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Non Local Weighted Fourth Order Equation in Dimension 4 with Non-linear Exponential Growth

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Abstract. In this work, we study the weighted Kirchhoff problem

$$\begin{cases} g\big(\int_B (w(x)|\Delta u|^2)\,dx\big)[\Delta(w(x)\Delta u)] = f(x,u) & \text{in } B, \\ u > 0 & \text{in } B, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

where B is the unit ball of $\mathbb{R}^4$, $w(x) = \big(\log \frac{e}{|x|}\big)^{\beta}$, the singular logarithm weight in Adam's embedding, g is a continuous positive function on $\mathbb{R}^+$. The nonlinearities are critical growth in view of Adam's inequalities. We prove the existence of a positive ground state solution using mountain pass method combined with a concentration compactness result. The associated energy function does not satisfy the condition of compactness. We provide a new condition for growth and we stress its importance to check the min-max compactness level.

1. Introduction

In this paper, we consider the non local fourth order elliptic equation

$$\begin{cases} g\big(\int_B (w(x)|\Delta u|^2)\,dx\big)\Delta(w(x)\Delta u) = f(x,u) & \text{in } B, \\ u > 0 & \text{in } B, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $B = B(0,1)$ is the unit open ball in $\mathbb{R}^4$, $f(x,t)$ is a radial function with respect to x, the weight $w(x)$ is given by

$$w(x) = \left(\log \frac{e}{|x|}\right)^{\beta}, \quad \beta \in (0,1). \tag{1.2}$$

The Kirchhoff function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a positive continuous function which will be specified later.

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In 1883, Kirchhoff [18] studied the following parabolic problem

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\,dx\right)\frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

The parameters in equation (1.3) have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. These kinds of problems have physical motivations. Indeed, the Kirchhoff operator $G\big(\big(\int_B|\nabla u|^2\,dx\big)\big)\Delta u$ also appears in the nonlinear vibration equation, namely

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}-G\big(\int_B|\nabla u|^2\,dx\big)\Delta u=f(x,u) & \text{in } B\times(0,T),\\ u>0 & \text{in } B\times(0,T),\\ u=0 & \text{on } \partial B,\\ u(x,0)=u_0(x) & \text{in } B,\\ \frac{\partial u}{\partial t}(x,0)=u_1(x) & \text{in } B\end{cases}$$

which have focused the attention of several researchers, mainly as a result of the work of Lions [19]. Non-local problems also arise in other areas, e.g. biological systems (where the function u describes a process that depends on the average of itself for example, population density), see e.g. [3, 4] and the references therein.

Second order Kirchhoff's classical equation has been extensively studied. We refer to the work of Chipot [11, 12], Corrêa et al. [18] and their references. We mention that Figueiredo and Severo [16] studied the following problem

$$\begin{cases}-m\big(\int_B|\nabla u|^2\,dx\big)\Delta u=f(x,u) & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, the nonlinearity f behaves like $\exp(\alpha t^2)$ as $t\to+\infty$, for some $\alpha>0$. $m\colon(0,+\infty)\to(0,+\infty)$ is a continuous function satisfying some conditions. The authors proved that this problem has a positive ground state solution. The existence result was proved by combining minimax techniques and Trudinger–Moser inequality.

Recently, Sitong Chen, Xianhua Tang and Jiuyang Wei [9], studied the last problem. They have developed some new approaches to estimate precisely the minimax level of the energy functional and prove the existence of Nehari-type ground-state solutions and nontrivial solutions for the above problem.

It should be noted that, recently, the following nonhomogeneous Kirchhoff–Schrödinger equation

$$-M\left(\int_{\mathbb{R}^2} |\nabla u|^2 + \xi(|x|)u^2\,dx\right)(-\Delta u + \xi(|x|)u) = Q(x)g(u) + \varepsilon h(x),$$
$$u(x) \to 0 \quad \text{as } |x| \to +\infty$$

has been studied in [2], where ε is a positive parameter, $M\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\xi, Q\colon (0,+\infty) \to \mathbb{R}$ are continuous functions that satisfy some mild conditions. The nonlinearity $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and behaves like $\exp(\alpha t^2)$ as $t \to +\infty$, for some $\alpha > 0$. The authors proved the existence of at least two weak solutions for this equation by combining the Mountain Pass Theorem and Ekeland's Variational Principle.

We point out that recently, weighted logarithmic second order elliptic equations are studied. We cite the following problem [8]

$$\begin{cases} -\operatorname{div}(\nu(x)\nabla u) = f(x,u) & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

with the weight $\nu(x) = \log\left(\frac{e}{|x|}\right)$ and where the function $f(x,t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\left\{e^{\alpha t^2}\right\}$ as $t \to +\infty$, for some $\alpha > 0$.

Also, recently, Deng et al. [13] and Zhang [22] studied the following problem

$$\begin{cases} -\operatorname{div}(\rho(x)|\nabla u|^{N-2}\nabla u) = f(x,u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $N \geq 2$, the function $f(x,t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\left\{e^{\alpha t^{N/(N-1)}}\right\}$ as $t \to +\infty$, for some $\alpha > 0$. Also, we mention that Baraket et al. [6] studied the following non-autonomous weighted elliptic equations

$$\begin{cases} -\operatorname{div}(\rho(x)|\nabla u|^{N-2}\nabla u) + \xi(x)|u|^{N-2}u = f(x,u) & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where B is the unit ball of $\mathbb{R}^N$, $N > 2$, $f(x,t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\left\{e^{\alpha t^{N/(N-1)}}\right\}$ as $t \to +\infty$, for some $\alpha > 0$. $\xi\colon B \to \mathbb{R}$ is a positive continuous function satisfying some conditions. The weight $\rho(x)$ is given by $\rho(x) = \left(\log \frac{e}{|x|}\right)^{N-1}$.

In the latter work cited, the authors proved that there is a non-trivial solution using Mountain Pass Theorem and weighted Trudinger–Moser inequalities [7]. In order to motivate our study, we begin by giving a brief survey on Adam's inequalities. For bounded domains $\Omega \subset \mathbb{R}^4$, in [1, 21] the authors proved the following Adams' inequality

$$\sup_{u\in S}\int_\Omega \left(e^{\alpha u^2}-1\right)dx<+\infty \quad\Longleftrightarrow\quad \alpha\le 32\pi^2,$$

where $S=\left\{u\in W_0^{2,2}(\Omega)\;\middle|\;\left(\int_\Omega|\Delta u|^2\,dx\right)^{1/2}\le 1\right\}$. This last result opened the way to study fourth-order problems with subcritical or critical nonlinearity. We cite the work of Sani [16]

$$\Delta^2 u+V(x)u=f(x,u)\quad\text{in } H^2(\mathbb{R}^4).$$

Inspired by the last work cited above, we study the existence of a positive ground state solutions when the nonlinear terms have the critical exponential growth in the sense of Adams' inequalities [23]. Our approach is variational methods such as the Mountain Pass Theorem combining with a concentration compactness result. Let $\Omega\subset\mathbb{R}^4$ be a bounded domain and $w\in L^1(\Omega)$ be a nonnegative function. We introduce the Sobolev space

$$W_0^{2,2}(\Omega,w)=\operatorname{cl}\left\{u\in C_0^\infty(\Omega)\;\middle|\;\int_\Omega w(x)|\Delta u|^2\,dx<\infty\right\}.$$

We will focus on radial functions and consider the subspace

$$W_{0,\mathrm{rad}}^{2,2}(\Omega,w)=\operatorname{cl}\left\{u\in C_{0,\mathrm{rad}}^\infty(\Omega)\;\middle|\;\int_\Omega w(x)|\Delta u|^2\,dx<\infty\right\}.$$

The choice of the weight and the space $W_{0,\mathrm{rad}}^{2,2}(B,w)$ are motivated by the following exponential inequality.

Theorem 1.1. [23] *Let $\beta\in(0,1)$ and let w be given by (1.2), then*

$$\sup_{\substack{u\in W_{0,\mathrm{rad}}^{2,2}(B,w)\\ \int_B|\Delta u|^2w(x)\,dx\le 1}}\int_B e^{\alpha|u|^{2/(1-\beta)}}\,dx<+\infty \quad\Longleftrightarrow\quad \alpha\le\alpha_\beta=4[8\pi^2(1-\beta)]^{1/(1-\beta)}. \tag{1.4}$$

Let $\gamma:=2/(1-\beta)$, in view of inequality (1.4), we say that f has critical growth at $+\infty$ if there exists some $\alpha_0>0$,

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{e^{\alpha s^\gamma}}=0,\ \ \forall\,\alpha \text{ such that } \alpha_0<\alpha \text{ and } \lim_{s\to+\infty}\frac{|f(x,s)|}{e^{\alpha s^\gamma}}=+\infty,\ \ \forall\,\alpha<\alpha_0.$$

To study the solvability of problem (1.1), we consider the subspace

$$\mathbf{X}=\left\{u\in W_{0,\mathrm{rad}}^{2,2}(B,w)\;\middle|\;\int_B w(x)|\Delta u|^2\,dx<\infty\right\}$$

endowed with the norm

$$\|u\|=\left(\int_B w(x)|\Delta u|^2\,dx\right)^{1/2}.$$

We note that this norm is issued from the product scalar

$$\langle u, v\rangle = \int_B w(x)\Delta u \cdot \Delta v \, dx.$$

Let us now state our results. We define the function

$$G(t) = \int_0^t g(s)\, ds,$$

where the function g is continuous on $\mathbb{R}^+$ and verifies

(G_1) There exists $g_0 > 0$ such that $g(t) \geq g_0$ for all $t \geq 0$ and

$$G(t+s) \geq G(t) + G(s), \quad \forall\, s, t \geq 0;$$

(G_2) There exists constants $a_1, a_2 > 0$ and $t_1 > 0$ such that for some $\delta \in \mathbb{R}$,

$$g(t) \leq a_1 + a_2 t^{\delta}, \quad \forall\, t \geq t_1;$$

(G_3) $\frac{g(t)}{t}$ is nonincreasing for $t > 0$.

