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Ventcel-type Transmission Conditions for the Scattering of a Time-harmonic Wave Problem with Accuracy up to Order 3

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Abstract. This work deals with the asymptotic behaviour of the electric field in the transverse magnetic (TM) mode, propagating in a bidimensional heterogeneous medium, composed by a homogeneous linear dielectric isotropic material surrounded by a thin layer of thickness ε (destined to tend to 0) and embedded in an ambient medium. Using the tools of multiscale analysis, an asymptotic expansion of the solution u^ε to the Helmholtz problem with respect to the thickness ε is derived. As a consequence, Ventcel-type transmission conditions on the limit interface Γ are obtained modelling the effect of the thin layer with accuracy up to $O(\varepsilon^3)$. A particular choice of the interface Γ leads to a well-posed Ventcel's problem.

1. Introduction

In the era of the 21st century and digital culture, new computers, like the supercomputer Fugaku, that can make up to 415.5 petaflops ($10^{15} \times 415.5$ floating-point operations per second) have seen the day. Consequently, precision and accuracy are in high demand which pushes to take into consideration of small parameters that were neglected in the past (for example, the thickness of cell membrane, the paint coating a plane, or the anechoic covers of submarines). Unfortunately, they generate more and more complex models (see [18,23]).

Such problems, posed in domains with thin layers, can be solved by boundary or finite element methods (see [10,19]). However, these methods may cause numerical instabilities and a significant computing time when the thickness of the thin layer goes to zero. To bypass these difficulties, we use asymptotic methods to replace the thin layer with an interface and the effect of the thin layer with nonstandard transmission conditions called impedance transmission conditions or Ventcel-type transmission conditions in the Russian literature when they involve tangential differential operators of order greater or equal to that of the interior differential operator. In the last decade, similar problems have been extensively studied in numerous papers (see [1, 9, 14–17, 21, 22, 24]).

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This paper is a continuation of [8]. The latter deals with the asymptotic behaviour of the solution to the Helmholtz equation, in a domain of $\mathbb{R}^3$, with a thin layer of thickness δ. The authors used an asymptotic expansion of the solution to model the effect of the thin layer by a problem with Ventcel-type transmission conditions, with accuracy up to $O(\delta^2)$.

In this work, we derive and justify, with the similar aforementioned techniques, an approximation of the solution to the Helmholtz equation in a bidimensional domain, with a thin layer of thickness ε (destined to tend to 0), with accuracy up to $O(\varepsilon^3)$. We use a framework (see [5, 8]) which consists in considering an interface Γ, dividing the thin layer into two thin layers of constant thickness, and an appropriate choice of its position leads to an approximate well-posed Ventcel's problem; unlike [7] where the Γ interface is outside the thin layer. Furthermore, due to the lack of coercivity of Ventcel's problem, we investigate several equivalent transmission conditions to be able to use Rellich's lemma to ensure the uniqueness of the approximate solution and, therefore, its existence via an alternative of Fredholm.

The present paper is organized as follows. In Section 2, we state the model of the considered scattering problem. We state also the existence, the uniqueness, and a stability estimate of the exact solution of the boundary-value problem relative to the thickness of the thin layer. In Section 3, we recall some definitions and notations from differential geometry of curves. Section 4 is devoted to the asymptotic expansion of the solution with respect to the thin layer up to any order. We calculate the first three terms of a formal asymptotic expansion, in addition to an error estimate justifying the ansatz. Further details are given in Appendix A.

The main contribution of this paper is presented in Sections 5 and 6. We derive an approximate model, with Ventcel-type transmission conditions modelling the effect of the thin layer, allowing to approximate the solution far from the thin layer and to deduce an approximation in the vicinity of the thin shell. The well-posedness of Ventcel's problem will also be proved, while, in Section 6, an error estimate will be established.

2. Statement of the model problem

Let Ω_∞ be an open heterogeneous exterior domain of $\mathbb{R}^2$ in which the wave propagates, such that its complement is a $\mathcal{C}^\infty$ compact manifold of $\mathbb{R}^2$ with boundary Γ_i representing the obstacle.

The domain Ω_∞ is made of three sub-domains: an open bounded subset Ω_-^ε with regular boundary consisting of two disjoint parts Γ_i and Γ_-^ε; a thin layer Ω_m^ε coating Ω_-^ε on the side Γ_-^ε, of thickness $\varepsilon > 0$, sufficiently small; and an exterior domain Ω_+^ε of $\mathbb{R}^2$ with boundary Γ_+^ε that represents the medium free of material within and defined by

$\Omega_+^\varepsilon = \Omega_\infty \setminus \left(\Omega_-^\varepsilon \cup \overline{\Omega_m^\varepsilon}\right)$ (see Figure 2.1).

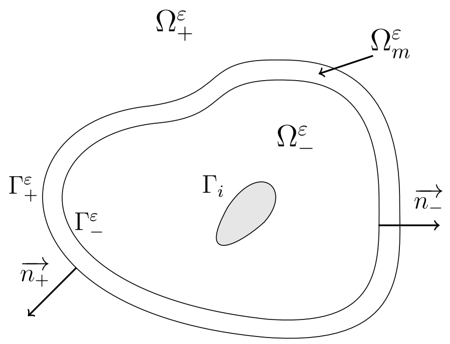

Figure 2.1: Geometry of the problem.

Define the two piecewise strictly positive constant functions α_ε and k_ε by

$$\alpha_\varepsilon(x) = \begin{cases} \alpha_+ & \text{if } x \in \Omega_+^\varepsilon, \\ 1 & \text{if } x \in \Omega_m^\varepsilon, \\ \alpha_- & \text{if } x \in \Omega_-^\varepsilon \end{cases} \quad \text{and} \quad k_\varepsilon(x) = \begin{cases} k_+ & \text{if } x \in \Omega_+^\varepsilon, \\ k_m & \text{if } x \in \Omega_m^\varepsilon, \\ k_- & \text{if } x \in \Omega_-^\varepsilon. \end{cases} \tag{2.1}$$

In (2.1), α_ε and k_ε^2 describe respectively the contrast and the refractive properties of the mediums Ω_-^ε and Ω_m^ε relative to the exterior propagation domain Ω_+^ε. We also assume that all the constants $\alpha_\pm$, $k_\pm^2$ and k_m^2 are independent of ε and $\alpha_\pm \in]1, +\infty[$ or $\alpha_\pm \in]0, 1[$.

Under the aforementioned assumptions, we look at the asymptotic behaviour as $\varepsilon \to 0$ of the solution u^ε to the Helmholtz problem

$$\begin{cases} \Delta u^\varepsilon + k_\varepsilon^2 u^\varepsilon = 0 & \text{in } \Omega_-^\varepsilon \cup \Omega_m^\varepsilon \cup \Omega_+^\varepsilon, & \text{(2.2a)} \\ u^\varepsilon = 0 & \text{on } \Gamma_i, & \text{(2.2b)} \\ \lim\limits_{|x| \to +\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)(u^\varepsilon - u_{\text{inc}}) = 0 & & \text{(2.2c)} \end{cases}$$

with transmission conditions for the Dirichlet and Neumann traces on the interfaces Γ_+^ε and Γ_-^ε

$$\begin{cases} u_\pm^\varepsilon = u_m^\varepsilon & \text{on } \Gamma_\pm^\varepsilon, & \text{(2.2d)} \\ \alpha_\pm \partial_{\boldsymbol{n}_\pm} u_\pm^\varepsilon = \partial_{\boldsymbol{n}_\pm} u_m^\varepsilon & \text{on } \Gamma_\pm^\varepsilon, & \text{(2.2e)} \end{cases}$$

where $\partial_{\boldsymbol{n}_+}$ and $\partial_{\boldsymbol{n}_-}$ denote the derivatives in the direction of the unit normal vectors $\boldsymbol{n}_+$ and $\boldsymbol{n}_-$ to Γ_+^ε and Γ_-^ε respectively (see Figure 2.1); u_+^ε, u_m^ε, and u_-^ε are the restrictions of u^ε respectively to the domains Ω_+^ε, Ω_m^ε, Ω_-^ε and u_{inc} is the incident wave defined by $u_{\text{inc}} = e^{ik_+(x \cdot d)}$, with d being a unit vector of $\mathbb{R}^2$ giving the direction of the plane wave u_{inc}.

We shall adopt similar arguments to those used in [5,8]. It consists in considering an artificial interface Γ parallel to $\Gamma_\pm^\varepsilon$, dividing Ω_m^ε into two thin layers $\Omega_{m,1}^\varepsilon$ and $\Omega_{m,2}^\varepsilon$ of thickness $d_1\varepsilon$ and $d_2\varepsilon$ respectively, where d_1 and d_2 are positive real numbers satisfying $d_1 + d_2 = 1$ (see Figure 2.2).

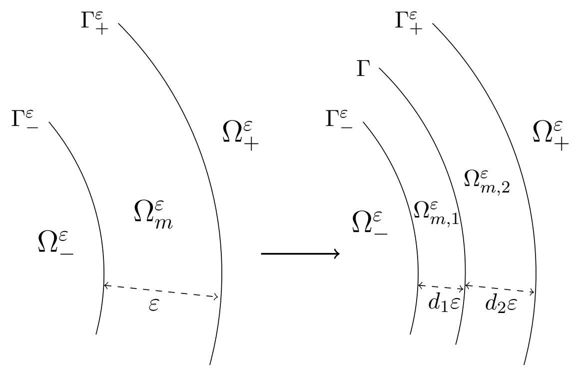

Figure 2.2: A zoom on the thin layer, with and without the Γ interface.

We will first determine an asymptotic expansion of the solution u^ε with respect to the thickness ε up to any order using d_1 and d_2 as parameters. We then provide Ventcel-type transmission conditions on the interface Γ modelling the effect of the thin layer with accuracy up to $O(\varepsilon^3)$. The determination of the constants d_1 and d_2 in the second step and therefore the position of Γ, which is not, necessary, the thin layer's midline, will ensure the existence and the uniqueness of the approximate solution.

The following theorem answers the question of existence, uniqueness, and gives a uniform estimate of the solution u^ε with respect to ε, for which a proof can be found in [6,8].

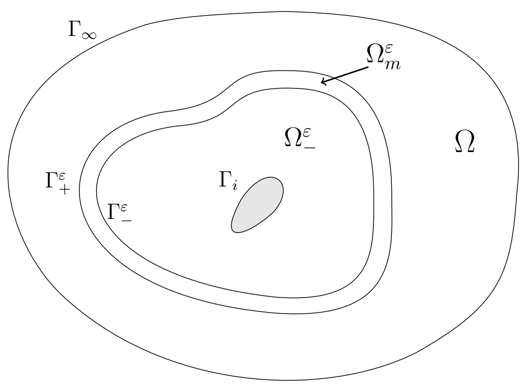

Figure 2.3: The Ω set.

Theorem 2.1. *Problem* (2.2) *admits a unique solution* u^ε *in* $H^1_{\mathrm{loc}}(\overline{\Omega}_\infty)$. *Furthermore,* *there exists a constant c independent of* ε *such that*

$$\|u^\varepsilon\|_{H^1(\Omega)} \leq c,$$

where Ω *is a bounded smooth domain with boundary* Γ_i *and a smooth curve denoted by* Γ_∞ *enclosing the obstacle as well as the thin layer* Ω_m^ε *(see Figure 2.3).*

3. Differential geometry tools

The goal of this section is to define and to collect the main features of differential geometry to formulate our problem in a fixed domain (independent of ε). This technique is a key tool to determine the asymptotic expansion of the solution u^ε.

3.1. Parameterization of Γ

Let Γ be a regular parameterized closed curve through the $\mathcal{C}^\infty$ map γ defined by

$$\begin{array}{rcl} \gamma : (0, l_\Gamma) & \rightarrow & \Gamma \subset \mathbb{R}^2 \\ t & \mapsto & \gamma(t) = (\gamma_1(t), \gamma_2(t)), \end{array}$$

where l_Γ is the length of Γ and t is the arc length of γ. The tangent and normal unit vectors $\boldsymbol{\tau}(t)$ and $\boldsymbol{n}(t)$ to Γ at $\gamma(t)$ are given by

$$\boldsymbol{\tau}(t) := \frac{d\gamma(t)}{dt} = (n_2(t), -n_1(t)), \quad \boldsymbol{n}(t) := (n_1(t), n_2(t)).$$

We recall Frénet's formulas defining the curvature $c(t)$ of Γ at point $\gamma(t)$ (see [13])

$$\frac{d\boldsymbol{\tau}(t)}{dt} = -c(t)\boldsymbol{n}, \quad \frac{d\boldsymbol{n}(t)}{dt} = c(t)\boldsymbol{\tau}.$$

3.2. Parameterization of Ω_m^ε

Let $I_\varepsilon = (-d_1\varepsilon, d_2\varepsilon)$. We parameterize the thin shell Ω_m^ε by the manifold $(0, l_\Gamma) \times I_\varepsilon$ through the mapping ψ defined by

$$\begin{array}{rcl} (0, l_\Gamma) \times I_\varepsilon & \stackrel{\psi}{\rightarrow} & \Omega_m^\varepsilon \\ (t, \eta) & \mapsto & x := \gamma(t) + \eta \boldsymbol{n}(t). \end{array}$$

As well-known [13], if the thickness of Ω_m^ε is small enough, ψ is a $\mathcal{C}^\infty$-diffeomorphism of manifolds. To each function v defined on Ω_m^ε, we associate the function $\widetilde{v}$ defined on $(0, l_\Gamma) \times I_\varepsilon$ by

$$\widetilde{v}(t, \eta) := v(x), \quad x = \psi(t, \eta),$$

then we have

$$\nabla_{t,\eta}\widetilde{v}(t,\eta) = \begin{pmatrix} \partial_t \widetilde{v}(t,\eta) \\ \partial_\eta \widetilde{v}(t,\eta) \end{pmatrix} = \begin{pmatrix} (1+\eta c(t))n_2(t) & -(1+\eta c(t))n_1(t) \\ n_1(t) & n_2(t) \end{pmatrix} \begin{pmatrix} \partial_{x_1} v(x) \\ \partial_{x_2} v(x) \end{pmatrix},$$

so

$$\nabla v(x) = \begin{pmatrix} \partial_{x_1} v(x) \\ \partial_{x_2} v(x) \end{pmatrix} = \begin{pmatrix} \frac{n_2(t)}{1+\eta c(t)} & n_1(t) \\ \frac{-n_1(t)}{1+\eta c(t)} & n_2(t) \end{pmatrix} \begin{pmatrix} \partial_t \widetilde{v}(t,\eta) \\ \partial_\eta \widetilde{v}(t,\eta) \end{pmatrix}.$$

Hence the expression of the Laplacian in the variables (t,η) is

$$\Delta v = \left[\partial_\eta^2 + \frac{c(t)}{1+\eta c(t)}\partial_\eta + \frac{1}{1+\eta c(t)}\partial_t\left(\frac{1}{1+\eta c(t)}\partial_t\right)\right]\widetilde{v}.$$

As usual, due to the dependence of Ω_m^ε on the thickness parameter ε, we cannot give an asymptotic expansion of u^ε in powers of ε. Therefore, we transform Ω_m^ε into a fixed domain independent of ε. We introduce the scaling $s = \eta/\varepsilon$, and the interval $I = (-d_1, d_2)$ such that the $\mathcal{C}^\infty$-diffeomorphism Φ, defined by

$$\begin{array}{cccc} \Omega_m := (0, l_\Gamma) \times I & \xrightarrow{\Phi} & \Omega_m^\varepsilon \\ (t,s) & \mapsto & x := \gamma(t) + \varepsilon s \boldsymbol{n}(t), \end{array}$$

parameterizes the thin shell Ω_m^ε. To any function v defined on Ω_m^ε, we associate the function V defined on Ω_m through

$$V(t,s) := v(x), \quad x = \Phi(t,s).$$

Hence the expression of the Laplacian in the variables (t,s) is

$$\begin{aligned} \Delta v &= \varepsilon^{-2}\left[\partial_s^2 V + \varepsilon\frac{c(t)}{1+\varepsilon s c(t)}\partial_s V + \frac{\varepsilon^2}{1+\varepsilon s c(t)}\partial_t\left(\frac{1}{1+\varepsilon s c(t)}\partial_t\right)V\right] \\ &= \varepsilon^{-2}\left(\partial_s^2 V - \sum_{j=1}^{N}\varepsilon^j A_j V + \varepsilon^{N+1} T_N V\right), \end{aligned} \tag{3.1}$$

where T_N is a bounded operator with respect to ε. In particular

$$A_1 = -c(t)\partial_s, \quad A_2 = s c^2(t)\partial_s - \partial_t^2, \quad A_3 = -s^2 c^3(t)\partial_s + 2sc(t)\partial_t^2 + sc'(t)\partial_t.$$

Remark 3.1. For any function u defined in a neighbourhood of Γ, we denote, for convenience, by $u|_\Gamma$ the trace of u on Γ indifferently in local coordinates or cartesian coordinates.

