$, as T is closed under addition. We proceed by induction on p. For $p = 2$, then $\gcd(t_1, t_2) = 1$, so $1 = n_1t_1 - n_2t_2$. If $n \geq N = n_2t_2^2$ then $n = Qt_2 + R$ with $Q \geq n_2t_2$ and $0 \leq R < t_2$. So $n = (Q - n_2R)t_2 + Rn_1t_1$. Assuming it holds for p we prove it for $p+1$. Consider that $\gcd(t_1, \ldots, t_{p+1}) = 1$, and $\gcd(t_1, \ldots, t_p) = k$, then $\gcd(k, t_{p+1}) = 1$. Denote

$$T_2 = \{n_1k + n_2t_{p+1} \mid n_1, n_2 \in \mathbb{N}_0\} \quad \text{and} \quad T_p = \left\{ r_1\frac{t_1}{k} + \cdots + r_p\frac{t_p}{k} \,\middle|\, r_1, \ldots, r_p \in \mathbb{N}_0 \right\}.$$

Clearly, T_2 and T_p are both closed under addition. By hypothesis, let N_1, with respect to T_2, be the number such that all $n \geq N_1$ implies $n = n_1k + n_2t_{p+1}$ for some $n_1, n_2 \in \mathbb{N}_0$, and let N_2, with respect to T_p, be the number such that all $n \geq N_2$ implies

$$n = m_1\frac{t_1}{k} + \cdots + m_p\frac{t_p}{k}$$

for some $m_1, \ldots, m_p \in \mathbb{N}_0$. Take $N = N_1 + N_2k$. For all $n \geq N$, $n - N_2k = n_1k + n_2t_{p+1}$, this implies that

$$\begin{aligned} n &= n_1k + n_2t_{p+1} + N_2k = (n_1 + N_2)k + n_2t_{p+1} \\ &= \left(m_1\frac{t_1}{k} + \cdots + m_p\frac{t_p}{k}\right)k + n_2t_{p+1} = m_1t_1 + \cdots + m_pt_p + n_2t_{p+1}. \end{aligned}$$

This completes our induction. □

Definition 5.5. [7] Let $J = (j_1, j_2, \ldots)$ be a sequence of integers greater than or equal to 2. Let X_J be the Cantor set of all sequences $(a_1, a_2, \ldots)$ where $a_i \in \{0, 1, \ldots, j_i - 1\}$ for all i. Define the adding machine map $f_J \colon X_J \to X_J$ by

$$f_J(a_1, a_2, \ldots) = (a_1, a_2, \ldots) + (1, 0, 0, \ldots),$$

where addition is defined componentwise $\bmod j_i$, with carrying to the right.

Lemma 5.6. [22, Theorem 2.1.3] *Let a_1, a_2 be non-negative relatively prime integers. Let*

$$g(a_1, a_2) = a_1a_2 - a_1 - a_2,$$

then, for any $N > g(a_1, a_2)$ is representable as a non-negative integer combination of m and n, that is, there are $p_1, p_2 \in \mathbb{N}_0$ such that $N = p_1a_1 + p_2a_2$.

Proof of Theorem 1.4. Let $\varepsilon > 0$ and $k_\varepsilon \geq 1$, k_ε as in Lemma 5.1. Then X is divided into k_ε equivalence classes for $\sim_\varepsilon$. In fact, for any $x, z \in X$ with $x \sim_\varepsilon z$, let us suppose that

$$(x_0 = x, x_1, x_2, \dots, x_{k_\varepsilon}, x_{k_\varepsilon+1}, x_{k_\varepsilon+2}, \dots, x_{2k_\varepsilon}, \dots, x_{nk_\varepsilon} = z)$$

is a (w, ε)-chain from x to z of length nk_ε. It is clear that

$$\begin{gathered} x_0 \sim_\varepsilon x_{k_\varepsilon} \sim_\varepsilon x_{2k_\varepsilon} \sim_\varepsilon \cdots \sim_\varepsilon x_{nk_\varepsilon}, \\ x_1 \sim_\varepsilon x_{k_\varepsilon+1} \sim_\varepsilon x_{2k_\varepsilon+1} \sim_\varepsilon \cdots \sim_\varepsilon x_{(n-1)k_\varepsilon+1}, \\ \vdots \\ x_{k_\varepsilon-1} \sim_\varepsilon x_{2k_\varepsilon-1} \sim_\varepsilon x_{3k_\varepsilon-1} \sim_\varepsilon \cdots \sim_\varepsilon x_{nk_\varepsilon-1}. \end{gathered}$$

