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Orlicz–Hardy Weak Martingale Spaces for Two-parameter

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Abstract. In this paper, we investigate several two-parameter weak Orlicz–Hardy martingale spaces generated by the p-convex and q-concave functions, and establish their atomic decomposition theorems. Using the atomic decomposition, we obtain a sufficient condition for the boundedness of a sublinear operator defined on the two-parameter weak Orlicz–Hardy martingale spaces. Furthermore, the dual spaces of the two-parameter weak Orlicz–Hardy martingale spaces are considered.

1. Introduction

The classical martingale theory was systematically studied by Garsia [2], Long [13], Weisz [20] and more. In particular, Weisz [21, 22] established the weak atom decomposition theorems of weak martingale Hardy spaces and obtained some interesting martingale inequalities. The corresponding Banach-valued versions were studied by Hou and Ren [4]. For more information about weak martingale Hardy spaces, we refer the readers to [3,6,15].

As an important generalization of Hardy martingale space, the Orlicz–Hardy martingale space has been extensively investigated in the past few years. Liu et al. [11] studied the weak Orlicz spaces associated with convex function Φ and discussed their applications in the martingale theory. Miyamoto et al. [18] investigated the martingale Orlicz–Hardy spaces, in which, some martingale inequalities and duality were established by the help of atomic decompositions, and a John–Nirenberg inequality was obtained when the stochastic basis is regular. Recently, Jiao et al. [8] extended the results in [18] to the weak type setting. In addition, one can refer [5, 7, 9, 12, 24] for some recent progress on the weak Orlicz–Hardy martingale spaces.

In this paper, we focus our attention on two-parameter martingale. Recall that multi-parameter martingale was studied in a few papers (see [20] and the references therein).

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Cairoli [1] extended one-parameter Doob's inequality to multi-parameter case. Metraux proved the two-parameter Burkholder–Gundy's inequality in [17]. Very recently, Weisz [23] characterized the dual spaces of the multi-parameter martingale Hardy Lorentz spaces by the help of atomic decomposition and a John–Nirenberg inequality was generalized for these martingale spaces. Lu [14] investigated the two-parameter martingale Orlicz–Hardy spaces, in which some new martingale inequalities and duality of these martingale spaces were established. For multi-parameter martingales, the proofs are not usually the analogues of that of the one-parameter martingales, they demand some new thoughts.

Inspired by [8], it is natural to study the two-parameter weak Orlicz–Hardy martingale spaces. It should be emphasized that Φ is essentially a concave function in [8, 14, 18] and Φ is assumed to be convex in [11]. But in this paper, we investigate several two-parameter weak Orlicz–Hardy martingale spaces generated by a more extensive class of functions, namely, the p-convex and q-concave function; see its definitions in Section 2.

The paper is organized as follows. In Section 2, some basic concepts and the definition of two-parameter weak Orlicz–Hardy martingale spaces will be introduced. Section 3 is devoted to establishing the atomic decomposition for the two-parameter weak Orlicz–Hardy martingale space wH^s_Φ; see Theorem 3.2. In Section 4, as an application, a sufficient condition for a sublinear operator defined on the two-parameter weak Orlicz–Hardy martingale spaces to be bounded is given; see Theorems 4.1 and 4.2. The duality of the two-parameter weak Orlicz–Hardy martingale spaces $w\mathscr{H}^s_\Phi$ is considered in the last Section; see Theorem 5.2.

We conclude this section with some conventions. Throughout the paper, $\mathbb{Z}$ and $\mathbb{N}$ denote the integer set and non-negative integer set, respectively. C stands for a positive constant, which can vary from line to line. The symbol $f \approx g$ implies that there exist two positive constants C_1 and C_2 such that $C_1 g \le f \le C_2 g$. We write $\chi(A)$ for the characteristic function of the set A.

2. Preliminaries

In this section, we give some basic notions and knowledge that will be used in the sequel.

2.1. Weak Orlicz spaces

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be an Orlicz function. That is, Φ is a non-negative, non-decreasing and continuous function on $[0,\infty)$ satisfying $\lim_{t\to+0}\Phi(t) = \Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. Denote by $\mathcal{O}$ the set of all Orlicz functions on $[0,\infty)$. In this paper, Φ is not generally assumed to be convex, except we mention it especially.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. For an Orlicz function $\Phi \in \mathcal{O}$, the Orlicz space $L_\Phi(\Omega, \mathcal{A}, P)$ (briefly by L_Φ) is defined as the collection of all measurable functions f

satisfying $\|f\|_{L_\Phi} < \infty$, where

$$\|f\|_{L_\Phi} := \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f|}{\lambda}\right) dP \le 1\right\}.$$

It is well known that L_Φ equipped with this norm becomes a Banach space when Φ is convex. In particular, if $\Phi(t) = t^p$ $(0 < p < \infty)$, then L_Φ returns to the usual L_p space with the norm (or quasi-norm) $\|\cdot\|_p$. Moreover, one can easily check that for any measurable set $F \in \mathcal{A}$ such that $P(F) \neq 0$, we have

$$\|\chi(F)\|_{L_\Phi} = \frac{1}{\Phi^{-1}\left(\frac{1}{P(F)}\right)}.$$

Let $\Phi \in \mathcal{O}$. Then we define the weak Orlicz space wL_Φ as the space of all measurable functions f relative to $(\Omega, \mathcal{A}, P)$ for which

$$\|f\|_{wL_\Phi} := \inf\left\{\lambda > 0 : \sup_{t>0} \Phi\left(\frac{t}{\lambda}\right) P(|f| > t) \le 1\right\}$$

is finite. By a simple calculation, the following equivalences

$$\|f\|_{wL_\Phi} = \sup_{t>0} t\|\chi(|f| > t)\|_{L_\Phi} \approx \sup_{k\in\mathbb{Z}} 2^k\|\chi(|f| > 2^k)\|_{L_\Phi}$$

hold. Especially, if $\Phi(t) = t^p$ $(0 < p < \infty)$, then wL_Φ becomes the usual weak L_p space wL_p with the following quasi-norm

$$\|f\|_{wL_p} := \sup_{t>0} tP(|f| > t)^{1/p}.$$

We say that a function $\Phi\colon [0,\infty) \to [0,\infty)$ satisfies the Δ_2-condition, written as $\Phi \in \Delta_2$, if there exists a positive constant C such that

$$\Phi(2t) \le C\Phi(t) \quad \text{for any } t > 0.$$

We say that a function $\Phi\colon [0,\infty) \to [0,\infty)$ belongs to Δ_0, denoted by $\Phi \in \Delta_0$, if

$$\lim_{c\to 0} \sup_{t>0} \frac{\Phi(ct)}{\Phi(t)} = 0.$$

For instance, if Φ is convex, or $\Phi(t) = t^p$ $(0 < p < \infty)$ then $\Phi \in \Delta_0$. Let $\Phi \in \mathcal{O}$. In [10] the authors proved that the weak Orlicz space wL_Φ is a complete quasi-normed space when $\Phi \in \Delta_2$ and $\Phi \in \Delta_0$.

Let $\Phi \in \mathcal{O}$ and let $0 < p \le q < \infty$. We say that Φ is a p-convex and q-concave function if the function $t \mapsto \Phi(t^{1/p})$, $t > 0$ is convex while the function $t \mapsto \Phi(t^{1/q})$, $t > 0$ is concave.

The following lemma is used frequently in this paper.

