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Composition Operators on Hilbert Spaces of Dirichlet Series

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Abstract. Motivated by a theorem of Gordon and Hedenmalm in 1999, the study of composition operators acting on various scales of function spaces of Dirchlet series has arisen intensive interest. In this paper, we characterize the boundedness of composition operators induced by specific Dirichlet series symbols from Bergman space to Hardy space of Dirichlet series.

1. Introduction

Let $S = S(G)$ be the class of all analytic self-maps of a domain G of the complex plane $\mathbb{C}$ and $H(G)$ denote the space of all analytic functions on G. Each $\varphi \in S$ induces a composition operator $C_\varphi \colon H(G) \to H(G)$ defined as follows:

$$C_\varphi f := f \circ \varphi.$$

With regard to the theory of composition operators acting on analytic function spaces, Gordon and Hedenmalm [11] initiated the study of composition operators on the Hardy space of Dirichlet series $\mathscr{H}^2$. A Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ belongs to $\mathscr{H}^2$ if

$$\|f\|_{\mathscr{H}^2}^2 := \sum_{n=1}^\infty |a_n|^2 < \infty.$$

By the Cauchy–Schwarz inequality, we observe that the elements of $\mathscr{H}^2$ are analytic in the half-plane $\mathbb{C}_{1/2}$, where for any real number θ,

$$\mathbb{C}_\theta := \{s \in \mathbb{C} : \operatorname{Re} s > \theta\}.$$

The present paper is devoted to discussing the basic properties of composition operators between Bergman spaces and Hardy spaces of Dirichlet series.

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First of all, let us recall our function spaces to work on. Let $\{\omega_n\}_{n\geq 1}$ be a sequence of nonnegative numbers. The space $\mathscr{H}_\omega$ is defined by

$$\mathscr{H}_\omega = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \|f\|_{\mathscr{H}_\omega} = \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{\omega_n} \right)^{1/2} < \infty \right\}.$$

For $\alpha > 0$, the Bergman space $\mathscr{D}_\alpha$ consists of Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ such that

$$\|f\|_{\mathscr{D}_\alpha}^2 := \sum_{n=1}^{\infty} \frac{|a_n|^2}{[d(n)]^\alpha} < \infty,$$

where $d(n)$ denotes the number of divisors of the positive integer n. Due to $d(n) = \mathcal{O}(n^\epsilon)$ for every $\epsilon > 0$ (see [12, Theorem 315]), one readily observes that $\mathscr{D}_\alpha$ is a space of analytic functions in $\mathbb{C}_{1/2}$ by Cauchy–Schwarz inequality.

The algebra $\mathscr{H}^\infty$ of bounded Dirichlet series on the right half-plane is defined in [15] by Maurizi and Queffélec. We shall denote by $\|\cdot\|_\infty$ the norm on $\mathscr{H}^\infty$

$$\|f\|_\infty := \sup_{\operatorname{Re} s > 0} |f(s)|.$$

As is well known, the main result in [8, 18] demonstrated that composition operator C_φ is bounded on $\mathscr{H}^2$ if and only if φ is a member of the following class.

Definition 1.1. The Gordon–Hedenmalm class, denoted $\mathscr{G}$, is the class of analytic functions $\varphi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ which can be expressed as

$$\varphi(s) = c_0 s + \psi(s),$$

where c_0 is a nonnegative integer, the Dirichlet series ψ converges uniformly in $\mathbb{C}_\epsilon$ for every $\epsilon > 0$ and has the following properties:

(a) If $c_0 > 0$, then either $\psi \equiv 0$ or $\psi(\mathbb{C}_0) \subseteq \mathbb{C}_0$;

(b) If $c_0 = 0$, then $\psi(\mathbb{C}_0) \subseteq \mathbb{C}_{1/2}$.

From then on, the research of composition operators on various scales of function spaces of Dirichlet series has aroused great interest, see for example [1–5, 7, 10, 16, 17]. In particular, Bailleul and Bervig [2] extended the boundedness of composition operator to Bergman spaces. Namely, for $\alpha > 0$, a function $\varphi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ induces a bounded composition operator $C_\varphi\colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$, where

$$\beta = \begin{cases} 2^\alpha - 1 & \text{if } c_0 = 0, \\ \alpha & \text{if } c_0 \geq 1 \end{cases}$$

if and only if $\varphi \in \mathscr{G}$. In this paper, we follow the line of research to give some sufficient and necessary conditions for the boundedness of composition operators induced by specific Dirichlet series symbols from Bergman space $\mathscr{D}_\alpha$ to Hardy space $\mathscr{H}^2$. To simplify the following description, we use $\mathscr{G}_0$ to denote the subclass of $\mathscr{G}$ of $c_0 = 0$.

We need several notations to state our results. Let f be a Dirichlet series which converges uniformly in $\mathbb{C}_\epsilon$ with $\epsilon > 0$. For $w \neq f(+\infty)$, the mean counting function is defined by

$$\mathscr{M}_f(w) = \lim_{\delta \to 0^+} \lim_{T \to \infty} \frac{\pi}{T} \sum_{\substack{s \in f^{-1}(\{w\}) \\ \delta < \operatorname{Re} s < \infty \\ |\operatorname{Im} s| \leq T}} \operatorname{Re} s.$$

In this paper, we give the following sufficient condition for the boundedness of composition operators.