Denote these equivalence classes as $[x_0]_{\sim_\varepsilon}, [x_1]_{\sim_\varepsilon}, \dots, [x_{k_\varepsilon-1}]_{\sim_\varepsilon}$, respectively. For any $x \in X$, we deduce that $x \in [x_i]_{\sim_\varepsilon}$ for some $0 \leq i \leq k_\varepsilon - 1$ since G is chain transitive. And we have $x_i \nsim_\varepsilon x_{i+1}$ for each $i = 0, \dots, k_\varepsilon - 1$ since (x_i, x_{i+1}) is an ε-chain of length 1. Consequently, X is divided into k_ε equivalence classes.

Obviously, G cyclically permutes k_ε equivalence classes, that is,

$$[f_j(x_i)]_{\sim_\varepsilon} = [x_{i+1 \bmod k_\varepsilon}]_{\sim_\varepsilon}$$

for all $j = 0, \dots, m-1$ and $i = 0, \dots, k_\varepsilon - 1$. Moreover, every equivalence class is invariant under G^{k_ε}, that is, $f_w([x_i]_{\sim_\varepsilon}) \subset [x_i]_{\sim_\varepsilon}$ for all $i = 0, \dots, k_\varepsilon - 1$ and $w \in F_m^+$ with $|w| = k_\varepsilon$. Since $[f_w(x_i)]_{\sim_\varepsilon} = [x_i]_{\sim_\varepsilon}$, we only prove $f_w([x_i]_{\sim_\varepsilon}) \subset [f_w(x_i)]_{\sim_\varepsilon}$. For any $x \in [x_i]_{\sim_\varepsilon}$, we have $f_w(x) \sim_\varepsilon f_w(x_i)$ since $x \sim_\varepsilon f_w(x)$, $x \sim_\varepsilon x_i$ and $x_i \sim_\varepsilon f_w(x_i)$.

The quantity k_ε is nondecreasing as $\varepsilon \to 0$, and in fact k_{ε_2} divides k_{ε_1} if $\varepsilon_1 \leq \varepsilon_2$, since an ε_1-chain is an ε_2-chain. Either k_ε stabilizes at some k, or it grows without bound. We consider the three cases separately.

Case 1*: k_ε stabilizes at $k = 1$.* Then there is only one $\sim$ equivalence class, and G is chain mixing. Indeed, for any $\varepsilon > 0$ and any $x \in X$, $T_\varepsilon(x)$ is closed under addition. Since $\gcd(T_\varepsilon(x)) = 1$, there exist $r_1, s_1 \in T_\varepsilon(x)$ such that $\gcd(r_1, s_1) = 1$ by Lemma 5.4. This implies that there exist (w', ε)-chain and (w'', ε)-chain from x to itself of lengths $|w'| = r_1$ and $|w''| = s_1$, respectively. We can get an ε-chain from x to itself of length N for any $N > g(r_1, s_1)$ by Lemma 5.6. By compactness, there is a $p \in \mathbb{N}$ such that $\bigcup_{i=1}^{p} B(x_i, \varepsilon/2) = X$, let the length of an $\varepsilon/2$-chain from x_i to x_j be M_{ij}, and $M = \max_{1\le i,j\le p} M_{ij}$, then for between any two points in X there is an ε-chain of length less than or equal to M. Hence, for any $y \in X$, any $n > g(r_1, s_1) + M$, there is a (w, ε)-chain of length $|w| = n$ from x to y. Therefore, G is chain mixing.

