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Error Analysis of Nonconforming Virtual Element Method for Stokes Problem with Low Regularity

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Abstract. In this paper, the nonconforming virtual element method is used to solve the Stokes problem where the velocity and pressure are allowed to have the low regularity. With the help of an enriching operator, the consistency error is estimated under the low regularity condition. Then the optimal error estimates are obtained for the velocity and pressure approximations, which implies that the nonconforming virtual element method has the good convergence even for the Stokes problem with the low regularity.

1. Introduction

Stokes equation is a motion equation describing the momentum conservation of viscous incompressible fluid and is widely used in fluid mechanics. It is very difficult and complicated to solve the Stokes equation except for some specific conditions.

In recent years, research on the Stokes equation has attracted extensive attention at home and abroad. As well known, the finite element method (FEM) is an important method for solving the Stokes equation [10]. When the Stokes equation is discretized by FEM, the approximate spaces of velocity and pressure should satisfy Babuška–Brezzi condition [2, 7, 13, 22] in order to guarantee the stability of FEM. For Stokes problem, [25] presented VPVnet to achieve lower regularity requirements without considering Babuška–Brezzi condition.

There have been many studies on conforming and nonconforming FEMs for Stokes problem. When the Stokes problem is solved by the conforming FEMs, the error estimate of a numerical solution is bounded by the approximation errors of their spaces by Cea's lemma [12]. The approximation errors are bounded by the interpolation errors, which can be estimated under the low regularity $(\boldsymbol{u}, p) \in \boldsymbol{H}^{1+s}(\Omega) \times H^s(\Omega)$ with any $s \geq 0$ where $\boldsymbol{u}$ and p are the velocity and the pressure, respectively. For example, the general finite element approximation of Stokes equation solution for incompressible viscous fluid is given in [18].

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Compared with the conforming FEMs, the nonconforming FEMs are easier to satisfy Babuška–Brezzi condition for the Stokes problem [29], but the error estimate of a finite element solution is more troublesome, which is bounded by the approximation error plus the consistency error. The approximation error is consistent with the conforming counterparts. The consistency error is usually transformed to some jump flux terms along the edges of element, which needs the regularity $(\boldsymbol{u}, p) \in \boldsymbol{H}^{1+s}(\Omega) \times H^s(\Omega)$ with $s > 1/2$ at least [10, 12, 17], and this is along with some potential difficulties for solutions with lower regularity. In order to obtain the error estimates for the nonconforming FEM under low regularity condition, the so-called quasi-interpolation operator was used to estimate the consistency error in [4, 28].

Virtual element method (VEM), which is introduced as a generalization of classical FEM in [5], has some advantages compared with standard FEMs. VEMs are more convenient to deal with partial differential equations in complex geometric domains or high-regularity admissible spaces [14, 23, 30], and are suitable for polygonal or polyhedral meshes and have the characteristics of high flexibility in mesh processing and avoiding explicit shape function construction. Polygonal or polyhedral meshes have gained considerable attention in the field of scientific computing, in part because of their flexibility in dealing with complex regions or regions with curved boundaries, and the detailed analysis is given in [16, 26].

The error from conforming VEMs is usually bounded by three parts: the interpolation error, the local polynomial approximation error for the exact solution and the approximation error of the right-hand side. The error estimates for the first two terms can be estimated under low regularity i.e., in space $\boldsymbol{H}^{1+s}(\Omega)$ with any $s \geq 0$, and the error estimate for the third term only needs sufficiently smooth data, refer to [6, 11]. Thus, the error estimates can be obtained under the low regularity requirement. The conforming VEMs for the Stokes problem are developed in [15, 18, 21]. Compared with the conforming VEMs, the error estimates for nonconforming VEMs are more troublesome, which have an extra consistency error term.

At present, nonconforming VEM has won the attention of everyone and applied to the Stokes problem. For example, the nonconforming formulation of VEM for the steady Stokes problem is presented in [13] where the authors show that the nonconforming VEM is inf-sup stable and establish the optimal priori error estimates for the velocity and pressure in providing high-order accurate approximations. The divergence-free nonconforming VEM for the Stokes problem has been presented in [31] and converges at an optimal rate. However, the estimation on the consistency error in the works [13, 31] still needs the high regularity requirement i.e., $(\boldsymbol{u}, p) \in \boldsymbol{H}^{1+s}(\Omega) \times H^s(\Omega)$ with any $s > 1/2$ at least.

What's more, the solution region of Stokes equation is often irregular and complex in practice. In particular, when the solution region has a concave angle, such as the L-shaped region, the regularity of solution for Stokes equation is very low near the concave angle. Therefore, the research on the nonconforming VEM of the Stokes equation under the low regularity condition has very important application.

Based on the above facts, in this paper we carry on the convergence analysis of the nonconforming VEM for the Stokes problem under the low regularity condition. We note that in [23] a special enriching operator, which connects nonconforming VE space with the same order conforming VE space, is introduced to estimate the error on the nonconforming VEMs for 2nd and 4th elliptic problems. Therein, the error analysis is very complicated. So in this paper, we change the definition of the enriching operator a little bit, so that it maps the nonconforming space to the corresponding conforming space with one order higher. Then some terms from the consistency error will vanish which greatly simplifies the estimation on the consistency error term [24]. For more information about enriching operators, interested readers can refer to [9, 19, 20, 27]. Then we obtain the optimal convergence of nonconforming VEM for the Stokes problem under the low regularity condition in two dimensions.

The paper is organized as follows. In Section 2, we state the steady Stokes problem and give its weak form. In Section 3, we give the definitions of local and global spaces to prepare for the subsequent error estimations. In Section 4, we give the discrete form of the Stokes problem. In Section 5, we introduce a special enriching operator for estimating the consistency error. In Section 6, we estimate the consistency error and obtain the optimal estimates for the velocity and pressure approximations.

Throughout the paper, we assume that Ω is a polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$. For a positive integer m, let S be any given open subset of Ω, we use the standard definitions and notations of Sobolev spaces $H^m(S)$ and $H_0^m(S)$ with the corresponding norm $\|\cdot\|_{m,S}$ and seminorm $|\cdot|_{m,S}$. Besides, $(\cdot,\cdot)_S$ and $\|\cdot\|_S$ denote the usual integral inner product and the corresponding norm of $L^2(S)$, respectively. For the vector-valued functions or spaces, we use bold symbols, such as $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{H}^m(\Omega)$, $\boldsymbol{H}_0^m(\Omega)$, $\boldsymbol{P}_k(S)$, etc.

2. The model problem

We are concerned with the Stokes problem with the unknown fields $\boldsymbol{u}$ and p satisfying

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We will refer to $\boldsymbol{u}$ and p as velocity and pressure, respectively, and $\boldsymbol{f}$ is the given body force.

