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Continuity of Generalized Riesz Potentials for Double Phase Functionals with Variable Exponents over Metric Measure Spaces

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Abstract. Our aim in this paper is to deal with the continuity of generalized Riesz potentials $I_{\rho,\tau}f$ of functions in Morrey spaces $L^{\Phi,\nu(\cdot),\kappa}(X)$ of double phase functionals with variable exponents over bounded non-doubling metric measure spaces. What is new in this paper is that ρ depends on $x \in X$.

1. Introduction

Let (X, d, μ) be a metric measure space, where X is a bounded set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. We often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball in X centered at x with radius r and $d_X = \sup\{d(x, y) : x, y \in X\}$. We assume that

$$\mu(\{x\}) = 0$$

for $x \in X$ and $0 < \mu(B(x, r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. We do not assume that μ has a so-called doubling condition. Recall that a Radon measure μ is said to be doubling if there exists a constant $c_0 > 0$ such that $\mu(B(x, 2r)) \le c_0\mu(B(x, r))$ for all $x \in \operatorname{supp}(\mu)$ $(= X)$ and $r > 0$ (see [2]). For the Gauss measure space, see [11]. Otherwise μ is said to be non-doubling. For examples of non-doubling metric measure spaces we refer to [22, 28].

We consider the family (ρ) of all functions ρ satisfying the following conditions: $\rho(x, r)$: $X \times (0, \infty) \to (0, \infty)$ is a measurable function such that there exist constants $0 < k < 1$, $0 < k_1 < k_2$ and $C_\rho > 0$ such that

$$\sup_{kr \le s \le r} \rho(x, s) \le C_\rho \int_{k_1 r}^{k_2 r} \rho(x, s)\,\frac{ds}{s} \tag{1.1}$$

for all $r > 0$ and there exists a constant $C > 0$ such that

$$\int_0^{\max\{1,2k_2\}d_X} \rho(x, s)\,\frac{ds}{s} \le C \tag{1.2}$$

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for all $x \in X$. What is new in this paper is that ρ depends on $x \in X$. We do not assume the doubling condition on ρ.

We can include a variety of examples of ρ satisfying (1.1) and (1.2) as will be seen in Remark 4.3 and Example 4.4 below.

For $\tau \geq 1$ and a function $\rho \in (\rho)$, we define the generalized Riesz potential $I_{\rho,\tau} f$ for a locally integrable function f on X by

$$I_{\rho,\tau} f(x) = \int_X \frac{\rho(x, d(x,y)) f(y)}{\mu(B(x, \tau d(x,y)))} \, d\mu(y)$$

(see e.g. [27, 32]). The operator $I_{\rho,\tau}$ is also called the generalized fractional integral operator. When $X = \mathbf{R}^N$, $\mu = dx$, $I_{\rho,1} f(x)$ is equal to $I_\rho f(x) = \int_X \frac{\rho(x,|x-y|) f(y)}{|x-y|^N} \, dy$. When $\rho(x,r) = \rho(r)$, $I_\rho f$ was first introduced by Nakai [21]. See also [9]. If $X = \mathbf{R}^N$, $\mu = dx$ and $\rho(x,r) = r^{\alpha(x)}$ with $0 < \inf_{x \in \mathbf{R}^N} \alpha(x) \leq \sup_{x \in \mathbf{R}^N} \alpha(x) < N$, then $I_{\rho,1} f(x)$ is equal to $U_{\alpha(x)} f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha(x)-N} f(y) \, dy$.

Double phase problems have been studied intensively in variable exponent analysis and regularity theory of PDEs by many mathematicians (see e.g. [1, 4–6, 8, 13, 17, 33]).

In the previous paper [23], we considered the case $\widetilde{\Phi}(x,t)$ is a double phase functional given by

$$\widetilde{\Phi}(x,t) = t^p + (b(x)t)^q,$$

where $1 < p < q$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ (cf. [5]). In [23] we studied the continuity of Riesz potentials $\widetilde{I}_{\rho,\tau} f$ of functions in Morrey spaces $L^{\widetilde{\Phi},\nu,\kappa}(X)$ of the double phase functionals $\widetilde{\Phi}(x,t)$ when ρ does not depend on $x \in X$, where

$$\widetilde{I}_{\rho,\tau} f(x) = \int_X \frac{\rho(d(x,y)) f(y)}{\mu(B(x, \tau d(x,y)))} \, d\mu(y).$$

We refer to [24] for the Euclidean case. See also [15, Theorem 4.1] and [16, Theorem 4.1].

As in [13, 24], we consider the case $\Phi(x,t)$ as a double phase functional given by

$$\Phi(x,t) = t^{p(x)} + (b(x)t)^{q(x)},$$

where $p(x) < q(x)$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ (cf. [3, 26]).

In this paper, we shall extend [23, 24] from the case ρ does not depend on $x \in X$ to the case ρ depends on $x \in X$. In fact, we show the continuity of generalized Riesz potential $I_{\rho,\tau} f$ of functions f in Morrey spaces $L^{\Phi,\nu(\cdot),\kappa}(X)$ of the double phase functionals $\Phi(x,t)$ over bounded non-doubling metric measure spaces X (see Theorem 4.1), as an extension of [23, Theorem 1] and [24, Theorem 2.2]. Our key lemma is Lemma 3.2.

We refer to [25, 27, 29, 32] for the boundedness of $I_{\rho,\tau} f$, to [10] for Gagliardo–Nirenberg inequality for $I_{\rho,\tau} f$ and to e.g. [7, 9, 21] for the boundedness of $I_\rho f$.

Throughout this paper, let C denote various constants independent of the variables in question.

2. Preliminaries

Let $p(\cdot)$ be a measurable functions on X such that

(P1) $1 \le p^- := \inf_{x\in X} p(x) \le \sup_{x\in X} p(x) =: p^+ < \infty$,

(P2) $p(\cdot)$ is log-Hölder continuous on X, namely

$$|p(x) - p(y)| \le \frac{C_p}{\log(e + 1/d(x,y))}, \quad x, y \in X$$

with a constant $C_p \ge 0$.

Let $\nu(\cdot)$ be a measurable functions on X such that

$$0 < \nu^- := \inf_{x\in X} \nu(x) \le \sup_{x\in X} \nu(x) =: \nu^+ < \infty.$$

For $\kappa \ge 1$, the Morrey space with variable exponents $L^{p(\cdot),\nu(\cdot),\kappa}(X)$ is the family of measurable functions f on X satisfying

$$L^{p(\cdot),\nu(\cdot),\kappa}(X) = \left\{ f \in L^1_{\mathrm{loc}}(X) \,\middle|\, \sup_{\substack{x\in X\\ 0<r<d_X}} \frac{r^{\nu(x)}}{\mu(B(x,\kappa r))} \int_{B(x,r)} |f(y)|^{p(y)}\, d\mu(y) < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot),\nu(\cdot),\kappa}(X)} = \inf \left\{ \lambda > 0 \,\middle|\, \sup_{\substack{x\in X\\ 0<r<d_X}} \frac{r^{\nu(x)}}{\mu(B(x,\kappa r))} \int_{B(x,r)} \left(\frac{|f(y)|}{\lambda} \right)^{p(y)} d\mu(y) \le 1 \right\}$$

(cf. see [19]). When $p(\cdot) = p$ and $\nu(\cdot) = \nu$, we see that the definition of $L^{p,\nu,\kappa}(X)$ does not depend on κ as long as X is the Euclidean space and $\kappa > 1$ (see [18, 31]) and that $L^{p,\nu,\kappa}(X)$ can depend on κ (see [30]).