As a consequence of (G_3), a simple calculation shows that

$$\frac{1}{2}G(t) - \frac{1}{4}g(t)t \text{ is nondecreasing for } t \geq 0.$$

Consequently, one has

$$\frac{1}{2}G(t) - \frac{1}{4}g(t)t \geq 0, \quad \forall\, t \geq 0. \tag{1.5}$$

A typical example of a function g fulfilling the conditions (G_1), (G_2) and (G_3) is given by

$$g(t) = g_0 + at, \quad g_0,\, a > 0.$$

Another example is given by $g(t) = 1 + \ln(1+t)$.

Furthermore, we suppose that $f(x,t)$ has critical growth and satisfies the following hypothesis:

(H_1) The non-linearity $f\colon \overline{B} \times \mathbb{R} \to \mathbb{R}$ is positive, continuous, radial in x, and $f(x,t) = 0$ for $t \leq 0$.

(H_2) There exist $t_0 > 0$ and $M_0 > 0$ such that for all $t > t_0$ and for all $x \in B$ we have

$$0 < F(x,t) \leq M_0 f(x,t),$$

where

$$F(x,t) = \int_0^t f(x,s)\, ds.$$

(H$_3$) For each $x \in B$, $\frac{f(x,t)}{t^3}$ is increasing for $t > 0$.

(H$_4$) $\displaystyle\lim_{t\to\infty} \frac{f(x,t)t}{e^{\alpha_0 t^\gamma}} \geq \gamma_0$ uniformly in x with $\gamma_0 > \dfrac{1024(1-\beta)g\big(\big(\frac{\alpha_\beta}{\alpha_0}\big)^{2/\gamma}\big)}{\alpha_0^{1-\beta}}$.

The condition (H$_2$) implies that for any $\varepsilon > 0$, there exists a real $t_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon t f(x,t), \quad \forall\, |t| > t_\varepsilon, \quad \text{uniformly in } x \in B. \tag{1.6}$$

Also, we have that the condition (H$_3$) leads to

$$\lim_{t\to 0} \frac{f(x,t)}{t^\theta} = 0 \quad \text{for all } 0 \leq \theta < 3. \tag{1.7}$$

The asymptotic condition (H$_4$) would be crucial to identify the min-max level of the energy associated to the problem (1.1). We give an example of f. Let $f(t) = F'(t)$ with $F(t) = \frac{t^4}{4} + t^4 e^{\alpha_0 t^\gamma}$. A simple calculation shows that f verifies the conditions (H$_1$), (H$_2$), (H$_3$) and (H$_4$).

It will be said that u is a solution of the problem (1.1), if u is a weak solution in the following sense.

Definition 1.2. A function u is called a solution of (1.1) if $u \in \mathbf{X}$ and

$$g(\|u\|^2) \int_B (w(x)\Delta u \Delta\varphi)\, dx = \int_B f(x,u)\varphi\, dx \quad \text{for all } \varphi \in \mathbf{X}.$$

The energy functional $\mathcal{J} \colon \mathbf{X} \to \mathbb{R}$, also known as the Euler–Lagrange functional associated to (1.1), is defined by

$$\mathcal{J}(u) = \frac{1}{2} G(\|u\|^2) - \int_B F(x,u)\, dx, \tag{1.8}$$

where

$$F(x,t) = \int_0^t f(x,s)\, ds.$$

Definition 1.3. A solution u is a ground state solution of the problem (1.1) if u is a solution and

$$\mathcal{J}(u) = r \ \text{ with } \ r = \inf_{u\in\mathcal{S}} \mathcal{J}(u) \ \text{ where } \ \mathcal{S} = \{u \in \mathbf{X} : \mathcal{J}'(u) = 0, u \neq 0\},$$

and

$$\mathcal{J}'(u)\varphi = g(\|u_n\|^2) \left(\int_B (\omega(x)\Delta u \Delta\varphi)\, dx - \int_B f(x,u)\varphi\, dx \right), \quad \varphi \in \mathbf{X}.$$

It is quite clear that finding weak solutions to the problem (1.1) is equivalent to finding non-zero critical points of the functional $\mathcal{J}$ over $\mathbf{X}$.

The major difficulties in this problem lies first in the concurrence between the growths of g and f and second in the loss of compactness of the energy. To avoid the first difficulty, many authors usually assume that g is increasing or bounded (see [3,4,10,16]). For the loss of compactness, we use suitable Adam's functions and prove concentration compactness result.

Our result is as follows:

Theorem 1.4. *Assume that $f(x,t)$ has a critical growth at $+\infty$ for some α_0 and satisfies the conditions* (H_1), (H_2), (H_3) *and* (H_4). *If in addition* (G_1), (G_2) *and* (G_3) *are satisfied, then the problem* (1.1) *has a positive ground state solution.*

To the best of our knowledge, the present papers results have not been covered yet in the literature.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about functional space. In Section 3, we give some useful lemmas for the compactness analysis. In Section 4, we prove that the energy $\mathcal{J}$ satisfies the two geometric properties. Section 5 is devoted to estimate the min-max level of the energy. Finally, we conclude with the proofs of the main results in Section 6.

Throughout this paper, the constant C may change from a line to another and we sometimes index the constants in order to show how they change.

2. Weighted Lebesgue and Sobolev spaces setting

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $w \in L^1(\Omega)$ be a nonnegative function. To deal with weighted operator, we need to introduce some functional spaces $L^p(\Omega, w)$, $W^{m,p}(\Omega, w)$, $W_0^{m,p}(\Omega, w)$ and some of their properties that will be used later. Let $S(\Omega)$ be the set of all measurable real-valued functions defined on Ω and two measurable functions are considered as the same element if they are equal almost everywhere.

Following Drabek et al. and Kufner in [14,17], the weighted Lebesgue space $L^p(\Omega, w)$ is defined as follows:

$$L^p(\Omega, w) = \left\{ u\colon \Omega \to \mathbb{R} \text{ measurable} \;\middle|\; \int_\Omega w(x)|u|^p \, dx < \infty \right\}$$

for any real number $1 \leq p < \infty$. This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \left(\int_\Omega w(x)|u|^p \, dx \right)^{1/p}.$$

For $w(x) = 1$, one finds the standard Lebesgue space $L^p(\Omega)$ endowed with the norm $\|u\|_p = \big(\int_\Omega |u|^p \, dx \big)^{1/p}$.

For $m \geq 2$, let w be a given family of weight functions w_τ, $|\tau| \leq m$,

$$w = \{w_\tau(x) \mid x \in \Omega, |\tau| \leq m\}.$$

In [14], the corresponding weighted Sobolev space was defined as

$$\begin{aligned} W^{m,p}(\Omega, w) = \big\{u \in L^p(\Omega) \mid D^\tau u \in L^p(\Omega) \text{ for all } |\tau| \leq m-1, \\ D^\tau u \in L^p(\Omega, w) \text{ for all } |\tau| = m\big\} \end{aligned}$$

endowed with the following norm

$$\|u\|_{W^{m,p}(\Omega,w)} = \left(\sum_{|\tau| \leq m-1} \int_\Omega |D^\tau u|^p \, dx + \sum_{|\tau| = m} \int_\Omega |D^\tau u|^p w(x) \, dx \right)^{1/p}.$$

If we suppose also that $w(x) \in L^1_{\mathrm{loc}}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{m,p}(\Omega, w)$ and we can introduce the space

$$W_0^{m,p}(\Omega, w)$$

as the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega, w)$. Moreover, the following embedding

$$W^{m,p}(\Omega, w) \hookrightarrow\hookrightarrow W^{m-1,p}(\Omega)$$

is compact. Also, $(L^p(\Omega, w), \|\cdot\|_{p,w})$ and $(W^{m,p}(\Omega, w), \|\cdot\|_{W^{m,p}(\Omega,w)})$ are separable, reflexive Banach spaces provided that $w(x)^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)$. Then the space $\mathbf{X}$ is a Banach and reflexive space. The space $\mathbf{X}$ is endowed with the norm

$$\|u\| = \left(\int_B w(x) |\Delta u|^2 \, dx \right)^{1/2}$$

which is equivalent to the following norm (see Lemma 3.1)

$$\|u\|_{W^{2,2}_{0,\mathrm{rad}}(B,w)} = \left(\int_B u^2 \, dx + \int_B |\nabla u|^2 \, dx + \int_B |\Delta u|^2 w(x) \, dx \right)^{1/2}.$$

We also have the continuous embedding

$$\mathbf{X} \hookrightarrow L^q(B) \quad \text{for all } q \geq 1.$$

Moreover, $\mathbf{X}$ is compactly embedded in $L^q(B)$ for all $q \geq 1$ (see Lemma 3.1).

3. Preliminary for the compactness analysis

In this section, we will derive several technical lemmas for our use later. First we begin by the radial lemma.