The corresponding variational formulation of problem (2.1) is to find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that for $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ it holds that

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = (\boldsymbol{f},\boldsymbol{v}), & \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ b(\boldsymbol{u},q) = 0, & \forall\, q \in L_0^2(\Omega), \end{cases} \tag{2.2}$$

where the bilinear forms $a(\,\cdot\,,\cdot\,)$ and $b(\,\cdot\,,\cdot\,)$ are defined by $a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$ and $b(\boldsymbol{v},q) = -\int_\Omega q \operatorname{div} \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$, respectively, and $L_0^2(\Omega) = \big\{q \in L^2(\Omega) \;\big|\; \int_\Omega q\, \mathrm{d}\boldsymbol{x} = 0\big\}$.

The well-posedness of (2.2) follows from the coercivity of the bilinear form $a(\,\cdot\,,\cdot\,)$ on the kernel of the bilinear form $b(\,\cdot\,,\cdot\,)$ and the inf-sup condition, refer to [8].

3. Nonconforming virtual element

For any fixed $h > 0$, we introduce a finite decomposition (the mesh) $\mathcal{T}_h$ of the domain Ω into nonoverlapping simple polygonal elements with maximum size h. Let $\mathcal{E}_h$ denote the set of all mesh edges in $\mathcal{T}_h$ of Ω. For any $E \in \mathcal{T}_h$, $\boldsymbol{n}_E$ denotes the unit outward normal vector along the boundary ∂E. $\boldsymbol{n}_e$ denotes the unit normal of an edge $e \in \mathcal{E}_h$, whose orientation is chosen arbitrarily but fixed for internal edges and coinciding with the outward normal of Ω for boundary edges, h_e denotes the length of $e \in \mathcal{E}_h$ and h_E denotes the diameter of E.

Throughout the paper, we use the short symbol $a \lesssim b$ for the inequality $a \le Cb$ with the constant C independent of a, b and the mesh size.

H0 (Mesh assumption). We assume that there exists a constant $\rho > 0$ such that

- for every element E of $\mathcal{T}_h$ and every edge $e \subseteq \partial E$, it holds that $h_e \ge \rho h_E$;
- every element E of $\mathcal{T}_h$ is star-shaped with respect to a ball of radius ρh_E.

For an internal edge e shared by $E_1, E_2 \in \mathcal{T}_h$ such that $\boldsymbol{n}_e$ points from E_1 to E_2, we define the jump and average of function $\boldsymbol{v}$ through the edge e by $[\boldsymbol{v}]|_e = \boldsymbol{v}_1 - \boldsymbol{v}_2$, and $\{\boldsymbol{v}\}|_e = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$, where $\boldsymbol{v}_i = \boldsymbol{v}|_{E_i}$, $i = 1, 2$. Then, we can get that $[\boldsymbol{u}\boldsymbol{v}]|_e = \boldsymbol{u}_1^+\boldsymbol{v}_1^+ - \boldsymbol{u}_2^-\boldsymbol{v}_2^- = \{\boldsymbol{u}\}[\boldsymbol{v}] + [\boldsymbol{u}]\{\boldsymbol{v}\}$.

For the boundary edge e, there exists $E_1 \in \mathcal{T}_h$ such that $e \subseteq \partial E_1 \cap \partial\Omega$, and we define the jump and average of function $\boldsymbol{v}$ through the edge e by $[\boldsymbol{v}]|_e = \boldsymbol{v}|_e$, and $\{\boldsymbol{v}\}|_e = \boldsymbol{v}|_e/2$. Then, we can also get that $[\boldsymbol{u}\boldsymbol{v}]|_e = \{\boldsymbol{u}\}[\boldsymbol{v}] + [\boldsymbol{u}]\{\boldsymbol{v}\}$, refer to [23].

For any $E \in \mathcal{T}_h$, $\mathbb{P}_k(E)$ denotes the space consisting of polynomials of order k or less. For $k \ge 1$, we define the local vector nonconforming VE space on the element E by

$$\boldsymbol{V}_h^k(E) = \big\{\boldsymbol{v} \in \boldsymbol{H}^1(E) \mid \Delta \boldsymbol{v} \in \boldsymbol{P}_{k-2}(E), (\boldsymbol{n}_E \cdot \nabla \boldsymbol{v})|_e \in \boldsymbol{P}_{k-1}(e), \forall\, e \subseteq \partial E\big\}$$

with the convention that $\boldsymbol{P}_{-1}(E) = \{\boldsymbol{0}\}$. Obviously it holds that $\boldsymbol{P}_k(E) \subseteq \boldsymbol{V}_h^k(E)$.

For $k \geq 1$, the same discussion as in [30] reveals that the dimension of $\boldsymbol{V}_h^k(E)$ is

$$N_E = 2nk + k(k-1).$$

Denote by $\mathcal{M}_l^*(E)$, $l \in \mathbb{N}$, the set of scaled monomials

$$\mathcal{M}_l^*(E) = \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_E}{h_E} \right)^{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}| = l \right\},$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is a multi-index, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$, $\boldsymbol{x}^\alpha = \boldsymbol{x}_1^{\alpha_1} \boldsymbol{x}_2^{\alpha_2}$ and $\boldsymbol{x}_E$ is the barycenter of E. Furthermore, we define $\mathcal{M}_k(E) = \bigcup_{l \leq k} \mathcal{M}_l^*(E)$, a basis of the polynomial space $\mathbb{P}_k(E)$ whose dimension is $N_{2,k}$. The vector version of $\mathcal{M}_k(E)$ is denoted by $\boldsymbol{\mathcal{M}}_k(E)$.

The degrees of freedom (DOF) for $\boldsymbol{V}_h^k(E)$ can be chosen as:

- for $k \geq 1$, the moments of degree $(k-1)$ on each edge

$$\frac{1}{|e|} \int_e \boldsymbol{v}_h \cdot \boldsymbol{q} \, \mathrm{d}\boldsymbol{s}, \quad \boldsymbol{q} \in \boldsymbol{\mathcal{M}}_{k-1}(e), \ e \subseteq \partial E, \tag{3.1}$$

- for $k > 1$, the moments of degree $(k-2)$ inside each element

$$\frac{1}{|E|} \int_E \boldsymbol{v}_h \cdot \boldsymbol{q} \, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{q} \in \boldsymbol{\mathcal{M}}_{k-2}(E). \tag{3.2}$$

According to [1], we have the unisolvence of DOF as follows.

Lemma 3.1. *For $k \geq 1$, the DOF* (3.1)–(3.2) *are unislovent for the space $\boldsymbol{V}_h^k(E)$.*

A subspace of the broken Sobolev space is defined by

$$\boldsymbol{H}^1(\mathcal{T}_h) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid \boldsymbol{v}|_E \in \boldsymbol{H}^1(E), \forall\, E \in \mathcal{T}_h \right\},$$

and, for $k \geq 1$, $\boldsymbol{H}^{1,nc}(\mathcal{T}_h)$ is defined by

$$\boldsymbol{H}^{1,nc}(\mathcal{T}_h) = \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h) \;\middle|\; \int_e [\boldsymbol{v}] \cdot \boldsymbol{q} \, \mathrm{d}\boldsymbol{s} = 0, \forall\, \boldsymbol{q} \in \boldsymbol{P}_{k-1}(e), \forall\, e \in \mathcal{E}_h \right\}.$$

According to the local DOF (3.1)–(3.2), we define the global vector nonconforming VE space with order $k \geq 1$ by

$$\boldsymbol{V}_h^k = \left\{ \boldsymbol{v}_h \in \boldsymbol{H}^{1,nc}(\mathcal{T}_h) \mid \boldsymbol{v}_h|_E \in \boldsymbol{V}_h^k(E), \forall\, E \in \mathcal{T}_h \right\}.$$

We note that $\boldsymbol{V}_h^k \nsubseteq \boldsymbol{H}_0^1(\Omega)$.