We consider a function

$$\Phi(x,t)\colon X \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1) and (Φ2):

(Φ1) $\Phi(\,\cdot\,,t)$ is measurable on X for each $t \ge 0$ and $\Phi(x,\cdot\,)$ is convex on $[0,\infty)$ for each $x \in X$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \Phi(x,1) \le A_1 \quad \text{for all } x \in X.$$

For $\kappa \ge 1$, the Musielak–Orlicz–Morrey space $L^{\Phi,\nu(\cdot),\kappa}(X)$ is defined by

$$L^{\Phi,\nu(\cdot),\kappa}(X) = \left\{ f \in L^1_{\mathrm{loc}}(X) \;\middle|\; \sup_{\substack{x\in X\\ 0<r<d_X}} \frac{r^{\nu(x)}}{\mu(B(x,\kappa r))} \int_{B(x,r)} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) d\mu(y) < \infty \text{ for some } \lambda > 0 \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi,\nu(\cdot),\kappa}(X)} = \inf\left\{ \lambda > 0 \;\middle|\; \sup_{\substack{x\in X\\ 0<r<d_X}} \frac{r^{\nu(x)}}{\mu(B(x,\kappa r))} \int_{B(x,r)} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) d\mu(y) \le 1 \right\}$$

(see [12, 20]).

Let $q(\cdot)$ be a measurable function on X such that

(Q1) $1 \le q^- := \inf_{x\in X} q(x) \le \sup_{x\in X} q(x) =: q^+ < \infty$,

(Q2) $q(\cdot)$ is log-Hölder continuous on X, namely

$$|q(x) - q(y)| \le \frac{C_q}{\log(e + 1/d(x,y))}, \quad x, y \in X$$

with a constant $C_q \ge 0$.

In what follows, set

$$\Phi(x,t) = t^{p(x)} + (b(x)t)^{q(x)},$$

where $p(x) < q(x)$ and $b(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$ (cf. [5]).

3. Lemmas

Let's begin with the following lemma.

Lemma 3.1. (see [16, Lemma 2.1] or [14, Lemma 2.7]) *There exists a constant $C > 0$ such that*

$$\frac{r^{\nu(x)/p(x)}}{\mu(B(x,\kappa r))} \int_{B(x,r)} |f(y)|\, d\mu(y) \le C$$

for all $x \in X$, $0 < r < d_X$ and measurable functions f on X with $\|f\|_{L^{p(\cdot),\nu(\cdot),\kappa}(X)} \le 1$.

We give an estimate inside and outside balls.

Lemma 3.2. *Let $\beta \in \mathbf{R}$, $\iota > 0$ and $\rho_1 \in (\rho)$. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot),\nu(\cdot),\kappa}(X)} \le 1$. If $1 \le \kappa < \tau$, then there exists a constant $C > 0$ such that*

$$\int_{B(x,r)} \frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y) \le C \int_0^{k_2\iota r} t^{-\nu(x)/p(x)+\beta}\rho_1(x,t)\,\frac{dt}{t} \tag{3.1}$$

and

$$\int_{X\setminus B(x,r)} \frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y) \le C \int_{k_1\iota r}^{2k_2\iota d_X} t^{-\nu(x)/p(x)+\beta}\rho_1(x,t)\,\frac{dt}{t} \tag{3.2}$$

for all $x \in X$ and $0 < r \le d_X$.

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot),\nu(\cdot),\kappa}(X)} \le 1$. Take $\gamma \in \mathbf{R}$ such that $1 < \gamma \le \min\{1/k, \tau/\kappa, 2\}$. If $y \in B(x,\gamma^j r) \setminus B(x,\gamma^{j-1}r)$ for $j \in \mathbf{Z}$, then we see from (1.1) that

$$\begin{aligned}\frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} &\le \frac{\max\{1,\gamma^{-\beta}\}(\gamma^j r)^{\beta}}{\mu(B(x,\tau\gamma^{j-1}r))} \sup_{\gamma^{j-1}\iota r \le s \le \gamma^j \iota r} \rho_1(x,s)\\ &\le \frac{\max\{1,\gamma^{-\beta}\}(\gamma^j r)^{\beta}}{\mu(B(x,\tau\gamma^{j-1}r))} \sup_{k\gamma^{j}\iota r \le s \le \gamma^j \iota r} \rho_1(x,s)\\ &\le \frac{C_{\rho_1}\max\{1,\gamma^{-\beta}\}(\gamma^j r)^{\beta}}{\mu(B(x,\kappa\gamma^{j}r))} \int_{\gamma^j k_1\iota r}^{\gamma^j k_2 \iota r} \rho_1(x,s)\,\frac{ds}{s}\end{aligned}$$

since $\gamma \le \min\{1/k, \tau/\kappa\}$. By Lemma 3.1, we obtain

$$\begin{aligned}&\int_{B(x,\gamma^j r)\setminus B(x,\gamma^{j-1}r)} \frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y)\\ &\le C_{\rho_1}\max\{1,\gamma^{-\beta}\}(\gamma^j r)^{\beta} \int_{\gamma^j k_1\iota r}^{\gamma^j k_2 \iota r} \rho_1(x,s)\,\frac{ds}{s} \cdot \frac{1}{\mu(B(x,\kappa\gamma^j r))} \int_{B(x,\gamma^j r)} f(y)\,d\mu(y)\\ &\le C_1 C_{\rho_1}\max\{1,2^{-\beta}\}(\gamma^j r)^{-\nu(x)/p(x)+\beta} \int_{\gamma^j k_1\iota r}^{\gamma^j k_2 \iota r} \rho_1(x,s)\,\frac{ds}{s}\\ &\le C_1 C_{\rho_1}\max\{1,2^{-\beta}\}\\ &\quad \times \max\left\{(\iota k_1)^{\nu(x)/p(x)-\beta}, (\iota k_2)^{\nu(x)/p(x)-\beta}\right\} \int_{\gamma^j k_1\iota r}^{\gamma^j k_2 \iota r} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s}\\ &\le C_2 \int_{\gamma^j k_1\iota r}^{\gamma^j k_2 \iota r} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s}\end{aligned}$$

for $j \in \mathbf{Z}$, where

$$C_2 = C_1 C_{\rho_1}\max\{1,2^{-\beta}\}\max\left\{(\iota k_1)^{\nu^+/p^- -\beta}, (\iota k_1)^{\nu^-/p^+ -\beta}, (\iota k_2)^{\nu^+/p^- -\beta}, (\iota k_2)^{\nu^-/p^+ -\beta}\right\}.$$