Theorem 1.2. *Let $\alpha > 0$ and $\varphi \in \mathscr{G}_0$. If $\operatorname{Im} \varphi$ is bounded on $\mathbb{C}_0$ and $\mathscr{M}_\varphi(s) = \mathcal{O}(\operatorname{Re} s - 1/2)^{2^\alpha}$ for $\operatorname{Re} s \to (1/2)^+$, then C_φ is bounded from $\mathscr{D}_\alpha$ to $\mathscr{H}^2$.*

As for the necessary condition, we obtain the following result.

Theorem 1.3. *Let $\alpha > 0$ and $\varphi \in \mathscr{G}_0$. If C_φ is bounded from $\mathscr{D}_\alpha$ to $\mathscr{H}^2$, then $\mathscr{M}_\varphi(s) = \mathcal{O}(\operatorname{Re} s - 1/2)^{2^\alpha}$ for $\operatorname{Re} s \to (1/2)^+$.*

The structure of the paper is organized as follows. In Section 2, we recall some notations and basic facts to be used in the sequel. Section 3 is devoted to the proof of Theorems 1.2 and 1.3.

Notations. Throughout this paper, we use the letter C to denote absolute constants which may change at every appearance but do not depend on the essential parameters. We write $A \lesssim B$ or equivalently $B \gtrsim A$ if there exists an inessential constant C such that $A \leq CB$. Similarly, we use the notation $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminaries

In this section, we collect some preliminaries that will be needed in the sequel.

2.1. Reproducing kernel

Let $\alpha > 0$. The reproducing kernel of $\mathscr{D}_\alpha$ at w in $\mathbb{C}_{1/2}$ is given by

$$K_\alpha(s, w) := \zeta_\alpha(s + \overline{w}) = \sum_{n=1}^{\infty} [d(n)]^\alpha n^{-(s+\overline{w})}.$$

We recall from [19, pp. 240–241] that

$$\zeta_\alpha(s) := [\zeta(s)]^{2\alpha} \prod_{j=1}^{\infty} \left\{ (1-p_j^{-s})^{2\alpha} (1 + 2^\alpha p_j^{-s} + 3^\alpha p_j^{-2s} + \cdots) \right\} = [\zeta(s)]^{2\alpha} \Phi_\alpha(s), \tag{2.1}$$

where ζ is the Riemann zeta-function, p_j denotes the j-th prime and the Euler product $\Phi_\alpha(s)$ converges absolutely in $\mathbb{C}_{1/2}$ with $\Phi_\alpha(1) \neq 0$. Note that

$$\|K_\alpha(\,\cdot\,, w)\|_{\mathscr{D}_\alpha}^2 = \zeta_\alpha(2 \operatorname{Re} w).$$

Furthermore, if $\{w_n\}$ is any sequence of $\mathbb{C}_{1/2}$ such that $\operatorname{Re} w_n \to (1/2)^+$, then $\frac{K_\alpha(\,\cdot\,,w_n)}{\|K_\alpha(\,\cdot\,,w_n)\|_{\mathscr{D}_\alpha}} \to 0$ uniformly in $\mathbb{C}_{1/2+\epsilon}$ for every $\epsilon > 0$ (see [6]).

Using the reproducing kernels, we have the following estimates for point evaluations.

Lemma 2.1. *Let $\alpha > 0$. Then for every $f \in \mathscr{D}_\alpha$ and $\operatorname{Re} s > 1/2$,*

$$|f(s)| \leq \sqrt{\zeta_\alpha(2 \operatorname{Re} s)} \|f\|_{\mathscr{D}_\alpha}.$$

Proof. A straightforward calculation based on reproducing kernel and duality yields

$$|f(s)|^2 = \left| \langle f, K_\alpha(\,\cdot\,, s) \rangle_{\mathscr{D}_\alpha} \right|^2 \leq \zeta_\alpha(2 \operatorname{Re} s) \|f\|_{\mathscr{D}_\alpha}^2. \qquad \square$$

2.2. Carleson measure

To state our results in terms of Carleson measure, we review the definition of non-conformal spaces. For $\beta > 0$, the non-conformal Bergman space $D_\beta(\mathbb{C}_{1/2})$ is the space of all analytic functions f in $\mathbb{C}_{1/2}$ such that

$$\|f\|_{D_\beta(\mathbb{C}_{1/2})}^2 := \int_{1/2}^{\infty} \int_{\mathbb{R}} |f(\delta + it)|^2 (\delta - 1/2)^{\beta - 1} \, dt d\delta < \infty.$$

For simplicity, let H denote a Hilbert space of functions in $\mathbb{C}_{1/2}$ and μ be a positive Borel measure on $\mathbb{C}_{1/2}$. We say that μ is a Carleson measure for H if there exists an absolute constant C such that

$$\int_{\mathbb{C}_{1/2}} |f(s)|^2 \, d\mu(s) \leq C \|f\|_H^2, \quad f \in H.$$

The infimum of all possible C in this inequality is said to be Carleson norm of μ with respect to H.