Lemma 3.1. *Let u be a radially symmetric function in $C_0^2(B)$. Then we have*

(i) [23]
$$|u(x)| \leq \frac{1}{2\sqrt{2}\pi}\frac{\left|\left|\log\left(\frac{e}{|x|}\right)\right|^{1-\beta}-1\right|^{1/2}}{\sqrt{1-\beta}}\int_B w(x)|\Delta u|^2\,dx$$
$$\leq \frac{1}{2\sqrt{2}\pi}\frac{\left|\left|\log\left(\frac{e}{|x|}\right)\right|^{1-\beta}-1\right|^{1/2}}{\sqrt{1-\beta}}\|u\|^2.$$

(ii) *The norms $\|\cdot\|$ and $\|u\|_{W^{2,2}_{0,\mathrm{rad}}(B,w)} = \left(\int_B u^2\,dx + \int_B |\nabla u|^2\,dx + \int_B |\Delta u|^2 w(x)\,dx\right)^{1/2}$ are equivalents.*

(iii) *The following embedding*
$$\mathbf{X} \hookrightarrow L^q(B) \quad \textit{for all } q \geq 1$$
is continuous.

(iv) $\mathbf{X}$ *is compactly embedded in $L^q(B)$ for all $q \geq 2$.*

Proof. (i) See [23].

(ii) By Poincaré inequality, for all $u \in W^{1,2}_{0,\mathrm{rad}}(B)$,
$$\int_B |u|^2 \leq C\int_B |\nabla u|^2.$$
Using the Green formula, we get
$$\int_B |\nabla u|^2 = \int_B \nabla u \nabla u = -\int_B u\Delta u + \underbrace{\int_{\partial B} u\frac{\partial u}{\partial n}}_{=0} \leq \left|\int_B u\Delta u\right|.$$
By Young inequality, we get for all $\varepsilon > 0$,
$$\left|\int_B u\Delta u\right| \leq \frac{1}{2\varepsilon}\int_B |\Delta u|^2 + \frac{\varepsilon}{2}\int_B |u|^2 \leq \frac{1}{2\varepsilon}\int_B w(x)|\Delta u|^2 + \frac{\varepsilon}{2}\int_B |u|^2.$$
Hence
$$\left(1-\frac{\varepsilon}{2}C^2\right)\int_B |\nabla u|^2 \leq \frac{1}{2\varepsilon}\int_B |\Delta u|^2 w(x),$$
then
$$\left(\int_B u^2\,dx + \int_B |\nabla u|^2\,dx + \int_B |\Delta u|^2 w(x)\,dx\right)^{1/2} \leq C\int_B |\Delta u|^2 w(x)\,dx \leq C\|u\|^2.$$
Then (ii) follows.

(iii) Since $w(x) \geq 1$, then following embeddings
$$\mathbf{X} \hookrightarrow W^{2,2}_{0,\mathrm{rad}}(B,w) \hookrightarrow W^{2,2}_{0,\mathrm{rad}}(B) \hookrightarrow L^q(B), \quad \forall\, q \geq 2$$
are continuous, and we have that $\mathbf{X} \hookrightarrow L^1(B)$ is continuous from (i).

(iv) Since $W^{2,2}_0(B,w) \hookrightarrow\hookrightarrow W^{1,1}(B)$ is compact, then (iv) follows. This concludes the lemma. □

In the next, we give the following useful lemma.

Lemma 3.2. [15] *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain and $f\colon \overline{\Omega} \times \mathbb{R}$ a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x, u_n)u_n|\, dx \leq C,$$

where C is a positive constant, then

$$f(x, u_n) \to f(x, u) \quad \text{in } L^1(\Omega).$$

In the sequel, we prove a concentration compactness result of Lions type [20].

Theorem 3.3. *Let $(u_k)_k$ be a sequence in $\mathbf{X}$. Suppose that $\|u_k\| = 1$, $u_k \rightharpoonup u$ weakly in $\mathbf{X}$, $u_k(x) \to u(x)$ a.e. $x \in B$, and $u \not\equiv 0$. Then*

$$\sup_k \int_B e^{p\alpha_\beta |u_k|^\gamma}\, dx < +\infty, \quad \text{where } \alpha_\beta = 4[8\pi^2(1-\beta)]^{1/(1-\beta)}$$

for all $1 < p < U(u)$, where $U(u)$ is given by

$$U(u) := \begin{cases} \frac{1}{(1-\|u\|^2)^{\gamma/2}} & \text{if } \|u\| < 1, \\ +\infty & \text{if } \|u\| = 1. \end{cases}$$

Proof. Since $\|u\| \leq \lim_k \|u_k\| = 1$, we will split the evidence into two cases.

Case 1*:* $\|u\| < 1$. We assume by contradiction for some $p_1 < U(u)$, we have

$$\sup_k \int_B \exp\left(\alpha_\beta p_1 u_k^\gamma\right) dx = +\infty.$$

Set

$$B_\lambda^k = \{x \in B : u_k(x) \geq \lambda\}$$

where λ is a constant that we will choose later. Let $v_k = u_k - \lambda$. We have

$$(1+a)^q \leq (1+\varepsilon)a^q + \left(1 - \frac{1}{(1+\varepsilon)^{1/(q-1)}}\right)^{1-q}, \quad \forall\, a \geq 0,\ \forall\, \varepsilon > 0,\ \forall\, q > 1. \tag{3.1}$$

So, using (3.1), we get

$$\begin{aligned} |u_k|^\gamma &= |u_k - \lambda + \lambda|^\gamma \leq (|u_k - \lambda| + |\lambda|)^\gamma \\ &\leq (1+\varepsilon)|u_k - \lambda|^\gamma + \left(1 - \frac{1}{(1+\varepsilon)^{1/(\gamma-1)}}\right)^{1-\gamma} |\lambda|^\gamma \leq (1+\varepsilon)v_k^\gamma + C(\varepsilon, \gamma)\lambda^\gamma. \end{aligned} \tag{3.2}$$

We have

$$\int_B \exp\left(\alpha_\beta p_1 u_k^\gamma\right) dx = \int_{B_\lambda^k} \exp\left(\alpha_\beta p_1 u_k^\gamma\right) dx + \int_{B \setminus B_\lambda^k} \exp\left(\alpha_\beta p_1 u_k^\gamma\right) dx$$

$$\leq \int_{B_\lambda^k} \exp\left(\alpha_\beta p_1 u_k^\gamma\right) dx + c\exp\left(\alpha_\beta p_1 \lambda^\gamma\right)$$
$$\leq \int_{B_\lambda^k} \exp\left(\alpha_\beta p_1 u_k^\gamma\right) dx + c(\lambda, \gamma, |B|),$$

and then

$$\sup_k \int_{B_\lambda^k} \exp\left(\alpha_\beta p_1 u_k^\gamma\right) dx = +\infty.$$

By (3.2) we have

$$\int_{B_\lambda^k} \exp\left(\alpha_\beta p_1 u_k^\gamma\right) dx \leq \exp\left(\alpha_\beta p_1 C(\varepsilon,\gamma)\lambda^\gamma\right) \int_{B_\lambda^k} \exp\left((1+\varepsilon)\alpha_\beta p_1 v_k^\gamma\right) dx.$$

Since $p_1 < U(u)$, there exists $\widetilde{p}_1$ such that $\widetilde{p}_1 = (1+\varepsilon)p_1 < U(u)$. Thus

$$\sup_k \int_{B_\lambda^k} \exp\left(\widetilde{p}_1 \alpha_\beta v_k^\gamma\right) dx = +\infty.$$

Now, we define

$$T^\lambda(u) = \min\{\lambda, u\} \quad \text{and} \quad T_\lambda(u) = u - T^\lambda(u)$$

and choose λ such that

$$\frac{1-\|u\|^2}{1-\|T^\lambda u\|^2} > \left(\frac{\widetilde{p}_1}{U(u)}\right)^{2/\gamma}. \tag{3.3}$$

We claim that

$$\limsup_k \int_{B_\lambda^k} \omega(x)|\Delta v_k|^2\, dx < \left(\frac{1}{\widetilde{p}_1}\right)^{2/\gamma}.$$

If this is not the case, then up to a subsequence, we get

$$\int_{B_\lambda^k} \omega(x)|\Delta v_k|^2\, dx = \int_B \omega(x)|\Delta T_\lambda u_k|^2\, dx \geq \left(\frac{1}{\widetilde{p}_1}\right)^{2/\gamma} + o_k(1).$$

Thus

$$\left(\frac{1}{\widetilde{p}_1}\right)^{2/\gamma} + \int_B \omega(x)|\Delta T^\lambda u_k|^2\, dx + o_k(1)$$
$$\leq \int_B \omega(x)|\Delta T_\lambda u_k|^2\, dx + \int_{B\setminus B_\lambda^k} \omega(x)|\Delta u_k|^2\, dx$$
$$= \int_{B_\lambda^k} \omega(x)|\Delta u_k|^2\, dx + \int_{B\setminus B_\lambda^k} \omega(x)|\Delta u_k|^2\, dx = 1.$$

For $\lambda > 0$ fixed, $T^\lambda u_k$ is also bounded in $\mathbf{X}$. Hence, up to a subsequence, $T^\lambda u_k \to T^\lambda u$ in $\mathbf{X}$ and $T^\lambda u_k \to T^\lambda u$ almost everywhere in B. By the lower semicontinuity of the norm in $\mathbf{X}$ and the above inequality, we have

$$\widetilde{p}_1 \geq \frac{1}{\left(1-\liminf_{k\to+\infty}\|T^\lambda u_k\|^2\right)^{\gamma/2}} \geq \frac{1}{\left(1-\|T^\lambda u\|^2\right)^{\gamma/2}},$$

combining with (3.3), we obtain

$$\widetilde{p}_1 \geq \frac{1}{\left(1-\|T^\lambda u\|^2\right)^{\gamma/2}} > \frac{\widetilde{p}_1}{U(u)} \frac{1}{\left(1-\|T^\lambda u\|^2\right)^{\gamma/2}} = \widetilde{p}_1,$$

which is a contradiction. Therefore

$$\limsup_k \int_{B_\lambda^k} \omega(x)|\Delta v_k|^2\,dx < \left(\frac{1}{\widetilde{p}_1}\right)^{2/\gamma}.$$

By Adam's inequality (1.4), we deduce that

$$\sup_k \int_{B_\lambda^k} \exp\left(\widetilde{p}_1 \alpha_\beta v_k^\gamma\right) dx < +\infty$$

which is also a contradiction. The proof is finished for this case.