Therefore we obtain

$$\begin{aligned}
&\int_{B(x,r)} \frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y) \\
&= \sum_{j=0}^{\infty} \int_{B(x,\gamma^{-j}r)\setminus B(x,\gamma^{-j-1}r)} \frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y) \\
&\le C_2 \sum_{j=0}^{\infty} \int_{\gamma^{-j}k_1\iota r}^{\gamma^{-j}k_2\iota r} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s}.
\end{aligned}$$

Let j_0 be the smallest integer such that $k_2/k_1 \le \gamma^{j_0}$. Then we have

$$\begin{aligned}
\int_{B(x,r)} \frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y) &\le C_2 \sum_{j=0}^{\infty} \int_{\gamma^{-j-j_0}k_2\iota r}^{\gamma^{-j}k_2\iota r} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s} \\
&\le j_0 C_2 \int_0^{k_2\iota r} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s},
\end{aligned}$$

which proves (3.1).

Let j_1 be the smallest integer such that $d_X \le \gamma^{j_1} r$. Then we obtain

$$\begin{aligned}
&\int_{X\setminus B(x,r)} \frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y) \\
&= \sum_{j=1}^{j_1} \int_{B(x,\gamma^{j}r)\setminus B(x,\gamma^{j-1}r)} \frac{d(x,y)^{\beta}\rho_1(x,\iota d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y) \\
&\le C_2 \sum_{j=1}^{j_1} \int_{\gamma^{j}k_1\iota r}^{\gamma^{j}k_2\iota r} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s} \\
&\le C_2 \sum_{j=1}^{j_1} \int_{\gamma^{j-j_0}k_2\iota r}^{\gamma^{j}k_2\iota r} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s} \\
&\le j_0 C_2 \int_{k_1\iota r}^{\gamma k_2\iota d_X} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s} \\
&\le j_0 C_2 \int_{k_1\iota r}^{2k_2\iota d_X} s^{-\nu(x)/p(x)+\beta}\rho_1(x,s)\,\frac{ds}{s},
\end{aligned}$$

which proves (3.2). □

Here note that $2k_2\iota d_X$ in (3.2) can be replaced by $ak_2\iota d_X$ with $a > 1$.

4. Continuity of generalized Riesz potentials

Before we state our theorem we consider the following conditions:

($\rho\mu$) there are constants $\eta_1 > 0$, $\eta_2 > 0$, $\iota_1 > 0$, $\iota_2 \ge 1$, $\sigma_1 > 1$ and $c_1 > 0$ such that

$$\left|\frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(x,d(z,y))}{\mu(B(z,\tau d(z,y)))}\right| \le c_1 \frac{d(x,z)^{\eta_1}}{d(x,y)^{\eta_2}} \frac{\rho(x,\iota_1 d(x,y))}{\mu(B(x,\iota_2 d(x,y)))} \tag{4.1}$$

whenever $d(x,z) \le d(x,y)/\sigma_1$,

($\rho 1$) there are functions $h(x,z)\colon X \times X \to [0,\infty)$ and $\widetilde{\rho} \in (\rho)$ and constants $\iota_3 > 0$, $\iota_4 > 0$, $\sigma_2 > 1$ and $c_2 > 0$ such that

$$|\rho(x,d(z,y)) - \rho(z,d(z,y))| \le c_2 h(x,z)\big\{\widetilde{\rho}(x,\iota_3 d(x,y)) + \widetilde{\rho}(z,\iota_4 d(z,y))\big\} \tag{4.2}$$

whenever $d(x,z) \le d(x,y)/\sigma_2$.

Let $\sigma = \max\{\sigma_1,\sigma_2\}$. For $x,z \in X$ and $0 < r \le d_X$, we consider the functions

$$\begin{aligned}\psi_1(x,z,r) = &\int_0^{k_2\sigma r} t^{-\nu(x)/p(x)+\theta}\rho(x,t)\,\frac{dt}{t} + \int_0^{k_2\sigma r} t^{-\nu(x)/q(x)}\rho(x,t)\,\frac{dt}{t}\\ &+ \int_0^{k_2(\sigma+1)r} t^{-\nu(z)/p(z)+\theta}\rho(z,t)\,\frac{dt}{t} + \int_0^{k_2(\sigma+1)r} t^{-\nu(z)/q(z)}\rho(z,t)\,\frac{dt}{t}\\ &+ r^\theta \int_{k_1(\sigma-1)r}^{2k_2 d_X} t^{-\nu(z)/p(z)}\rho(z,t)\,\frac{dt}{t}\end{aligned}$$

and

$$\begin{aligned}\psi_2(x,z,r) = r^{\eta_1} &\int_{k_1\sigma\iota_1 r}^{2k_2\iota_1 d_X} t^{-\nu(x)/p(x)+\theta-\eta_2}\rho(x,t)\,\frac{dt}{t}\\ &+ r^{\eta_1}\int_{k_1\sigma\iota_1 r}^{2k_2\iota_1 d_X} t^{-\nu(x)/q(x)-\eta_2}\rho(x,t)\,\frac{dt}{t}.\end{aligned}$$

Further we set

$$\begin{aligned}&\psi_3(x,z,r)\\ &= h(x,z)\int_{k_1\sigma\iota_3 r}^{2k_2\iota_3 d_X} t^{-\nu(x)/p(x)+\theta}\widetilde{\rho}(x,t)\,\frac{dt}{t} + h(x,z)\int_{k_1\sigma\iota_3 r}^{2k_2\iota_3 d_X} t^{-\nu(x)/q(x)}\widetilde{\rho}(x,t)\,\frac{dt}{t}\\ &\quad + h(x,z)\int_{k_1(\sigma-1)\iota_4 r}^{2k_2\iota_4 d_X} t^{-\nu(z)/p(z)+\theta}\widetilde{\rho}(z,t)\,\frac{dt}{t} + h(x,z)\int_{k_1(\sigma-1)\iota_4 r}^{2k_2\iota_4 d_X} t^{-\nu(z)/q(z)}\widetilde{\rho}(z,t)\,\frac{dt}{t}\end{aligned}$$

for $x,z \in X$ and $0 < r \le d_X$.

We prove the following theorem, as an extension of [23, Theorem 1] and [24, Theorem 2.2]. See also [15, Theorem 4.1] and [16, Theorem 4.1].

Theorem 4.1. *Assume that ρ satisfies ($\rho\mu$) and ($\rho 1$). If $1 \le \kappa < \min\{\tau(1-1/\sigma)-1/\sigma, \iota_2\}$, then there exists a constant $C > 0$ such that*

$$\big|b(x)I_{\rho,\tau}f(x) - b(z)I_{\rho,\tau}f(z)\big| \le C\sum_{k=1}^{3}\psi_k(x,z,d(x,z))$$

for all $x, z \in X$ *with* $\psi_1(x, z, d(x, z)) < \infty$ *and measurable functions* f *on* X *with* $\|f\|_{L^{\Phi,\nu(\cdot),\kappa}(X)} \le 1$.