Lemma 2.1. *Suppose that $0 < p \leq q < \infty$ and $\Phi \in \mathcal{O}$ is p-convex and q-concave. Then the following statements hold.*

(i) *the functions $\frac{\Phi(t)}{t^p}$, $\frac{\Phi^{-1}(t)}{t^{1/q}}$ are non-decreasing on $(0, \infty)$ and the functions $\frac{\Phi(t)}{t^q}$, $\frac{\Phi^{-1}(t)}{t^{1/p}}$ are non-increasing on $(0, \infty)$;*

(ii) *for $0 \leq \lambda \leq 1$, we have*

$$\lambda^q \Phi(t) \leq \Phi(\lambda t) \leq \lambda^p \Phi(t), \quad \lambda^{1/p} \Phi^{-1}(t) \leq \Phi^{-1}(\lambda t) \leq \lambda^{1/q} \Phi^{-1}(t), \quad t \geq 0;$$

for $\lambda \geq 1$, we have

$$\lambda^p \Phi(t) \leq \Phi(\lambda t) \leq \lambda^q \Phi(t), \quad \lambda^{1/q} \Phi^{-1}(t) \leq \Phi^{-1}(\lambda t) \leq \lambda^{1/p} \Phi^{-1}(t), \quad t \geq 0;$$

(iii) *$\Phi \in \Delta_2$ and $\Phi \in \Delta_0$.*

Proof. (i) Note that the proof of the monotonicity of the functions involved the index q is similar to the one of the index p, it suffices to show that the function $\frac{\Phi(t)}{t^p}$ is non-decreasing while the function $\frac{\Phi^{-1}(t)}{t^{1/p}}$ is non-increasing. Since Φ is p-convex, that is, $\Phi(t^{1/p})$ is convex, one can conclude that $\Phi((\lambda t)^{1/p}) \geq \lambda \Phi(t^{1/p})$ for any $\lambda \geq 1$, $t > 0$. Hence,

$$\frac{\Phi((\lambda t)^{1/p})}{\lambda t} \geq \frac{\lambda \Phi(t^{1/p})}{\lambda t} = \frac{\Phi(t^{1/p})}{t},$$

which implies that the function $\frac{\Phi(t^{1/p})}{t}$ is increasing on $(0, \infty)$ and it follows immediately that $\frac{\Phi(t)}{t^p}$ is increasing on $(0, \infty)$. Substituting $\Phi^{-1}(t)$ for t one can obtain that $\frac{\Phi^{-1}(t)}{t^{1/p}}$ is decreasing on $(0, \infty)$.

(ii) From the monotonicity of the functions in (i) we can get the desired inequalities.

(iii) Let $\lambda = 2$. From (ii) we get $\Phi(2t) \leq 2^q \Phi(t)$, that is, $\Phi \in \Delta_2$. Using (ii) again, one can conclude that $\sup_{t>0} \frac{\Phi(ct)}{\Phi(t)} \leq c^p$ for any $0 < c < 1$ and thus

$$\lim_{c \to 0} \sup_{t>0} \frac{\Phi(ct)}{\Phi(t)} = 0,$$

namely, $\Phi \in \Delta_0$. □

Remark 2.2. Let $0 < p \leq q < \infty$. It follows from Lemma 2.1(iii) that the weak Orlicz space wL_Φ is a complete quasi-normed space for every p-convex and q-concave function $\Phi \in \mathcal{O}$.

Proposition 2.3. *Let $0 < p \leq q < 2$ and let $\Phi \in \mathcal{O}$ be p-convex and q-concave. Then $L_2 \subset L_\Phi \subset wL_\Phi$.*

Proof. Let $f \in L_2$. Then by Lemma 2.1 we get

$$\begin{aligned}\int_\Omega \Phi\left(\frac{|f|}{\|f\|_2}\right) dP &= \int_{\{|f|\le\|f\|_2\}} \Phi\left(\frac{|f|}{\|f\|_2}\right) dP + \int_{\{|f|>\|f\|_2\}} \Phi\left(\frac{|f|}{\|f\|_2}\right) dP \\ &\le \int_{\{|f|\le\|f\|_2\}} \Phi(1)\, dP + \int_{\{|f|>\|f\|_2\}} \left(\frac{|f|}{\|f\|_2}\right)^q \Phi(1)\, dP \\ &\le \Phi(1) + \frac{\Phi(1)}{\|f\|_2^q}\|f\|_q^q \\ &\le 2\Phi(1).\end{aligned}$$

Denote $C_0 = \max\{2\Phi(1), 1\}$. Then, applying Lemma 2.1 again, we have

$$\int_\Omega \Phi\left(\frac{|f|}{C_0^{1/p}\|f\|_2}\right) dP \le \frac{1}{C_0}\int_\Omega \Phi\left(\frac{|f|}{\|f\|_2}\right) dP \le \frac{2\Phi(1)}{C_0} \le 1,$$

which implies

$$\|f\|_{L_\Phi} \le C_0\|f\|_2.$$

Assume that $f \in L_\Phi$. Then

$$\Phi\left(\frac{t}{\|f\|_{L_\Phi}}\right) P(|f| > t) \le \int_{\{|f|>t\}} \Phi\left(\frac{|f|}{\|f\|_{L_\Phi}}\right) dP \le \int_\Omega \Phi\left(\frac{|f|}{\|f\|_{L_\Phi}}\right) dP \le 1,$$

which means $L_\Phi \subset wL_\Phi$. The proof is complete. □

Definition 2.4. Let $\Phi \in \mathcal{O}$. A measurable function $f \in wL_\Phi$ is said to have absolutely continuous norm if

$$\lim_{P(A)\to 0} \|f\chi(A)\|_{wL_\Phi} = 0.$$

Denote by $w\mathscr{L}_\Phi$ the set of all $f \in wL_\Phi$ having the absolutely continuous norm. That is,

$$w\mathscr{L}_\Phi := \left\{ f \in wL_\Phi : \lim_{P(A)\to 0} \|f\chi(A)\|_{wL_\Phi} = 0 \right\}.$$

Remark 2.5. (1) It was shown in [10] that not all elements in wL_Φ have absolutely continuous norm, even if $\Phi \in \Delta_2$ (see [10, Example 2.5]).

(2) It was also proved in [10, Lemma 2.5] that $w\mathscr{L}_\Phi$ is a closed subspace of wL_Φ when $\Phi \in \Delta_2$. Furthermore, one can conclude that $L_2 \subset L_\Phi \subset w\mathscr{L}_\Phi$ for every p-convex and q-concave $(0 < p \le q < 2)$ function $\Phi \in \mathcal{O}$. Indeed, it only needs to note the fact that the Orlicz space L_Φ has absolutely continuous norm when $\Phi \in \Delta_2$ (see [19]), then the conclusion follows from Lemma 2.1 and Proposition 2.3.

The following proposition is a generalization of Lebesgue dominated convergence theorem in wL_Φ space. We will apply it to state the convergence in wL_Φ (see Remark 3.6).

Proposition 2.6. [10, Theorem 3.2] *Let $\Phi \in \mathcal{O}$ be p-convex and q-concave for $0 < p \le q < \infty$, $f_n, f \in wL_\Phi$, $g \in w\mathscr{L}_\Phi$ and $|f_n| \le g$. If f_n converges to f almost everywhere, then*

$$\lim_{n\to\infty} \|f_n - f\|_{wL_\Phi} = 0.$$

2.2. Two-parameter martingales

Let $\mathbb{N}$ be the set of all non-negative integers and let $\mathbb{N}^2$ be its double Descartes product $\mathbb{N}\times\mathbb{N}$. We denote by (n_1, n_2) (or simply by n, if there is no confusion) the non-negative integer pair from $\mathbb{N}^2$. The first and the second coordinates of a pair $n \in \mathbb{N}^2$ are written as n_1 and n_2, respectively. For $n = (n_1, n_2)$, let $n-1 := (n_1-1, n_2-1)$. The partial ordering on $\mathbb{N}^2$ is defined as follows: for two arbitrary pairs $n = (n_1, n_2), m = (m_1, m_2) \in \mathbb{N}^2$ we say that $n \le m$ if $n_1 \le m_1$ and $n_2 \le m_2$. If $n \le m$ and $n \ne m$ $(n, m \in \mathbb{N}^2)$, then we write $n < m$. Furthermore, the notation $n \ll m$ indicates that both $n_1 < m_1$ and $n_2 < m_2$ hold. Besides, if $n \le m$ (respectively, $n \ll m$) is not true, we denote by $n \nleq m$ (respectively, $n \not\ll m$).

Two non-negative integer pairs $n, m \in \mathbb{N}^2$ are called incomparable if neither $n \le m$ nor $m \le n$ holds. For two arbitrary sets $K, L \subset \mathbb{N}^2$ whose elements are incomparable we say that $K \le L$ (respectively, $K \ll L$) if, for every $n \in L$, there exists $m \in K$ such that $m \le n$ (respectively, $m \ll n$). The infimum of a set $K \subset \mathbb{N}^2$ is defined as

$$\inf K := \{m \in K : \text{there does not exist any } n \in K \text{ such that } n < m\}.$$

Here, we adopt the convention $\inf \emptyset = \infty$. For any two subsets $K, L \subset \mathbb{N}^2$ we say that $K \le L$ (respectively, $K \ll L$) if $\inf K \le \inf L$ (respectively, $\inf K \ll \inf L$). In addition, if $K \le L$ (respectively, $K \ll L$) is false, we write $K \nleq L$ (respectively, $K \not\ll L$).