Case 2*:* $\|u\| = 1$. We can then proceed as in Case 1 and obtain

$$\sup_k \int_{B_\lambda^k} \exp\left(\widetilde{p}_1 \alpha_\beta v_k^\gamma\right) dx = +\infty$$

where $\widetilde{p}_1 = (1+\varepsilon)p_1$. Then we have

$$\limsup_k \int_{B_\lambda^k} \omega(x)|\Delta v_k|^2\,dx = \limsup_k \int_B \omega(x)|\Delta T_\lambda u_k|^2\,dx \geq \left(\frac{1}{\widetilde{p}_1}\right)^{2/\gamma},$$

thus

$$\|T^\lambda u\|^2 \leq \liminf_k \int_B \omega(x)|\Delta T^\lambda u_k|^2\,dx \leq 1 - \limsup_k \int_B \omega(x)|\Delta T_\lambda u_k|^2\,dx \leq 1 - \left(\frac{1}{\widetilde{p}_1}\right)^{2/\gamma}.$$

On the other hand, since $\|u\| = 1$, we can take λ large enough such that

$$\|T^\lambda u\|^2 > 1 - \frac{1}{3}\left(\frac{1}{\widetilde{p}_1}\right)^{2/\gamma}$$

which is a contradiction, and the proof is completed for this case. □

4. The mountain pass geometry of the energy

Since the nonlinearity $f(x,t)$ is critical or subcritical at $+\infty$, there exist positive constants $a, C > 0$ and there exists $t_2 > 1$ such that

$$|f(x,t)| \leq C e^{a t^\gamma}, \quad \forall\, |t| > t_2. \tag{4.1}$$

So the functional $\mathcal{J}$ defined by (1.8), is well defined and of class C^1.

In order to prove the existence of a ground state solution of the problem (1.1), we will prove the existence of a nonzero critical point of the functional $\mathcal{J}$ by using the theorem introduced by Ambrosetti and Rabinowitz in [5] (Mountain Pass Theorem) without the Palais–Smale condition.

Theorem 4.1. [5] *Let E be a Banach space and $J\colon E \to \mathbb{R}$ a $\mathcal{C}^1$ functional satisfying $J(0) = 0$. Suppose that there exist $\rho, \overline{\beta_0} > 0$ and $e \in E$ with $\|e\| > \rho$ such that*

$$\inf_{\|u\|=\rho} J(u) \geq \beta_0 \quad \textit{and} \quad J(e) \leq 0.$$

Then there is a sequence $(u_n) \subset E$ such that

$$J(u_n) \to \overline{c} \quad \textit{and} \quad J'(u_n) \to 0,$$

where

$$\overline{c} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \geq \overline{\beta_0} \;\; \textit{and} \;\; \Gamma := \{\gamma \in C([0,1], E) \mid \gamma(0) = 0 \;\textit{and}\; \gamma(1) = e\}.$$

The number $\overline{c}$ is called mountain pass level or minimax level of the functional J.

Before starting the proof of the geometric properties for the functional $\mathcal{J}$, it follows from the continuous embedding $\mathbf{X} \hookrightarrow L^q(B)$ for all $q \geq 1$, that there exists a constant $C > 0$ such that $\|u\|_{2q} \leq c\|u\|$ for all $u \in \mathbf{X}$.

In the next lemmas, we prove that the functional $\mathcal{J}$ has the mountain pass geometry of Theorem 4.1.

Lemma 4.2. *Suppose that f has critical growth at $+\infty$. In addition if* (H_1), (H_3) *and* (G_1) *hold, then there exist $\rho, \beta_0 > 0$ such that $\mathcal{J}(u) \geq \beta_0$ for all $u \in \mathbf{X}$ with $\|u\| = \rho$.*

Proof. It follows from (1.7) that there exists $\delta_0 > 0$,

$$F(x,t) \leq \epsilon|t|^2 \quad \text{for } |t| < \delta_0.$$

From (H_3), (4.1) and for all $q > 2$, there exist a positive constant $C > 0$ such that

$$F(x,t) \leq C|t|^q e^{at^\gamma}, \quad \forall\, |t| > \delta_1.$$

So, using the continuity of F, we get

$$F(x,t) \leq \epsilon|t|^2 + C|t|^q e^{at^\gamma} \quad \text{for all } t \in \mathbb{R}.$$

Since

$$\mathcal{J}(u) = \frac{1}{2}G(\|u\|^2) - \int_B F(x,u)\,dx,$$

we get from (G_1) and (V_1),

$$\mathcal{J}(u) \geq \frac{g_0}{2}\|u\|^2 - \varepsilon\int_B |u|^2\,dx - C\int_B |u|^q e^{au^\gamma}\,dx.$$

From the Hölder inequality, we obtain

$$\mathcal{J}(u) \geq \frac{g_0}{2}\|u\|^2 - \varepsilon \int_B |u|^2\,dx - C\left(\int_B e^{a|u|^\gamma}\,dx\right)^{1/2}\|u\|_{2q}^q.$$

From Theorem 1.1, if we choose $u \in \mathbf{X}$ such that

$$a\|u\|^\gamma \leq \alpha_{\beta_0}, \tag{4.2}$$

we get

$$\int_B e^{a|u|^\gamma}\,dx = \int_B e^{a\|u\|^\gamma\left(\frac{|u|}{\|u\|}\right)^\gamma}\,dx < +\infty.$$

On the other hand, $\|u\|_{2q} \leq C_1\|u\|$, so for fixed ϵ such that $\frac{g_0}{2C_1} > \epsilon$,

$$\mathcal{J}(u) \geq \frac{g_0}{2}\|u\|^2 - \varepsilon C_1\|u\|^2 - C\|u\|^q = \|u\|^2\left(\frac{g_0}{2} - \varepsilon C_1 - C\|u\|^{q-2}\right)$$

for all $u \in \mathbf{X}$ satisfying (4.2). Since $2 < q$, we can choose $\rho = \|u\| \leq \left(\frac{\alpha_\beta}{a}\right)^{1/\gamma}$ and for ϵ such that $\frac{g_0}{2C_1} > \epsilon$, there exists $\beta_0 = \rho^2\big((g_0/2-\varepsilon)C_1 - C\rho^{q-2}\big) > 0$ with $\mathcal{J}(u) \geq \beta_0 > 0$. □

By the following lemma, we prove the second geometric property for the functional $\mathcal{J}$.

Lemma 4.3. *Suppose that* (H_1), (H_2) *and* (G_2) *hold. Then there exists* $e \in \mathbf{X}$ *with* $\mathcal{J}(e) < 0$ *and* $\|e\| = \rho$.

Proof. From the condition (G_2), for all $t \geq t_1$, we have that

$$G(t) \leq \begin{cases} a_0 + a_1 t + \frac{a_2}{\delta+1}t^{\delta+1} & \text{if } \delta \neq -1,\\ b_0 + a_1 t + a_2 \ln t & \text{if } \delta = -1, \end{cases} \tag{4.3}$$

where $a_0 = \int_0^{t_1} g(t)\,dt - a_1 t_1 - a_2\frac{t_1^{\delta+1}}{\delta+1}$ and $b_0 = \int_0^{t_1} g(t)\,dt - a_1 t_1 - a_2 \ln t_1$. It follows from the condition (H_2) that

$$f(x,t) = \frac{\partial}{\partial t}F(x,t) \geq \frac{1}{M}F(x,t)$$

for all $t \geq t_0$. So

$$F(x,t) \geq Ce^{t/M}, \quad \forall\, t \geq t_0. \tag{4.4}$$

In particular, for $p > \max(2, 2(\delta+1))$ there exist C_1 and C_2 such that

$$F(x,t) \geq C_1|t|^p - C_2, \quad \forall\, t \in \mathbb{R},\ x \in B.$$

Next, one arbitrarily picks $\overline{u} \in \mathbf{X}$ such that $\|\overline{u}\| = 1$. Thus from (4.3) and (4.4), for all $t \geq t_1$,

$$\mathcal{J}(t\overline{u}) \leq \begin{cases} \frac{a_0}{2} + \frac{a_1}{2}t^2 + \frac{a_2}{2(\delta+1)}t^{2(\delta+1)} - C_1\|\overline{u}\|_p^p t^p - \frac{\pi^2}{2}C_2 & \text{if } \delta \neq -1,\\ \frac{b_0}{2} + \frac{a_1}{2}t^2 + \frac{a_2}{2}\ln^2 t - C_1\|\overline{u}\|_p^p t^p - \frac{\pi^2}{2}C_2 & \text{if } \delta = -1. \end{cases}$$

Therefore

$$\lim_{t\to+\infty}\mathcal{J}(t\overline{u})=-\infty.$$

We take $e=\bar{t}\overline{u}$ for some $\bar{t}>0$ large enough. So Lemma 4.3 follows. □

5. The minimax estimate of the energy

According to Lemmas 4.2 and 4.3, let

$$d_*:=\inf_{\gamma\in\Lambda}\max_{t\in[0,1]}\mathcal{J}(\gamma(t))>0$$

where

$$\Lambda:=\{\gamma\in C([0,1],\mathbf{X})\mid\gamma(0)=0\text{ and }\mathcal{J}(\gamma(1))<0\}.$$