4. The asymptotic analysis

This section is dedicated to a multiscale expansion for the solution u^ε to Problem (2.2) in power of ε. We derive a hierarchy of equations defined in a fixed domain (independent of ε). Then we give the first three terms of the asymptotic expansions. Further details on the derivation of these terms are given in Appendix A. We conclude by the convergence theorem justifying our ansatz.

4.1. Hierarchy of equations

As well known (see [3,11,12,26]), it is impossible to determine an asymptotic expansion in power of ε uniformly on the whole domain Ω_∞. This is due to boundary layer phenomena in the vicinity of the thin layer.

Therefore, we consider two asymptotic expansions: exterior asymptotic expansions corresponding to the expansion of the solution u^ε restricted to Ω_+^ε and to Ω_-^ε, written in cartesian coordinates $x = (x_1, x_2)$ (macroscopic scale) and given by the ansatz

$$u_+^\varepsilon = \sum_{n\geq 0} \varepsilon^n u_+^n \quad \text{in } \Omega_+^\varepsilon, \tag{4.1}$$

$$u_-^\varepsilon = \sum_{n\geq 0} \varepsilon^n u_-^n \quad \text{in } \Omega_-^\varepsilon, \tag{4.2}$$

where the terms u_+^n and u_-^n $(n \in \mathbb{N})$ are independent of ε and respectively defined on $\Omega_+ := \Omega_+^\varepsilon \cup \Gamma_2^\varepsilon \cup \Omega_{m,2}^\varepsilon$ and on $\Omega_- := \Omega_-^\varepsilon \cup \Gamma_1^\varepsilon \cup \Omega_{m,1}^\varepsilon$. They satisfy

$$\begin{cases} \Delta u_+^n + k_+^2 u_+^n = 0 & \text{in } \Omega_+, \\ \Delta u_-^n + k_-^2 u_-^n = 0 & \text{in } \Omega_-, \\ u_-^n = 0 & \text{on } \Gamma_i, \\ \lim_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)(u_+^n - \delta_{0,n} u_{\text{inc}}) = 0 \end{cases} \tag{4.3}$$

in which $\delta_{0,n}$ indicates the Kronecker symbol. And an interior expansion corresponding to the asymptotic expansion of u^ε restricted to Ω_m^ε, written in local coordinates (t, s) (microscopic scale) and defined by the ansatz

$$u_m^\varepsilon(x_1, x_2) = U_m^\varepsilon(t, s) = \sum_{n\geq 0} \varepsilon^n U_m^n(t, s) \quad \text{in } \Omega_m, \tag{4.4}$$

where U_m^n are independent of ε.

Using a Taylor expansion in the normal variable, we infer formally

$$\begin{aligned} u_-^\varepsilon|_{\Gamma_-^\varepsilon} &= u_-^0|_\Gamma + \varepsilon\big(u_-^1|_\Gamma - d_1\partial_{\boldsymbol{n}} u_-^0|_\Gamma\big) + \varepsilon^2 \left(u_-^2|_\Gamma - d_1\partial_{\boldsymbol{n}} u_-^1|_\Gamma + \frac{d_1^2}{2}\partial_{\boldsymbol{n}}^2 u_-^0|_\Gamma \right) + \cdots, \\ \alpha_-\partial_{\boldsymbol{n}} u_-^\varepsilon|_{\Gamma_-^\varepsilon} &= \alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma + \varepsilon\big(\alpha_-\partial_{\boldsymbol{n}} u_-^1|_\Gamma - \alpha_- d_1 \partial_{\boldsymbol{n}}^2 u_-^0|_\Gamma\big) \\ &\quad + \varepsilon^2 \left(\alpha_-\partial_{\boldsymbol{n}} u_-^2|_\Gamma - \alpha_- d_1 \partial_{\boldsymbol{n}}^2 u_-^1|_\Gamma + \alpha_- \frac{d_1^2}{2}\partial_{\boldsymbol{n}}^3 u_-^0|_\Gamma \right) + \cdots, \end{aligned}$$

and

$$\begin{aligned} u_+^\varepsilon|_{\Gamma_+^\varepsilon} &= u_+^0|_\Gamma + \varepsilon\big(u_+^1|_\Gamma + d_2\partial_{\boldsymbol{n}} u_+^0|_\Gamma\big) + \varepsilon^2 \left(u_+^2|_\Gamma + d_2\partial_{\boldsymbol{n}} u_+^1|_\Gamma + \frac{d_2^2}{2}\partial_{\boldsymbol{n}}^2 u_+^0|_\Gamma \right) + \cdots, \\ \alpha_+\partial_{\boldsymbol{n}} u_+^\varepsilon|_{\Gamma_+^\varepsilon} &= \alpha_+\partial_{\boldsymbol{n}} u_+^0|_\Gamma + \varepsilon\big(\alpha_+\partial_{\boldsymbol{n}} u_+^1|_\Gamma + \alpha_+ d_2 \partial_{\boldsymbol{n}}^2 u_+^0|_\Gamma\big) \\ &\quad + \varepsilon^2 \left(\alpha_+\partial_{\boldsymbol{n}} u_+^2|_\Gamma + \alpha_+ d_2 \partial_{\boldsymbol{n}}^2 u_+^1|_\Gamma + \alpha_+ \frac{d_2^2}{2}\partial_{\boldsymbol{n}}^3 u_+^0|_\Gamma \right) + \cdots. \end{aligned}$$

Transmission conditions (2.2d) imply that

$$\begin{aligned} &U_m^0(t,-d_1)+\varepsilon U_m^1(t,-d_1)+\varepsilon^2 U_m^2(t,-d_1)+\cdots \\ &=u_-^0|_\Gamma+\varepsilon\big(u_-^1|_\Gamma-d_1\partial_{\boldsymbol n}u_-^0|_\Gamma\big)+\varepsilon^2\left(u_-^2|_\Gamma-d_1\partial_{\boldsymbol n}u_-^1|_\Gamma+\frac{d_1^2}{2}\partial_{\boldsymbol n}^2u_-^0|_\Gamma\right)+\cdots, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &U_m^0(t,d_2)+\varepsilon U_m^1(t,d_2)+\varepsilon^2 U_m^2(t,d_2)+\cdots \\ &=u_+^0|_\Gamma+\varepsilon\big(u_+^1|_\Gamma+d_2\partial_{\boldsymbol n}u_+^0|_\Gamma\big)+\varepsilon^2\left(u_+^2|_\Gamma+d_2\partial_{\boldsymbol n}u_+^1|_\Gamma+\frac{d_2^2}{2}\partial_{\boldsymbol n}^2u_+^0|_\Gamma\right)+\cdots, \end{aligned} \tag{4.6}$$

and the condition (2.2e) gives

$$\begin{aligned} &\varepsilon^{-1}\partial_sU_m^0(t,-d_1)+\partial_sU_m^1(t,-d_1)+\varepsilon\partial_sU_m^2(t,-d_1)+\varepsilon^2\partial_sU_m^3(t,-d_1)+\cdots \\ &=\alpha_-\partial_{\boldsymbol n}u_-^0|_\Gamma+\varepsilon\big(\alpha_-\partial_{\boldsymbol n}u_-^1|_\Gamma-\alpha_-d_1\partial_{\boldsymbol n}^2u_-^0|_\Gamma\big) \\ &\quad+\varepsilon^2\left(\alpha_-\partial_{\boldsymbol n}u_-^2|_\Gamma-\alpha_-d_1\partial_{\boldsymbol n}^2u_-^1|_\Gamma+\alpha_-\frac{d_1^2}{2}\partial_{\boldsymbol n}^3u_-^0|_\Gamma\right)+\cdots, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\varepsilon^{-1}\partial_sU_m^0(t,d_2)+\partial_sU_m^1(t,d_2)+\varepsilon\partial_sU_m^2(t,d_2)+\varepsilon^2\partial_sU_m^3(t,d_2)+\cdots \\ &=\alpha_+\partial_{\boldsymbol n}u_+^0|_\Gamma+\varepsilon\big(\alpha_+\partial_{\boldsymbol n}u_+^1|_\Gamma+\alpha_+d_2\partial_{\boldsymbol n}^2u_+^0|_\Gamma\big) \\ &\quad+\varepsilon^2\left(\alpha_+\partial_{\boldsymbol n}u_+^2|_\Gamma+\alpha_+d_2\partial_{\boldsymbol n}^2u_+^1|_\Gamma+\alpha_+\frac{d_2^2}{2}\partial_{\boldsymbol n}^3u_+^0|_\Gamma\right)+\cdots. \end{aligned} \tag{4.8}$$

Otherwise, inserting Expansion (4.4) in (2.2a), using (3.1) and matching the same powers of ε, we get, for all $(t,s)\in(0,l_\Gamma)\times I$, the hierarchy of equations

$$\partial_s^2U_m^0=0, \tag{4.9}$$

$$\partial_s^2U_m^1=A_1U_m^0, \tag{4.10}$$

$$\partial_s^2U_m^2=A_1U_m^1+A_2U_m^0-k_m^2U_m^0, \tag{4.11}$$

$$\partial_s^2U_m^3=A_1U_m^2+A_2U_m^1+A_3U_m^0-k_m^2U_m^1, \tag{4.12}$$

$$\vdots$$

4.2. Calculation of the first three terms

We give in this paragraph the first three terms of the asymptotic expansions of the solution u^ε. More details are given in Appendix A.

By solving (4.9)–(4.12), with the help of transmission conditions (4.5)–(4.8), the terms (u_-^n,u_+^n), $0\le n\le 2$, are solutions of the following boundary-value problems

$$\begin{cases} \Delta u_+^n+k_+^2u_+^n=0 & \text{in } \Omega_+, \\ \Delta u_-^n+k_-^2u_-^n=0 & \text{in } \Omega_-, \\ u_-^n=0 & \text{on } \Gamma_i, \\ \lim_{|x|\to+\infty}\sqrt{|x|}(\partial_{|x|}-ik_+)(u_+^n-\delta_{0,n}u_{\text{inc}})=0 \end{cases}$$

with transmission conditions on Γ:

At order 0:

$$u_+^0 = u_-^0, \quad \alpha_+ \partial_{\boldsymbol{n}} u_+^0 = \alpha_- \partial_{\boldsymbol{n}} u_-^0.$$

At order 1:

$$\begin{aligned} u_+^1 - u_-^1 &= \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{2\alpha_+\alpha_-}(\alpha_+\partial_{\boldsymbol{n}} u_+^0 + \alpha_-\partial_{\boldsymbol{n}} u_-^0), \\ \alpha_+\partial_{\boldsymbol{n}} u_+^1 - \alpha_-\partial_{\boldsymbol{n}} u_-^1 &= \frac{d_1\alpha_- + d_2\alpha_+ - 1}{2}(\partial_t^2 u_+^0 + \partial_t^2 u_-^0) \\ &\quad + \frac{d_1\alpha_- k_-^2 + d_2\alpha_+ k_+^2 - k_m^2}{2}(u_+^0 + u_-^0). \end{aligned}$$

At order 2:

$$\begin{aligned} & u_+^2 - u_-^2 \\ &= \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{2\alpha_+\alpha_-}(\alpha_+\partial_{\boldsymbol{n}} u_+^1 + \alpha_-\partial_{\boldsymbol{n}} u_-^1) \\ &\quad + \frac{d_1\alpha_+\alpha_-^2 - d_2\alpha_+^2\alpha_- + d_1 d_2(\alpha_+^2 - \alpha_-^2) + d_2\alpha_- - d_1\alpha_+}{4\alpha_+\alpha_-}(\partial_t^2 u_+^0 + \partial_t^2 u_-^0) \\ &\quad + c(t)\frac{(d_1 - d_2)\alpha_-\alpha_+ + d_2^2\alpha_- - d_1^2\alpha_+}{2(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-)}(u_+^1 - u_-^1) \\ &\quad + \frac{(d_1 k_-^2\alpha_- - d_2 k_+^2\alpha_+)\alpha_+\alpha_- + d_1 d_2(k_+^2\alpha_+^2 - k_-^2\alpha_-^2) + k_m^2(d_2\alpha_- - d_1\alpha_+)}{4\alpha_+\alpha_-}(u_+^0 + u_-^0), \end{aligned}$$

$$\begin{aligned} & \alpha_+\partial_{\boldsymbol{n}} u_+^2 - \alpha_-\partial_{\boldsymbol{n}} u_-^2 \\ &= \frac{(d_2\alpha_+ + d_1\alpha_- - 1)}{2}(\partial_t^2 u_+^1 + \partial_t^2 u_-^1) \\ &\quad + \frac{(d_2\alpha_+ k_+^2 + d_1\alpha_- k_-^2 - k_m^2)}{2}(u_-^1 + u_+^1) \\ &\quad + \frac{\alpha_+ d_1 k_m^2 - \alpha_- d_2 k_m^2 - \alpha_+^2 d_1 d_2 k_+^2 + \alpha_-^2 d_1 d_2 k_-^2 + \alpha_+^2\alpha_- d_2 k_+^2 - \alpha_+\alpha_-^2 d_1 k_-^2}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)}(u_+^1 - u_-^1) \\ &\quad + \left(\frac{\alpha_+ d_2^2 k_+^2 - \alpha_- d_1^2 k_-^2 + d_1 k_m^2 - d_2 k_m^2}{4}\right) c(t)(u_+^0 + u_-^0) \\ &\quad + \left(\frac{d_2 - d_1 + \alpha_- d_1^2 - \alpha_+ d_2^2}{4}\right) c(t)(\partial_t^2 u_+^0 + \partial_t^2 u_-^0) \\ &\quad + \left(\frac{d_2 - d_1 + \alpha_- d_1^2 - \alpha_+ d_2^2}{4}\right) c'(t)(\partial_t u_+^0 + \partial_t u_-^0) \\ &\quad + \left(\frac{(d_2\alpha_+ - d_1\alpha_-)(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-) + (d_2 - d_1)\alpha_+\alpha_- + d_1\alpha_+ - d_2\alpha_-}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)}\right) \\ &\qquad \times (\partial_t^2 u_+^1 - \partial_t^2 u_-^1). \end{aligned}$$

Moreover $(U_m^n)_{0\le n\le 2}$ is determined by

$$U_m^0(t,s) = u_-^0|_\Gamma = u_+^0|_\Gamma, \quad \forall\,(t,s)\in\Omega_m, \tag{4.13}$$

$$U_m^1(t,s) = u_-^1|_\Gamma + (\alpha_- s + d_1\alpha_- - d_1)\partial_{\boldsymbol{n}} u_-^0|_\Gamma, \quad \forall\,(t,s)\in\Omega_m, \tag{4.14}$$

$$\begin{aligned} U_m^2(t,s) &= u_-^2|_\Gamma + (\alpha_- s + d_1\alpha_- - d_1)\partial_{\boldsymbol{n}} u_-^1|_\Gamma \\ &\quad - \left(\frac{d_1^2}{2}k_-^2 - d_1^2\alpha_- k_-^2 - d_1\alpha_- s k_-^2 + \frac{d_1^2}{2}k_m^2 + \frac{s^2}{2}k_m^2 + d_1 k_m^2 s\right) u_-^0|_\Gamma \\ &\quad - \left(\frac{s^2}{2} + d_1 s + d_1^2 - d_1^2\alpha_- - d_1\alpha_- s\right)\partial_t^2 u_-^0|_\Gamma \\ &\quad - \left(\frac{s^2}{2}\alpha_- - \frac{d_1^2}{2}\alpha_- + \frac{d_1^2}{2}\right) c(t)\partial_{\boldsymbol{n}} u_-^0|_\Gamma. \end{aligned} \tag{4.15}$$

We have derived the first three terms of the asymptotic expansions (4.1), (4.2) and (4.4). We can continue up to any order since the data are smooth enough. We now come to a convergence theorem justifying the ansatz (4.1), (4.2) and (4.4) by estimating the error resulting from the truncation of the expansions after a finite number of terms. A complete proof giving theorems of existence and uniqueness of the series $(u_-^n)_n$, $(u_+^n)_n$ and $(U_m^n)_n$ and an error estimate can be found in [6,8]. Let $N\in\mathbb{N}$ and $\widetilde{\Omega}_+^\varepsilon$ be a domain of $\mathbb{R}^2$ defined by $\widetilde{\Omega}_+^\varepsilon := \Omega_+^\varepsilon \cap \Omega$. We set

$$u_-^{\varepsilon,(N)} := \sum_{n=0}^{N} \varepsilon^n u_-^n, \quad u_+^{\varepsilon,(N)} := \sum_{n=0}^{N} \varepsilon^n u_+^n \quad \text{and} \quad u_m^{\varepsilon,(N)} := \sum_{n=0}^{N} \varepsilon^n u_m^n,$$

where $u_m^n(x) := U_m^n(t,s)$, $\forall\, x = \Phi(t,s)\in\Omega_m^\varepsilon$.