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N}^2)$ be an increasing sequence of σ-algebras relative to the partial ordering on $\mathbb{N}^2$ such that

$$\mathcal{A} = \sigma\left(\bigcup_{n\in\mathbb{N}^2} \mathcal{F}_n\right).$$

The expectation operator and the conditional expectation operator with respect to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively.

A function sequence $f = (f_n, n \in \mathbb{N}^2)$ is called a two-parameter *martingale* with respect to $(\mathcal{F}_n, n \in \mathbb{N}^2)$ if

(1) for all $n \in \mathbb{N}^2$, $f_n \in L_1$;

(2) for every $n \in \mathbb{N}^2$, f_n is $\mathcal{F}_n$ measurable;

(3) for all $n \le m$, $\mathbb{E}_n f_m = f_n$.

Denote by $\mathcal{M}$ the set of all martingales $f = (f_n, n \in \mathbb{N}^2)$ relative to $(\mathcal{F}_n, n \in \mathbb{N}^2)$.

Let $0 < p \le \infty$. For any two-parameter martingale $f = (f_n, n \in \mathbb{N}^2)$, define

$$\|f\|_p := \sup_{n\in\mathbb{N}^2} \|f_n\|_p.$$

If $\|f\|_p < \infty$, then f is said to be an *L_p-bounded* martingale.

We say that the stochastic basis $\mathcal{F}$ is *regular* if there exists a positive number R such that for all non-negative martingales $(f_n, n \in \mathbb{N}^2)$,

$$f_{n_1,n_2} \leq R f_{n_1-1,n_2}, \quad f_{n_1,n_2} \leq R f_{n_1,n_2-1}, \quad n \in \mathbb{N}^2.$$

The *martingale differences* of a two-parameter martingale $f = (f_n, n \in \mathbb{N}^2)$ are defined by

$$d_n f := \begin{cases} 0 & \text{if } n_1 = 0 \text{ or } n_2 = 0, \\ f_{n_1,n_2} - f_{n_1-1,n_2} - f_{n_1,n_2-1} + f_{n_1-1,n_2-1} & \text{else.} \end{cases}$$

It is clear that $(d_n f, n \in \mathbb{N}^2)$ is an adapted process such that $d_n f \in L_1$ $(n \in \mathbb{N}^2)$ and

$$\mathbb{E}_n d_m f = 0, \quad m \nleq n. \tag{2.1}$$

Conversely, if $(d_n, n \in \mathbb{N}^2)$ is a sequence of adapted and integrable functions which satisfies the formula above then $(f_n, n \in \mathbb{N}^2)$ is a martingale, where

$$f_n = \sum_{m \leq n} d_m.$$

We say that a function ν which maps Ω into the set of subspaces of $\mathbb{N}^2 \cup \{\infty\}$ is a two-parameter *stopping time* with respect to $(\mathcal{F}_n, n \in \mathbb{N}^2)$ if

(i) for every $\omega \in \Omega$, the set $\nu(\omega)$ consists of incomparable non-negative integer pairs;

(ii) for any $n \in \mathbb{N}^2$,

$$\{\omega \in \Omega : n \in \nu(\omega)\} =: \{n \in \nu\} \in \mathcal{F}_n.$$

For example, if H is a Borel set and $(f_n, n \in \mathbb{N}^2)$ is an adapted sequence then it is easy to see that

$$\nu(\omega) := \inf\{n \in \mathbb{N}^2 : f_n(\omega) \in H\}$$

is a stopping time. Moreover, if ν is a stopping time then one can conclude that

$$\{\nu \not\ll n\} \in \mathcal{F}_{n-1}, \quad n \in \mathbb{N}^2, \tag{2.2}$$

since

$$\{\nu \ll n\} = \bigcup_{m \leq n-1} \{m \in \nu\}, \quad n \in \mathbb{N}^2.$$

The collection of all stopping times relative to $(\mathcal{F}_n, n \in \mathbb{N}^2)$ is denoted by $\mathcal{T}$.

Suppose that ν is a two-parameter stopping time and $f = (f_n, n \in \mathbb{N}^2)$ is a two-parameter martingale adapted to the same filtration. Then we can define the *stopped martingale* $f^\nu = (f_n^\nu, n \in \mathbb{N}^2)$ as

$$f_n^\nu := \sum_{m \leq n} \chi(\nu \not\ll m) d_m f.$$

In fact, one can use (2.1) and (2.2) to verify that the definition above is well defined.

2.3. Two-parameter weak Orlicz–Hardy martingale spaces

For $f \in \mathcal{M}$, we define the maximal function, the quadratic variation and the conditional quadratic variation of f by

$$M_n(f) = \sup_{m \le n} |f_m|, \quad M(f) = \sup_{m \in \mathbb{N}^2} |f_m|,$$
$$S_n(f) = \left(\sum_{m \le n} |d_m f|^2 \right)^{1/2}, \quad S(f) = \left(\sum_{m \in \mathbb{N}^2} |d_m f|^2 \right)^{1/2},$$
$$s_n(f) = \left(\sum_{m \le n} \mathbb{E}_{m-1} |d_m f|^2 \right)^{1/2}, \quad s(f) = \left(\sum_{m \in \mathbb{N}^2} \mathbb{E}_{m-1} |d_m f|^2 \right)^{1/2},$$

respectively.

Further on, for $\Phi \in \mathcal{O}$, we define the two-parameter weak Orlicz–Hardy martingale spaces as

$$wH_\Phi = \{ f \in \mathcal{M} : \|f\|_{wH_\Phi} = \|M(f)\|_{wL_\Phi} < \infty \},$$
$$wH_\Phi^S = \{ f \in \mathcal{M} : \|f\|_{wH_\Phi^S} = \|S(f)\|_{wL_\Phi} < \infty \},$$
$$wH_\Phi^s = \{ f \in \mathcal{M} : \|f\|_{wH_\Phi^s} = \|s(f)\|_{wL_\Phi} < \infty \}.$$

Remark 2.7. If $\|\cdot\|_{wL_\Phi}$ is replaced by $\|\cdot\|_{L_\Phi}$ in the definition above, then we obtain the corresponding two-parameter martingale Orlicz–Hardy spaces H_Φ, H_Φ^S and H_Φ^s defined by Lu [14]. With the purpose of discussing the duality, we define the following martingale space

$$w\mathscr{H}_\Phi^s := \{ f \in \mathcal{M} : s(f) \in w\mathscr{L}_\Phi \}.$$

From Remark 2.5 we know that $w\mathscr{H}_\Phi^s$ is also a closed subspace of wH_Φ^s, if $\Phi \in \Delta_2$. Similarly, if $\Phi \in \Delta_2$, then $w\mathscr{H}_\Phi$ and $w\mathscr{H}_\Phi^S$ are the closed subspaces of wH_Φ and wH_Φ^S, respectively.

3. Atomic decompositions

In order to establish the atomic decomposition of the two-parameter weak Orlicz–Hardy martingale spaces, we recall the definition of (Φ, q) atoms first.

Definition 3.1. [14, Definition 3.1] Let $\Phi \in \mathcal{O}$ and $q \in (1, \infty]$. A measurable function $a \in L_q$ is said to be a (Φ, q) atom if there exists a stopping time $\nu \in \mathcal{T}$ such that

(i) $a_n = \mathbb{E}_n a = 0$ if $\nu \not\ll n$,

(ii) $\|M(a)\|_q \le \frac{P(\nu \ne \infty)^{1/q}}{\|\chi(\nu \ne \infty)\|_{L_\Phi}}$.