We are going to estimate the minimax value of the functional $\mathcal{J}$. The idea is to construct a sequence of functions $(v_n)\in\mathbf{X}$, and estimate $\max\{\mathcal{J}(tv_n):t\geq 0\}$. For this goal, let consider the following Adam's function defined for all $n\geq 3$ by

$$w_n(x)=\begin{cases}\Big(\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\Big)^{1/\gamma}-\frac{|x|^{2(1-\beta)}}{2\big(\frac{\alpha_\beta}{4n}\big)^{1/\gamma}\big(\log(e\sqrt[4]{n})\big)^{(\gamma-1)/\gamma}}\\ \quad+\frac{1}{2\big(\frac{\alpha_\beta}{4}\big)^{1/\gamma}\big(\log(e\sqrt[4]{n})\big)^{(\gamma-1)/\gamma}} & \text{if } 0\leq|x|\leq 1/\sqrt[4]{n},\\ \frac{\big(\log\big(\frac{e}{|x|}\big)\big)^{1-\beta}}{\big(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\big)^{1/\gamma}} & \text{if } 1/\sqrt[4]{n}\leq|x|\leq 1/2,\\ \zeta_n & \text{if } 1/2\leq|x|\leq 1,\end{cases}$$

where $\zeta_n\in C_0^\infty(B)$ is such that

$$\zeta_n\Big|_{x=1/2}=\frac{1}{\big(\frac{\alpha_\beta}{16}\log(e^4n)\big)^{1/\gamma}}(\log 2e)^{1-\beta},\quad \frac{\partial\zeta_n}{\partial x}\Big|_{x=1/2}=\frac{-2(1-\beta)}{\big(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\big)^{1/\gamma}}(\log(2e))^{-\beta},$$

$$\zeta n\Big|_{\partial B}=\frac{\partial\zeta_n}{\partial x}\Big|_{\partial B}=0\quad\text{and}\quad \xi_n,\nabla\xi_n,\Delta\xi_n\ \text{ are all }\ o\bigg(\frac{1}{\log(e\sqrt[4]{n})}\bigg).$$

Let $v_n(x)=\frac{w_n}{\|w_n\|}$. We have, $v_n\in\mathbf{X}$, $\|v_n\|^2=1$. We compute $\Delta w_n(x)$, we get

$$\Delta w_n(x)=\begin{cases}\frac{-(1-\beta)(4-2\beta)|x|^{-2\beta}}{\big(\frac{\alpha_\beta}{4n}\big)^{1/\gamma}\big(\log(e\sqrt[4]{n})\big)^{(\gamma-1)/\gamma}} & \text{if } 0\leq|x|\leq 1/\sqrt[4]{n},\\ \frac{-(1-\beta)\big(\log\big(\frac{e}{|x|}\big)\big)^{-\beta}\big(2+\beta\big(\log\frac{e}{|x|}\big)^{-1}\big)}{\big(\frac{\alpha_\beta}{4}\log(e\sqrt[4]{n})\big)^{1/\gamma}} & \text{if } 1/\sqrt[4]{n}\leq|x|\leq 1/2,\\ \Delta\zeta_n & \text{if } 1/2\leq|x|\leq 1.\end{cases}$$

So

$$\|\Delta w_n\|_{2,w}^2=\underbrace{2\pi^2\int_0^{1/\sqrt[4]{n}}r^3|\Delta w_n(x)|^2\big(\log\frac{e}{r}\big)^\beta\,dr}_{I_1}+\underbrace{2\pi^2\int_{1/\sqrt[4]{n}}^{1/2}r^3|\Delta w_n(x)|^2\big(\log\frac{e}{r}\big)^\beta\,dr}_{I_2}$$

$$+ \underbrace{2\pi^2 \int_{1/2}^{1} r^3 |\Delta w_n(x)|^2 \big(\log \frac{e}{r}\big)^{\beta}\, dr}_{I_3},$$

we have

$$\begin{aligned}
I_1 &= 2\pi^2 \frac{(1-\beta)^2(4-2\beta)^2}{\big(\frac{\alpha_\beta}{4n}\big)^{2/\gamma}\big(\log(e\sqrt[4]{n})\big)^{2(\gamma-1)/\gamma}} \int_0^{1/\sqrt[4]{n}} r^{3-4\beta}\left(\log\frac{e}{r}\right)^{\beta} dr \\
&= 2\pi^2 \frac{(1-\beta)^2(4-2\beta)^2}{\big(\frac{\alpha_\beta}{4n}\big)^{2/\gamma}\big(\log(e\sqrt[4]{n})\big)^{2(\gamma-1)/\gamma}} \left[\frac{r^{4-4\beta}}{4-4\beta}\left(\log\frac{e}{r}\right)^{\beta}\right]_0^{1/\sqrt[4]{n}} \\
&\quad + 2\pi^2 \frac{\beta(1-\beta)^2(4-2\beta)^2}{\big(\frac{\alpha_\beta}{4n}\big)^{2/\gamma}\big(\log(e\sqrt[4]{n})\big)^{2(\gamma-1)/\gamma}} \int_0^{1/\sqrt[4]{n}} \frac{r^{4-4\beta}}{4-4\beta}\left(\log\frac{e}{r}\right)^{\beta-1} dr \\
&= o\left(\frac{1}{\log e\sqrt[4]{n}}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 &= 2\pi^2 \frac{(1-\beta)^2}{\big(\frac{\alpha_\beta}{4}\big)^{2/\gamma}\big(\log(e\sqrt[4]{n})\big)^{2/\gamma}} \int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{1/2}} \frac{1}{r}\left(\log\frac{e}{r}\right)^{-\beta}\left(2+\beta\left(\log\frac{e}{r}\right)^{-1}\right)^2 dr \\
&= -2\pi^2 \frac{(1-\beta)^2}{\big(\frac{\alpha_\beta}{4}\big)^{2/\gamma}\big(\log(e\sqrt[4]{n})\big)^{2/\gamma}} \\
&\quad \times \left[\frac{\beta^2}{-1-\beta}\left(\log\frac{e}{r}\right)^{-\beta-1} + 4\left(\log\frac{e}{r}\right)^{-\beta} + \frac{4}{1-\beta}\left(\log\frac{e}{r}\right)^{1-\beta}\right]_{1/\sqrt[4]{n}}^{1/2} \\
&= 1 + o\left(\frac{1}{\big(\log e\sqrt[4]{n}\big)^{2/\gamma}}\right)
\end{aligned}$$

and $I_3 = o\big(\frac{1}{(\log e\sqrt[4]{n})^{2/\gamma}}\big)$. Then

$$\|\Delta w_n\|_{2,w}^2 = 1 + o\left(\frac{1}{\big(\log e\sqrt[4]{n}\big)^{2/\gamma}}\right).$$

Also, for $0 \le |x| \le 1/\sqrt[4]{n}$,

$$v_n^{\gamma}(x) \ge \left(\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\right) + o(1).$$

5.1. Estimate of the energy $\mathcal{J}$

We are now going to prove the desired estimate.

Lemma 5.1. *Assume that* (G_1), (G_2) *and* (H_4) *hold, then*

$$d_* < \frac{1}{2} G\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right).$$

Proof. We have $v_n \geq 0$ and $\|v_n\| = 1$. Then from Lemma 4.3, $\mathcal{J}(tv_n) \to -\infty$ as $t \to +\infty$. As a consequence,

$$d \leq \max_{t\geq 0} \mathcal{J}(tv_n).$$

We argue by contradiction and suppose that for all $n \geq 1$,

$$\max_{t\geq 0} \mathcal{J}(tv_n) \geq \frac{1}{2} G\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right).$$

Since $\mathcal{J}$ possesses the mountain pass geometry, for any $n \geq 1$, there exists $t_n > 0$ such that

$$\max_{t\geq 0} \mathcal{J}(tv_n) = \mathcal{J}(t_n v_n) \geq \frac{1}{2} G\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right).$$

Using the fact that $F(x,t) \geq 0$ for all $(x,t) \in B \times \mathbb{R}$, we get

$$G(t_n^2) \geq G\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right).$$

On one hand, the condition (G_1) implies that $G\colon [0,+\infty) \to [0,+\infty)$ is an increasing bijection. So

$$t_n^2 \geq \left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}.$$

On the other hand,

$$\frac{d}{dt} J(tv_n)\Big|_{t=t_n} = g(t_n^2)t_n - \int_B f(x, t_n v_n) v_n \, dx = 0,$$

that is

$$g(t_n^2)t_n^2 = \int_B f(x, t_n v_n) t_n v_n \, dx. \tag{5.1}$$

Now, we claim that the sequence (t_n) is bounded in $(0,+\infty)$. Indeed, it follows from (H_4) that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x,t)t \geq (\gamma_0 - \varepsilon) e^{\alpha_0 t^\gamma}, \ \ \forall\, |t| \geq t_\varepsilon, \ \text{ uniformly in } x \in B, \tag{5.2}$$

$$t_n^2 = \int_B f(x, t_n v_n) t_n v_n \, dx \geq \int_{0\leq |x| \leq 1/\sqrt[4]{n}} f(x, t_n v_n) t_n v_n \, dx.$$

Since

$$\frac{t_n}{\|w_n\|} \left(\frac{\log e\sqrt[4]{n}}{\alpha_\beta}\right)^{1/\gamma} \to \infty \quad \text{as } n \to +\infty,$$

then it follows from (5.2) that for all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$t_n^2 \geq (\gamma_0 - \varepsilon) \int_{0\leq |x| \leq 1/\sqrt[4]{n}} e^{\alpha_0 t_n^\gamma v_n^\gamma} \, dx.$$