Case 2*: $k = k_\varepsilon$ for sufficiently small ε.* Then the equivalence relation $\sim$ is the same as $\sim_\varepsilon$. Thus there are k equivalence classes, G^k cycles among the classes periodically, and each class is invariant under G^k. An argument similar to that for Case 1 show that $([x_i]_\sim, G^k)$ is chain mixing for each $i = 0, \ldots, k-1$. By uniform continuity, pick $\varepsilon' < \varepsilon/k$ small enough that for any (w, ε')-chain of length $|w| = k$ for G, denotes $(x_0, \ldots, x_n)$, we have $d(f_{\overline{w}}(x_0), x_k) < \varepsilon$. Notice that $k_{\varepsilon'} = k$ as $\varepsilon' < \varepsilon$, then $\gcd(T_{\varepsilon'}(x)) = k$ for any $x \in [x_i]_\sim$. By Lemma 5.4, there exist $r_2, s_2 \in T_{\varepsilon'}(x)$ such that $\gcd(r_2, s_2) = k$. This implies that there are (w', ε')-chain and (w'', ε')-chain from x to itself of lengths $|w'| = r_2$ and $|w''| = s_2$ for G, respectively. Put $r_2 = ak$ and $s_2 = bk$, then $\gcd(a, b) = 1$. By Lemma 5.6, for any N with $N > g(a, b) := ab - a - b$, there are $p, q \in \mathbb{N}_0$ such that $N = pa + qb$, hence there is an ε'-chain from x to itself of length kN. Since $[x_i]_\sim$ is closed, there is $p' \ge 1$ such that $\bigcup_{j=1}^{p'} B(z_j, \varepsilon'/2) \supset [x_i]_\sim$ where $z_j \in [x_i]_\sim$ for all $j = 1, \ldots, p'$. For $1 \le j, r \le p'$, there is a (w_{jr}, ε')-chain from z_j to z_r of length $|w_{jr}| = kM_{j,r}$ for G. Let $M = \max_{1\le j,r\le p'} M_{j,r}$, then between any two points in $[x_i]_\sim$ there is an ε'-chain for G of length equal to ck for some $1 \le c \le M$. We claim that for any $n > g(a, b) + M$ and any $y \in [x_i]_\sim$, there is an ε-chain from x to y of the length n for G^k. Indeed, notice that there is a (w''', ε')-chain from x to y of length ck with $1 \le c \le M$ for G. Since $n - c > g(a, b)$, there are $p_1, q_1 \in \mathbb{N}_0$ such that $n - c = p_1 a + q_1 b$ by Lemma 5.6. Then from the above structure, a (w, ε')-chain for G of length nk, $(x_0 = x, \ldots, x_{nk} = y)$, is naturally formed, where

$$w = \underbrace{w' \ldots w'}_{p_1}\underbrace{w'' \ldots w''}_{q_1} w''' := i_0 \ldots i_{k-1} i_k \ldots i_{2k-1} i_{2k} \ldots i_{nk-1}.$$

Then, we have

$$d\big(f_{\overline{i_{jk}\cdots i_{(j+1)k-1}}}(x_{jk}), x_{(j+1)k}\big) < \varepsilon$$

for all $j = 0, \ldots, n-1$. Moreover, there is $u = u_0 \ldots u_{n-1} \in F^+_{m^k}$ such that $(f)_{u_j} = f_{\overline{i_{jk}\cdots i_{(j+1)k-1}}}$ for all $j = 0, \ldots, n-1$, this means that $(x_0, x_k, x_{2k}, \ldots, x_{nk})$ is a (u, ε)-chain of length $|u| = n$ for G^k. Consequently, $([x_i]_\sim, G^k)$ is chain mixing.