Theorem 3.2. *Let $0 < p \leq q < 2$ and let $\Phi \in \mathcal{O}$ be p-convex and q-concave. If the martingale $f = (f_n, n \in \mathbb{N}^2) \in wH_\Phi^s$ then there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(\Phi, 2)$ atoms with respect to the stopping times $(\nu_k)_{k\in\mathbb{Z}}$ and a sequence of positive numbers $(\mu_k)_{k\in\mathbb{Z}} \in l_\infty$ such that*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \quad a.e., \quad \forall\, n \in \mathbb{N}^2 \tag{3.1}$$

and

$$\sup_{k\in\mathbb{Z}} \mu_k \leq C\|f\|_{wH_\Phi^s}. \tag{3.2}$$

Conversely, if the martingale f has a decomposition of type (3.1) *then $f \in wH_\Phi^s$ and*

$$\|f\|_{wH_\Phi^s} \approx \inf \sup_{k\in\mathbb{Z}} \mu_k, \tag{3.3}$$

where the infimum is taken over all decompositions of f of the form (3.1)*.*

Proof. Assume that $f = (f_n, n \in \mathbb{N}^2) \in wH_\Phi^s$. Let

$$F_k = \{s(f) > 2^k\}$$

and consider the following stopping times for all $k \in \mathbb{Z}$,

$$\nu_k = \inf\{n \in \mathbb{N}^2 : \mathbb{E}_n \chi(F_k) > 1/2\}.$$

It is easy to obtain that (see [20, Page 82])

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}) \quad \text{and} \quad f_n^{\nu_{k+1}} - f_n^{\nu_k} = \sum_{m\leq n} d_m f \chi(\nu_k \ll m \not\gg \nu_{k+1})$$

hold. Set

$$\mu_k = 4\sqrt{2} \cdot 2^{k+1} \|\chi(\nu_k \neq \infty)\|_{L_\Phi} = \frac{4\sqrt{2} \cdot 2^{k+1}}{\Phi^{-1}\big(1/P(\nu_k \neq \infty)\big)} \quad \text{and} \quad a_n^k = \frac{f_n^{\nu_{k+1}} - f_n^{\nu_k}}{\mu_k}$$

(set $a_n^k = 0$ if $\mu_k = 0$). It is clear that for arbitrary fixed $k \in \mathbb{Z}$, $a^k := (a_n^k, n \in \mathbb{N}^2)$ is a martingale. Furthermore, we can see that

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k a_n^k \quad \text{a.e.}$$

for all $n \in \mathbb{N}^2$.

Further on let us check that a^k is a $(\Phi, 2)$ atom relative to ν_k. It's obvious that $a_n^k = 0$ for each fixed $k \in \mathbb{Z}$ if $\nu_k \not\ll n$, which confirms Definition 3.1(i). To see (ii), we should prove that

$$\mathbb{E}[(M(a^k))^2] \leq \frac{P(\nu_k \neq \infty)}{\|\chi(\nu_k \neq \infty)\|_{L_\Phi}^2}.$$

By [20, Proposition 3.4] and the definition of a^k it suffices to show that

$$\mathbb{E}(f^{\nu_{k+1}} - f^{\nu_k})^2 \leq 2 \cdot (2^{k+1})^2 P(\nu_k \neq \infty).$$

Note that L_2 and H_2^s are isometric, for the inequality above, we only need to verify that

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}^2} \mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\right) \leq 2 \cdot (2^{k+1})^2 P(\nu_k \neq \infty). \tag{3.4}$$

Since $\{\nu_k \ll n \not\gg \nu_{k+1}\} \in \mathcal{F}_{n-1}$, we divide the left side of (3.4) into the following two parts:

$$(G) = \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(\mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\chi(F_{k+1}^c)\big)$$

and

$$(H) = \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(\mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\chi(F_{k+1})\big).$$

Consequently,

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}^2} \mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\right) = (G) + (H). \tag{3.5}$$

It is easy to check that

$$(G) \leq (2^{k+1})^2 P(\nu_k \neq \infty) \tag{3.6}$$

and

$$(H) = \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(\mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\mathbb{E}_{n-1}\chi(F_{k+1})\big).$$

By the definition of ν_{k+1} one can conclude that, if $\nu_{k+1} \not\ll n$, then $\mathbb{E}_{n-1}\chi(F_{k+1}) \leq 1/2$. Hence,

$$(H) \leq \frac{1}{2}\mathbb{E}\left(\sum_{n\in\mathbb{N}^2} \mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\right). \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we get (3.4). Thus a^k is truly a $(\Phi, 2)$ atom relative to ν_k. Of course, $a^k := (a_n^k, n \in \mathbb{N}^2)$ is L_2-bounded. Denote its limit still by a^k then $a_n^k = \mathbb{E}_n a^k$ $(n \in \mathbb{N}^2)$. Consequently, (3.1) holds.

Now we verify that (3.2) also holds. By the Chebyshev inequality and [20, Proposition 3.4], we obtain (see Lu [14, Page 40])

$$P(\nu_k \neq \infty) \leq 64P(F_k).$$

Note that $P(F_k) = P(s(f) > 2^k)$ and Φ is non-decreasing, by Lemma 2.1, we obtain

$$\begin{aligned}
\mu_k &= 4\sqrt{2}\cdot 2^{k+1}\|\chi(\nu_k \neq \infty)\|_{L_\Phi} = 4\sqrt{2}\cdot 2^{k+1}\frac{1}{\Phi^{-1}(1/P(\nu_k\neq\infty))} \\
&\le 4\sqrt{2}\cdot 2^{k+1}\frac{1}{\Phi^{-1}(1/(64P(F_k)))} = 4\sqrt{2}\cdot 2^{k+1}\frac{1}{\Phi^{-1}(1/(64P(s(f)>2^k)))} \\
&= 4\sqrt{2}\cdot 2^{k+1}\frac{\left(\frac{1}{64P(s(f)>2^k)}\right)^{1/p}}{\Phi^{-1}(1/(64P(s(f)>2^k)))}\left(\frac{1}{64P(s(f)>2^k)}\right)^{-1/p} \\
&\le 4\sqrt{2}\cdot 2^{k+1}\frac{\left(\frac{1}{P(s(f)>2^k)}\right)^{1/p}}{\Phi^{-1}(1/(P(s(f)>2^k)))}\left(\frac{1}{64P(s(f)>2^k)}\right)^{-1/p} \\
&\le C2^k\frac{1}{\Phi^{-1}(1/P(s(f)>2^k))} = C2^k\|\chi(s(f)>2^k)\|_{L_\Phi} \le C\|f\|_{wH^s_\Phi}.
\end{aligned}$$

Taking the supremum of all $k\in\mathbb{Z}$, we get (3.2).

To prove the converse part, we need the following lemmas.

Lemma 3.3. [16, Theorem 10.1] *Let φ_1, φ_2, φ be Orlicz functions.*

(1) *If for some $C>0$,*

$$\varphi_1^{-1}(u)\varphi_2^{-1}(u) \le C\varphi^{-1}(u) \quad \text{for all } u\ge 0$$

and $x\in L_{\varphi_1}$, $y\in L_{\varphi_2}$, then the product $xy\in L_\varphi$ and

$$\|xy\|_{L_\varphi} \le 2C\|x\|_{L_{\varphi_1}}\|y\|_{L_{\varphi_2}}.$$

(2) *If for some $D>0$,*

$$\varphi^{-1}(u) \le D\varphi_1^{-1}(u)\varphi_2^{-1}(u) \quad \text{for all } u\ge 0$$

and $x\in L_\varphi$, then there are $x_i\in L_{\varphi_i}$ $(i=1,2)$ such that $x_1x_2=|x|$ and

$$\|x_1\|_{L_{\varphi_1}}\|x_2\|_{L_{\varphi_2}} \le D\|x\|_{L_\varphi}.$$

Lemma 3.4. *Suppose that $0<p\le q<2$ and $\Phi\in\mathcal{O}$ is p-convex and q-concave. Let $1<L<\infty$. Then*

$$\mu_k^L\|[s(a^k)]^L\|_{L_\Phi} \le C2^{kL}\|\chi(\nu_k\neq\infty)\|_{L_\Phi}.$$

Proof. Consider the Orlicz function Ψ satisfying the condition

$$\Psi^{-1}(u) = u^{-L/2}\Phi^{-1}(u), \quad u>0.$$

Since $s(a^k)=0$ on the set $\{\nu_k=\infty\}$, using Lemma 3.3 we obtain

$$\begin{aligned}\mu_k^L\|[s(a^k)]^L\|_{L_\Phi}&=\mu_k^L\|s(a^k)^L\chi(\nu_k\neq\infty)\|_{L_\Phi}\le C\mu_k^L\|s(a^k)^L\|_{2/L}\|\chi(\nu_k\neq\infty)\|_{L_\Psi}\\&=C\mu_k^L\|s(a^k)\|_2^L\|\chi(\nu_k\neq\infty)\|_{L_\Psi}\le C\mu_k^L\|M(a^k)\|_2^L\|\chi(\nu_k\neq\infty)\|_{L_\Psi}\\&\le C\mu_k^L\left(\frac{\|\chi(\nu_k\neq\infty)\|_2}{\|\chi(\nu_k\neq\infty)\|_{L_\Phi}}\right)^L\|\chi(\nu_k\neq\infty)\|_{L_\Psi}\\&=C2^{kL}\|\chi(\nu_k\neq\infty)\|_2^L\|\chi(\nu_k\neq\infty)\|_{L_\Psi}\\&=C2^{kL}\|\chi(\nu_k\neq\infty)\|_{2/L}\|\chi(\nu_k\neq\infty)\|_{L_\Psi}\le C2^{kL}\|\chi(\nu_k\neq\infty)\|_{L_\Phi}.\qquad\square\end{aligned}$$

Remark 3.5. If we replace the operator s with S and M in Lemma 3.4 respectively, the conclusion also holds.