Using the condition (G_2), (5.1) and (5.2), for n large enough, we get

$$\begin{aligned}
a_1 t_n^2 + a_2 t_n^{2+2w} &\geq g(t_n^2)t_n^2 \geq (\gamma_0 - \varepsilon) \int_{0\leq|x|\leq 1/\sqrt[4]{n}} e^{\alpha_0 t_n^\gamma v_n^\gamma}\, dx \\
&\geq 2\pi^2(\gamma_0-\varepsilon)\int_0^{1/\sqrt[4]{n}} r^3 e^{\alpha_0 t_n^\gamma \left(\left(\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\right)+o(1)\right)}\, dr \\
&= 2\pi^2(\gamma_0-\varepsilon) e^{\alpha_0 t_n^\gamma \left(\left(\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\right)+o(1)\right)}.
\end{aligned}$$

There holds

$$1 \geq 2\pi^2(\gamma_0-\varepsilon) e^{\alpha_0 t_n^\gamma \left(\left(\frac{\log(e\sqrt[4]{n})}{\alpha_\beta}\right)+o(1)\right) - \log 4n \log(a_1 t_n^2) - \log(a_2 t_n^{2+2w})}.$$

As a direct result, (t_n) is a bounded sequence. We must note that, if

$$\lim_{n\to+\infty} t_n^\gamma > \frac{\alpha_\beta}{\alpha_0}, \tag{5.3}$$

then we get a contradiction with the boundedness of (t_n). Indeed if (5.3) is accurate, then there exists some $\delta > 0$ such that for n large enough,

$$t_n^\gamma \geq \delta + \frac{\alpha_\beta}{\alpha_0}.$$

Then the right-hand side of (5.3) tends to infinity which contradicts the boundedness of (t_n). Consequently (5.3) can not hold, and we get

$$\lim_{n\to+\infty} t_n^2 = \left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}. \tag{5.4}$$

We claim that (5.4) leads to a contradiction with (H_5). Indeed, let us introduce the sets

$$A_n = \{x \in B \mid t_n v_n \geq t_\varepsilon\} \quad \text{and} \quad C_n = B \setminus A_n,$$

where t_ε is given in (5.2). We have

$$\begin{aligned}
g(t_n^2)t_n^2 &= \int_B f(x,t_nv_n)t_nv_n\, dx = \int_{\mathcal{A}_n} f(x,t_nv_n)t_nv_n\, dx + \int_{\mathcal{C}_n} f(x,t_nv_n)t_nv_n \\
&\geq (\gamma_0-\varepsilon)\int_{\mathcal{A}_n} e^{\alpha_0 t_n^\gamma v_n^\gamma}\, dx + \int_{\mathcal{C}_n} f(x,t_nv_n)t_nv_n\, dx \\
&= (\gamma_0-\varepsilon)\int_B e^{\alpha_0 t_n^\gamma v_n^\gamma}\, dx - (\gamma_0-\varepsilon)\int_{\mathcal{C}_n} e^{\alpha_0 t_n^\gamma v_n^\gamma}\, dx + \int_{\mathcal{C}_n} f(x,t_nv_n)t_nv_n\, dx.
\end{aligned}$$

Since $v_n \to 0$ a.e. in B, $\chi_{\mathcal{C}_n} \to 1$ a.e. in B, therefore using the dominated convergence theorem, we get

$$\int_{\mathcal{C}_n} f(x,t_nv_n)t_nv_n\, dx \to 0 \quad \text{and} \quad \int_{\mathcal{C}_n} e^{\alpha_0 t_n^\gamma v_n^\gamma}\, dx \to \frac{\pi^2}{2}.$$

On the other hand,

$$\int_B e^{\alpha_0 t_n^\gamma v_n^\gamma}\,dx \geq \int_{1/\sqrt[4]{n}\leq|x|\leq 1/2} e^{\alpha_0 t_n^\gamma v_n^\gamma}\,dx + \int_{\mathcal{C}_n} e^{\alpha_0 t_n^\gamma v_n^\gamma}\,dx.$$

Then

$$\lim_{n\to+\infty} g(t_n^2)t_n^2 = g\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right)\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma} \geq (\gamma_0-\varepsilon)\lim_{n\to+\infty}\int_{1/\sqrt[4]{n}\leq|x|\leq 1/2} e^{\alpha_0 t_n^\gamma v_n^\gamma}\,dx.$$

Using the fact that

$$t_n^2 \geq \left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma},$$

we get

$$\begin{aligned}\lim_{n\to+\infty} g(t_n^2)t_n^2 &\geq \lim_{n\to+\infty}(\gamma_0-\varepsilon)\int_B e^{\alpha_0 t_n^\gamma v_n^\gamma}\,dx\\ &\geq \lim_{n\to+\infty}(\gamma_0-\varepsilon)2\pi^2\int_{1/\sqrt[4]{n}}^{1/2} r^3 e^{\frac{4\left(\log\frac{e}{r}\right)^2}{\log(e\sqrt[4]{n})\|w_n\|^\gamma}}\,dr.\end{aligned}$$

Making the change of variable

$$s = \frac{4\log\frac{e}{r}}{\log(e\sqrt[4]{n})\|w_n\|^\gamma},$$

we get

$$\begin{aligned}\lim_{n\to+\infty} g(t_n^2)t_n^2 &\geq \lim_{n\to+\infty}(\gamma_0-\varepsilon)\int_B e^{\alpha_0 t_n^\gamma v_n^\gamma}\,dx\\ &\geq \lim_{n\to+\infty} 2\pi^2(\gamma_0-\varepsilon)\frac{\|w_n\|^\gamma\log(e\sqrt[4]{n})}{4}e^4\int_{\frac{4\log 2e}{\|w_n\|^\gamma\log(e\sqrt[4]{n})}}^{\frac{4}{\|w_n\|^\gamma}} e^{\frac{\|w_n\|^\gamma\log(e\sqrt[4]{n})}{4}(s^2-4s)}\,ds\\ &\geq \lim_{n\to+\infty} 2\pi^2(\gamma_0-\varepsilon)\frac{\|w_n\|^\gamma\log(e\sqrt[4]{n})}{4}e^4\int_{\frac{4\log 2e}{\|w_n\|^\gamma\log(e\sqrt[4]{n})}}^{\frac{4}{\|w_n\|^\gamma}} e^{-\frac{\|w_n\|^\gamma\log(e\sqrt[4]{n})}{4}4s}\,ds\\ &= \lim_{n\to+\infty}(\gamma_0-\varepsilon)\frac{\pi^2}{2}e^4\big(-e^{-4\log e\sqrt[4]{n}} + e^{-4\log(2e)}\big)\\ &= (\gamma_0-\varepsilon)\frac{\pi^2 e^{4(1-\log 2e)}}{2} = (\gamma_0-\varepsilon)\frac{\pi^2}{32}.\end{aligned}$$

It follows that

$$g\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right)\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma} \geq (\gamma_0-\varepsilon)\frac{\pi^2}{32}$$

for all $\varepsilon > 0$. So

$$\gamma_0 \leq \frac{1024(1-\beta)g\big(\big(\frac{\alpha_\beta}{\alpha_0}\big)^{2/\gamma}\big)}{\alpha_0^{1-\beta}}.$$

This contradicts (H_4) and the lemma is proved. $\square$

6. Proof of main result

First we begin by some crucial lemmas. Now, we consider the Nehari manifold associated to the functional $\mathcal{J}$, namely

$$\mathcal{N} = \{u \in \mathbf{X} : \langle \mathcal{J}'(u), u\rangle = 0, u \neq 0\},$$

and the number $c = \inf_{u\in\mathcal{N}} \mathcal{J}(u)$. We have the following lemmas.

Lemma 6.1. [16] *Assume that the condition* (H_3) *holds, then for each* $x \in B$*,*

$$tf(x,t) - 4F(x,t) \quad \textit{is increasing for } t > 0.$$

In particular, $tf(x,t) - 4F(x,t) \geq 0$ *for all* $(x,t) \in B \times [0,+\infty)$.

Proof. Assume that $0 < t < s$. For each $x \in B$, we have

$$\begin{aligned} tf(x,t) - 4F(x,t) &= \frac{f(x,t)}{t^3}t^4 - 4F(x,s) + 4\int_t^s f(x,\nu)\,d\nu \\ &< \frac{f(x,t)}{s^3}t^4 - 4F(x,s) + \frac{f(x,s)}{s^3}(s^4 - t^4) \\ &= sf(x,s) - 4F(x,s). \end{aligned}$$

□

Lemma 6.2. *If* (G_3) *and* (H_3) *are satisfied then* $d_* \leq c$.