The global DOF for $\boldsymbol{V}_h^k$ can be chosen as:

- for $k \geq 1$, the moments of degree $(k-1)$ on each edge

$$\frac{1}{|e|}\int_e \boldsymbol{v}_h \cdot \boldsymbol{q}\,\mathrm{d}\boldsymbol{s}, \quad \boldsymbol{q} \in \boldsymbol{\mathcal{M}}_{k-1}(e),\ e \in \mathcal{E}_h, \tag{3.3}$$

- for $k > 1$, the moments of degree $(k-2)$ inside each element

$$\frac{1}{|E|}\int_E \boldsymbol{v}_h \cdot \boldsymbol{q}\,\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{q} \in \boldsymbol{\mathcal{M}}_{k-2}(E),\ E \in \mathcal{T}_h. \tag{3.4}$$

For any function $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, we define the interpolation $I_h\boldsymbol{v} \in \boldsymbol{V}_h^k$ by requiring that the values of DOF (3.3)–(3.4) of $I_h\boldsymbol{v}$ are equal to the corresponding ones of $\boldsymbol{v}$, i.e., $\mathrm{DOF}(I_h\boldsymbol{v}) = \mathrm{DOF}(\boldsymbol{v})$. Then, we have the following interpolation error estimates [3,13].

Lemma 3.2. *For any $\boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$ with $1 \leq s \leq k+1$, we have the estimates*

$$\|\boldsymbol{v} - I_h\boldsymbol{v}\|_{0,E} + h_E|\boldsymbol{v} - I_h\boldsymbol{v}|_{1,E} \lesssim h_E^s|\boldsymbol{v}|_{s,E}, \quad \forall\, E \in \mathcal{T}_h.$$

According to [10], for any $\boldsymbol{v} \in \boldsymbol{H}^s(E)$, there exists a polynomial $\boldsymbol{v}_\pi \in \boldsymbol{P}_k(E)$ such that

$$|\boldsymbol{v} - \boldsymbol{v}_\pi|_{m,E} \leq Ch_E^{s-m}|\boldsymbol{v}|_{s,E}, \quad \forall\, E \in \mathcal{T}_h, \tag{3.5}$$

where $0 \leq m \leq s \leq k+1$.

4. The VEM discretization

In order to approximate the pressure, we use the standard space of piecewise polynomial of degree up to $k-1$ with respect to the domain partition $\mathcal{T}_h$:

$$Q_h^{k-1} = \big\{q_h \in L_0^2(\Omega) \mid q_h|_E \in \mathbb{P}_{k-1}(E), \forall\, E \in \mathcal{T}_h\big\}.$$

Note that, by definition, all functions in Q_h^{k-1} have global mean zero.

A VEM of order k will be defined by two finite dimensional function spaces $\boldsymbol{V}_h^k$ and Q_h^{k-1} of discrete trial velocity and pressure fields and bilinear forms $a_h\colon \boldsymbol{V}_h^k \times \boldsymbol{V}_h^k \to \mathbb{R}$ and $b_h\colon \boldsymbol{V}_h^k \times Q_h^{k-1} \to \mathbb{R}$ as the discrete counterparts of $a(\,\cdot\,,\cdot\,)$ and $b(\,\cdot\,,\cdot\,)$, respectively.

For any element $E \in \mathcal{T}_h$, we denote by Π_{k-1}^E the L^2-projection from $L^2(E)$ to $\mathbb{P}_{k-1}(E)$, which is defined by finding $\Pi_{k-1}^E v_h \in \mathbb{P}_{k-1}(E)$ satisfying

$$(\Pi_{k-1}^E v_h, m_\alpha)_E = (v_h, m_\alpha)_E, \quad \forall\, m_\alpha \in \mathcal{M}_{k-1}(E),$$

where $v_h \in L^2(E)$. For vector-valued functions, it acts on each component, i.e., for any $\boldsymbol{v}_h \in \boldsymbol{L}^2(E)$, $\Pi_{k-1}^E\boldsymbol{v}_h \in \boldsymbol{P}_{k-1}(E)$. Let $\Pi_{k-1}^0|_E = \Pi_{k-1}^E$ for $E \in \mathcal{T}_h$. Obviously Π_{k-1}^0 is the interpolation from $L_0^2(\Omega)$ to Q_h^{k-1}.

The choice of DOF in space $\boldsymbol{V}_h^k(E)$ ensures that the operator Π_{k-1}^E when applied to $\nabla \boldsymbol{v}_h$ or $\operatorname{div} \boldsymbol{v}_h$ for a vector $\boldsymbol{v}_h$ in $\boldsymbol{V}_h^k(E)$ is computable using only the DOF of $\boldsymbol{v}_h$. According to the definition, for any $E \in \mathcal{T}_h$ and $\boldsymbol{v} \in \boldsymbol{L}^2(E)$, we have the orthogonal decomposition and optimal approximation of L^2-projection operator Π_{k-1}^E

$$\|\boldsymbol{v}\|_E^2 = \|\boldsymbol{v} - \Pi_{k-1}^E \boldsymbol{v}\|_E^2 + \|\Pi_{k-1}^E \boldsymbol{v}\|_E^2, \quad \|\boldsymbol{v} - \Pi_{k-1}^E \boldsymbol{v}\|_E = \inf_{\boldsymbol{q} \in \boldsymbol{P}_{k-1}(E)} \|\boldsymbol{v} - \boldsymbol{q}\|_E.$$

According to [13], we have the following lemma.

Lemma 4.1. *For any $E \in \mathcal{T}_h$, $\boldsymbol{v} \in \boldsymbol{H}^s(E)$ with $1 \le s \le k$, it holds that*

$$\|\boldsymbol{v} - \Pi_{k-1}^E \boldsymbol{v}\|_{0,E} + h_E |\boldsymbol{v} - \Pi_{k-1}^E \boldsymbol{v}|_{1,E} \le C h_E^s |\boldsymbol{v}|_{s,E}.$$