Remark 4.2. Let $x, z \in X$ with $x \neq z$ and $\psi_1(x, z, d(x, z)) < \infty$. Then note that

$$\int_0^{k_2\sigma d(x,z)} t^{-\nu(x)/p(x)+\theta}\rho(x,t)\,\frac{dt}{t} + \int_0^{k_2\sigma d(x,z)} t^{-\nu(x)/q(x)}\rho(x,t)\,\frac{dt}{t}$$
$$+ \int_0^{k_2(\sigma+1)d(x,z)} t^{-\nu(z)/p(z)+\theta}\rho(z,t)\,\frac{dt}{t} + \int_0^{k_2(\sigma+1)d(x,z)} t^{-\nu(z)/q(z)}\rho(z,t)\,\frac{dt}{t} < \infty.$$

Let f be a nonnegative measurable function f on X with $\|f\|_{L^{\Phi,\nu(\cdot),\kappa}(X)} \le 1$. By Lemma 3.2 and (1.2), we see that

$$\begin{aligned}
&\int_X \frac{d(x,y)^\theta\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y)\\
&= \int_{B(x,d(x,z))} \frac{d(x,y)^\theta\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y)\\
&\quad + \int_{X\setminus B(x,d(x,z))} \frac{d(x,y)^\theta\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y)\,d\mu(y)\\
&\le C\left\{\int_0^{k_2 d(x,z)} t^{-\nu(x)/p(x)+\theta}\rho(x,t)\,\frac{dt}{t} + \int_{k_1 d(x,z)}^{2k_2 d_X} t^{-\nu(x)/p(x)+\theta}\rho(x,t)\,\frac{dt}{t}\right\}\\
&\le C\left\{\int_0^{k_2\sigma d(x,z)} t^{-\nu(x)/p(x)+\theta}\rho(x,t)\,\frac{dt}{t} + d(x,z)^{-\nu(x)/p(x)}\int_0^{2k_2 d_X} \rho(x,t)\,\frac{dt}{t}\right\}\\
&< \infty
\end{aligned}$$

and that

$$\begin{aligned}
&\int_X \frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))}\{b(y)f(y)\}\,d\mu(y)\\
&= \int_{B(x,d(x,z))} \frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))}\{b(y)f(y)\}\,d\mu(y)\\
&\quad + \int_{X\setminus B(x,d(x,z))} \frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))}\{b(y)f(y)\}\,d\mu(y)\\
&\le C\left\{\int_0^{k_2 d(x,z)} t^{-\nu(x)/q(x)}\rho(x,t)\,\frac{dt}{t} + \int_{k_1 d(x,z)}^{2k_2 d_X} t^{-\nu(x)/q(x)}\rho(x,t)\,\frac{dt}{t}\right\}\\
&\le C\left\{\int_0^{k_2\sigma d(x,z)} t^{-\nu(x)/q(x)}\rho(x,t)\,\frac{dt}{t} + d(x,z)^{-\nu(x)/q(x)}\int_0^{2k_2 d_X} \rho(x,t)\,\frac{dt}{t}\right\}\\
&< \infty.
\end{aligned}$$

Hence

$$b(x)I_{\rho,\tau}f(x) \le \int_X \frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))}|b(x)-b(y)|f(y)\,d\mu(y)$$

$$
\begin{aligned}
&+ \int_X \frac{\rho(x, d(x,y))}{\mu(B(x, \tau d(x,y)))} b(y) f(y)\, d\mu(y) \\
&\le C \int_X \frac{d(x,y)^\theta \rho(x, d(x,y))}{\mu(B(x, \tau d(x,y)))} f(y)\, d\mu(y) \\
&+ \int_X \frac{\rho(x, d(x,y))}{\mu(B(x, \tau d(x,y)))} \{b(y) f(y)\}\, d\mu(y) < \infty.
\end{aligned}
$$

Similarly, we see that $b(z) I_{\rho,\tau} f(z) < \infty$, so that $\left|b(x) I_{\rho,\tau} f(x) - b(z) I_{\rho,\tau} f(z)\right|$ in Theorem 4.1 is well defined.

Proof of Theorem 4.1. We may assume that f is nonnegative on X. Let f be a nonnegative function on X such that $\|f\|_{L^{\Phi,\nu(\cdot),\kappa}(X)} \le 1$. Let $x, z \in X$ and set $r = d(x,z)$. First we estimate the following three terms:

$$
\begin{aligned}
I_1(x) &= b(x) \int_{B(x,\sigma r)} \frac{\rho(x, d(x,y))}{\mu(B(x, \tau d(x,y)))} f(y)\, d\mu(y), \\
I_2(z) &= b(z) \int_{B(z,(\sigma+1) r)} \frac{\rho(z, d(z,y))}{\mu(B(z, \tau d(z,y)))} f(y)\, d\mu(y)
\end{aligned}
$$

and

$$
I_3(z) = r^\theta \int_{X \setminus B(z,(\sigma-1) r)} \frac{\rho(z, d(z,y))}{\mu(B(z, \tau d(z,y)))} f(y)\, d\mu(y).
$$

For $I_1(x)$, we have

$$
\begin{aligned}
I_1(x) &\le \int_{B(x,\sigma r)} \frac{\rho(x, d(x,y))}{\mu(B(x, \tau d(x,y)))} |b(x) - b(y)| f(y)\, d\mu(y) \\
&\quad + \int_{B(x,\sigma r)} \frac{\rho(x, d(x,y))}{\mu(B(x, \tau d(x,y)))} b(y) f(y)\, d\mu(y) \\
&\le C \int_{B(x,\sigma r)} \frac{d(x,y)^\theta \rho(x, d(x,y))}{\mu(B(x, \tau d(x,y)))} f(y)\, d\mu(y) \\
&\quad + \int_{B(x,\sigma r)} \frac{\rho(x, d(x,y))}{\mu(B(x, \tau d(x,y)))} \{b(y) f(y)\}\, d\mu(y) \\
&= C I_{11}(x) + I_{12}(x).
\end{aligned}
$$

We obtain from (3.1),

$$
I_{11}(x) \le C \int_0^{k_2 \sigma r} t^{-\nu(x)/p(x) + \theta} \rho(x,t)\, \frac{dt}{t} \quad \text{and} \quad I_{12}(x) \le C \int_0^{k_2 \sigma r} t^{-\nu(x)/q(x)} \rho(x,t)\, \frac{dt}{t}
$$

since $1 \le \kappa < \tau$. For $I_3(z)$, we have by (3.2),

$$
I_3(z) \le C r^\theta \int_{k_1(\sigma-1) r}^{2 k_2 d_X} t^{-\nu(z)/p(z)} \rho(z,t)\, \frac{dt}{t}
$$

since $1 \le \kappa < \tau$. Therefore, we find

$$I_1(x) + I_2(z) + I_3(z) \le C\psi_1(x, z, r). \tag{4.3}$$

Next we estimate the following term:

$$I_4(z) = r^{\eta_1} b(x) \int_{X\setminus B(x,\sigma r)} \frac{d(x,y)^{-\eta_2}\rho(x, \iota_1 d(x,y))}{\mu(B(x, \iota_2 d(x,y)))} f(y)\, d\mu(y).$$