Proof. Let $\overline{u} \in \mathcal{N}$ and consider the function $\psi\colon (0,+\infty) \to \mathbb{R}$ defined by $\psi(t) = \mathcal{J}(t\overline{u})$. ψ is differentiable and we have

$$\psi'(t) = \langle \mathcal{J}'(t\overline{u}), \overline{u}\rangle = g(t^2\|\overline{u}\|^2)t\|\overline{u}\|^2 - \int_B f(x,t\overline{u})\overline{u}\,dx \quad \text{for all } t > 0.$$

Since $\overline{u} \in \mathcal{N}$, we have $\langle \mathcal{J}'(\overline{u}), \overline{u}\rangle = 0$ and therefore $g(\|\overline{u}\|^2)\|\overline{u}\|^2 = \int_B f(x,\overline{u})\overline{u}\,dx$. Hence,

$$\psi'(t) = t^3\|\overline{u}\|^4 \left(\frac{g(t^2\|\overline{u}\|^2)}{t^2\|\overline{u}\|^2} - \frac{g(\|\overline{u}\|^2)}{\|\overline{u}\|^2}\right) + t^3\int_B \left(\frac{f(x,\overline{u})}{\overline{u}^3} - \frac{f(x,t\overline{u})}{(t\overline{u})^3}\right) dx.$$

We have that $\psi'(1) = 0$. We also have by the conditions (G_3) and (H_3) that $\psi'(t) > 0$ for all $0 < t < 1$, $\psi'(t) \leq 0$ for all $t > 1$. It follows that

$$\mathcal{J}(\overline{u}) = \max_{t\geq 0} \mathcal{J}(t\overline{u}).$$

We define the function $\lambda\colon [0,1] \to \mathbf{X}$ such that $\lambda(t) = t\bar{t}\overline{u}$ with $\mathcal{J}(\bar{t}\overline{u}) < 0$. We have $\lambda \in \Lambda$, and hence

$$d_* \leq \max_{t\in[0,1]} \mathcal{J}(\lambda(t)) \leq \max_{t\geq 0} \mathcal{J}(t\overline{u}) = \mathcal{J}(\overline{u}).$$

Since $\overline{u} \in \mathcal{N}$ is arbitrary then $d_* \leq c$. □

Proof of Theorem 1.4. Since $\mathcal{J}$ possesses the mountain pass geometry, there exists $u_n \in \mathbf{X}$ such that

$$\mathcal{J}(u_n) = \frac{1}{2}G(\|u_n\|^2) - \int_B F(x,u_n)\,dx \to d_*, \quad n \to +\infty \tag{6.1}$$

and

$$|\langle \mathcal{J}'(u_n),\varphi\rangle| = \left| g(\|u_n\|^2)\left[\int_B w(x)\Delta u_n \cdot \nabla\varphi\,dx\right] - \int_B f(x,u_n)\varphi\,dx\right| \le \epsilon_n\|\varphi\| \tag{6.2}$$

for all $\varphi \in \mathbf{X}$, where $\epsilon_n \to 0$ as $n \to +\infty$. In order to obtain a ground state solution for problem (1.1), it is enough to show that there is $u \in \mathcal{N}$ such that $\mathcal{J}(u) = d_*$ ($d_* \le c \le r$). From (6.1) for all $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\frac{1}{N}G(\|u_n\|^2) \le C + \int_B F(x,u_n)\,dx.$$

From (1.6), for all $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$F(x,t) \le \epsilon t f(x,t) \quad \text{for all } |t| > t_\epsilon \text{ and uniformly in } x \in B.$$

It follows that

$$\frac{1}{2}G(\|u_n\|^2) \le C + \int_{|u_n|\le t_\epsilon} F(x,u_n)\,dx + \epsilon\int_B f(x,u_n)u_n\,dx.$$

From (6.2), we get

$$\frac{1}{4}g(\|u_n\|^2)\|u_n\|^2 \le \frac{1}{2}G(\|u_n\|^2) \le C_1 + \epsilon\epsilon_n\|u_n\| + \epsilon g(\|u_n\|^2)\|u_n\|^2$$

for some constant $C_1 > 0$.

Using (1.5) and the condition (G_1), for all ϵ with $0 < \epsilon < 1/4$, we get

$$g_0(1/4 - \epsilon)\|u_n\|^2 \le C_1 + \epsilon\epsilon_n\|u_n\|.$$

We deduce that the sequence (u_n) is bounded in $\mathbf{X}$. As consequence, there exists $u \in \mathbf{X}$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in $\mathbf{X}$, $u_n \to u$ strongly in $L^q(B)$, for all $q \ge 1$ and $u_n(x) \to u(x)$ a.e. in B. Furthermore, we have from (6.1) and (6.2), that

$$0 < \int_B f(x,u_n)u_n \le C \quad \text{and} \quad 0 < \int_B F(x,u_n) \le C.$$

By Lemma 3.2, we have

$$f(x,u_n) \to f(x,u) \quad \text{in } L^1(B) \text{ as } n \to +\infty.$$

It follows from (H_2) and the generalized Lebesgue dominated convergence theorem that

$$F(x,u_n) \to F(x,u) \quad \text{in } L^1(B) \text{ as } n \to +\infty.$$

So

$$\lim_{n\to+\infty} G(\|u_n\|^2) = 2\left(d_* + \int_B F(x,u)\,dx\right).$$

Next, we are going to make some claims.

Claim 1: $u \neq 0$. Indeed, we argue by contradiction and suppose that $u \equiv 0$. Therefore $\int_B F(x,u_n)\,dx \to 0$ and consequently we get

$$\frac{1}{2}G(\|u_n\|^2) \to d_* < \frac{1}{2}G\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right).$$

So there exist $n_0 \in \mathbb{N}$ and $\eta \in (0,1)$ such that $\alpha_0\|u_n\|^\gamma = (1-\eta)\alpha_\beta$ for all $n \geq n_0$. By (6.2), we have also

$$\left|g(\|u_n\|^2)\|u_n\|^2 - \int_B f(x,u_n)u_n\,dx\right| \leq C\epsilon_n.$$

First we claim that there exists $q > 1$ such that

$$\int_B |f(x,u_n)|^q\,dx \leq C. \tag{6.3}$$

So

$$g(\|u_n\|^2)\|u_n\|^2 \leq C\epsilon_n + \left(\int_B |f(x,u_n)|^q\,dx\right)^{1/q}\left(\int_B |u_n|^{q'}\right)^{1/q'}$$

where q' is the conjugate of q. Since (u_n) converge to $u=0$ in $L^{q'}(B)$,

$$\lim_{n\to+\infty} g(\|u_n\|^2)\|u_n\|^2 = 0.$$

From the condition (G_1), we obtain

$$\lim_{n\to+\infty} \|u_n\|^2 = 0.$$

Therefore $\mathcal{J}(u_n) \to 0$ which is in contradiction with $d > 0$.

For the proof of the claim (6.3), since f has critical growth, for every $\epsilon > 0$ and $q > 1$ there exists $t_\epsilon > 0$ and $C > 0$ such that for all $|t| \geq t_\epsilon$, we have

$$|f(x,t)|^q \leq Ce^{\alpha_0(\epsilon+1)t^\gamma}.$$

Consequently,

$$\begin{aligned}\int_B |f(x,u_n)|^q\,dx &= \int_{\{|u_n|\leq t_\epsilon\}} |f(x,u_n)|^q\,dx + \int_{\{|u_n|>t_\epsilon\}} |f(x,u_n)|^q\,dx\\ &\leq 2\pi^2 \max_{B\times[-t_\epsilon,t_\epsilon]} |f(x,t)|^q + C\int_B e^{\alpha_0(\epsilon+1)|u_n|^\gamma}\,dx.\end{aligned}$$

Since there exist $n_0 \in \mathbb{N}$ and $\eta \in (0,1)$ such that $\alpha_0 \|u_n\|^\gamma = (1-\eta)\alpha_\beta$ for all $n \geq n_0$, then

$$\alpha_0(1+\epsilon)\left(\frac{|u_n|}{\|u_n\|}\right)^\gamma \|u_n\|^\gamma \leq (1+\epsilon)(1-\eta)\alpha_\beta.$$

We choose $\epsilon > 0$ small enough to get

$$\alpha_0(1+\epsilon)\|u_n\|^\gamma \leq \alpha_\beta.$$

Therefore the second integral is uniformly bounded in view of (1.4).

Claim 2: $u > 0$. Indeed, since (u_n) is bounded, up to a subsequence, $\|u_n\| \to \rho > 0$. In addition, $\mathcal{J}'(u_n) \to 0$ leads to

$$g(\rho^2)\left[\int_B w(x)\Delta u \cdot \Delta\varphi\right] = \int_B f(x,u)\varphi\,dx, \quad \forall\,\varphi \in \mathbf{X}.$$

By taking $\varphi = u^-$ with $w^\pm = \max(\pm w, 0)$, we get $\|u^-\|^2 = 0$ and so $u = u^+ \geq 0$. Since the nonlinearity has critical growth at $+\infty$ and from Adam's inequality (1.4), $f(\,\cdot\,, u) \in L^p(B)$ for all $p \geq 1$. So, by elliptic regularity and Sobolev embedding, $u \in C(B)$.

Define $B_0 = \{x \in B : u(x) = 0\}$. The set $B_0 = \emptyset$. Indeed, suppose by contradiction that $B_0 \neq \emptyset$. Since $f(x,u) \geq 0$, by Harnack inequality we can deduce that B_0 is an open and closed set of B. In virtue of the connectedness of B, we reach a contradiction. Hence Claim 2 is proved.