We define the computable H^1-projection operator $\Pi_k^\nabla : \boldsymbol{V}_h^k(E) \to \boldsymbol{P}_k(E)$ by finding $\Pi_k^\nabla \boldsymbol{v} \in \boldsymbol{P}_k(E)$ satisfying

$$\begin{aligned} (\nabla \Pi_k^\nabla \boldsymbol{v}, \nabla \boldsymbol{m}_\alpha)_E &= (\nabla \boldsymbol{v}, \nabla \boldsymbol{m}_\alpha)_E, \quad \forall\, \boldsymbol{m}_\alpha \in \boldsymbol{\mathcal{M}}_k(E), \\ \int_{\partial E} \Pi_k^\nabla \boldsymbol{v}\, \mathrm{d}\boldsymbol{s} &= \int_{\partial E} \boldsymbol{v}\, \mathrm{d}\boldsymbol{s}, \end{aligned}$$

where $\boldsymbol{v} \in \boldsymbol{V}_h^k(E)$. Since system of equations above has a unique solution, it is easy to see that $\Pi_k^\nabla \boldsymbol{v} = \boldsymbol{v}$ for any $\boldsymbol{v} \in \boldsymbol{P}_k(E)$. According to the definition, for any $E \in \mathcal{T}_h$ and $\boldsymbol{v} \in \boldsymbol{V}_h^k(E)$, we have the orthogonal decomposition and boundedness of H^1-projection operator Π_k^∇

$$|\boldsymbol{v}|_{1,E}^2 = |\boldsymbol{v} - \Pi_k^\nabla \boldsymbol{v}|_{1,E}^2 + |\Pi_k^\nabla \boldsymbol{v}|_{1,E}^2, \quad |\Pi_k^\nabla \boldsymbol{v}|_{1,E} \le |\boldsymbol{v}|_{1,E}.$$

For convenience, the restriction of bilinear form $a(\,\cdot\,,\cdot\,)$ and $b(\,\cdot\,,\cdot\,)$ on element $E \in \mathcal{T}_h$ are defined by

$$a^E(\boldsymbol{u}, \boldsymbol{v}) = \int_E \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}, \quad b^E(\boldsymbol{v}, q) = -\int_E q \operatorname{div} \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1(E),\ q \in L^2(E).$$

For each element $E \in \mathcal{T}_h$, the local bilinear form $a_h^E(\,\cdot\,,\cdot\,)$ on $\boldsymbol{V}_h^k(E) \times \boldsymbol{V}_h^k(E)$ is defined by

$$a_h^E(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_E \nabla \Pi_k^\nabla \boldsymbol{u}_h : \nabla \Pi_k^\nabla \boldsymbol{v}_h\, \mathrm{d}\boldsymbol{x} + S_h^E\big((I - \Pi_k^\nabla)\boldsymbol{u}_h, (I - \Pi_k^\nabla)\boldsymbol{v}_h\big),$$

where the bilinear form $S_h^E(\,\cdot\,,\cdot\,)$ is given by

$$S_h^E\big((I - \Pi_k^\nabla)\boldsymbol{u}_h, (I - \Pi_k^\nabla)\boldsymbol{v}_h\big) = \sum_{i=1}^{N_E} \mathcal{X}_i\big((I - \Pi_k^\nabla)\boldsymbol{u}_h\big) \cdot \mathcal{X}_i\big((I - \Pi_k^\nabla)\boldsymbol{v}_h\big),$$

where $\mathcal{X}_i$, $i = 1, 2, \ldots, N_E$ is the operator associated to the i-th local DOF.

According to [5], the bilinear term $S_h^E(\,\cdot\,,\cdot\,)$ satisfies

$$c_* a^E(\boldsymbol{v}_h, \boldsymbol{v}_h) \le S_h^E(\boldsymbol{v}_h, \boldsymbol{v}_h) \le c^* a^E(\boldsymbol{v}_h, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h^k(E) \cap \operatorname{Ker}(\Pi_k^\nabla).$$

The local bilinear form $b_h^E(\cdot,\cdot)$ on $\boldsymbol{V}_h^k(E)\times Q_h^{k-1}$ is defined by

$$b_h^E(\boldsymbol{v}_h, q_h) = -\int_E q_h \Pi_{k-1}^E \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x}.$$

We define the right-hand side by

$$(\boldsymbol{f}_h, \boldsymbol{v}_h) = \sum_{E\in\mathcal{T}_h} (\boldsymbol{f}_h, \boldsymbol{v}_h)_E$$

with

$$(\boldsymbol{f}_h, \boldsymbol{v}_h)_E = \begin{cases} (\Pi_0^E \boldsymbol{f}, \overline{\boldsymbol{v}}_h)_E, & k=1, \\ (\Pi_{k-2}^E \boldsymbol{f}, \boldsymbol{v}_h)_E, & k\geq 2, \end{cases}$$

where $\overline{\boldsymbol{v}}_h = \frac{1}{|\partial E|}\int_{\partial E} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{s}$. Moreover, refer to [30], assume $\boldsymbol{f}\in \boldsymbol{H}^{k-1}(\Omega)$, we have

$$\|\boldsymbol{f}-\boldsymbol{f}_h\|_* \triangleq \sup_{\forall\, \boldsymbol{v}\in \boldsymbol{V}_h^k, \boldsymbol{v}\neq \boldsymbol{0}} \frac{(\boldsymbol{f}-\boldsymbol{f}_h, \boldsymbol{v})}{|\boldsymbol{v}|_{1,h}} \lesssim h^k |\boldsymbol{f}|_{k-1}. \tag{4.1}$$

With the above preparations, we obtain that the nonconforming VE discretization of problem (2.2) is to find $(\boldsymbol{u}_h, p_h)\in \boldsymbol{V}_h^k \times Q_h^{k-1}$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}_h, \boldsymbol{v}_h), & \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h^k, \\ b_h(\boldsymbol{u}_h, q_h) = 0, & \forall\, q_h\in Q_h^{k-1}, \end{cases} \tag{4.2}$$

where $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{E\in\mathcal{T}_h} a_h^E(\boldsymbol{u}_h, \boldsymbol{v}_h)$, $b_h(\boldsymbol{u}_h, q_h) = \sum_{E\in\mathcal{T}_h} b_h^E(\boldsymbol{u}_h, q_h)$. Definition of $a_h(\cdot,\cdot)$ guarantees the following *polynomial consistency* and *stability* properties, refer to [13, 14].

Lemma 4.2 (Polynomial consistency and stability)**.** (i) *Polynomial consistency: If $\boldsymbol{u}_h$ or $\boldsymbol{v}_h$, or both, belong to $\boldsymbol{P}_k(E)$, the bilinear form $a_h(\cdot,\cdot)$ satisfies*

$$a_h^E(\boldsymbol{u}_h, \boldsymbol{v}_h) = a^E(\boldsymbol{u}_h, \boldsymbol{v}_h).$$

(ii) *Stability: There exist two positive constants α_* and α^* independent of h such that, for all $\boldsymbol{v}_h\in \boldsymbol{V}_h^k(E)$, the bilinear form $a_h(\cdot,\cdot)$ satisfies*

$$\alpha_* a^E(\boldsymbol{v}_h, \boldsymbol{v}_h) \leq a_h^E(\boldsymbol{v}_h, \boldsymbol{v}_h) \leq \alpha^* a^E(\boldsymbol{v}_h, \boldsymbol{v}_h). \tag{4.3}$$

We present the inf-sup condition that is introduced in the following lemma, refer to [8, 13].