Theorem 4.1 (Convergence theorem)**.** *For all integers N, there exists a constant c independent of ε such that*

$$\left\|u_-^\varepsilon - u_-^{\varepsilon,(N)}\right\|_{H^1(\Omega_-^\varepsilon)} + \varepsilon^{1/2}\left\|u_m^\varepsilon - u_m^{\varepsilon,(N)}\right\|_{H^1(\Omega_m^\varepsilon)} + \left\|u_+^\varepsilon - u_+^{\varepsilon,(N)}\right\|_{H^1(\widetilde{\Omega}_+^\varepsilon)} \le c\varepsilon^{N+1}.$$

5. Approximate transmission conditions

In this section, which is the main part of the paper, we model the effect of the thin layer by a problem with Ventcel-type transmission conditions with accuracy up to $O(\varepsilon^3)$.

In the spirit of [5,8], we truncate the series (4.1) and (4.2) keeping only the first three terms and neglect all the terms of order greater than or equal to 3. This yields a candidate $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$, solution of the following problem, to approximate the exact solution u^ε far from the thin layer

$$
\text{(5.1a)}\quad \begin{cases} \Delta u_+^{\varepsilon,ap} + k_+^2 u_+^{\varepsilon,ap} = 0 & \text{in } \Omega_+, \\ \Delta u_-^{\varepsilon,ap} + k_-^2 u_-^{\varepsilon,ap} = 0 & \text{in } \Omega_-, \\ u_-^{\varepsilon,ap} = 0 & \text{on } \Gamma_i, \\ \lim_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)(u_+^{\varepsilon,ap} - u_{\text{inc}}) = 0 & \end{cases}
$$

with the following transmission conditions on Γ:

$$
\text{(5.1b)}\quad \begin{aligned} u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap} &= \varepsilon\mathcal{A}_1(\alpha_+\partial_{\boldsymbol{n}} u_+^{\varepsilon,ap} + \alpha_-\partial_{\boldsymbol{n}} u_-^{\varepsilon,ap}) + \varepsilon^2\mathcal{A}_2(\partial_t^2 u_+^{\varepsilon,ap} + \partial_t^2 u_-^{\varepsilon,ap}) \\ &\quad + \varepsilon\mathcal{A}_3(u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}) + \varepsilon^2\mathcal{A}_4(u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}), \end{aligned}
$$

$$
\text{(5.1c)}\quad \begin{aligned} \alpha_+\partial_{\boldsymbol{n}} u_+^{\varepsilon,ap} - \alpha_-\partial_{\boldsymbol{n}} u_-^{\varepsilon,ap} &= (\varepsilon\mathcal{B}_1 + \varepsilon^2\mathcal{B}_2)(\partial_t^2 u_+^{\varepsilon,ap} + \partial_t^2 u_-^{\varepsilon,ap}) \\ &\quad + (\varepsilon\mathcal{B}_3 + \varepsilon^2\mathcal{B}_4)(u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}) + \varepsilon\mathcal{B}_5(u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}) \\ &\quad + \varepsilon^2\mathcal{B}_6(\partial_t u_+^{\varepsilon,ap} + \partial_t u_-^{\varepsilon,ap}) + \varepsilon\mathcal{B}_7(\partial_t^2 u_+^{\varepsilon,ap} - \partial_t^2 u_-^{\varepsilon,ap}) \end{aligned}
$$

in which

$$
\text{(5.2)}\quad \mathcal{A}_1 = \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{2\alpha_+\alpha_-},
$$

$$
\text{(5.3)}\quad \mathcal{A}_2 = \frac{d_1\alpha_+\alpha_-^2 - d_2\alpha_+^2\alpha_- + d_1d_2(\alpha_+^2 - \alpha_-^2) + d_2\alpha_- - d_1\alpha_+}{4\alpha_+\alpha_-},
$$

$$
\text{(5.4)}\quad \mathcal{A}_3 = c(t)\frac{(d_1 - d_2)\alpha_+\alpha_- + d_2^2\alpha_- - d_1^2\alpha_+}{2(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-)},
$$

$$
\text{(5.5)}\quad \mathcal{A}_4 = \frac{(d_1k_-^2\alpha_- - d_2k_+^2\alpha_+)\alpha_+\alpha_- + d_1d_2(k_+^2\alpha_+^2 - k_-^2\alpha_-^2) + k_m^2(d_2\alpha_- - d_1\alpha_+)}{4\alpha_+\alpha_-},
$$

$$
\text{(5.6)}\quad \mathcal{B}_1 = \frac{d_2\alpha_+ + d_1\alpha_- - 1}{2},
$$

$$
\text{(5.7)}\quad \mathcal{B}_2 = \frac{d_2 - d_1 + \alpha_-d_1^2 - \alpha_+d_2^2}{4}c(t),
$$

$$
\text{(5.8)}\quad \mathcal{B}_3 = \frac{d_2\alpha_+k_+^2 + d_1\alpha_-k_-^2 - k_m^2}{2},
$$

$$
\text{(5.9)}\quad \mathcal{B}_4 = \frac{\alpha_+d_2^2k_+^2 - \alpha_-d_1^2k_-^2 + d_1k_m^2 - d_2k_m^2}{4}c(t),
$$

$$
\text{(5.10)}\quad \begin{aligned} \mathcal{B}_5 &= \frac{\alpha_+d_1k_m^2 - \alpha_-d_2k_m^2 - \alpha_+^2d_1d_2k_+^2 + \alpha_-^2d_1d_2k_-^2 + \alpha_+^2\alpha_-d_2k_+^2 - \alpha_+\alpha_-^2d_1k_-^2}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)} \\ &= -\frac{\mathcal{A}_4}{\mathcal{A}_1}, \end{aligned}
$$

$$
\text{(5.11)}\quad \mathcal{B}_6 = \left(\frac{d_2 - d_1 + \alpha_-d_1^2 - \alpha_+d_2^2}{4}\right)c'(t) = \partial_t\mathcal{B}_2,
$$

$$
\text{(5.12)}\quad \begin{aligned} \mathcal{B}_7 &= \frac{(d_2\alpha_+ - d_1\alpha_-)(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-) + (d_2 - d_1)\alpha_+\alpha_- + d_1\alpha_+ - d_2\alpha_-}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)} \\ &= -\frac{\mathcal{A}_2}{\mathcal{A}_1}. \end{aligned}
$$

Note that $\mathcal{A}_1 \neq 0$ since we have assumed that $\alpha_\pm \in]1,+\infty[$ or $\alpha_\pm \in]0,1[$.

Once the approximate solution $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$ is determined, we deduce, from (4.13)–(4.15), an approximation $u_m^{\varepsilon,ap}$ of the part u_m^ε of the exact solution u^ε defined in Ω_m^ε by

$$
\begin{aligned}
u_m^{\varepsilon,ap}(x) := U_m^{\varepsilon,ap}(t,s) &:= u_-^{\varepsilon,ap}|_\Gamma + \varepsilon(\alpha_- s + d_1\alpha_- - d_1)\partial_{\boldsymbol{n}} u_-^{\varepsilon,ap}|_\Gamma \\
&- \varepsilon^2\left(\frac{d_1^2}{2}k_-^2 - d_1^2\alpha_- k_-^2 - d_1\alpha_- s k_-^2 + \frac{d_1^2}{2}k_m^2 + \frac{s^2}{2}k_m^2 + d_1 k_m^2 s\right) u_-^{\varepsilon,ap}|_\Gamma \\
&- \varepsilon^2\left(\frac{s^2}{2} + d_1 s + d_1^2 - d_1^2\alpha_- - d_1\alpha_- s\right)\partial_t^2 u_-^{\varepsilon,ap}|_\Gamma \\
&- \varepsilon^2\left(\frac{s^2}{2}\alpha_- - \frac{d_1^2}{2}\alpha_- + \frac{d_1^2}{2}\right)c(t)\partial_{\boldsymbol{n}} u_-^{\varepsilon,ap}|_\Gamma, \quad \forall\, x = \Phi(t,s) \in \Omega_m^\varepsilon.
\end{aligned} \tag{5.13}
$$

Theorem 5.1. *Problem* (5.1) *admits at most one solution.*

Proof. Let us consider the homogeneous problem associated with Problem (5.1):

$$
\begin{cases}
\Delta u_+^{\varepsilon,ap} + k_+^2 u_+^{\varepsilon,ap} = 0 & \text{in } \Omega_+, \quad (5.14)\\
\Delta u_-^{\varepsilon,ap} + k_-^2 u_-^{\varepsilon,ap} = 0 & \text{in } \Omega_-, \quad (5.15)\\
u_-^{\varepsilon,ap} = 0 & \text{on } \Gamma_i, \quad (5.16)\\
\lim\limits_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)u_+^{\varepsilon,ap} = 0 & \quad (5.17)
\end{cases}
$$

with transmission conditions on Γ:

$$
\begin{aligned}
u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap} &= \varepsilon\mathcal{A}_1(\alpha_+\partial_{\boldsymbol{n}} u_+^{\varepsilon,ap} + \alpha_-\partial_{\boldsymbol{n}} u_-^{\varepsilon,ap}) + \varepsilon^2\mathcal{A}_2(\partial_t^2 u_+^{\varepsilon,ap} + \partial_t^2 u_-^{\varepsilon,ap}) \\
&\quad + \varepsilon\mathcal{A}_3(u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}) + \varepsilon^2\mathcal{A}_4(u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}),
\end{aligned} \tag{5.18}
$$

$$
\begin{aligned}
\alpha_+\partial_{\boldsymbol{n}} u_+^{\varepsilon,ap} - \alpha_-\partial_{\boldsymbol{n}} u_-^{\varepsilon,ap} &= (\varepsilon\mathcal{B}_1 + \varepsilon^2\mathcal{B}_2)(\partial_t^2 u_+^{\varepsilon,ap} + \partial_t^2 u_-^{\varepsilon,ap}) \\
&\quad + (\varepsilon\mathcal{B}_3 + \varepsilon^2\mathcal{B}_4)(u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}) + \varepsilon\mathcal{B}_5(u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}) \\
&\quad + \varepsilon^2\mathcal{B}_6(\partial_t u_+^{\varepsilon,ap} + \partial_t u_-^{\varepsilon,ap}) + \varepsilon\mathcal{B}_7(\partial_t^2 u_+^{\varepsilon,ap} - \partial_t^2 u_-^{\varepsilon,ap}).
\end{aligned} \tag{5.19}
$$

Standard regularity results for elliptic problems (see [2]) ensure that $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap}) \in \mathcal{C}^\infty(\overline{\Omega}_-) \times \mathcal{C}^\infty(\overline{\Omega}_+)$. Let B_R be the ball with centre O and radius R sufficiently large to enclose Ω_- and let Ω_R be the domain of $\mathbb{R}^2$ defined by $\Omega_R := B_R \cap \Omega_+$. Multiplying (5.14) and (5.15) by $\overline{u_+^{\varepsilon,ap}}$ and $\overline{u_-^{\varepsilon,ap}}$ respectively and using Green's formula, we get

(5.20)

$$
\begin{aligned}
&\alpha_- \int_{\Omega_-} |\nabla u_-^{\varepsilon,ap}|^2\, d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} |u_-^{\varepsilon,ap}|^2\, d\Omega_- + \alpha_+ \int_{\Omega_R} |\nabla u_+^{\varepsilon,ap}|^2\, d\Omega_R \\
&- \alpha_+ k_+^2 \int_{\Omega_R} |u_+^{\varepsilon,ap}|^2\, d\Omega_R - \int_\Gamma \frac{\varepsilon\mathcal{A}_3 - 1}{2\varepsilon\mathcal{A}_1}|u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}|^2\, d\Gamma
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}\int_\Gamma (\varepsilon\mathcal{B}_3 + \varepsilon^2\mathcal{B}_4)\big(|u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}|^2\big)\, d\Gamma - \frac{1}{2}\int_\Gamma (\varepsilon\mathcal{B}_1 + \varepsilon^2\mathcal{B}_2)\big(|\partial_t u_-^{\varepsilon,ap} + \partial_t u_+^{\varepsilon,ap}|^2\big)\, d\Gamma \\
&+ \varepsilon\mathcal{B}_5 \int_\Gamma \big(|u_+^{\varepsilon,ap}|^2 - |u_-^{\varepsilon,ap}|^2\big)\, d\Gamma - \varepsilon\mathcal{B}_7 \int_\Gamma \big(|\partial_t u_+^{\varepsilon,ap}|^2 - |\partial_t u_-^{\varepsilon,ap}|^2\big)\, d\Gamma \\
= {}& \alpha_+ \int_{S_R} \partial_{\mathbf{R}} u_+^{\varepsilon,ap} \overline{u_+^{\varepsilon,ap}}\, dS_R,
\end{aligned}
$$

where S_R denotes the circle with centre O and radius R. Taking the imaginary part of (5.20), we obtain

$$
\Im\left(\int_{S_R} \partial_{\mathbf{R}} u_+^{\varepsilon,ap} \overline{u_+^{\varepsilon,ap}}\, dS_R\right) = 0.
$$

It follows from radiation condition (5.17) and Rellich's lemma [25] that $u_+^{\varepsilon,ap} = 0$ in Ω_+.