Now we show the converse part of Theorem 3.2. It needs to prove that if the martingale $f=(f_n,n\in\mathbb{N}^2)$ has a decomposition as (3.1), where $\{\mu_k\}_{k\in\mathbb{Z}}$ and $\{a^k\}_{k\in\mathbb{Z}}$ are just the same as the statement in Theorem 3.2, then $f\in wH_\Phi^s$ and (3.3) holds. Set $I=\sup_{k\in\mathbb{Z}}\mu_k<\infty$ and for an arbitrary $k_0\in\mathbb{Z}$, let

$$f=\sum_{k\in\mathbb{Z}}\mu_ka^k=\sum_{k\le k_0-1}\mu_ka^k+\sum_{k\ge k_0}\mu_ka^k=:F_1+F_2.$$

As s is sublinear, we get $s(f)\le s(F_1)+s(F_2)$ and

$$s(F_1)\le\sum_{k\le k_0-1}\mu_ks(a^k),\quad s(F_2)\le\sum_{k\ge k_0}\mu_ks(a^k).$$

We now estimate $s(F_1)$ and $s(F_2)$, respectively. Note that $s(a^k)=0$ on the set $\{\nu_k=\infty\}$, we have $\{s(a^k)>0\}\subset\{\nu_k\neq\infty\}$. Hence,

$$\{s(F_2)>2^{k_0}\}\subset\{s(F_2)>0\}\subset\bigcup_{k=k_0}^\infty\{s(a^k)>0\}\subset\bigcup_{k=k_0}^\infty\{\nu_k\neq\infty\}.$$

Consequently,

$$\begin{aligned}&\mathbb{E}\left(\Phi\left(\frac{2^{k_0}\chi(s(F_2)>2^{k_0})}{I}\right)\right)\\&=\mathbb{E}\left(\Phi\left(\frac{2^{k_0}}{I}\right)\chi(s(F_2)>2^{k_0})\right)\le\sum_{k\ge k_0}\mathbb{E}\left(\Phi\left(\frac{2^{k_0}}{I}\right)\chi(\nu_k\neq\infty)\right)\\&=\sum_{k\ge k_0}\mathbb{E}\left(\Phi\left(\frac{2^{k_0}\chi(\nu_k\neq\infty)}{I}\right)\right)\le\sum_{k\ge k_0}\mathbb{E}\left(\Phi\left(\frac{2^{k_0}\chi(\nu_k\neq\infty)}{C\cdot 2^k\|\chi(\nu_k\neq\infty)\|_{L_\Phi}}\right)\right)\\&\le C^{-p}\sum_{k\ge k_0}2^{p(k_0-k)}\mathbb{E}\left(\Phi\left(\frac{\chi(\nu_k\neq\infty)}{\|\chi(\nu_k\neq\infty)\|_{L_\Phi}}\right)\right)\le C^{-p}\sum_{k\ge k_0}2^{p(k_0-k)}=C,\end{aligned}$$

where the third "$\leq$" above is due to Lemma 2.1. Thus, by the definition of L_Φ norm we obtain

$$2^{k_0}\|\chi(s(F_2) > 2^{k_0})\|_{L_\Phi} = \|2^{k_0}\chi(s(F_2) > 2^{k_0})\|_{L_\Phi} \leq CI.$$

This implies $\|s(F_2)\|_{wL_\Phi} \leq CI$.

Set $1 < L < \infty$, $0 < \lambda < 1 - 1/L$, and denote by L' the conjugate number of L such that $1/L + 1/L' = 1$. By Hölder's inequality we have

$$\begin{aligned} s(F_1) &\leq \sum_{k\leq k_0-1} \mu_k s(a^k) = \sum_{k\leq k_0-1} 2^{k\lambda}\cdot 2^{-k\lambda}\mu_k s(a^k) \\ &\leq \left(\sum_{k\leq k_0-1} 2^{k\lambda L'}\right)^{1/L'} \left(\sum_{k\leq k_0-1} 2^{-k\lambda L}\mu_k^L (s(a^k))^L\right)^{1/L} \\ &\leq \sum_{k\leq k_0-1} 2^{k\lambda} \left(\sum_{k\leq k_0-1} 2^{-k\lambda L}\mu_k^L (s(a^k))^L\right)^{1/L} \\ &\leq C2^{k_0\lambda} \left(\sum_{k\leq k_0-1} 2^{-k\lambda L}\mu_k^L (s(a^k))^L\right)^{1/L}. \end{aligned}$$

Applying Lemma 3.4, we have

$$\begin{aligned} &\|\chi(s(F_1) > 2^{k_0})\|_{L_\Phi} \\ &\leq \left\|\chi(s(F_1) > 2^{k_0})\left(\frac{s(F_1)}{2^{k_0}}\right)^L\right\|_{L_\Phi} \leq C2^{k_0L(\lambda-1)}\left\|\sum_{k\leq k_0-1} 2^{-k\lambda L}\mu_k^L (s(a^k))^L\right\|_{L_\Phi} \\ &\leq C2^{k_0L(\lambda-1)}\sum_{k\leq k_0-1} 2^{-k\lambda L}\mu_k^L\|s(a^k)^L\|_{L_\Phi} \leq C2^{k_0L(\lambda-1)}\sum_{k\leq k_0-1} 2^{-k\lambda L}2^{kL}\|\chi(\nu_k\neq\infty)\|_{L_\Phi} \\ &= C2^{k_0L(\lambda-1)}\sum_{k\leq k_0-1} 2^{kL(1-\lambda)}2^{-k}\cdot 2^k\|\chi(\nu_k\neq\infty)\|_{L_\Phi} \\ &\leq I\cdot C2^{k_0L(\lambda-1)}\sum_{k\leq k_0-1} 2^{k[L(1-\lambda)-1]} = I\cdot C2^{-k_0}. \end{aligned}$$

Thus one can conclude that

$$2^{k_0}\|\chi(s(F_1) > 2^{k_0})\|_{L_\Phi} \leq CI,$$

that is, $\|s(F_1)\|_{wL_\Phi} \leq CI$. Note the fact that $\|\cdot\|_{wL_\Phi}$ is a quasi-norm when Φ satisfies the condition in Theorem 3.2. So we obtain

$$\|s(f)\|_{wL_\Phi} \leq C\big(\|s(F_1)\|_{wL_\Phi} + \|s(F_2)\|_{wL_\Phi}\big) \leq CI.$$

Consequently, $f \in wH_\Phi^s$ and (3.3) holds. This completes the proof of Theorem 3.2. □

Remark 3.6. If $f \in w\mathscr{H}_\Phi^s$ in Theorem 3.2, then in addition to (3.1) and (3.2), we have the following convergence result:

$$\text{the sum } \sum_{k=i}^{j} \mu_k a^k \text{ converges to } f \text{ in } wH_\Phi^s \text{ as } j \to \infty,\ i \to -\infty.$$