Lemma 4.3 (Inf-sup condition)**.** *There exists a strictly positive constant β independent of h such that for every q_h in Q_h^{k-1} there exists a vector $\boldsymbol{v}_h$ in $\boldsymbol{V}_h^k$ such that*

$$\frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geq \beta \|q_h\|. \tag{4.4}$$

The stability (4.3) of $a_h(\cdot,\cdot)$ implies the continuity and coercivity, which together with the inf-sup condition (4.4) implies that the discrete problem (4.2) is well-posed.

In order to carry out the error analysis, we introduce the broken H^1-norm on $\boldsymbol{V}_h^k$ by setting

$$|\boldsymbol{v}_h|_{1,h} = \left(\sum_{E\in\mathcal{T}_h} |\boldsymbol{v}_h|_{1,E}^2\right)^{1/2}.$$

5. Enriching operator

Since the nonconforming VE is not C^0 continuous, there exists a consistency error term defined by $W_h(\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) - a(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p)$ where $\boldsymbol{v}\in\boldsymbol{V}_h^k$ when we carry on convergence analysis.

As usual, the consistency error is transformed to the jump flux terms on edges as follows: for $(\boldsymbol{u},p)\in\boldsymbol{H}^{1+s}(\Omega)\times\boldsymbol{H}^s(\Omega)$ with any $s>1/2$,

$$\begin{aligned}(\boldsymbol{f},\boldsymbol{v}) - a(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p) &= \sum_{E\in\mathcal{T}_h}\left(-\int_{\partial E}\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}_E}\boldsymbol{v}\,\mathrm{d}\boldsymbol{s} + \int_{\partial E} p\boldsymbol{v}\cdot\boldsymbol{n}_E\,\mathrm{d}\boldsymbol{s}\right)\\ &= \sum_{e\in\mathcal{E}_h}\left(-\int_e\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}_e}[\boldsymbol{v}]\,\mathrm{d}\boldsymbol{s} + \int_e p[\boldsymbol{v}\cdot\boldsymbol{n}_e]\,\mathrm{d}\boldsymbol{s}\right).\end{aligned}$$

Then the right-hand terms on the last equation above can be estimated by using the weak continuity of nonconforming VE. There also exist many cases with lower regularity. So, to estimate the consistency error term under the condition of lower regularity, we introduce an enriching operator to reduce the high regularity requirement in the section. We first define a conforming VE space by

$$\boldsymbol{V}_h^{k+1,c}(E) = \left\{\boldsymbol{v}\in\boldsymbol{H}^1(E)\mid \Delta\boldsymbol{v}\in\boldsymbol{P}_{k-1}(E), \boldsymbol{v}|_e\in\boldsymbol{P}_{k+1}(e), e\subseteq\partial E\right\}$$

and

$$\boldsymbol{V}_h^{k+1,c} = \left\{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\mid \boldsymbol{v}|_E\in\boldsymbol{V}_h^{k+1,c}(E), E\in\mathcal{T}_h\right\}.$$

The DOF for $\boldsymbol{V}_h^{k+1,c}$ can be chosen as:

- values of $\boldsymbol{v}_h(a)$, $a\in\mathcal{V}_h$,
- the moments $\frac{1}{|e|}\int_e \boldsymbol{v}_h\cdot\boldsymbol{q}\,\mathrm{d}\boldsymbol{s}$, $\boldsymbol{q}\in\boldsymbol{\mathcal{M}}_{k-1}(e)$, $\forall\, e\in\mathcal{E}_h$,
- the moments $\frac{1}{|E|}\int_E \boldsymbol{v}_h\cdot\boldsymbol{q}\,\mathrm{d}\boldsymbol{x}$, $\boldsymbol{q}\in\boldsymbol{\mathcal{M}}_{k-1}(E)$, $E\in\mathcal{T}_h$,

where $\mathcal{V}_h$ is the set of vertices in mesh.

We construct an enriching operator E_h^k from $\boldsymbol{V}_h^k$ to $\boldsymbol{V}_h^{k+1,c}$ by averaging [9,19,20,23,27], which maps elements in the nonconforming VE space to the conforming VE space with one order higher. For an interior vertex p of VE space associated with the decomposition $\mathcal{T}_h$, let $\mathcal{T}_p$ be the set of all elements in $\mathcal{T}_h$ sharing the vertex p.

For $\boldsymbol{v} \in \boldsymbol{V}_h^k$, we define $E_h^k \boldsymbol{v} \in \boldsymbol{V}_h^{k+1,c}$ by

$$\begin{cases} E_h^k \boldsymbol{v}(p) = \frac{1}{|\mathcal{T}_p|} \sum_{E \in \mathcal{T}_p} \boldsymbol{v}|_E(p), & \forall \text{ interior mesh vertex } p, \\ E_h^k \boldsymbol{v} = \frac{1}{|e|} \int_e \boldsymbol{v} \cdot \boldsymbol{q} \, \mathrm{d}\boldsymbol{s}, & \boldsymbol{q} \in \boldsymbol{\mathcal{M}}_{k-1}(e), \ e \in \mathcal{E}_h, \\ E_h^k \boldsymbol{v} = \frac{1}{|E|} \int_E \boldsymbol{v} \cdot \boldsymbol{q} \, \mathrm{d}\boldsymbol{x}, & \boldsymbol{q} \in \boldsymbol{\mathcal{M}}_{k-1}(E), \ E \in \mathcal{T}_h, \end{cases}$$

where $|\mathcal{T}_p|$ is the cardinality of $\mathcal{T}_p$. For all boundary vertex p, set $E_h^k \boldsymbol{v}(p) = 0$.

For any $\boldsymbol{v}_h \in \boldsymbol{V}_h^k$, we have $E_h^k \boldsymbol{v}_h \in \boldsymbol{V}_h^{k+1,c} \subseteq \boldsymbol{H}_0^1(\Omega)$, so we can obtain that it satisfies

$$a(\boldsymbol{u}, E_h^k \boldsymbol{v}_h) + b(E_h^k \boldsymbol{v}_h, p) = (\boldsymbol{f}, E_h^k \boldsymbol{v}_h). \tag{5.1}$$

Further, to derive the estimates related to the enriching operator E_h^k, we introduce the following lemma, refer to [19, 20, 23].

Lemma 5.1. *For any $\boldsymbol{v}_h \in \boldsymbol{V}_h^k$, we have the estimates*

$$|\boldsymbol{v}_h - E_h^k \boldsymbol{v}_h|_{1,h} \lesssim |\boldsymbol{v}_h|_{1,h}, \quad \sum_{E \in \mathcal{T}_h} h_E^{-2} \|\boldsymbol{v}_h - E_h^k \boldsymbol{v}_h\|_{0,E}^2 \lesssim |\boldsymbol{v}_h|_{1,h}^2.$$

6. Medius error analysis

With the above preparations, we estimate the consistency error.