Problem (5.14)–(5.19) is thus reduced to

$$
\begin{cases}
\Delta u_-^{\varepsilon,ap} + k_-^2 u_-^{\varepsilon,ap} = 0 & \text{in } \Omega_-, \\
u_-^{\varepsilon,ap} = 0 & \text{on } \Gamma_i
\end{cases}
$$

with the following transmission conditions on Γ:

$$
\begin{aligned}
u_-^{\varepsilon,ap} &= -\varepsilon\mathcal{A}_1(\alpha_-\partial_{\boldsymbol{n}} u_-^{\varepsilon,ap}) - \varepsilon^2\mathcal{A}_2(\partial_t^2 u_-^{\varepsilon,ap}) + \varepsilon\mathcal{A}_3(u_-^{\varepsilon,ap}) - \varepsilon^2\mathcal{A}_4(u_-^{\varepsilon,ap}), \\
\alpha_-\partial_{\boldsymbol{n}} u_-^{\varepsilon,ap} &= -(\varepsilon\mathcal{B}_1 + \varepsilon^2\mathcal{B}_2)(\partial_t^2 u_-^{\varepsilon,ap}) - (\varepsilon\mathcal{B}_3 + \varepsilon^2\mathcal{B}_4)(u_-^{\varepsilon,ap}) + \varepsilon\mathcal{B}_5(u_-^{\varepsilon,ap}) \\
&\quad - \varepsilon^2\mathcal{B}_6\partial_t u_-^{\varepsilon,ap} + \varepsilon\mathcal{B}_7(\partial_t^2 u_-^{\varepsilon,ap}).
\end{aligned}
$$

This implies that

$$
\left[\left(\frac{1}{\mathcal{A}_1} - \varepsilon\frac{\mathcal{A}_3}{\mathcal{A}_1} - \varepsilon^2\mathcal{B}_3 - \varepsilon^3\mathcal{B}_4\right) I - (\varepsilon^2\mathcal{B}_1 + \varepsilon^3\mathcal{B}_2)\partial_t^2 - \varepsilon^3\mathcal{B}_6\partial_t\right] u_-^{\varepsilon,ap}|_\Gamma = 0. \tag{5.21}
$$

Multiplying (5.21) by $\overline{u_-^{\varepsilon,ap}|_\Gamma}$ and integrating over Γ, we obtain

$$
\int_\Gamma \left(\frac{1}{\mathcal{A}_1} - \varepsilon\frac{\mathcal{A}_3}{\mathcal{A}_1} - \varepsilon^2\mathcal{B}_3 - \varepsilon^3\mathcal{B}_4\right) |u_-^{\varepsilon,ap}|^2\, d\Gamma + \int_\Gamma (\varepsilon^2\mathcal{B}_1 + \varepsilon^3\mathcal{B}_2)|\partial_t u_-^{\varepsilon,ap}|^2\, d\Gamma = 0.
$$

As $\alpha_\pm \in]1, +\infty[$ or $\alpha_\pm \in]0, 1[$, then $\mathcal{A}_1\mathcal{B}_1 > 0$, and since ε is small enough, there exists a positive constant C such that

$$
\begin{aligned}
C\left(\|u_-^{\varepsilon,ap}\|^2_{L^2(\Gamma)} + \|\partial_t u_-^{\varepsilon,ap}\|^2_{L^2(\Gamma)}\right) &\leq \int \left|\frac{1}{\mathcal{A}_1} - \varepsilon\frac{\mathcal{A}_3}{\mathcal{A}_1} - \varepsilon^2\mathcal{B}_3 - \varepsilon^3\mathcal{B}_4\right| |u_-^{\varepsilon,ap}|^2\, d\Gamma \\
&\quad + \varepsilon^2 \int_\Gamma |\mathcal{B}_1 + \varepsilon\mathcal{B}_2||\partial_t u_-^{\varepsilon,ap}|^2\, d\Gamma \\
&= 0.
\end{aligned}
$$

This leads to $u_-^{\varepsilon,ap}|_\Gamma = 0$. Then, we obtain

$$\begin{cases} \Delta u_-^{\varepsilon,ap} + k_-^2 u_-^{\varepsilon,ap} = 0 & \text{in } \Omega_-, \\ u_-^{\varepsilon,ap} = 0 & \text{on } \Gamma, \\ \partial_{\boldsymbol{n}} u_-^{\varepsilon,ap} = 0 & \text{on } \Gamma, \\ u_-^{\varepsilon,ap} = 0 & \text{on } \Gamma_i. \end{cases}$$

Well-know arguments of uniqueness of the solution of this type of problems (see [20]) leads to $u_-^{\varepsilon,ap} = 0$ in Ω_-, which proves the uniqueness of the solution $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$. □

To show the existence of the solution $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$, we transform Problem (5.1) into a pseudodifferential equation set on Γ. Therefore, we introduce the Steklov–Poincaré operators (see [3]) (called also Dirichlet-to-Neumann operators) T_- and T_+ defined from $H^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$ by $T_-\varphi := \alpha_- \partial_{\boldsymbol{n}} u_-|_\Gamma$, where u_- is the solution to the boundary-value problem

$$\begin{cases} \Delta u_- + k_-^2 u_- = 0 & \text{in } \Omega_-, \\ u_- = \varphi & \text{on } \Gamma, \\ u_- = 0 & \text{on } \Gamma_i, \end{cases}$$

and by $T_+\psi := \alpha_+ \partial_{-\boldsymbol{n}} u_+|_\Gamma$, where u_+ is the solution to the boundary-value problem

$$\begin{cases} \Delta u_+ + k_+^2 u_+ = 0 & \text{in } \Omega_+, \\ u_+ = \psi & \text{on } \Gamma, \\ \lim_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)u_+ = 0. \end{cases}$$

The Dirichlet-to-Neumann operators T_- and T_+ are elliptic pseudodifferential operators (see [27–29]) of real symbol of order 1.

Remark 5.2. The function u_- is defined only in the case where the constant k_-^2 does not belong to the spectrum of the closed operator $(-\Delta, H_0^1(\Omega_-))$. We will therefore assume that this condition holds.

The definition of The Steklov–Poincaré operators allows us to rewrite Problem (5.1) into an equivalent system of boundary equations: Find $(\omega, \varkappa) \in H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that

$$\Lambda \begin{pmatrix} \omega \\ \varkappa \end{pmatrix} = \begin{pmatrix} \varepsilon \mathcal{A}_1 g \\ -g \end{pmatrix}, \tag{5.22}$$

where ω and $\varkappa$ are the traces of $u_+^{\varepsilon,ap}$ and $u_-^{\varepsilon,ap}$ on Γ respectively,

$$g = T_+(u_{\text{inc}}|_\Gamma) + \alpha_+ \partial_{\boldsymbol{n}} u_{\text{inc}}|_\Gamma,$$

and $\Lambda = (\Lambda_{ij})_{1\le i,j\le 2}$ is a matrix of pseudodifferential operators defined by

$$\Lambda_{11} = (1-\varepsilon\mathcal{A}_3-\varepsilon^2\mathcal{A}_4)I+\varepsilon\mathcal{A}_1T_+-\varepsilon^2\mathcal{A}_2\partial_t^2, \tag{5.23}$$

$$\Lambda_{12} = (-1+\varepsilon\mathcal{A}_3-\varepsilon^2\mathcal{A}_4)I-\varepsilon\mathcal{A}_1T_--\varepsilon^2\mathcal{A}_2\partial_t^2, \tag{5.24}$$

$$\Lambda_{21} = (-\varepsilon\mathcal{B}_3-\varepsilon\mathcal{B}_5-\varepsilon^2\mathcal{B}_4)I-T_+-\varepsilon(\mathcal{B}_1+\mathcal{B}_7+\varepsilon\mathcal{B}_2)\partial_t^2-\varepsilon^2\mathcal{B}_6\partial_t, \tag{5.25}$$

$$\Lambda_{22} = (-\varepsilon\mathcal{B}_3+\varepsilon\mathcal{B}_5-\varepsilon^2\mathcal{B}_4)I-T_--\varepsilon(\mathcal{B}_1-\mathcal{B}_7+\varepsilon\mathcal{B}_2)\partial_t^2-\varepsilon^2\mathcal{B}_6\partial_t. \tag{5.26}$$

We are now in position to state the existence theorem. In the case $\alpha_+\neq\alpha_-$, we have

Theorem 5.3. *For any integer $k\ge 1$, if $\alpha_+\neq\alpha_-$ and $g\in H^{k-5/2}(\Gamma)$ then Problem* (5.1) *admits a unique solution $(u_-^{\varepsilon,ap},u_+^{\varepsilon,ap})\in H^k(\Omega_-)\times H^k_{\mathrm{loc}}(\overline{\Omega}_+)$.*

Proof. We set

$$\begin{aligned}\Lambda &= M_1^\varepsilon\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}+M_2^\varepsilon\begin{pmatrix} T_+ & 0\\ 0 & T_-\end{pmatrix}+M_3^\varepsilon\begin{pmatrix} -\partial_t^2 & 0\\ 0 & -\partial_t^2\end{pmatrix}+M_4^\varepsilon\begin{pmatrix} \partial_t & 0\\ 0 & \partial_t\end{pmatrix}\\ &= (M_1^\varepsilon-M_3^\varepsilon)\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}+M_2^\varepsilon\begin{pmatrix} T_+ & 0\\ 0 & T_-\end{pmatrix}+M_4^\varepsilon\begin{pmatrix} \partial_t & 0\\ 0 & \partial_t\end{pmatrix}+M_3^\varepsilon\begin{pmatrix} I-\partial_t^2 & 0\\ 0 & I-\partial_t^2\end{pmatrix},\end{aligned}$$

where

$$M_1^\varepsilon=\begin{pmatrix} 1-\varepsilon\mathcal{A}_3-\varepsilon^2\mathcal{A}_4 & -1+\varepsilon\mathcal{A}_3-\varepsilon^2\mathcal{A}_4\\ -\varepsilon\mathcal{B}_3-\varepsilon\mathcal{B}_5-\varepsilon^2\mathcal{B}_4 & -\varepsilon\mathcal{B}_3+\varepsilon\mathcal{B}_5-\varepsilon^2\mathcal{B}_4\end{pmatrix},\qquad M_2^\varepsilon=\begin{pmatrix} \varepsilon\mathcal{A}_1 & -\varepsilon\mathcal{A}_1\\ -1 & -1\end{pmatrix},$$

$$M_3^\varepsilon=\begin{pmatrix} \varepsilon^2\mathcal{A}_2 & \varepsilon^2\mathcal{A}_2\\ \varepsilon(\mathcal{B}_1+\mathcal{B}_7+\varepsilon\mathcal{B}_2) & \varepsilon(\mathcal{B}_1-\mathcal{B}_7+\varepsilon\mathcal{B}_2)\end{pmatrix},\qquad M_4^\varepsilon=\begin{pmatrix} 0 & 0\\ -\varepsilon^2\mathcal{B}_6 & -\varepsilon^2\mathcal{B}_6\end{pmatrix}.$$

The operator $I-\partial_t^2$ is an elliptic self-adjoint semibounded from bellow pseudodifferential operator of order 2 (see [4]), it is Fredholm with zero index and maps $H^s(\Gamma)$ to $H^{s-2}(\Gamma)$, for any $s\in\mathbb{R}$. As $\det(M_3^\varepsilon)\neq 0$, then the operator $M_3^\varepsilon\left(\begin{smallmatrix}\partial_t^2-I & 0\\ 0 & \partial_t^2-I\end{smallmatrix}\right):H^s(\Gamma)\times H^s(\Gamma)\to H^{s-2}(\Gamma)\times H^{s-2}(\Gamma)$ is Fredholm with zero index. Since ∂_t, T_+, and T_- are pseudodifferential operators of order 1, they map $H^s(\Gamma)$ to $H^{s-1}(\Gamma)$, and since the injection $H^{s-1}(\Gamma)\hookrightarrow H^{s-2}(\Gamma)$ is compact, $(M_1^\varepsilon-M_3^\varepsilon)\left(\begin{smallmatrix} I & 0\\ 0 & I\end{smallmatrix}\right)+M_2^\varepsilon\left(\begin{smallmatrix} T_+ & 0\\ 0 & T_-\end{smallmatrix}\right)+M_4^\varepsilon\left(\begin{smallmatrix} \partial_t & 0\\ 0 & \partial_t\end{smallmatrix}\right):H^s(\Gamma)\times H^s(\Gamma)\to H^{s-2}(\Gamma)\times H^{s-2}(\Gamma)$ is a compact operator. It follows that Λ is Fredholm with zero index, then the equivalence of System (5.22) to Problem (5.1) and Theorem 5.1 show that the uniqueness of $(u_-^{\varepsilon,ap},u_+^{\varepsilon,ap})$ implies that for any integer $k\ge 1$, if $(\varepsilon\mathcal{A}_1g,-g)\in H^{k-5/2}(\Gamma)\times H^{k-5/2}(\Gamma)$, there exists a unique solution $(\omega,\varkappa)\in H^{k-1/2}(\Gamma)\times H^{k-1/2}(\Gamma)$ of (5.22) which leads to the existence of a unique solution $(u_-^{\varepsilon,ap},u_+^{\varepsilon,ap})\in H^k(\Omega_-)\times H^k_{\mathrm{loc}}(\overline{\Omega}_+)$, as we wished. □

Remark 5.4. Note that Theorem 5.3 remains valid when $\alpha_+=\alpha_-$ and $d_1\neq 1/2$.

In the case $\alpha_+ = \alpha_-$ and $d_1 = d_2 = 1/2$, the matrix of pseudodifferential operators Λ defined by formulas (5.23)–(5.26) becomes

$$\begin{aligned}
\Lambda_{11} &= (1 - \varepsilon^2\mathcal{A}_4)I + \varepsilon\mathcal{A}_1 T_+,\\
\Lambda_{12} &= (-1 - \varepsilon^2\mathcal{A}_4)I - \varepsilon\mathcal{A}_1 T_-,\\
\Lambda_{21} &= (-\varepsilon\mathcal{B}_3 - \varepsilon\mathcal{B}_5 - \varepsilon^2\mathcal{B}_4)I - T_+ - \varepsilon\mathcal{B}_1\partial_t^2,\\
\Lambda_{22} &= (-\varepsilon\mathcal{B}_3 + \varepsilon\mathcal{B}_5 - \varepsilon^2\mathcal{B}_4)I - T_- - \varepsilon\mathcal{B}_1\partial_t^2.
\end{aligned}$$

Then, we have the following theorem.

Theorem 5.5. *For any integer $k \geq 1$, if $\alpha_+ = \alpha_-$, $g \in H^{k-3/2}(\Gamma)$ and*

$$\frac{\varepsilon^2\mathcal{A}_4 - 1}{\varepsilon\mathcal{A}_1} \notin \sigma(T_+), \tag{5.27}$$

then Problem (5.1) *admits a unique solution* $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap}) \in H^k(\Omega_-) \times H^k_{\mathrm{loc}}(\overline{\Omega}_+)$.