In fact,

$$f - \sum_{k=i}^{j} \mu_k a^k = (f - f^{\nu_{j+1}}) + f^{\nu_i}.$$

Note that

$$s^2(f - f^{\nu_{j+1}}) = \sum_{n \in \mathbb{N}^2} \mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_{j+1} \ll n) = s^2(f) - s^2(f^{\nu_{j+1}})$$

and

$$s^2(f^{\nu_i}) = \sum_{n \in \mathbb{N}^2} \mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_i \not\ll n),$$

we obtain

$$s(f - f^{\nu_{j+1}}),\ s(f^{\nu_i}) \le s(f) \quad \text{and} \quad s(f - f^{\nu_{j+1}}),\ s(f^{\nu_i}) \to 0 \ \text{ a.e. as } j \to \infty,\ i \to -\infty.$$

From Proposition 2.6 it follows that

$$\|s(f - f^{\nu_{j+1}})\|_{wL_\Phi},\ \|s(f^{\nu_i})\|_{wL_\Phi} \to 0 \quad \text{as } j \to \infty,\ i \to -\infty.$$

Consequently, by the sublinearity of s we have

$$\begin{aligned}\left\| f - \sum_{k=i}^{j} \mu_k a^k \right\|_{wH_\Phi^s} &= \|s(f - f^{\nu_{j+1}} + f^{\nu_i})\|_{wL_\Phi} \le \|s(f - f^{\nu_{j+1}}) + s(f^{\nu_i})\|_{wL_\Phi} \\ &\le C\big(\|s(f - f^{\nu_{j+1}})\|_{wL_\Phi} + \|s(f^{\nu_i})\|_{wL_\Phi}\big),\end{aligned}$$

and

$$\left\| f - \sum_{k=i}^{j} \mu_k a^k \right\|_{wH_\Phi^s} \to 0 \quad \text{as } j \to \infty,\ i \to -\infty.$$

Furthermore, note the facts that $L_2 = H_2^s \subset H_\Phi^s \subset wH_\Phi^s$ (see Proposition 2.3) and $a^k = (a_n^k, n \in \mathbb{N}^2)$ is L_2-bounded for every $k \in \mathbb{Z}$, thus $L_2 = H_2^s$ is dense in $w\mathscr{H}_\Phi^s$.

Remark 3.7. Let $0 < p \le q < 2$ and let $\Phi \in \mathcal{O}$ be p-convex and q-concave. If the martingale $f \in \mathcal{M}$ has a decomposition of type (3.1), where $\{\mu_k\}_{k\in\mathbb{Z}}$ and $\{a^k\}_{k\in\mathbb{Z}}$ are just the same as the statement in Theorem 3.2, then similar to the proof of the converse part of Theorem 3.2 we can prove that

$$\|f\|_{wH_\Phi^S} \le C \sup_{k\in\mathbb{Z}} \mu_k \quad \text{and} \quad \|f\|_{wH_\Phi} \le C \sup_{k\in\mathbb{Z}} \mu_k.$$

Indeed, we only need to replace the operator s with operators S and M in the proof of the converse part of Theorem 3.2, respectively.

If $\mathcal{F}$ is regular then the previous theorem can be shown for wH_Φ^S as well.

Theorem 3.8. *Suppose that $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N}^2)$ is regular. Then Theorem 3.2 also holds for wH_Φ^S.*

Proof. For every $k \in \mathbb{Z}$, set

$$F_k = \{S(f) > 2^k\} \quad \text{and} \quad \nu_k = \inf\left\{n \in \mathbb{N}^2 : \mathbb{E}_n \chi(F_k) > \frac{1}{2R^2}\right\},$$

where R is the regularity constant. Since L_2 is also isometric to H_2^S, we only need to modify the inequality (3.4) to the following one:

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}^2} |d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\right) \le 2 \cdot (2^{k+1})^2 P(\nu_k \neq \infty).$$

Accordingly, we define the formulas (G) and (H) as follows:

$$(G) = \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1}) \chi(S(f) \le 2^{k+1})\big)$$

and

$$(H) = \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1}) \chi(S(f) > 2^{k+1})\big).$$

It follows from the regularity of $\mathcal{F}$ that $|d_n f|^2 \le R^2 \mathbb{E}_{n-1} |d_n f|^2$, and we obtain

$$\begin{aligned}
(H) &\le \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(R^2 \mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1}) \chi(S(f) > 2^{k+1})\big) \\
&= R^2 \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(\mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1}) \mathbb{E}_{n-1}\chi(S(f) > 2^{k+1})\big) \\
&= R^2 \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(\mathbb{E}_{n-1}|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1}) \mathbb{E}_{n-1}\chi(F_{k+1})\big) \\
&\le R^2 \frac{1}{2R^2} \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(|d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\big) \\
&= \frac{1}{2}\mathbb{E}\left(\sum_{n\in\mathbb{N}^2} |d_n f|^2 \chi(\nu_k \ll n \not\gg \nu_{k+1})\right).
\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.2, so we omit it. □

4. Bounded operators on two parameter weak Orlicz–Hardy martingale spaces

As an application, in this section, we first obtain a sufficient condition for a sublinear operator to be bounded from two-parameter weak Orlicz–Hardy martingale space to usual weak Orlicz space by atomic decomposition. Immediately, some martingale inequalities are deduced by equipping this condition to the sublinear operators M, S and s, respectively.

Let T be an operator defined on a martingale space X and taking values in a measurable function space Y. T is said to be sublinear if for any martingale $f, g \in X$ and complex number λ, the following

$$|T(f+g)| \leq |Tf| + |Tg|, \quad |T(\lambda f)| = |\lambda||Tf|.$$

hold.

Now we give a theorem below without proof, since the proof is similar to that of Theorem 3.1 in [8].

Theorem 4.1. *Let $1 \leq r \leq 2$ and $T\colon L_r(\Omega) \to L_r(\Omega)$ be a bounded sublinear operator. If*

$$P(|Ta| > 0) \leq CP(\nu \neq \infty)$$

for all $(\Phi, 2)$ atoms a, where ν is the corresponding stopping time, then, for every p-convex and q-concave function $\Phi \in \mathcal{O}$ with $0 < p \leq q < r$,

$$\|Tf\|_{wL_\Phi} \leq C\|f\|_{wH_\Phi^s}, \quad f \in wH_\Phi^s.$$

The following result follows immediately from Theorem 3.8.

Theorem 4.2. *Suppose that $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N}^2)$ is regular. Let $1 \leq r \leq 2$ and let $T\colon L_r(\Omega) \to L_r(\Omega)$ be a bounded sublinear operator. If*

$$P(|Ta| > 0) \leq CP(\nu \neq \infty)$$

for all $(\Phi, 2)$ atoms a, where ν is the corresponding stopping time, then, for every p-convex and q-concave function $\Phi \in \mathcal{O}$ with $0 < p \leq q < r$,

$$\|Tf\|_{wL_\Phi} \leq C\|f\|_{wH_\Phi^S}, \quad f \in wH_\Phi^S.$$

Next we apply Theorems 4.1 and 4.2 to obtain some martingale inequalities.

Proposition 4.3. *Let $0 < p \leq q < 2$ and $\Phi \in \mathcal{O}$ be a p-convex and q-concave function. Then*

$$\|f\|_{wH_\Phi} \leq C\|f\|_{wH_\Phi^s}, \quad \|f\|_{wH_\Phi^S} \leq C\|f\|_{wH_\Phi^s} \tag{4.1}$$

hold for all $f \in \mathcal{M}$, namely, $wH_\Phi^s \subset wH_\Phi$, $wH_\Phi^s \subset wH_\Phi^S$. In addition, if $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N}^2)$ is regular, then

$$wH_\Phi^s = wH_\Phi^S \subset wH_\Phi. \tag{4.2}$$

Proof. First we show (4.1). Let $f \in wH_\Phi^s$. For the first inequality of (4.1), we consider the operator T in Theorem 4.1 to be the maximal operator M, that is, $Tf = Mf$. Obviously, M is sublinear and $\|Mf\|_2 \leq 4\|f\|_2$ (see [20, Proposition 3.4]). If a is a $(\Phi, 2)$ atom and ν is the stopping time associated with a, then

$$\{|Ta| > 0\} = \{|Ma| > 0\} \subset \{\nu \neq \infty\}$$

and hence $P(|Ta| > 0) \leq P(\nu \neq \infty)$. Since $q < 2$, it follows from Theorem 4.1 that

$$\|f\|_{wH_\Phi} = \|Tf\|_{wL_\Phi} \leq C\|f\|_{wH_\Phi^s}.$$

Similarly, considering the operator $Tf = Sf$ we get the second inequality of (4.1).