Claim 3: $g(\|u\|^2)\|u\|^2 \geq \int_B f(x,u)u\,dx$. We proceed by contradiction and we suppose that $g(\|u\|^2)\|u\|^2 < \int_B f(x,u)u\,dx$. Hence, $\langle \mathcal{J}'(u), u\rangle < 0$. The function $\psi\colon t \to \psi(t) = \langle \mathcal{J}'(tu), u\rangle$ is positive for t small enough. Indeed, from (1.7) and the critical growth of the non linearity f, for every $\epsilon > 0$, for every $q > 2$, there exist positive constants C and c such that

$$|f(x,t)| \leq \epsilon t + Ct^q e^{ct^\gamma}, \quad \forall\,(t,x) \in \mathbb{R} \times B.$$

Then using the condition (G_1), the last inequality and the Hölder inequality, we obtain

$$\begin{aligned}\psi(t) &= g(t^2\|u\|^2)t\|u\|^2 - \int_B f(x,tu)u\,dx\\ &\geq g_0 t\|u\|^2 - \epsilon t\int_B u^2\,dx - C\left(\int_B e^{2ct^\gamma u^\gamma}\,dx\right)^{1/2}\left(\int_B u^{2q}\,dx\right)^{1/2}.\end{aligned}$$

In view of (1.4) the integral $\int_B e^{2ct^\gamma u^\gamma}\,dx \leq \int_B e^{2ct^\gamma \frac{u^\gamma}{\|u\|^\gamma}\|u\|^\gamma}\,dx \leq C$, provided $t \leq \frac{1}{\|u\|}\left(\frac{\alpha_\beta}{2c}\right)^{1/\gamma}$. Using the radial Lemma 3.1 we get $\|u\|_{2q}^2 \leq C'\|u\|^q$. Then

$$\psi(t) \geq g_0 t\|u\|^2 - C_1\epsilon t\|u\|^2 - C_2\|u\|^q = \|u\|^2 t\big[(g_0 - C_1\epsilon) - C_2 t^{q-1}\|u\|^{q-2}\big].$$

We choose $\epsilon > 0$ such that $g_0 - C_1\epsilon > 0$ and since $q > 2$, for small t, we get $\psi\colon t \to \psi(t) = \langle \mathcal{J}'(tu), u\rangle > 0$. So there exists $\eta \in (0,1)$ such that $\psi(\eta u) = 0$. Therefore $\eta u \in \mathcal{N}$. Using

(1.5), the result of Lemma 6.1 and the semicontinuity of norm and Fatou's Lemma we get

$$\begin{aligned}
d_* \leq c \leq \mathcal{J}(\eta u) &= \mathcal{J}(\eta u) - \frac{1}{4}\langle \mathcal{J}'(\eta u), \eta u\rangle \\
&= \frac{1}{2}G(\|\eta u\|^2) - \frac{1}{4}g(\|\eta u\|^2)\|\eta u\|^2 + \frac{1}{4}\int_B \big(f(x,\eta u)\eta u - 4F(x,\eta u)\big)\,dx \\
&< \frac{1}{2}G(\|u\|^2) - \frac{1}{4}g(\|u\|^2)\|u\|^2 + \frac{1}{4}\int_B (f(x,u)u - 4F(x,u)) \\
&\leq \liminf_{n\to+\infty}\left[\frac{1}{2}G(\|u_n\|^2) - \frac{1}{4}g(\|u_n\|^2)\|u_n\|^2\right] \\
&\quad + \liminf_{n\to+\infty}\left[\frac{1}{4}\int_B \big(f(x,u_n)u_n - 4F(x,u_n)\big)\,dx\right] \\
&\leq \lim_{n\to+\infty}\left[\mathcal{J}(u_n) - \frac{1}{4}\langle \mathcal{J}'(u_n), u_n\rangle\right] = d_*,
\end{aligned}$$

which is absurd and the claim is well established.

On the other hand, by Claim 3, (1.7) and Lemma 6.1 we obtain

$$\mathcal{J}(u) \geq \frac{1}{2}G(\|u\|^2) - \frac{1}{4}g(\|u\|^2)\|u\|^2 + \frac{1}{4}\int_B [f(x,u) - 4F(x,u)]\,dx \geq 0.$$

We claim that $\mathcal{J}(u) = d_*$ and therefore we get

$$\lim_{n\to+\infty} G(\|u_n\|^2) = 2\left(d_* + \int_B F(x,u)\,dx\right) = G(\|u\|^2).$$

So $\|u_n\| \to \|u\|$. Now, using the semicontinuity of the norm and (6.1), we get

$$\mathcal{J}(u) \leq \frac{1}{2}\liminf_{n\to\infty} G(\|u_n\|^2) - \int_B F(x,u)\,dx = d_*.$$

Suppose that

$$\mathcal{J}(u) < d_*. \tag{6.4}$$

Then $\|u\|^2 < \rho^2$. In addition,

$$\frac{1}{2}G(\rho^2) = \frac{1}{2}\lim_{n\to+\infty} G(\|u_n\|^2) = d_* + \int_B F(x,u)\,dx,$$

which means that

$$\rho^2 = G^{-1}\left(2\left(d_* + \int_B F(x,u)\,dx\right)\right).$$

Set

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\rho}.$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in $\mathbf{X}$, $v \not\equiv 0$ and $\|v\| < 1$. So, by Theorem 3.3, we get

$$\sup_n \int_B e^{p\alpha_\beta |v_n|^\gamma}\,dx < \infty$$

for $1 < p < (1 - \|v\|^2)^{-\gamma/2}$.

On the other hand, by Claim 1, (1.7) and Lemma 5.1, we obtain

$$\mathcal{J}(u) \geq \frac{1}{2}G(\|u\|^2) - \frac{1}{4}g(\|u\|^2)\|u\|^2 + \frac{1}{4}\int_B [f(x,u) - 4F(x,u)]\,dx \geq 0. \tag{6.5}$$

From (6.5), Lemma 6.1 and the following equality

$$2d_* - 2\mathcal{J}(u) = G(\rho^2) - G(\|u\|^2),$$

we get

$$G(\rho^2) \leq 2d_* + G(\|u\|^2) < G\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right) + G(\|u\|^2).$$

Now, using the condition (G_1) one has

$$\rho^2 < G^{-1}\left(G\left(\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}\right) + G(\|u\|^2)\right) \leq \left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma} + \|u\|^2. \tag{6.6}$$

Since

$$\rho^2 = \frac{\rho^2 - \|u\|^2}{1 - \|v\|^2},$$

we deduce from (6.6) that

$$\rho^2 < \frac{\left(\frac{\alpha_\beta}{\alpha_0}\right)^{2/\gamma}}{1 - \|v\|^2}.$$

Then there exists $\delta \in (0, 1/2)$ such that $\rho^\gamma = (1 - 2\delta)\frac{\alpha_\beta/\alpha_0}{(1-\|v\|^2)^{\gamma/2}}$.

On one hand, we have this estimate $\int_B |f(x,u_n)|^q\,dx < C$. Indeed, for $\epsilon > 0$,

$$\begin{aligned}\int_B |f(x,u_n)|^q\,dx &= \int_{\{|u_n| \leq t_\epsilon\}} |f(x,u_n)|^q\,dx + \int_{\{|u_n| > t_\epsilon\}} |f(x,u_n)|^q\,dx \\ &\leq \pi \max_{B \times [-t_\epsilon, t_\epsilon]} |f(x,t)|^q + C\int_B e^{\alpha_0(1+\epsilon)|u_n|^\gamma}\,dx \\ &\leq C_\epsilon + C\int_B e^{\alpha_0(1+\epsilon)\|u_n\|^\gamma |v_n|^\gamma}\,dx \leq C,\end{aligned}$$

provided $\alpha_0(1+\epsilon)\|u_n\|^\gamma \leq p\alpha_\beta$ for p with $1 < p < (1 - \|v\|^2)^{-\gamma/2}$. On the other hand, since

$$\lim_{n\to+\infty} \|u_n\|^\gamma = \rho^\gamma,$$

then for n large enough, we get

$$\alpha_0(1+\epsilon)\lim_{n\to+\infty} \|u_n\|^\gamma \leq \alpha_0(1+\epsilon)\rho^\gamma \leq (1+\epsilon)(1-\delta)\frac{\alpha_\beta}{(1 - \|v\|^2)^{\gamma/2}}.$$

We choose $\epsilon > 0$ small enough such that $(1+\epsilon)(1-\delta) < 1$, which means, for n large enough,

$$\alpha_0(1+\epsilon)\|u_n\|^\gamma < \frac{\alpha_\beta}{(1 - \|v\|^2)^{\gamma/2}}.$$

So the sequence $(f(x,u_n))$ is bounded in L^q, $q>1$. Using the Hölder inequality, we deduce that

$$\begin{aligned}\left|\int_B f(x,u_n)(u_n-u)\,dx\right| &\le \left(\int_B |f(x,u_n)|^q\,dx\right)^{1/q}\left(\int_B |u_n-u|^{q'}\,dx\right)^{1/q'}\\ &\le C\left(\int_B |u_n-u|^{q'}\,dx\right)^{1/q'} \to 0 \quad \text{as } n\to+\infty,\end{aligned}$$

where $1/q+1/q'=1$. Since $\langle \mathcal{J}'(u_n), u_n-u\rangle = o_n(1)$, it follows that

$$g(\|u_n\|^2)\left[\int_B w(x)\Delta u_n(\Delta u_n-\Delta u)\right]\to 0.$$

On the other hand,

$$g(\|u_n\|^2)\left[\int_B w(x)\Delta u_n(\Delta u_n-\Delta u)\right] = g(\|u_n\|^2)\|u_n\|^2 - g(\|u_n\|^2)\left[\int_B w(x)\Delta u_n\cdot\Delta u\right].$$

Passing to the limit in the last equality, we get

$$g(\rho^2)\rho^2 - g(\rho^2)\|u\|^2 = 0,$$

therefore $\|u\|=\rho$ and $\|u_n\|\to\|u\|$. This is in contradiction with (6.4). It follows that $G(\rho^2)=G(\|u\|)$ and consequently $\mathcal{J}(u)=d_*$. Also,

$$g(\|u\|^2)\left[\int_B w(x)\Delta u\Delta\varphi\,dx\right] = \int_B f(x,u)\varphi\,dx,\quad \forall\,\varphi\in\mathbf{X}.$$

So u is a positive ground state solution of the problem (1.1). □

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