Proof. Let k be an integer in $\mathbb{N}^*$. In view of (5.27), Λ_{11}^{-1} is a well-defined pseudodifferential operator of order -1. Thus, from the first equation of System (5.22) we get

$$\omega = -\Lambda_{11}^{-1}\Lambda_{12}\varkappa + \varepsilon\mathcal{A}_1\Lambda_{11}^{-1}g,$$

then (5.22) is reduced to the equation

$$K\varkappa := (K_1 + K_2 + K_3)\varkappa = \theta,$$

where

$$\begin{aligned}
K_1 &= (-\varepsilon\mathcal{B}_3 + \varepsilon\mathcal{B}_5 - \varepsilon^2\mathcal{B}_4)I + (\varepsilon\mathcal{B}_3 + \varepsilon\mathcal{B}_5 + \varepsilon^2\mathcal{B}_4)\Lambda_{11}^{-1}\Lambda_{12},\\
K_2 &= -T_- + T_+\Lambda_{11}^{-1}\Lambda_{12},\\
K_3 &= -\varepsilon\mathcal{B}_1\partial_t^2\Lambda_{11}^{-1}(\Lambda_{11} - \Lambda_{12}),\\
\theta &= -g - \varepsilon\mathcal{A}_1\Lambda_{21}\Lambda_{11}^{-1}g.
\end{aligned}$$

Using the same arguments as in Theorem 5.3 based on Fredholm alternative, we shall prove that K is Fredholm with 0 index. Since K_1 and K_2 are pseudodifferential operators respectively of order 0 and 1, they map respectively $H^{k-1/2}(\Gamma)$ to $H^{k-1/2}(\Gamma)$ and $H^{k-1/2}(\Gamma)$ to $H^{k-3/2}(\Gamma)$. K_3 is a pseudodifferential operator of order 2, it maps $H^{k-1/2}(\Gamma)$ to $H^{k-5/2}(\Gamma)$. The injections $H^{k-1/2}(\Gamma) \hookrightarrow H^{k-5/2}(\Gamma)$ and $H^{k-3/2}(\Gamma) \hookrightarrow H^{k-5/2}(\Gamma)$ being compact and $\mathcal{B}_1 \neq 0$, the operator K defined from $H^{k-1/2}(\Gamma)$ to $H^{k-5/2}(\Gamma)$ is a compact perturbation of K_3. Thus, since ∂_t^2 is Fredholm with index 0, it follows that to show that K_3 is Fredhom with index 0, it remains to prove that the operator $\Lambda_{11} - \Lambda_{12} = T_+ + T_- + \frac{2}{\varepsilon\mathcal{A}_1}I$ defined from $H^{k-1/2}(\Gamma)$ to $H^{k-3/2}(\Gamma)$ is invertible. Let us consider the equation

$$\left(T_+ + T_- + \frac{2}{\varepsilon\mathcal{A}_1}I\right)\varphi = \psi, \quad \psi \in H^{k-3/2}(\Gamma),\ k \geq 1. \tag{5.28}$$

Using the definition of the operators T_+ and T_-, (5.28) is equivalent to the boundary-value problem

$$\begin{cases} \Delta u_+ + k_+^2 u_+ = 0 & \text{in } \Omega_+, \\ \Delta u_- + k_-^2 u_- = 0 & \text{in } \Omega_-, \\ u_- = 0 & \text{on } \Gamma_i, \\ u_- = u_+ & \text{on } \Gamma, \\ \alpha_- \partial_{\boldsymbol{n}} u_- - \alpha_+ \partial_{\boldsymbol{n}} u_+ + \frac{2}{\varepsilon \mathcal{A}_1} u_+ = \psi & \text{on } \Gamma, \\ \lim_{|x| \to +\infty} \sqrt{|x|}(\partial_{|x|} - ik_+) u_+ = 0, \end{cases} \tag{5.29}$$

where $\varphi = u_-|_\Gamma = u_+|_\Gamma$. Standard arguments based on Rellich's lemma and the Fredholm alternative show that, for all k in $\mathbb{N}^*$, if $\psi \in H^{k-3/2}(\Gamma)$, then Problem (5.29) admits a unique solution (u_-, u_+) in $H^k(\Omega_-) \times H^k_{\text{loc}}(\overline{\Omega}_+)$, and hence there exists a unique Dirichlet trace $\varphi \in H^{k-1/2}(\Gamma)$. As a consequence, the operator $T_+ + T_- + \frac{2}{\varepsilon \mathcal{A}_1} I$, defined from $H^{k-1/2}(\Gamma)$ to $H^{k-3/2}(\Gamma)$, is invertible. The end of the proof is the same as that of Theorem 5.3. □

Remark 5.6. As an alternative to Condition (5.27), we may replace the latter with

$$\frac{-1 - \varepsilon^2 \mathcal{A}_4}{\varepsilon \mathcal{A}_1} \notin \sigma(T_-). \tag{5.30}$$

We can easily verify that the proof of Theorem 5.5 with Condition (5.30) is almost the same.

6. Error estimation

We are now able to establish an approximate solution $u^{\varepsilon,ap}$ of the exact solution u^ε and an error estimate of the convergence of $u^{\varepsilon,ap}$ to u^ε. Intuitively, we define the approximate solution $u^{\varepsilon,ap}$, using $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$ and (5.13), by

$$u^{\varepsilon,ap} = \begin{cases} u_+^{\varepsilon,ap} & \text{in } \Omega_+^\varepsilon, \\ u_m^{\varepsilon,ap} & \text{in } \Omega_m^\varepsilon, \\ u_-^{\varepsilon,ap} & \text{in } \Omega_-^\varepsilon. \end{cases}$$

Then we have an optimal error estimate given by the following theorem.

Theorem 6.1. *There exists a constant c independent of ε such that*

$$\left\| u_-^\varepsilon - u_-^{\varepsilon,ap} \right\|_{H^1(\Omega_-^\varepsilon)} + \varepsilon^{1/2} \left\| u_m^\varepsilon - u_m^{\varepsilon,ap} \right\|_{H^1(\Omega_m^\varepsilon)} + \left\| u_+^\varepsilon - u_+^{\varepsilon,ap} \right\|_{H^1(\widetilde{\Omega}_+^\varepsilon)} \leq c\varepsilon^3.$$

Before starting the proof of Theorem 6.1, we need a stability result. Let $\mathbb{H}^1(\Omega)$ be the Hilbert space defined by

$$\mathbb{H}^1(\Omega) := \big\{v = (v_-, v_+) \in H^1(\Omega_-) \times H^1(\widetilde{\Omega}_+) \mid v_-|_{\Gamma_i} = 0\big\},$$

where $\widetilde{\Omega}_+ = \Omega_+ \cap \Omega$ and $b_\varepsilon(\,\cdot\,,\cdot\,)$ be the bilinear form on $\mathbb{H}^1(\Omega)$ defined by

$$\begin{aligned} b_\varepsilon(u, v) := {} & \alpha_- \int_{\Omega_-} \nabla u_- \cdot \nabla v_- \, d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} u_- v_- \, d\Omega_- \\ & + \alpha_+ \int_{\widetilde{\Omega}_+} \nabla u_+ \cdot \nabla v_+ \, d\widetilde{\Omega}_+ - \alpha_+ k_+^2 \int_{\widetilde{\Omega}_+} u_+ v_+ \, d\widetilde{\Omega}_+ \\ & - \frac{1}{2}\lambda_\varepsilon \int_\Gamma (u_- - u_+)(v_- - v_+) \, d\Gamma \\ & + \alpha_+ \langle T u_+, v_+ \rangle_{H^{-1/2}(\Gamma_\infty)\times H^{1/2}(\Gamma_\infty)}, \end{aligned}$$

where $\lambda_\varepsilon = O(\varepsilon^{-1})$ and T is the Steklov–Poincaré operator defined from $H^{1/2}(\Gamma_\infty)$ onto $H^{-1/2}(\Gamma_\infty)$ by $T\varphi := -\partial_{\boldsymbol{n}_{\widetilde{\Omega}_\infty}} w|_{\Gamma_\infty}$ in which $\widetilde{\Omega}_\infty$ is an exterior domain of $\mathbb{R}^2$ with boundary Γ_∞, $\boldsymbol{n}_{\widetilde{\Omega}_\infty}$ indicates the unit normal to Γ_∞ outwardly directed to Ω and w is the solution to the boundary-value problem

$$\begin{cases} \Delta w + k_+^2 w = 0 & \text{in } \widetilde{\Omega}_\infty, \\ w = \varphi & \text{on } \Gamma_\infty, \\ \lim_{|x|\to+\infty} |x|^{1/2}(\partial_{|x|} - ik_+)w = 0. \end{cases}$$

We have the following lemma, which proof can be founded in [6].

Lemma 6.2. *For all $h_\varepsilon \in (\mathbb{H}^1(\Omega))'$, there exists a positive constant c independent of ε such that the solution to the variational problem*

$$\textit{Find } u \in \mathbb{H}^1(\Omega),\ \forall\, v \in \mathbb{H}^1(\Omega), \textit{ such that } b_\varepsilon(u, v) = h_\varepsilon(v),$$

satisfies

$$\|u\|_{\mathbb{H}^1(\Omega)} \le c\varepsilon^{-1/2} \|h_\varepsilon\|_{(\mathbb{H}^1(\Omega))'}.$$

Proof of Theorem 6.1. According to

$$\big\|U_m^{\varepsilon,ap} - U_m^{\varepsilon,(2)}\big\|_{H^1(\Omega_m)} \le c\big(\big\|u_-^{\varepsilon,ap} - u_-^{\varepsilon,(2)}\big\|_{H^1(\Omega_-)} + \varepsilon^3 \|u_-^1\|_{H^1(\Omega_-)} + \varepsilon^3 \|u_-^2\|_{H^1(\Omega_-)}\big),$$

where c is a constant independent of ε, and convergence theorem, it suffices to estimate $\|u_-^{\varepsilon,ap} - u_-^{\varepsilon,(2)}\|_{H^1(\Omega_-)}$ and $\|u_+^{\varepsilon,ap} - u_+^{\varepsilon,(2)}\|_{H^1(\Omega_+)}$. As in [8], we derive asymptotic expansions for $u_-^{\varepsilon,ap}$ and $u_+^{\varepsilon,ap}$ through the ansatz

$$u_-^{\varepsilon,ap} = \sum_{n\ge0} \varepsilon^n w_-^n \quad \text{and} \quad u_+^{\varepsilon,ap} = \sum_{n\ge0} \varepsilon^n w_+^n, \tag{6.1}$$

where the terms w_-^n and w_+^n are independent of ε. Inserting (6.1) in Problem (5.1) and identifying the same powers of ε, we get the following hierarchy of boundary-value problems

$$\begin{cases} \Delta w_+^n + k_+^2 w_+^n = 0 & \text{in } \Omega_+, \\ \Delta w_-^n + k_-^2 w_-^n = 0 & \text{in } \Omega_-, \\ w_-^n = 0 & \text{on } \Gamma_i, \\ \lim_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)(w_+^n - \delta_{0,n} u_{\text{inc}}) = 0, \end{cases}$$

with transmission conditions on Γ:

$$\begin{aligned} w_+^n - w_-^n &= \mathcal{A}_1(\alpha_+ \partial_{\boldsymbol{n}} w_+^{n-1} + \alpha_- \partial_{\boldsymbol{n}} w_-^{n-1}) + \mathcal{A}_2(\partial_t^2 w_+^{n-2} + \partial_t^2 w_-^{n-2}) \\ &\quad + \mathcal{A}_3(w_+^{n-1} - w_-^{n-1}) + \mathcal{A}_4(w_+^{n-2} + w_-^{n-2}), \end{aligned}$$

$$\begin{aligned} &\alpha_+ \partial_{\boldsymbol{n}} w_+^n - \alpha_- \partial_{\boldsymbol{n}} w_-^n \\ &= \mathcal{B}_1(\partial_t^2 w_+^{n-1} + \partial_t^2 w_-^{n-1}) + \mathcal{B}_2(\partial_t^2 w_+^{n-2} + \partial_t^2 w_-^{n-2}) + \mathcal{B}_3(w_+^{n-1} + w_-^{n-1}) \\ &\quad + \mathcal{B}_4(w_+^{n-2} + w_-^{n-2}) + \mathcal{B}_5(w_+^{n-1} - w_-^{n-1}) + \mathcal{B}_6(\partial_t w_+^{n-2} + \partial_t w_-^{n-2}) \\ &\quad + \mathcal{B}_7(\partial_t^2 w_+^{n-1} - \partial_t^2 w_-^{n-1}) \end{aligned}$$

in which $(\mathcal{A}_i)_{1\le i\le 4}$ and $(\mathcal{B}_i)_{1\le i\le 7}$ are defined by formulas (5.2)–(5.12) with the convention that $w_-^{-1} = w_-^{-2} = w_+^{-1} = w_+^{-2} = 0$. A simple calculation shows that the terms (w_-^n, w_+^n) coincide with (u_-^n, u_+^n), for $n \in \{0, 1, 2\}$. Furthermore, each term in (6.1) is bounded in $\mathbb{H}^1(\Omega)$ (see [8, Theorem 4.1]).

Let R_w be the remainder obtained by truncating the asymptotic expansions (6.1) at order 4:

$$\begin{aligned} R_w|_{\Omega_-} &= R_{w_-} = u_-^{\varepsilon,ap} - w_-^0 - \varepsilon w_-^1 - \varepsilon^2 w_-^2 - \varepsilon^3 w_-^3 - \varepsilon^4 w_-^4, \\ R_w|_{\Omega_+} &= R_{w_+} = u_+^{\varepsilon,ap} - w_+^0 - \varepsilon w_+^1 - \varepsilon^2 w_+^2 - \varepsilon^3 w_+^3 - \varepsilon^4 w_+^4. \end{aligned}$$

Hence R_w is the solution of the following problem

$$\begin{cases} \Delta R_{w_+} + k_+^2 R_{w_+} = 0 & \text{in } \Omega_+, \\ \Delta R_{w_-} + k_-^2 R_{w_-} = 0 & \text{in } \Omega_-, \\ R_{w_-} = 0 & \text{on } \Gamma_i, \\ \lim_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+) R_{w_+} = 0 \end{cases}$$

with transmission conditions on Γ:

$$\begin{aligned} R_{w_+} - R_{w_-} &= \varepsilon \mathcal{A}_1(\alpha_+ \partial_{\boldsymbol{n}} R_{w_+} + \alpha_- \partial_{\boldsymbol{n}} R_{w_-}) + \varepsilon^2 \mathcal{A}_2(\partial_t^2 R_{w_+} + \partial_t^2 R_{w_-}) \\ &\quad + \varepsilon \mathcal{A}_3(R_{w_+} - R_{w_-}) + \varepsilon^2 \mathcal{A}_4(R_{w_+} + R_{w_-}) + \varepsilon^5 \mathcal{A}_1(\alpha_+ \partial_{\boldsymbol{n}} w_+^4 + \alpha_- \partial_{\boldsymbol{n}} w_-^4) \\ &\quad + \varepsilon^5 \mathcal{A}_2(\partial_t^2 w_+^3 + \partial_t^2 w_-^3) + \varepsilon^6 \mathcal{A}_2(\partial_t^2 w_+^4 + \partial_t^2 w_-^4) + \varepsilon^5 \mathcal{A}_3(w_+^4 - w_-^4) \\ &\quad + \varepsilon^5 \mathcal{A}_4(w_+^3 + w_-^3) + \varepsilon^6 \mathcal{A}_4(w_+^4 + w_-^4), \end{aligned}$$

$$
\begin{aligned}
&\alpha_+\partial_{\boldsymbol{n}} R_{w_+} - \alpha_-\partial_{\boldsymbol{n}} R_{w_-} \\
&= (\varepsilon\mathcal{B}_1 + \varepsilon^2\mathcal{B}_2)(\partial_t^2 R_{w_+} + \partial_t^2 R_{w_-}) + (\varepsilon\mathcal{B}_3 + \varepsilon^2\mathcal{B}_4)(R_{w_+} + R_{w_-}) \\
&\quad + \varepsilon\mathcal{B}_5(R_{w_+} - R_{w_-}) + \varepsilon^2\mathcal{B}_6(\partial_t R_{w_+} + \partial_t R_{w_-}) + \varepsilon\mathcal{B}_7(\partial_t^2 R_{w_+} - \partial_t^2 R_{w_-}) \\
&\quad + \mathcal{B}_1\varepsilon^5(\partial_t^2 w_+^4 + \partial_t^2 w_-^4) + \mathcal{B}_2\varepsilon^5(\partial_t^2 w_+^3 + \partial_t^2 w_-^3) + \mathcal{B}_2\varepsilon^6(\partial_t^2 w_+^4 + \partial_t^2 w_-^4) \\
&\quad + \mathcal{B}_3\varepsilon^5(w_+^4 + w_-^4) + \mathcal{B}_4\varepsilon^5(w_+^3 + w_-^3) + \mathcal{B}_4\varepsilon^6(w_+^4 + w_-^4) \\
&\quad + \mathcal{B}_5\varepsilon^5(w_+^4 - w_-^4) + \mathcal{B}_6\varepsilon^5(\partial_t w_+^3 + \partial_t w_-^3) + \mathcal{B}_6\varepsilon^6(\partial_t w_+^4 + \partial_t w_-^4) \\
&\quad + \mathcal{B}_7\varepsilon^5(\partial_t^2 w_+^4 - \partial_t^2 w_-^4).
\end{aligned}
$$