Then we have

$$\begin{aligned} I_4(x) &\le r^{\eta_1} \int_{X\setminus B(x,\sigma r)} \frac{d(x,y)^{-\eta_2}\rho(x, \iota_1 d(x,y))}{\mu(B(x, \iota_2 d(x,y)))} |b(x) - b(y)| f(y)\, d\mu(y) \\ &\quad + r^{\eta_1} \int_{X\setminus B(x,\sigma r)} \frac{d(x,y)^{-\eta_2}\rho(x, \iota_1 d(x,y))}{\mu(B(x, \iota_2 d(x,y)))} b(y) f(y)\, d\mu(y) \\ &\le Cr^{\eta_1} \int_{X\setminus B(x,\sigma r)} \frac{d(x,y)^{\theta-\eta_2}\rho(x, \iota_1 d(x,y))}{\mu(B(x, \iota_2 d(x,y)))} f(y)\, d\mu(y) \\ &\quad + r^{\eta_1} \int_{X\setminus B(x,\sigma r)} \frac{d(x,y)^{-\eta_2}\rho(x, \iota_1 d(x,y))}{\mu(B(x, \iota_2 d(x,y)))} \{b(y) f(y)\}\, d\mu(y) \\ &= CI_{41}(x) + I_{42}(x). \end{aligned}$$

Note from (3.2) that

$$I_{41}(x) \le Cr^{\eta_1} \int_{k_1\sigma\iota_1 r}^{2k_2\iota_1 d_X} t^{-\nu(x)/p(x)+\theta-\eta_2} \rho(x,t)\, \frac{dt}{t}$$

and that

$$I_{42}(x) \le Cr^{\eta_1} \int_{k_1\sigma\iota_1 r}^{2k_2\iota_1 d_X} t^{-\nu(x)/q(x)-\eta_2} \rho(x,t)\, \frac{dt}{t}$$

since $1 \le \kappa < \iota_2$. Therefore, we find

$$I_4(x) \le C\psi_2(x, z, r). \tag{4.4}$$

Finally we estimate the following two terms:

$$I_5(x,z) = b(x)h(x,z) \int_{X\setminus B(x,\sigma r)} \frac{\widetilde{\rho}(x, \iota_3 d(x,y))}{\mu(B(z, \tau d(z,y)))} f(y)\, d\mu(y)$$

and

$$I_6(x,z) = b(x)h(x,z) \int_{X\setminus B(z,(\sigma-1)r)} \frac{\widetilde{\rho}(z, \iota_4 d(z,y))}{\mu(B(z, \tau d(z,y)))} f(y)\, d\mu(y).$$

For $I_5(x,z)$, set $\tau' = \tau(1 - 1/\sigma) - 1/\sigma$. Note that

$$\left(1 - \frac{1}{\sigma}\right) d(x,y) \le d(z,y) \le \left(1 + \frac{1}{\sigma}\right) d(x,y) \tag{4.5}$$

and that

$$B(x, \tau' d(x,y)) \subset B(z, \tau d(z,y))$$

for $y \in X \setminus B(x, \sigma r)$. Hence, we have

$$\begin{aligned}
I_5(x,z) &\le b(x)h(x,z) \int_{X\setminus B(x,\sigma r)} \frac{\widetilde{\rho}(x, \iota_3 d(x,y))}{\mu(B(x, \tau' d(x,y)))} f(y)\, d\mu(y) \\
&\le h(x,z) \int_{X\setminus B(x,\sigma r)} \frac{\widetilde{\rho}(x, \iota_3 d(x,y))}{\mu(B(x, \tau' d(x,y)))} |b(x) - b(y)| f(y)\, d\mu(y) \\
&\quad + h(x,z) \int_{X\setminus B(x,\sigma r)} \frac{\widetilde{\rho}(x, \iota_3 d(x,y))}{\mu(B(x, \tau' d(x,y)))} b(y) f(y)\, d\mu(y) \\
&\le Ch(x,z) \int_{X\setminus B(x,\sigma r)} \frac{d(x,y)^\theta \widetilde{\rho}(x, \iota_3 d(x,y))}{\mu(B(x, \tau' d(x,y)))} f(y)\, d\mu(y) \\
&\quad + h(x,z) \int_{X\setminus B(x,\sigma r)} \frac{\widetilde{\rho}(x, \iota_3 d(x,y))}{\mu(B(x, \tau' d(x,y)))} \{b(y) f(y)\}\, d\mu(y) \\
&= CI_{51}(x,z) + I_{52}(x,z).
\end{aligned}$$

Note from (3.2) that

$$I_{51}(x,z) \le Ch(x,z) \int_{k_1\sigma\iota_3 r}^{2k_2\iota_3 d_X} t^{-\nu(x)/p(x)+\theta} \widetilde{\rho}(x,t)\, \frac{dt}{t}$$

and that

$$I_{52}(x,z) \le Ch(x,z) \int_{k_1\sigma\iota_3 r}^{2k_2\iota_3 d_X} t^{-\nu(x)/q(x)} \widetilde{\rho}(x,t)\, \frac{dt}{t}$$

since $1 \le \kappa < \tau'$. By (4.5) we have

$$\begin{aligned}
I_6(x,z) &\le h(x,z) \int_{X\setminus B(z,(\sigma-1)r)} \frac{\widetilde{\rho}(z, \iota_4 d(z,y))}{\mu(B(z, \tau d(z,y)))} |b(x) - b(y)| f(y)\, d\mu(y) \\
&\quad + h(x,z) \int_{X\setminus B(z,(\sigma-1)r)} \frac{\widetilde{\rho}(z, \iota_4 d(z,y))}{\mu(B(z, \tau d(z,y)))} b(y) f(y)\, d\mu(y) \\
&\le Ch(x,z) \int_{X\setminus B(z,(\sigma-1)r)} \frac{d(x,y)^\theta \widetilde{\rho}(z, \iota_4 d(z,y))}{\mu(B(z, \tau d(z,y)))} f(y)\, d\mu(y) \\
&\quad + h(x,z) \int_{X\setminus B(z,(\sigma-1)r)} \frac{\widetilde{\rho}(z, \iota_4 d(z,y))}{\mu(B(z, \tau d(z,y)))} b(y) f(y)\, d\mu(y) \\
&\le Ch(x,z) \int_{X\setminus B(z,(\sigma-1)r)} \frac{d(z,y)^\theta \widetilde{\rho}(z, \iota_4 d(z,y))}{\mu(B(z, \tau d(z,y)))} f(y)\, d\mu(y) \\
&\quad + h(x,z) \int_{X\setminus B(z,(\sigma-1)r)} \frac{\widetilde{\rho}(z, \iota_4 d(z,y))}{\mu(B(z, \tau d(z,y)))} \{b(y) f(y)\}\, d\mu(y) \\
&= CI_{61}(x,z) + I_{62}(x,z).
\end{aligned}$$