Now we check (4.2). Assume that the stochastic basis $\mathcal{F}$ is regular, and let $f \in wH_\Phi^S$. Considering the operator T in Theorem 4.2 to be the conditional quadratic variation operator s. Then by Theorem 4.2 we obtain the following inequality

$$\|f\|_{wH_\Phi^s} \leq C\|f\|_{wH_\Phi^S}.$$

Combining with (4.1), one can conclude that (4.2) holds. □

Remark 4.4. It should be noted that Proposition 4.3 can be proved directly with the help of the atomic decomposition theorems in Section 3. Indeed, let $f \in wH_\Phi^s$. Then by Theorem 3.2 there exists a decomposition such that (3.1) and (3.2) hold. Hence, it follows from Remark 3.7 that

$$C^{-1}\|f\|_{wH_\Phi} \leq \sup_{k\in\mathbb{Z}} \mu_k \leq C\|f\|_{wH_\Phi^s} \quad \text{and} \quad C^{-1}\|f\|_{wH_\Phi^S} \leq \sup_{k\in\mathbb{Z}} \mu_k \leq C\|f\|_{wH_\Phi^s},$$

which complete (4.1).

On the other hand, assume that the stochastic basis $\mathcal{F}$ is regular, and let $f \in wH_\Phi^S$. By Theorem 3.8, there exist a sequence of $(\Phi, 2)$ atoms $(a^k)_{k\in\mathbb{Z}}$ and a sequence of positive numbers of $(\mu_k)_{k\in\mathbb{Z}} \in l_\infty$ such that

$$\sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k = f_n \text{ a.e.} \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \mu_k \leq C\|f\|_{wH_\Phi^S}.$$

From the converse part of Theorem 3.2, we conclude that

$$C^{-1}\|f\|_{wH_\Phi^s} \leq \sup_{k\in\mathbb{Z}} \mu_k \leq C\|f\|_{wH_\Phi^S}.$$

That is, $wH_\Phi^S \subset wH_\Phi^s$. Combining with (4.1), we obtain (4.2).

5. On the dual spaces of $w\mathscr{H}_\Phi^s$

In this section, we first introduce the two-parameter weak generalized Campanato martingale space $w\mathcal{L}_{q,\varphi}$, which is similar to [8, Definition 0.1], then the dual space of the two-parameter weak martingale Orlicz–Hardy space $w\mathscr{H}_\Phi^s$ is characterized.

Definition 5.1. For $q \in [1, \infty)$ and a function $\varphi\colon (0,\infty) \to (0,\infty)$, let

$$w\mathcal{L}_{q,\varphi} := \left\{ f \in L_q : \|f\|_{w\mathcal{L}_{q,\varphi}} = \int_0^\infty \frac{t_\varphi^q(x)}{x}\, dx < \infty \right\},$$

where

$$t_\varphi^q(x) := \frac{1}{\varphi(x)} x^{-1/q} \sup_{P(\nu \neq \infty) \le x} \|f - f^\nu\|_q$$

and f^ν is the two-parameter stopped martingale with respect to the stopping time ν.

Now we are ready to describe the duality theorem.

Theorem 5.2. *Let $0 < p \le q < 2$ and let $\Phi \in \mathcal{O}$ be p-convex and q-concave. Then*

$$(w\mathscr{H}_\Phi^s)' = w\mathcal{L}_{2,\varphi},$$

where $\varphi(r) = \frac{1}{r\Phi^{-1}\left(\frac{1}{r}\right)}$.

Proof. Assume that $g \in w\mathcal{L}_{2,\varphi}$. Then $g \in H_2^s$. Define

$$l_g(f) = \mathbb{E}(fg), \quad f \in H_2^s.$$

Note that $H_2^s \subset w\mathscr{H}_\Phi^s$. By Theorem 3.2, there exist a sequence of $(\Phi, 2)$ atoms $(a^k)_{k\in\mathbb{Z}}$ with respect to the stopping times $(\nu_k)_{k\in\mathbb{Z}}$ and a sequence of positive numbers $(\mu_k)_{k\in\mathbb{Z}} \in l_\infty$ such that

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k \quad \text{a.e.}$$

It is easy to check that the last series converges to f in H_2^s as well. Therefore,

$$l_g(f) = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}(a^k g).$$

By Definition 3.1(i) of the atom a^k,

$$\mathbb{E}(a^k g) = \mathbb{E}[a^k(g - g^{\nu_k})].$$

Indeed,

$$\begin{aligned}\mathbb{E}(a^k g) &= \sum_{n\in\mathbb{N}^2} \mathbb{E}(d_n a^k d_n g) = \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(d_n a^k \chi(\nu_k \ll n) d_n g\big) \\ &= \sum_{n\in\mathbb{N}^2} \mathbb{E}\big(d_n a^k d_n (g - g^{\nu_k})\big) = \mathbb{E}[a^k(g - g^{\nu_k})].\end{aligned}$$

Recall that $\mu_k = 4\sqrt{2}\cdot 2^{k+1}\|\chi(\nu_k\neq\infty)\|_{L_\Phi}$. It follows from the Hölder inequality and Definition 3.1(ii) that

$$\begin{aligned}|l_g(f)| &= \left|\sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}[a^k(g-g^{\nu_k})]\right| \le \sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}|a^k(g-g^{\nu_k})|\\ &\le \sum_{k\in\mathbb{Z}}\mu_k\|a^k\|_2\|g-g^{\nu_k}\|_2 \le \sum_{k\in\mathbb{Z}}\mu_k\|M(a^k)\|_2\|g-g^{\nu_k}\|_2\\ &\le C\sum_{k\in\mathbb{Z}}2^k\|\chi(\nu_k\neq\infty)\|_{L_\Phi}\cdot\frac{P(\nu_k\neq\infty)^{1/2}}{\|\chi(\nu_k\neq\infty)\|_{L_\Phi}}\cdot\|g-g^{\nu_k}\|_2\\ &\le C\sum_{k\in\mathbb{Z}}2^kP(\nu_k\neq\infty)^{1/2}\|g-g^{\nu_k}\|_2.\end{aligned}$$

Note the facts that $P(\nu_k\neq\infty)\le 64P(F_k) = 64P(s(f)>2^k)$ and $\Phi\big(\frac{2^k}{\|f\|_{wH^s_\Phi}}\big)P(s(f)>2^k)\le 1$. It follows from Lemma 2.1 that

$$P(\nu_k\neq\infty)\le\frac{64}{\Phi\big(\frac{2^k}{\|f\|_{wH^s_\Phi}}\big)}\le\frac{1}{\Phi\big(64^{-1/p}\cdot\frac{2^k}{\|f\|_{wH^s_\Phi}}\big)}.$$

Denote $c_k = \frac{1}{\Phi\big(64^{-1/p}\cdot\frac{2^k}{\|f\|_{wH^s_\Phi}}\big)}$. Then $P(\nu_k\neq\infty)\le c_k$ and

$$\begin{aligned}|l_g(f)| &\le C\sum_{k\in\mathbb{Z}}2^kc_k^{1/2}\|g-g^{\nu_k}\|_2 \le C\|f\|_{wH^s_\Phi}\sum_{k\in\mathbb{Z}}64^{-1/p}\cdot\frac{2^k}{\|f\|_{wH^s_\Phi}}c_k^{1/2}\|g-g^{\nu_k}\|_2\\ &= C\|f\|_{wH^s_\Phi}\sum_{k\in\mathbb{Z}}\Phi^{-1}\left(\frac{1}{c_k}\right)c_k^{1/2}\|g-g^{\nu_k}\|_2\\ &\le C\|f\|_{wH^s_\Phi}\sum_{k\in\mathbb{Z}}\frac{1}{\varphi(c_k)}c_k^{-1/2}\sup_{P(\nu\neq\infty)\le c_k}\|g-g^{\nu}\|_2 = C\|f\|_{wH^s_\Phi}\sum_{k\in\mathbb{Z}}t^2_\varphi(c_k).\end{aligned}$$

Applying Lemma 2.1 once more, we obtain

$$\frac{c_{k+1}}{c_k} = \frac{\Phi\big(64^{-1/p}\cdot\frac{2^k}{\|f\|_{wH^s_\Phi}}\big)}{\Phi\big(64^{-1/p}\cdot\frac{2^{k+1}}{\|f\|_{wH^s_\Phi}}\big)}\le\left(\frac{2^k}{2^{k+1}}\right)^p = \left(\frac{1}{2}\right)^p.$$

Hence,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}t^2_\varphi(c_k) &= \sum_{k\in\mathbb{Z}}\frac{t^2_\varphi(c_k)(c_k-c_{k+1})}{c_k-c_{k+1}} \le \frac{1}{1-(1/2)^p}\sum_{k\in\mathbb{Z}}\frac{t^2_\varphi(c_k)(c_k-c_{k+1})}{c_k}\\ &\le C\int_0^\infty\frac{t^2_\varphi(x)}{x}\,dx = C\|g\|_{w\mathcal{L}_{2,\varphi}}.\end{aligned}$$

Consequently,

$$|l_g(f)|\le C\|f\|_{wH^s_\Phi}\|g\|_{w\mathcal{L}_{2,\varphi}},$$

that is, l_g is a continuous linear functional on H_2^s. Since H_2^s is dense in $w\mathscr{H}_\Phi^s$ (see Remark 3.6), l_g can be uniquely extended to a continuous linear functional on $w\mathscr{H}_\Phi^s$.