Case 3*: k_ε grows without bound as ε decreasing to* 0*.* Then the period of G's cycling goes to infinity as ε shrinks to 0. Let $\widetilde{K}_\varepsilon = X/{\sim_\varepsilon} = \{[x_0]_{\sim_\varepsilon}, \ldots, [x_{k_\varepsilon - 1}]_{\sim_\varepsilon}\}$ and $\widetilde{K} = X/{\sim}$ be the quotient spaces with the quotient topology. For $j = 0, \ldots, m-1$, we define the induced map on $\widetilde{K}_\varepsilon$ to be

$$\widetilde{f_j}\colon \widetilde{K}_\varepsilon \to \widetilde{K}_\varepsilon, \quad [x_i]_{\sim_\varepsilon} \mapsto [f_j(x_i)]_{\sim_\varepsilon}$$

for all $i = 0, \ldots, k_\varepsilon - 1$. Since $\widetilde{f}_j([x_i]_{\sim_\varepsilon}) = [f_j(x_i)]_{\sim_\varepsilon} = [x_{i+1 \bmod k_\varepsilon}]_{\sim_\varepsilon}$ for all $j = 0, \ldots, m-1$ and $i = 0, \ldots, k_\varepsilon - 1$, then every induced map $\widetilde{f}_j$ is the same, denoted as $\widetilde{f}_\varepsilon \colon \widetilde{K}_\varepsilon \to \widetilde{K}_\varepsilon$, $[x_i]_{\sim_\varepsilon} \mapsto [x_{i+1 \bmod k_\varepsilon}]_{\sim_\varepsilon}$ for all $i = 0, \ldots, k_\varepsilon - 1$. Similar to Theorem 6 of [23], we deduce that $(\widetilde{K}, \widetilde{f})$ is topological conjugate to an adding machine map f_J. Therefore, G factors onto an adding machine map f_J. □

Corollary 5.7. *Let X be connected and $G = \{f_0, \ldots, f_{m-1}\}$ where $f_i \colon X \to X$ is continuous for all $0 \le i \le m-1$. Then the following are equivalent:*

(1) *G is chain recurrence;*
(2) *G is chain transitive;*
(3) *G is totally chain transitive;*
(4) *G is chain mixing.*

Proof. It follows from the proof of Corollary 14 of [23]. Clearly, (4) ⇒ (3) ⇒ (2) ⇒ (1). By Theorem 1.4, X is connected and G is chain transitive, both of which imply that G is chain mixing. So it is enough to show that chain recurrence implies chain transitivity. Assume that G is chain recurrence and $\varepsilon > 0$. We say that x and y are ε-chain equivalent if there are (w', ε)-chain from x to y and (w'', ε)-chain from y to x for some $w', w'' \in F_m^+$. This is an equivalence relation as G is chain recurrence. By connectivity of X, so it suffices to show that this is an open equivalent relation. Consider that x and y are ε-chain equivalent. Choose $\delta \le \varepsilon/2$ such that $d(y, y') < \delta$ implies $d(f_i(y), f_i(y')) < \varepsilon/2$ for all $i = 0, \ldots, m-1$. It suffices to show that x is ε-chain equivalent to an arbitrary $y' \in B(y, \delta)$. Let $(x_0 = x, \ldots, x_n = y)$ be a (w', ε)-chain for some $w' \in F_m^+$ from x to y, and $(y_0 = y, \ldots, y_m = y)$ be a $(w'', \varepsilon/2)$-chain for some $w'' \in F_m^+$ from y to itself. Then

$$(x_0 = x, \ldots, x_n = y, y_1, \ldots, y_{m-1}, y')$$

is a $(w'w'', \varepsilon)$-chain from x to y'. Similarly, let $(z_0 = y, \ldots, z_r = x)$ be a (w''', ε)-chain for some $w''' \in F_m^+$. Then

$$(y', y_1, \ldots, y_m = y, z_1, \ldots, z_r = x)$$

is a $(w''w''', \varepsilon)$-chain from y' to x. This yields that x and y' are ε-chain equivalent. □

Remark 5.8. If X is connected and $m = 1$, by Corollary 5.7, this gives a generalization of Corollary 14 of [23].

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