So for all $v = (v_-, v_+) \in \mathbb{H}^1(\Omega)$, we get

$$
\begin{aligned}
&\alpha_- \int_{\Omega_-} \nabla R_{w_-}\cdot\nabla v_-\, d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} R_{w_-} v_-\, d\Omega_- + \alpha_+ \int_{\widetilde{\Omega}_+} \nabla R_{w_+}\cdot\nabla v_+\, d\widetilde{\Omega}_+ \\
&- \alpha_+ k_+^2 \int_{\widetilde{\Omega}_+} R_{w_+} v_+\, d\widetilde{\Omega}_+ - \int_\Gamma \frac{-1+\mathcal{A}_3\varepsilon}{2\mathcal{A}_1\varepsilon}(R_{w_+} - R_{w_-})(v_+ - v_-)\, d\Gamma \\
&+ \alpha_+ \langle T R_{w_+}, v_+\rangle_{H^{-1/2}(\Gamma_\infty)\times H^{1/2}(\Gamma_\infty)} \\
&= h_\varepsilon(v),
\end{aligned}
$$

where

$$
\begin{aligned}
&h_\varepsilon(v) \\
&= -\frac{\varepsilon}{2}\int_\Gamma (\mathcal{B}_1 + \varepsilon\mathcal{B}_2)(\partial_t^2 R_{w_+} + \partial_t^2 R_{w_-})(v_+ + v_-)\, d\Gamma - \frac{\mathcal{B}_3}{2}\varepsilon \int_\Gamma (R_{w_+} + R_{w_-})(v_+ + v_-)\, d\Gamma \\
&\quad - \frac{\mathcal{B}_7}{2}\varepsilon \int_\Gamma (\partial_t^2 R_{w_+} - \partial_t^2 R_{w_-})(v_+ + v_-)\, d\Gamma - \frac{\mathcal{B}_5}{2}\varepsilon \int_\Gamma (R_{w_+} - R_{w_-})(v_+ + v_-)\, d\Gamma \\
&\quad - \frac{\varepsilon^2}{2}\int_\Gamma \mathcal{B}_6(\partial_t R_{w_+} + \partial_t R_{w_-})(v_+ + v_-)\, d\Gamma + \frac{\mathcal{A}_2}{2\mathcal{A}_1}\varepsilon \int_\Gamma (\partial_t^2 R_{w_+} + \partial_t^2 R_{w_-})(v_+ - v_-)\, d\Gamma \\
&\quad + \frac{\mathcal{A}_4}{2\mathcal{A}_1}\varepsilon \int_\Gamma (R_{w_+} + R_{w_-})(v_+ - v_-)\, d\Gamma - \frac{\varepsilon^2}{2}\int_\Gamma \mathcal{B}_4(R_{w_+} + R_{w_-})(v_+ + v_-)\, d\Gamma \\
&\quad + \frac{1}{2}\varepsilon^4 \int_\Gamma (\alpha_+\partial_{\boldsymbol{n}} w_+^4 + \alpha_-\partial_{\boldsymbol{n}} w_-^4)(v_+ - v_-)\, d\Gamma + \frac{\mathcal{A}_2}{2\mathcal{A}_1}\varepsilon^4 \int_\Gamma (\partial_t^2 w_+^3 + \partial_t^2 w_-^3)(v_+ - v_-)\, d\Gamma \\
&\quad + \varepsilon^4 \int_\Gamma \frac{\mathcal{A}_3}{2\mathcal{A}_1}(w_+^4 - w_-^4)(v_+ - v_-)\, d\Gamma + \frac{\mathcal{A}_4}{2\mathcal{A}_1}\varepsilon^4 \int_\Gamma (w_+^3 + w_-^3)(v_+ - v_-)\, d\Gamma \\
&\quad - \frac{\varepsilon^5}{2}\int_\Gamma \mathcal{B}_4(w_+^3 + w_-^3)(v_+ + v_-)\, d\Gamma - \frac{\varepsilon^5}{2}\int_\Gamma \mathcal{B}_2(\partial_t^2 w_+^3 + \partial_t^2 w_-^3)(v_+ + v_-)\, d\Gamma \\
&\quad - \frac{\varepsilon^5}{2}\int_\Gamma \mathcal{B}_6(\partial_t w_+^3 + \partial_t w_-^3)(v_+ + v_-)\, d\Gamma - \frac{\mathcal{B}_1}{2}\varepsilon^5 \int_\Gamma (\partial_t^2 w_+^4 + \partial_t^2 w_-^4)(v_+ + v_-)\, d\Gamma \\
&\quad - \frac{\mathcal{B}_3}{2}\varepsilon^5 \int_\Gamma (w_+^4 + w_-^4)(v_+ + v_-)\, d\Gamma - \frac{\mathcal{B}_7}{2}\varepsilon^5 \int_\Gamma (\partial_t^2 w_+^4 - \partial_t^2 w_-^4)(v_+ + v_-)\, d\Gamma \\
&\quad - \frac{\mathcal{B}_5}{2}\varepsilon^5 \int_\Gamma (w_+^4 - w_-^4)(v_+ + v_-)\, d\Gamma + \frac{\mathcal{A}_2}{2\mathcal{A}_1}\varepsilon^5 \int_\Gamma (\partial_t^2 w_+^4 + \partial_t^2 w_-^4)(v_+ - v_-)\, d\Gamma
\end{aligned}
$$

$$+\frac{\mathcal{A}_4}{2\mathcal{A}_1}\varepsilon^5\int_\Gamma (w_+^4+w_-^4)(v_+-v_-)\,d\Gamma-\frac{\varepsilon^6}{2}\int_\Gamma \mathcal{B}_2(\partial_t^2 w_+^4+\partial_t^2 w_-^4)(v_++v_-)\,d\Gamma$$
$$-\frac{\varepsilon^6}{2}\int_\Gamma \mathcal{B}_4(w_+^4+w_-^4)(v_++v_-)\,d\Gamma-\frac{\varepsilon^6}{2}\int_\Gamma \mathcal{B}_6(\partial_t w_+^4+\partial_t w_-^4)(v_++v_-)\,d\Gamma.$$

By using Lemma 6.2, we obtain

$$\|R_w\|_{\mathbb{H}^1(\Omega)}\le c\varepsilon^{-1/2}\big[\varepsilon\|R_w\|_{\mathbb{H}^1(\Omega)}+\varepsilon^4\|w_3\|_{\mathbb{H}^1(\Omega)}+\varepsilon^4\|w_4\|_{\mathbb{H}^1(\Omega)}\big],$$

where c is a positive constant independent of ε. Therefore

$$\|R_w\|_{\mathbb{H}^1(\Omega)}\le \frac{c\varepsilon^{7/2}}{1-c\varepsilon^{1/2}}\big(\|w_3\|_{\mathbb{H}^1(\Omega)}+\|w_4\|_{\mathbb{H}^1(\Omega)}\big).$$

Since ε is very small, it follows that

$$\|R_w\|_{\mathbb{H}^1(\Omega)}\le c\varepsilon^3\big(\|w_3\|_{\mathbb{H}^1(\Omega)}+\|w_4\|_{\mathbb{H}^1(\Omega)}\big),$$

as we wished. □

7. Conclusion and perspective

In this work, we have derived an asymptotic expansion of the solution u^ε to Problem (2.2), with respect to the thickness ε, of the thin layer Ω_m^ε, up to any order using parameters d_1 and d_2. We have provided Ventcel-type transmission conditions on the interface Γ, modelling the effect of the thin layer, with accuracy up to $O(\varepsilon^3)$. In our analysis, we have shown that there exists an infinite number of the values of d_1 and d_2, i.e., of the position of the limit interface Γ, ensuring the existence and the uniqueness of the approximation solution. Finally, we have given a theorem on error estimation.

A natural question is whether such approximation results can be improved in order to have an error estimate of order greater than 3 and whether the above study can be extended to the cases $\alpha_- < 1 < \alpha_+$ or $\alpha_+ < 1 < \alpha_-$ and the case where the constants α_+ and α_- depend on ε. Another interesting forthcoming work is to consider Maxwell, elasticity or Eddy currents problems.

A. Calculation of the first three terms

A.1. Term of order 0

Equation (4.9) and the conditions (4.7) and (4.8) give

$$\partial_s U_m^0 = 0.$$

Using (4.5) and (4.6), we get

$$U_m^0(t,s) = u_-^0|_\Gamma = u_+^0|_\Gamma, \quad \forall\,(t,s) \in \Omega_m. \tag{A.1}$$

Equation (4.10) implies

$$\partial_s^2 U_m^1 = A_1 U_m^0 = -c(t)\partial_s U_m^0 = 0.$$

Using (4.7) and (4.8), we obtain

$$\partial_s U_m^1(t,s) = \alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma = \alpha_+\partial_{\boldsymbol{n}} u_+^0|_\Gamma, \quad \forall\,(t,s) \in \Omega_m. \tag{A.2}$$

Therefore, with (4.3), (A.1) and (A.2), (u_-^0, u_+^0) is a solution of the following problem

$$\begin{cases} \Delta u_+^0 + k_+^2 u_+^0 = 0 & \text{in } \Omega_+, \\ \Delta u_-^0 + k_-^2 u_-^0 = 0 & \text{in } \Omega_-, \\ u_+^0 = u_-^0 & \text{on } \Gamma, \\ \alpha_+\partial_{\boldsymbol{n}} u_+^0 = \alpha_-\partial_{\boldsymbol{n}} u_-^0 & \text{on } \Gamma, \\ u_-^0 = 0 & \text{on } \Gamma_i, \\ \lim_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)(u_+^0 - u_{\text{inc}}) = 0. \end{cases}$$

Note that the term (u_-^0, u_+^0) is nothing but the solution to the problem without the thin layer.

A.2. Term of order 1

The relation (A.2) with the condition (4.5) yield

$$U_m^1(t,s) = (\alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma)s + u_-^1|_\Gamma + d_1(\alpha_- - 1)\partial_{\boldsymbol{n}} u_-^0|_\Gamma, \quad \forall\,(t,s) \in \Omega_m.$$

So (4.6) gives

$$u_+^1|_\Gamma - u_-^1|_\Gamma = \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{2\alpha_+\alpha_-}(\alpha_+\partial_{\boldsymbol{n}} u_+^0|_\Gamma + \alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma). \tag{A.3}$$

From (4.11), we have

$$\partial_s^2 U_m^2 = A_1 U_m^1 + A_2 U_m^0 - k_m^2 U_m^0 = -c(t)\alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma - k_m^2 u_-^0|_\Gamma.$$

Using (4.7), we obtain

$$\partial_s U_m^2 = \left[-c(t)\alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma - k_m^2 u_-^0|_\Gamma\right](s+d_1) + \alpha_-\partial_{\boldsymbol{n}} u_-^1|_\Gamma - d_1\alpha_-\partial_{\boldsymbol{n}}^2 u_-^0|_\Gamma. \tag{A.4}$$

Now, from the condition (4.8) at order 1, we get

$$\begin{aligned} &\alpha_+\partial_{\boldsymbol{n}} u_+^1|_\Gamma - \alpha_-\partial_{\boldsymbol{n}} u_-^1|_\Gamma \\ &= \big[-c(t)\alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma - k_m^2 u_-^0|_\Gamma\big](d_2+d_1) - d_2\alpha_+\partial_{\boldsymbol{n}}^2 u_+^0|_\Gamma - d_1\alpha_-\partial_{\boldsymbol{n}}^2 u_-^0|_\Gamma \\ &= -c(t)\alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma - k_m^2 u_-^0|_\Gamma - d_2\alpha_+\partial_{\boldsymbol{n}}^2 u_+^0|_\Gamma - d_1\alpha_-\partial_{\boldsymbol{n}}^2 u_-^0|_\Gamma. \end{aligned} \tag{A.5}$$

As

$$\Delta = \frac{1}{1+\eta c(t)}\partial_t\left(\frac{1}{1+\eta c(t)}\partial_t\right) + \frac{c(t)}{1+\eta c(t)}\partial_\eta + \partial_\eta^2 \quad \text{and} \quad \Delta u_+^n + k_+^2 u_+^n = 0,$$

it follows

$$\frac{-\eta c'(t)}{[1+\eta c(t)]^3}\partial_t u_+^n + \frac{1}{[1+\eta c(t)]^2}\partial_t^2 u_+^n + \frac{c(t)}{1+\eta c(t)}\partial_\eta u_+^n + \partial_\eta^2 u_+^n + k_+^2 u_+^n = 0.$$

Taking the limit $\eta \to 0$, we obtain

$$\partial_t^2 u_+^n|_{\eta=0} + c(t)\partial_\eta u_+^n|_{\eta=0} + \partial_\eta^2 u_+^n|_{\eta=0} + k_+^2 u_+^n|_{\eta=0} = 0,$$

i.e.,

$$\partial_{\boldsymbol{n}}^2 u_+^n|_\Gamma = -\partial_t^2 u_+^n|_\Gamma - c(t)\partial_{\boldsymbol{n}} u_+^n|_\Gamma - k_+^2 u_+^n|_\Gamma, \quad \forall\, n \geq 0. \tag{A.6}$$

Similarly for u_-^n, we get

$$\partial_{\boldsymbol{n}}^2 u_-^n|_\Gamma = -\partial_t^2 u_-^n|_\Gamma - c(t)\partial_{\boldsymbol{n}} u_-^n|_\Gamma - k_-^2 u_-^n|_\Gamma, \quad \forall\, n \geq 0. \tag{A.7}$$

So (A.5) becomes

$$\begin{aligned} \alpha_+\partial_{\boldsymbol{n}} u_+^1|_\Gamma - \alpha_-\partial_{\boldsymbol{n}} u_-^1|_\Gamma = &\frac{d_1\alpha_- + d_2\alpha_+ - 1}{2}(\partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma) \\ &+ \frac{d_1\alpha_- k_-^2 + d_2\alpha_+ k_+^2 - k_m^2}{2}(u_+^0|_\Gamma + u_-^0|_\Gamma). \end{aligned} \tag{A.8}$$

Therefore, with (4.3), (A.3) and (A.8), (u_-^1, u_+^1) is a solution of the following problem

$$\begin{cases} \Delta u_+^1 + k_+^2 u_+^1 = 0 & \text{in } \Omega_+, \\ \Delta u_-^1 + k_-^2 u_-^1 = 0 & \text{in } \Omega_-, \\ u_-^1 = 0 & \text{on } \Gamma_i, \\ \lim_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)(u_+^1) = 0 & \end{cases}$$

with transmission conditions on Γ:

$$\begin{aligned} u_+^1 - u_-^1 &= \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{2\alpha_+\alpha_-}(\alpha_+\partial_{\boldsymbol{n}} u_+^0 + \alpha_-\partial_{\boldsymbol{n}} u_-^0), \\ \alpha_+\partial_{\boldsymbol{n}} u_+^1 - \alpha_-\partial_{\boldsymbol{n}} u_-^1 &= \frac{d_1\alpha_- + d_2\alpha_+ - 1}{2}(\partial_t^2 u_+^0 + \partial_t^2 u_-^0) \\ &\quad + \frac{d_1\alpha_- k_-^2 + d_2\alpha_+ k_+^2 - k_m^2}{2}(u_+^0 + u_-^0). \end{aligned}$$

A.3. Term of order 2

Using (A.4) and the condition (4.5), we obtain, $\forall\,(t,s)\in\Omega_m$,

$$U_m^2(t,s) = u_-^2|_\Gamma + (\alpha_- s + d_1\alpha_- - d_1)\partial_{\boldsymbol{n}} u_-^1|_\Gamma s - \left(\frac{s^2}{2} + d_1 s + \frac{d_1^2}{2}\right) c(t)\alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma$$
$$- \left(\frac{s^2}{2} + d_1 s + \frac{d_1^2}{2}\right)\partial_t^2 u_-^0 + \left(\frac{d_1^2}{2} - d_1^2\alpha_- - d_1\alpha_- s\right)\partial_{\boldsymbol{n}}^2 u_-^0|_\Gamma$$
$$- \left(\frac{d_1^2}{2} + \frac{s^2}{2} + d_1 s\right) k_m^2 u_-^0|_\Gamma.$$