Lemma 6.1. *Let $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^{s+1}(\Omega)$ $(0 < s \leq k)$ be the exact velocity solution to problem* (2.2)*. Let $p \in L_0^2(\Omega) \cap H^s(\Omega)$ be the exact pressure solution to problem* (2.2)*. For any $\boldsymbol{v} \in \boldsymbol{V}_h^k$, it holds that*

$$W_h(\boldsymbol{v}) \lesssim h^s \big(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s \big) |\boldsymbol{v}|_{1,h}.$$

Proof. According to (5.1), for any $\boldsymbol{v}_h \in \boldsymbol{V}_h^k$ and $q_h \in Q_h^{k-1}$ we obtain

$$\begin{aligned} W_h(\boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) - a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) \\ &= (\boldsymbol{f}, \boldsymbol{v} - E_h^k \boldsymbol{v}) - a(\boldsymbol{u}, \boldsymbol{v} - E_h^k \boldsymbol{v}) - b(\boldsymbol{v} - E_h^k \boldsymbol{v}, p) \\ &= (\boldsymbol{f}, \boldsymbol{v} - E_h^k \boldsymbol{v}) - a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{v} - E_h^k \boldsymbol{v}) - b(\boldsymbol{v} - E_h^k \boldsymbol{v}, p - q_h) \\ &\quad - a(\boldsymbol{v}_h, \boldsymbol{v} - E_h^k \boldsymbol{v}) - b(\boldsymbol{v} - E_h^k \boldsymbol{v}, q_h). \end{aligned}$$

We consider the term $a(\boldsymbol{v}_h, \boldsymbol{v} - E_h^k \boldsymbol{v})$ in the above equation firstly. By Green formula we obtain

$$\begin{aligned} a(\boldsymbol{v}_h, \boldsymbol{v} - E_h^k \boldsymbol{v}) &= \sum_{E \in \mathcal{T}_h} \int_E \nabla \boldsymbol{v}_h : \nabla (\boldsymbol{v} - E_h^k \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \\ &= - \sum_{E \in \mathcal{T}_h} \int_E \Delta \boldsymbol{v}_h \cdot (\boldsymbol{v} - E_h^k \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} + \sum_{E \in \mathcal{T}_h} \int_{\partial E} \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}_E} \cdot (\boldsymbol{v} - E_h^k \boldsymbol{v}) \, \mathrm{d}\boldsymbol{s}. \end{aligned} \tag{6.1}$$

By the definition of $\boldsymbol{V}_h^k$, we know that $\Delta \boldsymbol{v}_h \in \boldsymbol{P}_{k-2}(E)$ in E and $\frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}_E} \in \boldsymbol{P}_{k-1}(e)$ on edge e. By the definition of E_h^k, we have

$$\int_E (\boldsymbol{v} - \boldsymbol{E}_h^k \boldsymbol{v}) \cdot \boldsymbol{q}_{k-2} \,\mathrm{d}\boldsymbol{x} = 0, \quad \boldsymbol{q}_{k-2} \in \boldsymbol{P}_{k-2}(E), \tag{6.2}$$

$$\int_e (\boldsymbol{v} - \boldsymbol{E}_h^k \boldsymbol{v}) \cdot \boldsymbol{q}_{k-1} \,\mathrm{d}\boldsymbol{s} = 0, \quad \boldsymbol{q}_{k-1} \in \boldsymbol{P}_{k-1}(e). \tag{6.3}$$

This, together with (6.1), leads to $a(\boldsymbol{v}_h, \boldsymbol{v} - E_h^k \boldsymbol{v}) = 0$. Then, we consider the term $b(\boldsymbol{v} - E_h^k \boldsymbol{v}, q_h)$. By Green formula we obtain

$$\begin{aligned} b(\boldsymbol{v} - E_h^k \boldsymbol{v}, q_h) &= -\sum_{E\in\mathcal{T}_h} \int_E q_h \operatorname{div}(\boldsymbol{v} - E_h^k \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\ &= -\sum_{E\in\mathcal{T}_h} \left(-\int_E \nabla q_h \cdot (\boldsymbol{v} - E_h^k \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} + \int_{\partial E} q_h (\boldsymbol{v} - E_h^k \boldsymbol{v}) \cdot \boldsymbol{n}_E \,\mathrm{d}\boldsymbol{s} \right). \end{aligned}$$

By the definition of Q_h^{k-1}, we know that $q_h|_E \in \mathbb{P}_{k-1}(E)$. Observing (6.2)–(6.3), we can obtain that $b(\boldsymbol{v} - E_h^k \boldsymbol{v}, q_h) = 0$.

For the term $(\boldsymbol{f}, \boldsymbol{v} - E_h^k \boldsymbol{v})$, according to (6.2) and Lemmas 4.1 and 5.1, we obtain

$$\begin{aligned} (\boldsymbol{f}, \boldsymbol{v} - E_h^k \boldsymbol{v}) &= \sum_{E\in\mathcal{T}_h} \int_E \boldsymbol{f} \cdot (\boldsymbol{v} - E_h^k \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\ &= \sum_{E\in\mathcal{T}_h} \int_E (\boldsymbol{f} - \Pi_{k-1}^E \boldsymbol{f}) \cdot (\boldsymbol{v} - E_h^k \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\ &\lesssim \left(\sum_{E\in\mathcal{T}_h} h_E^2 \|\boldsymbol{f} - \Pi_{k-1}^E \boldsymbol{f}\|_{0,E}^2 \right)^{1/2} \left(\sum_{E\in\mathcal{T}_h} h_E^{-2} \|\boldsymbol{v} - E_h^k \boldsymbol{v}\|_{0,E}^2 \right)^{1/2} \\ &\lesssim h \cdot h^{s-1} |\boldsymbol{f}|_{s-1} |\boldsymbol{v}|_{1,h} \\ &= h^s |\boldsymbol{f}|_{s-1} |\boldsymbol{v}|_{1,h}. \end{aligned}$$

For the term $a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{v} - E_h^k \boldsymbol{v})$, according to Lemma 5.1, we obtain

$$\begin{aligned} a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{v} - E_h^k \boldsymbol{v}) &= \sum_{E\in\mathcal{T}_h} \int_E \nabla(\boldsymbol{u} - \boldsymbol{v}_h) : \nabla(\boldsymbol{v} - E_h^k \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \\ &\lesssim |\boldsymbol{u} - \boldsymbol{v}_h|_{1,h} |\boldsymbol{v} - E_h^k \boldsymbol{v}|_{1,h} \lesssim |\boldsymbol{u} - \boldsymbol{v}_h|_{1,h} |\boldsymbol{v}|_{1,h}. \end{aligned}$$

For the term $b(\boldsymbol{v} - E_h^k \boldsymbol{v}, p - q_h)$, according to Lemma 5.1 and by definition, we obtain

$$b(\boldsymbol{v} - E_h^k \boldsymbol{v}, p - q_h) = -\sum_{E\in\mathcal{T}_h} \int_E (p - q_h) \operatorname{div}(\boldsymbol{v} - E_h^k \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} \lesssim \|p - q_h\|_{0,\Omega} |\boldsymbol{v}|_{1,h}.$$