Note from (3.2) that

$$I_{61}(x,z) \le Ch(x,z)\int_{k_1(\sigma-1)\iota_4 r}^{2k_2\iota_4 d_X} t^{-\nu(z)/p(z)+\theta}\widetilde{\rho}(z,t)\,\frac{dt}{t}$$

and that

$$I_{62}(x,z) \le Ch(x,z)\int_{k_1(\sigma-1)\iota_4 r}^{2k_2\iota_4 d_X} t^{-\nu(z)/q(z)}\widetilde{\rho}(z,t)\,\frac{dt}{t}$$

since $1 \le \kappa < \tau$. Therefore, we find

$$I_5(x,z) + I_6(x,z) \le C\psi_3(x,z,r). \tag{4.6}$$

Note from (4.1) and (4.2),

$$\begin{aligned}
&\left|\frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(z,d(z,y))}{\mu(B(z,\tau d(z,y)))}\right| \\
&\le \left|\frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(x,d(z,y))}{\mu(B(z,\tau d(z,y)))}\right| + \left|\frac{\rho(x,d(z,y))}{\mu(B(z,\tau d(z,y)))} - \frac{\rho(z,d(z,y))}{\mu(B(z,\tau d(z,y)))}\right| \\
&\le C\left\{ r^{\eta_1}\frac{d(x,y)^{-\eta_2}\rho(x,\iota_1 d(x,y))}{\mu(B(x,\iota_2 d(x,y)))} + h(x,z)\frac{\widetilde{\rho}(x,\iota_3 d(x,y)) + \widetilde{\rho}(z,\iota_4 d(z,y))}{\mu(B(z,\tau d(z,y)))}\right\}
\end{aligned}$$

for $y \in X \setminus B(x,\sigma r)$, so that

$$\begin{aligned}
&\left|b(x)I_{\rho,\tau}f(x) - b(z)I_{\rho,\tau}f(z)\right| \\
&\le b(x)\int_{B(x,\sigma r)}\frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))}f(y)\,d\mu(y) + b(z)\int_{B(x,\sigma r)}\frac{\rho(z,d(z,y))}{\mu(B(z,\tau d(z,y)))}f(y)\,d\mu(y) \\
&\quad + |b(x)-b(z)|\int_{X\setminus B(x,\sigma r)}\frac{\rho(z,d(z,y))}{\mu(B(z,\tau d(z,y)))}f(y)\,d\mu(y) \\
&\quad + b(x)\int_{X\setminus B(x,\sigma r)}\left|\frac{\rho(x,d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(z,d(z,y))}{\mu(B(z,\tau d(z,y)))}\right| f(y)\,d\mu(y) \\
&\le C\left\{I_1(x) + I_2(z) + I_3(z) + I_4(x) + I_5(x,z) + I_6(x,z)\right\}.
\end{aligned}$$

Hence we obtain by (4.3), (4.4) and (4.6),

$$\left|b(x)I_{\rho,\tau}f(x) - b(z)I_{\rho,\tau}f(z)\right| \le C\sum_{k=1}^{3}\psi_k(x,z,r).$$

Thus we complete the proof. □

Remark 4.3. (1) If ρ satisfies the doubling condition, that is, there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\rho(x,r)}{\rho(x,s)} \le C$$

for $x \in X$ and $1/2 \le r/s \le 2$, then ρ satisfies (1.1) whenever $k = 1/2$ and $2k_1 = k_2$.

(2) If ρ is increasing in the second variable, then ρ satisfies (1.1) with $k = 1/2$, $k_1 = 1$ and $k_2 = 2$.

(3) If ρ is decreasing in the second variable, then ρ satisfies (1.1) with $k = 1/2$, $k_1 = 1/4$ and $k_2 = 1/2$.

Example 4.4. (i) Let $\alpha(\cdot)$ be a measurable function on X such that

$$0 < \alpha^- := \inf_{x \in X} \alpha(x) \le \sup_{x \in X} \alpha(x) =: \alpha^+ < \infty$$

and $\rho(x, r) = r^{\alpha(x)}$. Then ρ satisfies (1.1) and (1.2) with $k = 1/2$, $k_1 = 1$ and $k_2 = 2$ by Remark 4.3(1) or (2).

(ii) Let $x_0 \in X$ and $\rho(x, r) = (1 + d(x_0, x)/r)r^\alpha$ for some $\alpha > 0$. Then ρ satisfies (1.1) with $k = 1/2$, $k_1 = 1$ and $k_2 = 2$ by Remark 4.3(1). Further, if $\alpha > 1$, then

$$\int_0^1 \rho(x, s) \frac{ds}{s} \le (1 + d(x_0, x)) \int_0^1 s^{\alpha - 1} \frac{ds}{s} \le \frac{1 + d_X}{\alpha - 1},$$

so that ρ satisfies (1.2).

(iii) Let $\alpha > 0$ and let $A(\cdot)$ be a positive measurable function on X. Set

$$\rho(x, r) = \begin{cases} A(x) r^\alpha & \text{for } 0 < r < 1, \\ A(x) e^{-(r-1)} & \text{for } r \ge 1. \end{cases}$$

Then ρ satisfies (1.1) and (1.2) with $k = 1/2$, $k_1 = 1/4$ and $k_2 = 1/2$ by Remark 4.3(1) and (3). See [10].

(iv) Let $\rho(x, r) = \mu(B(x, \tau r))^\eta$ for some $0 < \eta < 1$ and $\tau \ge 1$. Then ρ satisfies (1.1) with $k = 1/2$, $k_1 = 1$ and $k_2 = 2$ by Remark 4.3(2). Further, if μ satisfies the upper Ahlfors condition $\mu(B(x, r)) \le C r^Q$ ($x \in X$, $r > 0$) for some $Q > 0$, then ρ satisfies (1.2). See [27, 32].

(v) Let $\alpha(\cdot)$ be as in (i) and let $\rho(x, r) = r^{\alpha(x)} e^{-a/r} (\log(e + 1/r))^\beta$ for $a \ge 0$ and $\beta \in \mathbf{R}$. Then ρ satisfies (1.1) and (1.2) with $k = 1/2$, $k_1 = 1$ and $k_2 = 2$. In fact, there exists a constant $C_1 > 0$ such that

$$r_1^{-\alpha^-/2} \rho(x, r_1) \le C_1 r_2^{-\alpha^-/2} \rho(x, r_2)$$

whenever $0 < r_1 < r_2$, so that

$$\sup_{r/2 \le s \le r} \rho(x, s) \le C_1 \rho(x, r) \le \frac{C_1^2}{\log 2} \int_r^{2r} \rho(x, s) \frac{ds}{s}$$

for all $r > 0$ and

$$\int_0^1 \rho(x, s) \frac{ds}{s} \le C_1 \rho(x, 1) \int_0^1 s^{\alpha^-/2} \frac{ds}{s} \le \frac{2C_1}{\alpha^-} e^{-a} (\log(e + 1))^\beta$$

for all $x \in X$.