As

$$U_m^2(t,d_2) = u_+^2|_\Gamma + d_2\partial_{\boldsymbol{n}} u_+^1|_\Gamma + \frac{d_2^2}{2}\partial_{\boldsymbol{n}}^2 u_+^0|_\Gamma,$$

we find

$$u_+^2|_\Gamma - u_-^2|_\Gamma = -\frac{1}{2}(c(t)\alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma + \partial_t^2 u_-^0 + k_m^2 u_-^0) - d_2\partial_{\boldsymbol{n}} u_+^1|_\Gamma$$
$$+ (\alpha_- - d_1)\partial_{\boldsymbol{n}} u_-^1|_\Gamma - \frac{d_2^2}{2}\partial_{\boldsymbol{n}}^2 u_+^0|_\Gamma + \left(\frac{d_1^2}{2} - d_1\alpha_-\right)\partial_{\boldsymbol{n}}^2 u_-^0|_\Gamma.$$

With (A.6) and (A.7), we get

$$\begin{aligned}
&u_+^2|_\Gamma - u_-^2|_\Gamma\\
&= -d_2\partial_{\boldsymbol{n}} u_+^1|_\Gamma + (\alpha_- - d_1)\partial_{\boldsymbol{n}} u_-^1|_\Gamma\\
&\quad + \frac{d_2^2}{2}(\partial_t^2 u_+^0|_\Gamma + c(t)\partial_{\boldsymbol{n}} u_+^0|_\Gamma + k_+^2 u_+^0|_\Gamma) - \frac{1}{2}(c(t)\alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma + \partial_t^2 u_-^0 + k_m^2 u_-^0)\\
&\quad - \left(\frac{d_1^2}{2} - d_1\alpha_-\right)(\partial_t^2 u_-^0|_\Gamma + c(t)\partial_{\boldsymbol{n}} u_-^0|_\Gamma + k_-^2 u_-^0|_\Gamma).
\end{aligned}$$

Using the transmission conditions (A.1), (A.2) and (A.8), we get

$$\begin{aligned}
&u_+^2|_\Gamma - u_-^2|_\Gamma\\
&= \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{2\alpha_+\alpha_-}(\alpha_+\partial_{\boldsymbol{n}} u_+^1|_\Gamma + \alpha_-\partial_{\boldsymbol{n}} u_-^1|_\Gamma)\\
&\quad + \frac{d_1\alpha_+\alpha_-^2 - d_2\alpha_+^2\alpha_- + d_1 d_2(\alpha_+^2 - \alpha_-^2) + d_2\alpha_- - d_1\alpha_+}{4\alpha_+\alpha_-}(\partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma)\\
&\quad + c(t)\frac{(d_1 - d_2)\alpha_-\alpha_+ + d_2^2\alpha_- - d_1^2\alpha_+}{4\alpha_+\alpha_-}(\alpha_+\partial_{\boldsymbol{n}} u_+^0|_\Gamma + \alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma)\\
&\quad + \frac{(d_1 k_-^2\alpha_- - d_2 k_+^2\alpha_+)\alpha_+\alpha_- + d_1 d_2(k_+^2\alpha_+^2 - k_-^2\alpha_-^2) + k_m^2(d_2\alpha_- - d_1\alpha_+)}{4\alpha_+\alpha_-}\\
&\qquad \times (u_+^0|_\Gamma + u_-^0|_\Gamma),
\end{aligned}$$

and with the help of (A.3), we obtain

(A.9)
$$
\begin{aligned}
&u_+^2|_\Gamma - u_-^2|_\Gamma \\
&= \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{2\alpha_+\alpha_-}(\alpha_+\partial_{\boldsymbol{n}} u_+^1|_\Gamma + \alpha_-\partial_{\boldsymbol{n}} u_-^1|_\Gamma) \\
&\quad + \frac{d_1\alpha_+\alpha_-^2 - d_2\alpha_+^2\alpha_- + d_1 d_2(\alpha_+^2 - \alpha_-^2) + d_2\alpha_- - d_1\alpha_+}{4\alpha_+\alpha_-}(\partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma) \\
&\quad + c(t)\frac{(d_1 - d_2)\alpha_-\alpha_+ + d_2^2\alpha_- - d_1^2\alpha_+}{2(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-)}(u_+^1|_\Gamma - u_-^1|_\Gamma) \\
&\quad + \frac{(d_1 k_-^2\alpha_- - d_2 k_+^2\alpha_+)\alpha_+\alpha_- + d_1 d_2(k_+^2\alpha_+^2 - k_-^2\alpha_-^2) + k_m^2(d_2\alpha_- - d_1\alpha_+)}{4\alpha_+\alpha_-} \\
&\quad\quad \times (u_+^0|_\Gamma + u_-^0|_\Gamma).
\end{aligned}
$$

Equation (4.12) implies that

(A.10)
$$
\begin{aligned}
&\partial_s U_m^3(t,s) \\
&= \left[2c^2(t)\Psi_1(t) + c(t)k_m^2 U_m^0 - \partial_t^2(\Psi_1(t)) + 3c(t)\partial_t^2 U_m^0 + c'(t)\partial_t U_m^0 - k_m^2\Psi_1(t)\right]\frac{s^2}{2} \\
&\quad - \left[c(t)\Psi_2(t) + \partial_t^2\Phi_1(t) + k_m^2\Phi_1(t)\right]s + \varphi(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_1(t) &= \alpha_-\partial_{\boldsymbol{n}} u_-^0|_\Gamma = \alpha_+\partial_{\boldsymbol{n}} u_+^0|_\Gamma, \\
\Psi_2(t) &= \alpha_+\partial_{\boldsymbol{n}} u_+^1|_\Gamma + (d_2 - d_2\alpha_+)\partial_t^2 u_+^0|_\Gamma + (d_2 k_m^2 - d_2 k_+^2\alpha_+)u_+^0|_\Gamma, \\
\Phi_1(t) &= u_-^1|_\Gamma + (\alpha_- - 1)d_1\partial_{\boldsymbol{n}} u_-^0|_\Gamma = u_+^1|_\Gamma + (1 - \alpha_+)d_2\partial_{\boldsymbol{n}} u_+^0|_\Gamma,
\end{aligned}
$$

and φ is a function defined on Γ independent of s. Using transmission condition (4.7) at order 2, we find

$$
\begin{aligned}
\varphi(t) &= \alpha_-\partial_{\boldsymbol{n}} u_-^2|_\Gamma - d_1\alpha_-\partial_{\boldsymbol{n}}^2 u_-^1|_\Gamma + \frac{\alpha_- d_1^2}{2}\partial_{\boldsymbol{n}}^3 u_-^0|_\Gamma \\
&\quad - \left[2c^2(t)\Psi_1(t) + c(t)k_m^2 U_m^0 - \partial_t^2(\Psi_1(t)) + 3c(t)\partial_t^2 U_m^0 + c'(t)\partial_t U_m^0 - k_m^2\Psi_1(t)\right]\frac{d_1^2}{2} \\
&\quad - \left[c(t)\Psi_2(t) + \partial_t^2\Phi_1(t) + k_m^2\Phi_1(t)\right]d_1.
\end{aligned}
$$

In the same way, using (A.10) and the transmission condition (4.8), we obtain

$$
\begin{aligned}
\varphi(t) &= \alpha_+\partial_{\boldsymbol{n}} u_+^2|_\Gamma + d_2\alpha_+\partial_{\boldsymbol{n}}^2 u_+^1|_\Gamma + \frac{\alpha_+ d_2^2}{2}\partial_{\boldsymbol{n}}^3 u_+^0|_\Gamma \\
&\quad - \left[2c^2(t)\Psi_1(t) + c(t)k_m^2 U_m^0 - \partial_t^2(\Psi_1(t)) + 3c(t)\partial_t^2 U_m^0 + c'(t)\partial_t U_m^0 - k_m^2\Psi_1(t)\right]\frac{d_2^2}{2} \\
&\quad + \left[c(t)\Psi_2(t) + \partial_t^2\Phi_1(t) + k_m^2\Phi_1(t)\right]d_2.
\end{aligned}
$$

Therefore

(A.11)
$$
\begin{aligned}
&\alpha_+ \partial_{\boldsymbol{n}} u^2_+|_\Gamma - \alpha_- \partial_{\boldsymbol{n}} u^2_-|_\Gamma \\
&= -d_1\alpha_- \partial^2_{\boldsymbol{n}} u^1_-|_\Gamma - d_2\alpha_+ \partial^2_{\boldsymbol{n}} u^1_+|_\Gamma - \frac{\alpha_+ d_2^2}{2}\partial^3_{\boldsymbol{n}} u^0_+|_\Gamma + \frac{\alpha_- d_1^2}{2}\partial^3_{\boldsymbol{n}} u^0_-|_\Gamma \\
&\quad - d_2 \partial_t^2 u^1_+|_\Gamma - d_2 k_m^2 u^1_+|_\Gamma - d_1 c\alpha_- \partial_{\boldsymbol{n}} u^1_-|_\Gamma - d_1 \partial_t^2 u^1_-|_\Gamma - d_1 k_m^2 u^1_-|_\Gamma - d_2 c\alpha_+ \partial_{\boldsymbol{n}} u^1_+|_\Gamma \\
&\quad - \left(\frac{d_1^2}{2}k_m^2\alpha_+ + d_1^2\alpha_+ c^2(t) - d_2^2 c^2(t)\alpha_+ - \frac{d_2^2}{2}k_m^2\alpha_+ + k_m^2 d_2^2 - k_m^2 d_1^2 \frac{\alpha_+}{\alpha_-}\right)\partial_{\boldsymbol{n}} u^0_+|_\Gamma \\
&\quad - \left(d_1^2\alpha_- k_-^2 - d_2^2\alpha_+ k_+^2 + \frac{d_2-d_1}{2}k_m^2\right) c(t) u^0_+|_\Gamma - \left(\frac{d_1-d_2}{2}\right) c'(t)\partial_t u^0_+|_\Gamma \\
&\quad - \left(d_1^2\alpha_- - d_2^2\alpha_+ + \frac{d_1}{2} - \frac{d_2}{2}\right) c(t)\partial_t^2 u^0_+|_\Gamma \\
&\quad - \left(d_2^2 - \frac{\alpha_+}{\alpha_-}d_1^2 + \frac{d_1-d_2}{2}\alpha_+\right)\partial_t^2(\partial_{\boldsymbol{n}} u^0_+|_\Gamma).
\end{aligned}
$$

Now, recall that

$$
\Delta u^n_+ + k_+^2 u^n_+ = \frac{-\eta c'(t)}{[1+\eta c(t)]^3}\partial_t u^n_+ + \frac{1}{[1+\eta c(t)]^2}\partial_t^2 u^n_+ + \frac{c(t)}{1+\eta c(t)}\partial_\eta u^n_+ + \partial_\eta^2 u^n_+ + k_+^2 u^n_+ = 0.
$$

The partial derivative with respect to η leads to

$$
\begin{aligned}
&\frac{-c'(t)[1+\eta c(t)]^3 + 3c(t)[1+\eta c(t)]^2 \eta c'}{[1+\eta c(t)]^6}\partial_t u^n_+ + \frac{\eta c'(t)}{[1+\eta c(t)]^3}\partial_\eta\partial_t u^n_+ \\
&- \frac{2c}{[1+\eta c(t)]^3}\partial_t^2 u^n_+ + \frac{1}{[1+\eta c(t)]^2}\partial_\eta\partial_t^2 u^n_+ - \frac{c^2(t)}{[1+\eta c(t)]^2}\partial_\eta u^n_+ \\
&+ \frac{c(t)}{1+\eta c(t)}\partial_\eta^2 u^n_+ + \partial_\eta^3 u^n_+ + k_+^2 \partial_\eta u^n_+ \\
&= 0.
\end{aligned}
$$

Taking the limit $\eta \to 0$, we obtain

$$
-c'(t)\partial_t u^n_+|_\Gamma - 2c(t)\partial_t^2 u^n_+|_\Gamma + \partial_{\boldsymbol{n}}\partial_t^2 u^n_+|_\Gamma - c^2(t)\partial_{\boldsymbol{n}} u^n_+|_\Gamma + c(t)\partial^2_{\boldsymbol{n}} u^n_+|_\Gamma + \partial^3_{\boldsymbol{n}} u^n_+|_\Gamma + k_+^2\partial_{\boldsymbol{n}} u^n_+|_\Gamma = 0,
$$

hence

(A.12)
$$
\partial^3_{\boldsymbol{n}} u^n_+|_\Gamma = c'(t)\partial_t u^n_+|_\Gamma + 2c\partial_t^2 u^n_+|_\Gamma - \partial_n \partial_t^2 u^n_+|_\Gamma + c^2(t)\partial_{\boldsymbol{n}} u^n_+|_\Gamma - c(t)\partial^2_{\boldsymbol{n}} u^n_+|_\Gamma - k^2\partial_{\boldsymbol{n}} u^n_+|_\Gamma.
$$

Using the identity (A.6), the relation (A.12) becomes

$$
\begin{aligned}
\partial^3_{\boldsymbol{n}} u^n_+|_\Gamma &= 3c(t)\partial_t^2 u^n_+|_\Gamma + (2c^2(t) - k_+^2)\partial_{\boldsymbol{n}} u^n_+|_\Gamma + c'(t)\partial_t u^n_+|_\Gamma \\
&\quad - \partial_{\boldsymbol{n}}\partial_t^2 u^n_+|_\Gamma + c(t)k_+^2 u^n_+|_\Gamma, \quad \forall\, n \geq 0.
\end{aligned} \tag{A.13}
$$

In the same way as for u_-^n, we find that

$$\begin{aligned}\partial_{\boldsymbol{n}}^3 u_-^n|_\Gamma &= 3c(t)\partial_t^2 u_-^n|_\Gamma + (2c^2(t) - k_-^2)\partial_{\boldsymbol{n}} u_-^n|_\Gamma + c'(t)\partial_t u_-^n|_\Gamma \\ &\quad - \partial_{\boldsymbol{n}}\partial_t^2 u_-^n|_\Gamma + c(t)k_-^2 u_-^n|_\Gamma, \quad \forall\, n \geq 0. \end{aligned} \tag{A.14}$$

Therefore, with (A.13) and (A.14) for $n = 0$, the transmission condition (A.11) becomes

$$\begin{aligned} &\alpha_+\partial_{\boldsymbol{n}} u_+^2|_\Gamma - \alpha_-\partial_{\boldsymbol{n}} u_-^2|_\Gamma \\ &= -d_1\alpha_-\partial_{\boldsymbol{n}}^2 u_-^1|_\Gamma - d_2\alpha_+\partial_{\boldsymbol{n}}^2 u_+^1|_\Gamma - d_2 c(t)\alpha_+\partial_{\boldsymbol{n}} u_+^1|_\Gamma \\ &\quad - d_2\partial_t^2 u_+^1|_\Gamma - d_2 k_m^2 u_+^1|_\Gamma - d_1 c(t)\alpha_-\partial_{\boldsymbol{n}} u_-^1|_\Gamma - d_1\partial_t^2 u_-^1|_\Gamma - d_1 k_m^2 u_-^1|_\Gamma \\ &\quad + \left(\frac{d_2^2\alpha_+}{2}k_+^2 - \frac{d_1^2\alpha_+}{2}k_-^2 - \frac{d_1}{2}k_m^2\alpha_+ + \frac{d_2}{2}k_m^2\alpha_+ - \frac{k_m^2 d_2^2\alpha_- - k_m^2 d_1^2\alpha_+}{\alpha_-}\right)\partial_{\boldsymbol{n}} u_+^0|_\Gamma \\ &\quad + \left(\frac{\alpha_+ d_2^2}{2}k_+^2 - \frac{\alpha_- d_1^2}{2}k_-^2 - \frac{d_2}{2}k_m^2 + \frac{d_1}{2}k_m^2\right) c(t) u_+^0|_\Gamma \\ &\quad + \left(\frac{d_2}{2} - \frac{d_1}{2} + \frac{\alpha_- d_1^2}{2} - \frac{\alpha_+ d_2^2}{2}\right) c(t)\partial_t^2 u_+^0|_\Gamma \\ &\quad + \left(\frac{\alpha_+ d_1^2 - \alpha_- d_2^2}{\alpha_-} - d_1\alpha_+ + d_2\alpha_+\right)\partial_t^2(\partial_{\boldsymbol{n}} u_+^0|_\Gamma) \\ &\quad + \left(-\frac{d_1}{2} + \frac{d_2}{2} - \frac{\alpha_+ d_2^2}{2} + \frac{\alpha_- d_1^2}{2}\right) c'(t)\partial_t u_+^0|_\Gamma. \end{aligned}$$