Summing up the results above and for any $\boldsymbol{v}_h \in \boldsymbol{V}_h^k$, $q_h \in Q_h^{k-1}$, we obtain

$$\begin{aligned} W_h(\boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v} - E_h^k \boldsymbol{v}) - a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{v} - E_h^k \boldsymbol{v}) - b(\boldsymbol{v} - E_h^k \boldsymbol{v}, p - q_h) \\ &\lesssim h^s |\boldsymbol{f}|_{s-1} |\boldsymbol{v}|_{1,h} + |\boldsymbol{u} - \boldsymbol{v}_h|_{1,h} |\boldsymbol{v}|_{1,h} + \|p - q_h\|_{0,\Omega} |\boldsymbol{v}|_{1,h}. \end{aligned}$$

Since $\boldsymbol{v}_h$ and q_h are arbitrary, we set $\boldsymbol{v}_h = I_h\boldsymbol{u}$ and $q_h = \Pi^0_{k-1}p$. Then Lemmas 3.2 and 4.1 imply

$$\begin{aligned} W_h(\boldsymbol{v}) &\lesssim \big(h^s|\boldsymbol{f}|_{s-1} + |\boldsymbol{u} - I_h\boldsymbol{u}|_{1,h} + \|p - \Pi^0_{k-1}p\|_{0,\Omega}\big)|\boldsymbol{v}|_{1,h} \\ &\lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s)|\boldsymbol{v}|_{1,h}, \end{aligned}$$

which concludes the proof. □

With the above preparations, we can obtain the following convergence theorems.

Theorem 6.2. *Let $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{H}^{s+1}(\Omega)$ $(0 < s \le k)$ be the exact velocity solution to problem* (2.2) *and $\boldsymbol{u}_h$ be the virtual element velocity solution to problem* (4.2)*. Let $p \in L^2_0(\Omega) \cap H^s(\Omega)$ be the exact pressure solution to problem* (2.2)*. Then, it holds that*

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s).$$

Proof. Using the triangle inequality, we obtain

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \le |\boldsymbol{u} - I_h\boldsymbol{u}|_{1,h} + |I_h\boldsymbol{u} - \boldsymbol{u}_h|_{1,h}. \tag{6.4}$$

Let $\boldsymbol{\phi} = \boldsymbol{u}_h - I_h\boldsymbol{u} \in \boldsymbol{V}^k_h$. Since $b_h(\boldsymbol{u}_h, q_h) = 0$ for any $q_h \in Q^{k-1}_h$, we obtain

$$\begin{aligned} b_h(\boldsymbol{\phi}, q_h) &= -b_h(I_h\boldsymbol{u}, q_h) = \sum_{E\in\mathcal{T}_h}\int_E q_h \operatorname{div} I_h\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \\ &= \sum_{E\in\mathcal{T}_h}\left(-\int_E \nabla q_h \cdot I_h\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} + \int_{\partial E} I_h\boldsymbol{u}\cdot\boldsymbol{n}_E q_h\,\mathrm{d}\boldsymbol{s}\right) \\ &= \sum_{E\in\mathcal{T}_h}\left(-\int_E \nabla q_h \cdot \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} + \int_{\partial E} \boldsymbol{u}\cdot\boldsymbol{n}_E q_h\,\mathrm{d}\boldsymbol{s}\right) \\ &= -b(\boldsymbol{u}, q_h) = 0. \end{aligned} \tag{6.5}$$

According to (6.5) and Lemma 4.2, we obtain

$$\begin{aligned} &|\boldsymbol{\phi}|^2_{1,h} \\ &\lesssim a_h(\boldsymbol{\phi},\boldsymbol{\phi}) = a_h(\boldsymbol{u}_h,\boldsymbol{\phi}) - a_h(I_h\boldsymbol{u},\boldsymbol{\phi}) \\ &= (\boldsymbol{f}_h,\boldsymbol{\phi}) - b_h(\boldsymbol{\phi}, p_h) - a_h(I_h\boldsymbol{u},\boldsymbol{\phi}) = (\boldsymbol{f}_h,\boldsymbol{\phi}) - a_h(I_h\boldsymbol{u},\boldsymbol{\phi}) \\ &= (\boldsymbol{f}_h - \boldsymbol{f},\boldsymbol{\phi}) + (\boldsymbol{f},\boldsymbol{\phi}) - \sum_{E\in\mathcal{T}_h} a^E_h(I_h\boldsymbol{u} - \boldsymbol{u}_\pi,\boldsymbol{\phi}) - \sum_{E\in\mathcal{T}_h} a^E(\boldsymbol{u}_\pi - \boldsymbol{u},\boldsymbol{\phi}) - a(\boldsymbol{u},\boldsymbol{\phi}) \\ &= (\boldsymbol{f}_h - \boldsymbol{f},\boldsymbol{\phi}) + (\boldsymbol{f},\boldsymbol{\phi}) - \sum_{E\in\mathcal{T}_h} a^E_h(I_h\boldsymbol{u} - \boldsymbol{u}_\pi,\boldsymbol{\phi}) - \sum_{E\in\mathcal{T}_h} a^E(\boldsymbol{u}_\pi - \boldsymbol{u},\boldsymbol{\phi}) - a(\boldsymbol{u},\boldsymbol{\phi}) \\ &\quad - b(\boldsymbol{\phi}, p) + b(\boldsymbol{\phi}, p - \Pi^0_{k-1}p) \\ &= (\boldsymbol{f}_h - \boldsymbol{f},\boldsymbol{\phi}) - \left(\sum_{E\in\mathcal{T}_h} a^E_h(I_h\boldsymbol{u} - \boldsymbol{u}_\pi,\boldsymbol{\phi}) + \sum_{E\in\mathcal{T}_h} a^E(\boldsymbol{u}_\pi - \boldsymbol{u},\boldsymbol{\phi})\right) \\ &\quad + b(\boldsymbol{\phi}, p - \Pi^0_{k-1}p) + ((\boldsymbol{f},\boldsymbol{\phi}) - a(\boldsymbol{u},\boldsymbol{\phi}) - b(\boldsymbol{\phi}, p)). \end{aligned} \tag{6.6}$$

For the first term on the right-hand side in (6.6), the inequality (4.1) leads to

$$(\boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{\phi}) \leq \|\boldsymbol{f} - \boldsymbol{f}_h\|_* |\boldsymbol{\phi}|_{1,h}.$$

For the second term on the right-hand side in (6.6), the stability of $a_h^E(\cdot,\cdot)$ implies

$$\sum_{E\in\mathcal{T}_h} a_h^E(I_h\boldsymbol{u} - \boldsymbol{u}_\pi, \boldsymbol{\phi}) + \sum_{E\in\mathcal{T}_h} a^E(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{\phi}) \lesssim \big(|\boldsymbol{u} - I_h\boldsymbol{u}|_{1,h} + |\boldsymbol{u} - \boldsymbol{u}_\pi|_{1,h}\big)|\boldsymbol{\phi}|_{1,h}.$$

For the third term on the right-hand side in (6.6), we have

$$b(\boldsymbol{\phi}, p - \Pi_{k-1}^0 p) = -\sum_{E\in\mathcal{T}_h} \int_E (p - \Pi_{k-1}^0 p)\operatorname{div}\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} \lesssim \|p - \Pi_{k-1}^0 p\|_{0,\Omega}|\boldsymbol{\phi}|_{1,h}.$$