5. Corollaries

In this section, we give consequences of Theorem 4.1.

Let $\alpha(\cdot)$ be a measurable function on X such that $0 < \alpha^- \le \alpha^+ < \infty$.

Remark 5.1. Let $\rho(x,r) = r^{\alpha(x)} e^{-a/r} (\log(e+1/r))^{\beta}$ for $a \ge 0$ and $\beta \in \mathbf{R}$. Then (ρ1) holds for $\iota_3 = 3/2$, $\iota_4 = 1$, $\sigma_2 = 2$, $h(x,z) = |\alpha(x) - \alpha(z)|$ and $\widetilde{\rho}(x,r) = r^{\alpha(x)} e^{-a/r} (\log(e+1/r))^{\beta+1}$.

In fact, we have by the mean value property

$$\begin{aligned}
&|\rho(x, d(z,y)) - \rho(z, d(z,y))| \\
&= e^{-a/d(z,y)} (\log(e + 1/d(z,y)))^{\beta} \left| d(z,y)^{\alpha(x)} - d(z,y)^{\alpha(z)} \right| \\
&\le e^{-a/d(z,y)} (\log(e + 1/d(z,y)))^{\beta} |\alpha(x) - \alpha(z)| \left(d(z,y)^{\alpha(x)} + d(z,y)^{\alpha(z)} \right) |\log d(z,y)| \\
&\le C h(x,z) \left\{ \widetilde{\rho}(x, d(z,y)) + \widetilde{\rho}(z, d(z,y)) \right\} \\
&\le C h(x,z) \left\{ \widetilde{\rho}(x, 3d(x,y)/2) + \widetilde{\rho}(z, d(z,y)) \right\}
\end{aligned}$$

whenever $d(x,z) \le d(x,y)/2$ since $d(x,y)/2 \le d(z,y) \le 3d(x,y)/2$ for all $x, z \in X$ with $d(x,z) \le d(x,y)/2$.

Remark 5.2. Let G be an open bounded set in $\mathbf{R}^N$. Let $\rho(x,r) = r^{\alpha(x)} e^{-a/r} (\log(e+1/r))^{\beta}$ for $a \ge 0$ and $\beta \in \mathbf{R}$.

(1) If $a = 0$, then ($\rho\mu$) holds for $\eta_1 = \eta_2 = \iota_1 = \iota_2 = 1$ and $\sigma_1 = 2$.

(2) If $a > 0$, then ($\rho\mu$) holds for $\eta_1 = 1$, $\eta_2 = 2$, $\iota_1 = 3/2$, $\iota_2 = 1$ and $\sigma_1 = 2$. We refer to [24, Remark 2.3].

We set

$$\psi_4(x,z) = d(x,z)^{\alpha(x)} \left(d(x,z)^{-\nu(x)/p(x)+\theta} + d(x,z)^{-\nu(x)/q(x)} \right)$$

and

$$\psi_5(x,z) = d(x,z)^{\alpha(z)} \left(d(x,z)^{-\nu(z)/p(z)+\theta} + d(x,z)^{-\nu(z)/q(z)} \right)$$

for $x, z \in X$.

As in the proof of [24, Corollary 3.1], we obtain the following corollary by Theorem 4.1.

Corollary 5.3. *Let $\rho(x,r) = r^{\alpha(x)} (\log(e+1/r))^{\beta}$ for $\beta \in \mathbf{R}$. Let X be a non-doubling metric measure space. Assume that ($\rho\mu$) holds. Suppose*

$$\inf_{x \in X} (\nu(x) - \alpha(x)p(x)) > 0, \quad \inf_{x \in X} (\nu(x) - (\alpha(x) + \theta - \eta_2)p(x)) > 0$$

and

$$\inf_{x \in X} ((\alpha(x) + \theta)p(x) - \nu(x)) > 0.$$

Further suppose

$$\inf_{x\in X}(\nu(x)-(\alpha(x)-\eta_2)q(x))>0 \quad \textit{and} \quad \inf_{x\in X}(\alpha(x)q(x)-\nu(x))>0.$$

If $1\le\kappa<\min\{\tau(1-1/\sigma)-1/\sigma,\iota_2\}$*, then there exists a constant* $C>0$ *such that*

$$\begin{aligned}&\big|b(x)I_{\rho,\tau}f(x)-b(z)I_{\rho,\tau}f(z)\big|\\ &\le C\Big[\big(\psi_4(x,z)+\psi_5(x,z)+\min\big\{d(x,z)^{\eta_1-\eta_2}\psi_4(x,z),d(x,z)^{\eta_1-\eta_2}\psi_5(x,z)\big\}\big)\\ &\qquad\times(\log(e+1/d(x,z)))^{\beta}+|\alpha(x)-\alpha(z)|\Big]\end{aligned}$$

for all $x,z\in X$ *and measurable functions* f *on* X *with* $\|f\|_{L^{\Phi,\nu(\cdot),\kappa}(X)}\le 1$.

Remark 5.4. The assumptions like $\inf_{x\in X}(\nu(x)-\alpha(x)p(x))>0$ in Corollary 5.3 were considered in [24, Corollary 3.1].

When $\rho(x,r)=r^{\alpha(x)}$, we write $I_{\rho,\tau}f=I_{\alpha(\cdot),\tau}f$, which is called the Riesz potential of variable order $\alpha(\cdot)$. If we take $\beta=0$ in Corollary 5.3, we obtain the next corollary.

Corollary 5.5. *Let* $\rho(x,r)=r^{\alpha(x)}$*. Let* X *be a non-doubling metric measure space. Assume that* $(\rho\mu)$ *holds. Suppose*

$$\inf_{x\in X}(\nu(x)-\alpha(x)p(x))>0,\quad \inf_{x\in X}(\nu(x)-(\alpha(x)+\theta-\eta_2)p(x))>0$$

and

$$\inf_{x\in X}((\alpha(x)+\theta)p(x)-\nu(x))>0.$$

Further suppose

$$\inf_{x\in X}(\nu(x)-(\alpha(x)-\eta_2)q(x))>0 \quad \textit{and} \quad \inf_{x\in X}(\alpha(x)q(x)-\nu(x))>0.$$

Assume that $\alpha(\cdot)$ *and* $\nu(\cdot)$ *are log-Hölder continuous on* X*. If* $1\le\kappa<\min\{\tau(1-1/\sigma)-1/\sigma,\iota_2\}$*, then there exists a constant* $C>0$ *such that*

$$\big|b(x)I_{\alpha(\cdot),\tau}f(x)-b(z)I_{\alpha(\cdot),\tau}f(z)\big|\le C\big\{\psi_4(x,z)+d(x,z)^{\eta_1-\eta_2}\psi_4(x,z)+|\alpha(x)-\alpha(z)|\big\}$$

for all $x,z\in X$ *and measurable functions* f *on* X *with* $\|f\|_{L^{\Phi,\nu(\cdot),\kappa}(X)}\le 1$.