Again, we use the identities

$$\begin{aligned} \partial_{\boldsymbol{n}}^2 u_+^1|_\Gamma &= -\partial_t^2 u_+^1|_\Gamma - c(t)\partial_{\boldsymbol{n}} u_+^1|_\Gamma - k_+^2 u_+^1|_\Gamma, \\ \partial_{\boldsymbol{n}}^2 u_-^1|_\Gamma &= -\partial_t^2 u_-^1|_\Gamma - c(t)\partial_{\boldsymbol{n}} u_-^1|_\Gamma - k_-^2 u_-^1|_\Gamma \end{aligned}$$

to obtain

$$\begin{aligned} &\alpha_+\partial_{\boldsymbol{n}} u_+^2|_\Gamma - \alpha_-\partial_{\boldsymbol{n}} u_-^2|_\Gamma \\ &= (d_2\alpha_+ - d_2)\partial_t^2 u_+^1|_\Gamma + (d_1\alpha_- - d_1)\partial_t^2 u_-^1|_\Gamma \\ &\quad + d_2\alpha_+ k_+^2 u_+^1|_\Gamma - d_2 k_m^2 u_+^1|_\Gamma - d_1 k_m^2 u_-^1|_\Gamma + d_1\alpha_- k_-^2 u_-^1|_\Gamma \\ &\quad + \left(\frac{d_2^2\alpha_+}{2}k_+^2 - \frac{d_1^2\alpha_+}{2}k_-^2 - \frac{d_1 - d_2}{2}k_m^2\alpha_+ - \frac{k_m^2 d_2^2\alpha_- - k_m^2 d_1^2\alpha_+}{\alpha_-}\right)\partial_{\boldsymbol{n}} u_+^0|_\Gamma \\ &\quad + \left(\frac{\alpha_+ d_2^2}{2}k_+^2 - \frac{\alpha_- d_1^2}{2}k_-^2 - \frac{d_2 - d_1}{2}k_m^2\right) c(t) u_+^0|_\Gamma \\ &\quad + \left(\frac{d_2 - d_1}{2} - \frac{\alpha_+ d_2^2 - \alpha_- d_1^2}{2}\right) c(t)\partial_t^2 u_+^0|_\Gamma \\ &\quad + \left(\frac{\alpha_+ d_1^2 - \alpha_- d_2^2}{\alpha_-} - d_1\alpha_+ + d_2\alpha_+\right)\partial_t^2(\partial_{\boldsymbol{n}} u_+^0|_\Gamma) \\ &\quad + \left(\frac{d_2 - d_1}{2} - \frac{\alpha_+ d_2^2 - \alpha_- d_1^2}{2}\right) c'(t)\partial_t u_+^0|_\Gamma. \end{aligned}$$

Upon using the transmission conditions

$$u^1_+|_\Gamma = u^1_-|_\Gamma + \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{\alpha_-}\partial_{\boldsymbol{n}} u^0_+|_\Gamma,$$
$$u^0_+|_\Gamma = u^0_-|_\Gamma \quad \text{and} \quad \alpha_+\partial_{\boldsymbol{n}} u^0_+|_\Gamma = \alpha_-\partial_{\boldsymbol{n}} u^0_-|_\Gamma,$$

we get

$$\begin{aligned}
&\alpha_+\partial_{\boldsymbol{n}} u^2_+|_\Gamma - \alpha_-\partial_{\boldsymbol{n}} u^2_-|_\Gamma \\
&= \frac{d_2\alpha_+ + d_1\alpha_- - 1}{2}(\partial_t^2 u^1_+|_\Gamma + \partial_t^2 u^1_-|_\Gamma) \\
&\quad + \frac{d_2\alpha_+ k_+^2 + d_1\alpha_- k_-^2 - k_m^2}{2}(u^1_-|_\Gamma + u^1_+|_\Gamma) \\
&\quad + \left(\frac{\alpha_+ d_1 k_m^2 - \alpha_- d_2 k_m^2 - \alpha_+^2 d_1 d_2 k_+^2 + \alpha_-^2 d_1 d_2 k_-^2 + \alpha_+^2\alpha_- d_2 k_+^2 - \alpha_+\alpha_-^2 d_1 k_-^2}{4\alpha_-\alpha_+}\right) \\
&\quad\quad \times (\alpha_+\partial_{\boldsymbol{n}} u^0_+|_\Gamma + \alpha_-\partial_{\boldsymbol{n}} u^0_-|_\Gamma) \\
&\quad + \left(\frac{-\alpha_- d_1^2 k_-^2 + \alpha_+ d_2^2 k_+^2 - d_2 k_m^2 + d_1 k_m^2}{4}\right) c(t)(u^0_+|_\Gamma + u^0_-|_\Gamma) \\
&\quad + \left(\frac{-d_1 + d_2 - \alpha_+ d_2^2 + \alpha_- d_1^2}{4}\right) c(t)(\partial_t^2 u^0_+|_\Gamma + \partial_t^2 u^0_-|_\Gamma) \\
&\quad + \left(\frac{-d_1 + d_2 - \alpha_+ d_2^2 + \alpha_- d_1^2}{4}\right) c'(t)(\partial_t u^0_+|_\Gamma + \partial_t u^0_-|_\Gamma) \\
&\quad + \frac{(d_2\alpha_+ - d_1\alpha_-)(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-) + (d_2 - d_1)\alpha_+\alpha_- + d_1\alpha_+ - d_2\alpha_-}{4\alpha_+\alpha_-} \\
&\quad\quad \times (\alpha_+\partial_t^2\partial_{\boldsymbol{n}} u^0_+|_\Gamma + \alpha_-\partial_t^2\partial_{\boldsymbol{n}} u^0_-|_\Gamma).
\end{aligned}$$

But

$$\frac{2\alpha_+\alpha_-}{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}(\partial_t^2 u^1_+|_\Gamma - \partial_t^2 u^1_-|_\Gamma) = \alpha_+\partial_t^2\partial_{\boldsymbol{n}} u^0_+|_\Gamma + \alpha_-\partial_t^2\partial_{\boldsymbol{n}} u^0_-|_\Gamma,$$

then

$$\begin{aligned}
&\alpha_+\partial_{\boldsymbol{n}} u^2_+|_\Gamma - \alpha_-\partial_{\boldsymbol{n}} u^2_-|_\Gamma \\
&= \frac{d_2\alpha_+ + d_1\alpha_- - 1}{2}\left[\partial_t^2 u^1_+|_\Gamma + \partial_t^2 u^1_-|_\Gamma\right] \\
&\quad + \frac{d_2\alpha_+ k_+^2 + d_1\alpha_- k_-^2 - k_m^2}{2}(u^1_-|_\Gamma + u^1_+|_\Gamma) \\
&\quad + \left(\frac{\alpha_+ d_1 k_m^2 - \alpha_- d_2 k_m^2 - \alpha_+^2 d_1 d_2 k_+^2 + \alpha_-^2 d_1 d_2 k_-^2 + \alpha_+^2\alpha_- d_2 k_+^2 - \alpha_+\alpha_-^2 d_1 k_-^2}{4\alpha_-\alpha_+}\right) \\
&\quad\quad \times (\alpha_+\partial_{\boldsymbol{n}} u^0_+|_\Gamma + \alpha_-\partial_{\boldsymbol{n}} u^0_-|_\Gamma) \\
&\quad + \left(\frac{\alpha_+ d_2^2 k_+^2 - \alpha_- d_1^2 k_-^2 + d_1 k_m^2 - d_2 k_m^2}{4}\right) c(t)(u^0_+|_\Gamma + u^0_-|_\Gamma)
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{d_2 - d_1 + \alpha_- d_1^2 - \alpha_+ d_2^2}{4}\right) c(t)(\partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma) \\
&+ \left(\frac{d_2 - d_1 + \alpha_- d_1^2 - \alpha_+ d_2^2}{4}\right) c'(t)(\partial_t u_+^0|_\Gamma + \partial_t u_-^0|_\Gamma) \\
&+ \frac{(d_2\alpha_+ - d_1\alpha_-)(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-) + (d_2 - d_1)\alpha_+\alpha_- + d_1\alpha_+ - d_2\alpha_-}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)} \\
&\times (\partial_t^2 u_+^1|_\Gamma - \partial_t^2 u_-^1|_\Gamma).
\end{aligned}
$$

Finally, using (A.3), we obtain

(A.15)

$$
\begin{aligned}
&\alpha_+\partial_{\boldsymbol{n}} u_+^2|_\Gamma - \alpha_-\partial_{\boldsymbol{n}} u_-^2|_\Gamma \\
&= \frac{d_2\alpha_+ + d_1\alpha_- - 1}{2}(\partial_t^2 u_+^1|_\Gamma + \partial_t^2 u_-^1|_\Gamma) \\
&+ \frac{d_2\alpha_+ k_+^2 + d_1\alpha_- k_-^2 - k_m^2}{2}(u_-^1|_\Gamma + u_+^1|_\Gamma) \\
&+ \left(\frac{\alpha_+ d_1 k_m^2 - \alpha_- d_2 k_m^2 - \alpha_+^2 d_1 d_2 k_+^2 + \alpha_-^2 d_1 d_2 k_-^2 + \alpha_+^2\alpha_- d_2 k_+^2 - \alpha_+\alpha_-^2 d_1 k_-^2}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)}\right) \\
&\times (u_+^1|_\Gamma - u_-^1|_\Gamma) \\
&+ \left(\frac{\alpha_+ d_2^2 k_+^2 - \alpha_- d_1^2 k_-^2 + d_1 k_m^2 - d_2 k_m^2}{4}\right) c(t)(u_+^0|_\Gamma + u_-^0|_\Gamma) \\
&+ \left(\frac{d_2 - d_1 + \alpha_- d_1^2 - \alpha_+ d_2^2}{4}\right) c(t)(\partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma) \\
&+ \left(\frac{d_2 - d_1 + \alpha_- d_1^2 - \alpha_+ d_2^2}{4}\right) c'(t)(\partial_t u_+^0|_\Gamma + \partial_t u_-^0|_\Gamma) \\
&+ \frac{(d_2\alpha_+ - d_1\alpha_-)(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-) + (d_2 - d_1)\alpha_+\alpha_- + d_1\alpha_+ - d_2\alpha_-}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)} \\
&\times (\partial_t^2 u_+^1|_\Gamma - \partial_t^2 u_-^1|_\Gamma).
\end{aligned}
$$

Therefore, with (4.3), (A.9) and (A.15), (u_-^2, u_+^2) is a solution of the following problem

$$
\begin{cases}
\Delta u_+^2 + k_+^2 u_+^2 = 0 & \text{in } \Omega_+, \\
\Delta u_-^2 + k_-^2 u_-^2 = 0 & \text{in } \Omega_-, \\
u_-^2 = 0 & \text{on } \Gamma_i, \\
\lim_{|x|\to+\infty} \sqrt{|x|}(\partial_{|x|} - ik_+)(u_+^2) = 0
\end{cases}
$$

with transmission conditions on Γ:

$$
\begin{aligned}
&u_+^2 - u_-^2 \\
&= \frac{\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+}{2\alpha_+\alpha_-}(\alpha_+\partial_{\boldsymbol{n}} u_+^1 + \alpha_-\partial_{\boldsymbol{n}} u_-^1)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{d_1\alpha_+\alpha_-^2 - d_2\alpha_+^2\alpha_- + d_1 d_2(\alpha_+^2 - \alpha_-^2) + d_2\alpha_- - d_1\alpha_+}{4\alpha_+\alpha_-}(\partial_t^2 u_+^0 + \partial_t^2 u_-^0)\\
&+ c(t)\frac{(d_1 - d_2)\alpha_-\alpha_+ + d_2^2\alpha_- - d_1^2\alpha_+}{2(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-)}(u_+^1 - u_-^1)\\
&+ \frac{(d_1 k_-^2\alpha_- - d_2 k_+^2\alpha_+)\alpha_+\alpha_- + d_1 d_2(k_+^2\alpha_+^2 - k_-^2\alpha_-^2) + k_m^2(d_2\alpha_- - d_1\alpha_+)}{4\alpha_+\alpha_-}(u_+^0 + u_-^0),
\end{aligned}
$$

$$
\begin{aligned}
&\alpha_+\partial_{\boldsymbol{n}} u_+^2 - \alpha_-\partial_{\boldsymbol{n}} u_-^2\\
&= \frac{d_2\alpha_+ + d_1\alpha_- - 1}{2}(\partial_t^2 u_+^1 + \partial_t^2 u_-^1)\\
&\quad + \frac{d_2\alpha_+ k_+^2 + d_1\alpha_- k_-^2 - k_m^2}{2}(u_-^1 + u_+^1)\\
&\quad + \left(\frac{\alpha_+ d_1 k_m^2 - \alpha_- d_2 k_m^2 - \alpha_+^2 d_1 d_2 k_+^2 + \alpha_-^2 d_1 d_2 k_-^2 + \alpha_+^2\alpha_- d_2 k_+^2 - \alpha_+\alpha_-^2 d_1 k_-^2}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)}\right)\\
&\quad \times (u_+^1 - u_-^1)\\
&\quad + \left(\frac{\alpha_+ d_2^2 k_+^2 - \alpha_- d_1^2 k_-^2 + d_1 k_m^2 - d_2 k_m^2}{4}\right) c(t)(u_+^0 + u_-^0)\\
&\quad + \left(\frac{d_2 - d_1 + \alpha_- d_1^2 - \alpha_+ d_2^2}{4}\right) c(t)(\partial_t^2 u_+^0 + \partial_t^2 u_-^0)\\
&\quad + \left(\frac{d_2 - d_1 + \alpha_- d_1^2 - \alpha_+ d_2^2}{4}\right) c'(t)(\partial_t u_+^0 + \partial_t u_-^0)\\
&\quad + \frac{(d_2\alpha_+ - d_1\alpha_-)(\alpha_+\alpha_- - d_1\alpha_+ - d_2\alpha_-) + (d_2 - d_1)\alpha_+\alpha_- + d_1\alpha_+ - d_2\alpha_-}{2(\alpha_+\alpha_- - d_2\alpha_- - d_1\alpha_+)}\\
&\quad \times (\partial_t^2 u_+^1 - \partial_t^2 u_-^1).
\end{aligned}
$$

Moreover, using (A.10) and the condition (4.5), the profile U_m^2 is given by

$$
\begin{aligned}
U_m^2(t,s) = {}& u_-^2|_\Gamma + (\alpha_- s + d_1\alpha_- - d_1)\partial_{\boldsymbol{n}} u_-^1|_\Gamma s\\
&- \left(\frac{d_1^2}{2}k_-^2 - d_1^2\alpha_- k_-^2 - d_1\alpha_- s k_-^2 + \frac{d_1^2}{2}k_m^2 + \frac{s^2}{2}k_m^2 + d_1 k_m^2 s\right) u_-^0|_\Gamma\\
&- \left(\frac{s^2}{2} + d_1 s + d_1^2 - d_1^2\alpha_- - d_1\alpha_- s\right)\partial_t^2 u_-^0\\
&- \left(\frac{s^2}{2}\alpha_- - \frac{d_1^2}{2}\alpha_- + \frac{d_1^2}{2}\right) c(t)\partial_{\boldsymbol{n}} u_-^0|_\Gamma.
\end{aligned}
$$

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