For the last term on the right-hand side in (6.6), we let $W_h(\boldsymbol{\phi}) = (\boldsymbol{f},\boldsymbol{\phi}) - a(\boldsymbol{u},\boldsymbol{\phi}) - b(\boldsymbol{\phi}, p)$. According to Lemma 6.1, we obtain

$$W_h(\boldsymbol{\phi}) \lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s)|\boldsymbol{\phi}|_{1,h}.$$

Summing up the results above and according to (3.5) and Lemma 3.2, we have

$$\begin{aligned}|\boldsymbol{\phi}|_{1,h}^2 &\lesssim \big(|\boldsymbol{u} - I_h\boldsymbol{u}|_{1,h} + |\boldsymbol{u} - \boldsymbol{u}_\pi|_{1,h} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_* + \|p - \Pi_{k-1}^0 p\|_{0,\Omega}\big)|\boldsymbol{\phi}|_{1,h} + W_h(\boldsymbol{\phi})\\ &\lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s)|\boldsymbol{\phi}|_{1,h}.\end{aligned}$$

So we have

$$|\boldsymbol{\phi}|_{1,h} \lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s).$$

This, together with (6.4) and Lemma 3.2, leads to the desired result. □

Theorem 6.3. *Let $p \in L_0^2(\Omega) \cap H^s(\Omega)$ $(0 < s \leq k)$ be the exact pressure solution to problem (2.2). Let $p_h \in Q_h^{k-1}$ be the virtual element pressure solution to problem (4.2). Then, it holds that*

$$\|p - p_h\| \lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s).$$

Proof. Using the triangle inequality, we obtain

$$\|p - p_h\| \leq \|p - \Pi_{k-1}^0 p\| + \|\Pi_{k-1}^0 p - p_h\|.$$

Firstly, estimate $\|p_h - \Pi_{k-1}^0 p\|$, and it is easy to see that $p_h - \Pi_{k-1}^0 p \in Q_h^{k-1}$. According to Lemma 4.3, we have

$$\|p_h - \Pi_{k-1}^0 p\| \lesssim \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h^k, \boldsymbol{v}_h\neq\boldsymbol{0}} \frac{b_h(\boldsymbol{v}_h, p_h - \Pi_{k-1}^0 p)}{|\boldsymbol{v}_h|_{1,h}}. \tag{6.7}$$

Let $\boldsymbol{v}_h$ be arbitrary, we estimate the term $b_h(\boldsymbol{v}_h, p_h - \Pi^0_{k-1}p)$:

$$
\begin{aligned}
b_h(\boldsymbol{v}_h, p_h - \Pi^0_{k-1}p) &= b_h(\boldsymbol{v}_h, p_h) - b_h(\boldsymbol{v}_h, \Pi^0_{k-1}p) \\
&= (\boldsymbol{f}_h, \boldsymbol{v}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, \Pi^0_{k-1}p) \\
&= (\boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{f}, \boldsymbol{v}_h) - a(\boldsymbol{u}, \boldsymbol{v}_h) + a(\boldsymbol{u}, \boldsymbol{v}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad - b(\boldsymbol{v}_h, p) + b(\boldsymbol{v}_h, p) - b_h(\boldsymbol{v}_h, \Pi^0_{k-1}p) \\
&= (\boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h) + W_h(\boldsymbol{v}_h) + a(\boldsymbol{u}, \boldsymbol{v}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad + \big(b(\boldsymbol{v}_h, p) - b_h(\boldsymbol{v}_h, \Pi^0_{k-1}p)\big).
\end{aligned} \tag{6.8}
$$

For the first term on the right-hand side in (6.8), according to (4.1), we have

$$
(\boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h) \lesssim \|\boldsymbol{f} - \boldsymbol{f}_h\|_* |\boldsymbol{v}_h|_{1,h} \lesssim h^s |\boldsymbol{f}|_{s-1} |\boldsymbol{v}_h|_{1,h}. \tag{6.9}
$$

According to Lemma 6.1, we have

$$
W_h(\boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) - a(\boldsymbol{u}, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p) \lesssim h^s(|\boldsymbol{f}|_{s-1} + |\boldsymbol{u}|_{s+1} + |p|_s)|\boldsymbol{v}_h|_{1,h}. \tag{6.10}
$$

Considering the term $a(\boldsymbol{u}, \boldsymbol{v}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$, for a polynomial $\boldsymbol{u}_\pi \in \boldsymbol{P}_k(E)$ with the estimates (3.5) and according to Lemma 4.2 and Theorem 6.2, we obtain

$$
\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}_h) - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \sum_{E \in \mathcal{T}_h} \big(a^E(\boldsymbol{u}, \boldsymbol{v}_h) - a^E_h(\boldsymbol{u}_h, \boldsymbol{v}_h)\big) \\
&= \sum_{E \in \mathcal{T}_h} \big(a^E(\boldsymbol{u} - \boldsymbol{u}_\pi, \boldsymbol{v}_h) - a^E_h(\boldsymbol{u}_h - \boldsymbol{u}_\pi, \boldsymbol{v}_h)\big) \\
&\lesssim (|\boldsymbol{u} - \boldsymbol{u}_\pi|_{1,h} + |\boldsymbol{u}_h - \boldsymbol{u}_\pi|_{1,h})|\boldsymbol{v}_h|_{1,h} \\
&\lesssim (|\boldsymbol{u} - \boldsymbol{u}_\pi|_{1,h} + |\boldsymbol{u} - \boldsymbol{u}_h|_{1,h})|\boldsymbol{v}_h|_{1,h} \\
&\lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s)|\boldsymbol{v}_h|_{1,h}.
\end{aligned} \tag{6.11}
$$

For the last term on the right-hand side in (6.8), by the definition of Π^0_{k-1}, we obtain

$$
\begin{aligned}
b(\boldsymbol{v}_h, p) - b_h(\boldsymbol{v}_h, \Pi^0_{k-1}p) &= -\sum_{E \in \mathcal{T}_h} \int_E p \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} + \sum_{E \in \mathcal{T}_h} \int_E \Pi^0_{k-1}p \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} \\
&= -\sum_{E \in \mathcal{T}_h} \int_E (p - \Pi^0_{k-1}p) \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} \\
&\lesssim \|p - \Pi^0_{k-1}p\|_{0,\Omega} |\boldsymbol{v}_h|_{1,h} \\
&\lesssim h^s |p|_s |\boldsymbol{v}_h|_{1,h}.
\end{aligned} \tag{6.12}
$$

Substituting (6.9)–(6.12) into (6.8), we obtain

$$
b_h(\boldsymbol{v}_h, p_h - \Pi^0_{k-1}p) \lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s)|\boldsymbol{v}_h|_{1,h}.
$$

This, together with (6.7), implies

$$\|p_h - \Pi^0_{k-1} p\| \lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s).$$

So, by the triangle inequality and Lemma 4.1, we obtain

$$\|p - p_h\| \lesssim h^s(|\boldsymbol{u}|_{s+1} + |\boldsymbol{f}|_{s-1} + |p|_s). \qquad \square$$

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