When $\rho(x,r)=r^{\alpha(x)}e^{-a/r}(\log(e+1/r))^{\beta}$, we obtain the next corollary by Theorem 4.1.

Corollary 5.6. *Let* $\rho(x,r)=r^{\alpha(x)}e^{-a/r}(\log(e+1/r))^{\beta}$ *for* $a>0$ *and* $\beta\in\mathbf{R}$*. Let* X *be a non-doubling metric measure space. Assume that* $(\rho\mu)$ *holds. If* $1\le\kappa<\min\{\tau(1-1/\sigma)-1/\sigma,\iota_2\}$*, then there exists a constant* $C>0$ *such that*

$$\big|b(x)I_{\rho,\tau}f(x)-b(z)I_{\rho,\tau}f(z)\big|\le C\big\{d(x,z)^{\theta}+d(x,z)^{\eta_1}+|\alpha(x)-\alpha(z)|\big\}$$

for all $x,z\in X$ *and measurable functions* f *on* X *with* $\|f\|_{L^{\Phi,\nu(\cdot),\kappa}(X)}\le 1$.

To get this, we note, for $b \in \mathbf{R}$, there exists a constant $c > 0$ such that

$$\int_0^r t^b e^{-a/t} (\log(e + 1/t))^\beta \frac{dt}{t} \le c r^\theta$$

for all $0 < r \le d_X$.

For the case $L^{p(\cdot),\nu(\cdot),\kappa}(X)$, we obtain the following corollaries. The following corollary is a consequence of Theorem 4.1 with $b(\cdot) \equiv 1$ and $\rho(x,r) = r^{\alpha(x)}(\log(e+1/r))^\beta$.

Corollary 5.7. *Let $\rho(x,r) = r^{\alpha(x)}(\log(e+1/r))^\beta$ for $\beta \in \mathbf{R}$. Let X be a non-doubling metric measure space. Assume that $(\rho\mu)$ holds. Suppose*

$$\inf_{x\in X}(\nu(x) - (\alpha(x) - \eta_2)p(x)) > 0 \quad \text{and} \quad \inf_{x\in X}(\alpha(x)p(x) - \nu(x)) > 0.$$

If $1 \le \kappa < \min\{\tau(1-1/\sigma) - 1/\sigma, \iota_2\}$, then there exists a constant $C > 0$ such that

$$\begin{aligned}
&\left|I_{\rho,\tau} f(x) - I_{\rho,\tau} f(z)\right| \\
&\le C\Big[\big(d(x,z)^{\alpha(x)-\nu(x)/p(x)} + d(x,z)^{\alpha(z)-\nu(z)/p(z)} \\
&\quad + \min\left\{d(x,z)^{\alpha(x)-\nu(x)/p(x)+\eta_1-\eta_2}, d(x,z)^{\alpha(z)-\nu(z)/p(z)+\eta_1-\eta_2}\right\}\big)(\log(e+1/d(x,z)))^\beta \\
&\quad + |\alpha(x) - \alpha(z)|\Big]
\end{aligned}$$

for all $x, z \in X$ and measurable functions f on X with $\|f\|_{L^{p(\cdot),\nu(\cdot),\kappa}(X)} \le 1$.

The next corollary is a consequence of Theorem 4.1 with $b(\cdot) \equiv 1$ and $\rho(x,r) = r^{\alpha(x)}e^{-a/r}(\log(e+1/r))^\beta$.

Corollary 5.8. *Let $\rho(x,r) = r^{\alpha(x)}e^{-a/r}(\log(e+1/r))^\beta$ for $a > 0$ and $\beta \in \mathbf{R}$. Let X be a non-doubling metric measure space. Assume that $(\rho\mu)$ holds. If $1 \le \kappa < \min\{\tau(1-1/\sigma) - 1/\sigma, \iota_2\}$, then there exists a constant $C > 0$ such that*

$$\left|I_{\rho,\tau} f(x) - I_{\rho,\tau} f(z)\right| \le C\left\{d(x,z)^{\eta_1} + |\alpha(x) - \alpha(z)|\right\}$$

for all $x, z \in X$ and measurable functions f on X with $\|f\|_{L^{p(\cdot),\nu(\cdot),\kappa}(X)} \le 1$.

The following corollary is the doubling metric measure case of Corollary 5.3.

Corollary 5.9. *Let $\rho(x,r) = r^{\alpha(x)}(\log(e+1/r))^\beta$ for $\beta \in \mathbf{R}$. Let X be a doubling metric measure space. Assume that $(\rho\mu)$ holds. Suppose*

$$\inf_{x\in X}(\nu(x) - \alpha(x)p(x)) > 0, \quad \inf_{x\in X}(\nu(x) - (\alpha(x) + \theta - \eta_2)p(x)) > 0$$

and

$$\inf_{x\in X}((\alpha(x) + \theta)p(x) - \nu(x)) > 0.$$

Further suppose

$$\inf_{x\in X}(\nu(x)-(\alpha(x)-\eta_2)q(x))>0 \quad \textit{and} \quad \inf_{x\in X}(\alpha(x)q(x)-\nu(x))>0.$$

Then there exists a constant $C>0$ such that

$$\begin{aligned}
&\big|b(x)I_{\rho,1}f(x)-b(z)I_{\rho,1}f(z)\big| \\
&\le C\Big[\big(\psi_4(x,z)+\psi_5(x,z)+\min\big\{d(x,z)^{\eta_1-\eta_2}\psi_4(x,z),d(x,z)^{\eta_1-\eta_2}\psi_5(x,z)\big\}\big) \\
&\qquad \times(\log(e+1/d(x,z)))^{\beta}+|\alpha(x)-\alpha(z)|\Big]
\end{aligned}$$

for all $x,z\in X$ and measurable functions f on X with $\|f\|_{L^{\Phi,\nu(\cdot),1}(X)}\le 1$.

The following corollary is the doubling metric measure case of Corollary 5.6.

Corollary 5.10. *Let $\rho(x,r)=r^{\alpha(x)}e^{-a/r}(\log(e+1/r))^{\beta}$ for $a>0$ and $\beta\in\mathbf{R}$. Let X be a doubling metric measure space. Assume that $(\rho\mu)$ holds. Then there exists a constant $C>0$ such that*

$$\big|b(x)I_{\rho,1}f(x)-b(z)I_{\rho,1}f(z)\big|\le C\big\{d(x,z)^{\theta}+d(x,z)^{\eta_1}+|\alpha(x)-\alpha(z)|\big\}$$

for all $x,z\in X$ and measurable functions f on X with $\|f\|_{L^{\Phi,\nu(\cdot),1}(X)}\le 1$.

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