For the converse part, let $l \in (w\mathscr{H}_\Phi^s)'$. Since $H_2^s \subset w\mathscr{H}_\Phi^s$, we have $l \in (H_2^s)'$ which implies that there exists $g \in H_2^s$ such that

$$l(f) = \mathbb{E}(fg), \quad f \in H_2^s.$$

Suppose that ν_k are the stopping times such that $P(\nu_k \neq \infty) \leq 2^{-k}$, $k \in \mathbb{Z}$. Let

$$a^k = \frac{\Phi^{-1}(2^k)(g - g^{\nu_k})}{(2^k)^{1/2}\|s(g - g^{\nu_k})\|_2}, \quad k \in \mathbb{Z}.$$

Obviously, $a_n^k = 0$ on the set $\{\nu_k \not< n\}$ for each fixed k. For any given $y > 0$, we can find $m \in \mathbb{Z}$ and $N \in \mathbb{N}$ satisfying $2^{m-1} \leq y < 2^m$ and $|m| \leq N$. Now let us define the martingales f^N, g^N and h^N by

$$f^N = \sum_{k=-N}^{N} a^k, \quad g^N = \sum_{k=-N}^{m-1} a^k, \quad h^N = \sum_{k=m}^{N} a^k,$$

respectively. Since Φ is p-convex and q-concave for $0 < p \leq q < 2$, one can conclude that $\Phi(t+s) \leq 4(\Phi(t) + \Phi(s))$ for any $t, s \geq 0$. In fact, by Lemma 2.1, we have

$$\Phi(t+s) \leq \Phi(2\max\{t,s\}) \leq 2^q\Phi(\max\{t,s\}) \leq 4(\Phi(t) + \Phi(s)).$$

Then, by the sublinearity of s, we get

$$\Phi(s(f^N)) \leq \Phi(s(g^N) + s(h^N)) \leq 4(\Phi(s(g^N)) + \Phi(s(h^N))).$$

Hence,

$$\begin{aligned} P(\Phi(s(f^N)) > 8y) &\leq P(\Phi(s(g^N)) + \Phi(s(h^N)) > 2y) \\ &\leq P(\Phi(s(g^N)) > y) + P(\Phi(s(h^N)) > y), \end{aligned}$$

that is,

$$P(s(f^N) > \Phi^{-1}(8y)) \leq P(s(g^N) > \Phi^{-1}(y)) + P(s(h^N) > \Phi^{-1}(y)).$$

Since

$$\|s(g^N)\|_2 \leq \sum_{k=-N}^{m-1} \|s(a^k)\|_2 \leq \sum_{k=-N}^{m-1} (2^{-k})^{1/2}\Phi^{-1}(2^k),$$

then by Lemma 2.1 we obtain

$$\begin{aligned} &P(s(g^N) > \Phi^{-1}(y)) \\ &\leq \frac{1}{(\Phi^{-1}(y))^2}\|s(g^N)\|_2^2 \leq \left(\sum_{k=-N}^{m-1} \frac{(2^{-k})^{1/2}\Phi^{-1}(2^k)}{\Phi^{-1}(y)}\right)^2 \leq \left(\sum_{k=-N}^{m-1} (2^{-k})^{1/2}\left(\frac{2^k}{y}\right)^{1/q}\right)^2 \\ &\leq y^{-2/q}\left(\sum_{k=-N}^{m-1} \left(2^{1/q-1/2}\right)^k\right)^2 \leq y^{-2/q}\left(\sum_{k=-\infty}^{m-1} \left(2^{1/q-1/2}\right)^k\right)^2 \leq C_1 y^{-1}. \end{aligned}$$

The last inequality above follows from $q<2$. Taking $C_1'=2\max\{C_1,1\}$. Then

$$\Phi\left(\frac{\Phi^{-1}(y)}{(C_1')^{1/p}}\right)P(s(g^N)>\Phi^{-1}(y))\le\frac{1}{C_1'}yP(s(g^N)>\Phi^{-1}(y))\le\frac{1}{2}.$$

Recall that $a_n^k=0$ on the set $\{\nu_k\not\ll n\}$ and $P(\nu_k\neq\infty)\le 2^{-k}$, we have

$$P(s(h^N)>\Phi^{-1}(y))\le\sum_{k=m}^{N}P(\nu_k\neq\infty)\le\sum_{k=m}^{\infty}2^{-k}=2^{1-m}\le 2y^{-1}$$

and

$$\Phi\left(\frac{\Phi^{-1}(y)}{4^{1/p}}\right)P(s(h^N)>\Phi^{-1}(y))\le\frac{1}{2}.$$

Taking $C=8^{1/p}\max\left\{(C_1')^{1/p},4^{1/p}\right\}$. Then

$$\begin{aligned}&\Phi\left(\frac{\Phi^{-1}(8y)}{C}\right)P(s(f^N)>\Phi^{-1}(8y))\\&\le\Phi\left(\frac{\Phi^{-1}(8y)}{C}\right)P(s(g^N)>\Phi^{-1}(y))+\Phi\left(\frac{\Phi^{-1}(8y)}{C}\right)P(s(h^N)>\Phi^{-1}(y))\\&\le\Phi\left(\frac{8^{1/p}\Phi^{-1}(y)}{8^{1/p}(C_1')^{1/p}}\right)P(s(g^N)>\Phi^{-1}(y))+\Phi\left(\frac{8^{1/p}\Phi^{-1}(y)}{8^{1/p}4^{1/p}}\right)P(s(h^N)>\Phi^{-1}(y))\\&\le\frac{1}{2}+\frac{1}{2}=1,\end{aligned}$$

which means $\|f^N\|_{wH_\Phi^s}\le C$. Since

$$\begin{aligned}l(f^N)&=\mathbb{E}(f^Ng)=\sum_{k=-N}^{N}\mathbb{E}(a^kg)=\sum_{k=-N}^{N}\mathbb{E}[a^k(g-g^{\nu_k})]=\sum_{k=-N}^{N}\frac{\Phi^{-1}(2^k)\|g-g^{\nu_k}\|_2^2}{(2^k)^{1/2}\|s(g-g^{\nu_k})\|_2}\\&=\sum_{k=-N}^{N}\frac{\Phi^{-1}(2^k)}{(2^k)^{1/2}}\|g-g^{\nu_k}\|_2=\sum_{k=-N}^{N}\frac{1}{\varphi(2^{-k})}(2^k)^{1/2}\|g-g^{\nu_k}\|_2,\end{aligned}$$

we get

$$\sum_{k=-N}^{N}\frac{1}{\varphi(2^{-k})}(2^{-k})^{-1/2}\|g-g^{\nu_k}\|_2=l(f^N)\le C\|l\|.$$

Let $N\to\infty$, while taking the supremum over all of such stopping times satisfying $P(\nu_k\neq\infty)\le 2^{-k}$, $k\in\mathbb{Z}$, we can immediately conclude that

$$\|g\|_{w\mathcal{L}_{2,\varphi}}=\int_0^\infty\frac{t_\varphi^2(x)}{x}\,dx\le C\sum_{k=-\infty}^{\infty}t_\varphi^2(2^{-k})\le C\|l\|.$$

The proof is finished. □

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