A Carleson square is a closed square in $\overline{\mathbb{C}_{1/2}}$ with one of its sides lying on the vertical line $\sigma = 1/2$. In this paper, we mainly consider

$$Q(\tau, \epsilon) = [1/2, 1/2 + \epsilon] \times [\tau - \epsilon/2, \tau + \epsilon/2],$$

where $0 < \epsilon < \infty$ and $\tau \in \mathbb{R}$.

In this paper, we restrict our attention to the spaces $\mathscr{D}_\alpha$, $D_\beta(\mathbb{C}_{1/2})$ for $\alpha, \beta > 0$. Let us start with the following well-known Carleson's characterization for $D_\beta(\mathbb{C}_{1/2})$. It can be found in [21].

Lemma 2.2. *Let $\beta > 0$ and μ be a Borel measure on $\mathbb{C}_{1/2}$. Then μ is a Carleson measure for $D_\beta(\mathbb{C}_{1/2})$ if and only if*

$$\mu(Q(\tau, \epsilon)) = \mathcal{O}(\epsilon^{\beta+1})$$

for every Carleson square $Q(\tau, \epsilon)$, where the implied constant is independent of τ.

Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and D denote the differentiation operator on Dirichlet series,

$$Df(s) := f'(s) = -\sum_{n=2}^{\infty} a_n (\log n) n^{-s}.$$

The following result shows the Carleson measure concerning differentiation operator.

Lemma 2.3. *Let $\alpha > 0$ and μ be a Borel measure on $\mathbb{C}_{1/2}$ with bounded support. Then the following assertions are equivalent:*

(1) *For every Carleson square $Q(\tau, \epsilon)$,*

$$\mu(Q(\tau, \epsilon)) = \mathcal{O}(\epsilon^{2^\alpha+2}),$$

where the implied constant is independent of τ.

(2) *$D\colon \mathscr{D}_\alpha \to L^2(\mu)$ is bounded.*

Auxiliary results will be needed to prove Lemma 2.3.

Lemma 2.4. *Let $\alpha > 0$, $\{\omega_n\}_{n\geq 1}$ be a sequence of nonnegative numbers with $\sum_{n\leq x} \omega_n \approx x(\log x)^{2^\alpha-1}$ and I be a bounded interval of $\mathbb{R}$. Then there exists an absolute constant C such that*

$$\int_{1/2}^{\theta} \int_I |f'(\delta + it)|^2 (\delta - 1/2)^{2^\alpha} \, dt d\delta \leq C \|f\|_{\mathscr{H}_\omega}^2$$

for every $f \in \mathscr{H}_\omega$ and $\theta > 1/2$.

Proof. Let $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ be a Dirichlet polynomial. For $\delta > 1/2$, we calculate by duality and Cauchy–Schwarz inequality

$$\begin{aligned} \int_I |f'(\delta + it)|^2 \, dt &= \sup_g \left| \int_I \sum_{n=2}^{N} a_n (\log n) n^{-\delta} n^{-it} g(t) \, dt \right|^2 \\ &= \sup_g \left| \sum_{n=2}^{N} a_n (\log n) n^{-\delta} \widehat{g}(\log n) \right|^2 \qquad (2.2) \\ &\leq \sum_{n=2}^{N} \frac{|a_n|^2}{\omega_n} \frac{(\log n)^{1+2^\alpha}}{n^{2\delta-1}} \sup_g \sum_{n=2}^{N} \frac{|\widehat{g}(\log n)|^2 \omega_n}{(\log n)^{2^\alpha-1} n}, \end{aligned}$$

where the supremum is taken over all $g \in L^2(I)$ of norm ≤ 1 and $\widehat{g}$ is the Fourier transform of g:

$$\widehat{g}(\xi) = \int_I e^{-it\xi} g(t)\, dt,$$

which extends to an entire function of exponential type ≤ 1 (refer to [13, p. 25]). For such $\widehat{g}$, a inequality of Plancherel–Pólya (see [20, pp. 99–100]) shows that there exists an absolute constant C_1 such that

$$\int_{-\infty}^{\infty} |\widehat{g}'(x)|^2\, dx \le C_1 \int_{-\infty}^{\infty} |\widehat{g}(x)|^2\, dx. \tag{2.3}$$

Moreover, for $\xi \in (k, k+1)$, we have

$$\begin{aligned}
\|\widehat{g}'\|_{L^2(k,k+1)} &= \left(\int_k^{k+1} |\widehat{g}'(x)|^2\, dx \right)^{1/2} \ge \left(\int_k^{\xi} |\widehat{g}'(x)|^2\, dx \right)^{1/2} \\
&= \left(\int_k^{\xi} \left| \int_I (-it) e^{-itx} g(t)\, dt \right|^2 dx \right)^{1/2} \ge \int_k^{\xi} \left| \int_I (-it) e^{-itx} g(t)\, dt \right| dx \\
&\ge \left| \int_k^{\xi} \int_I (-it) e^{-itx} g(t)\, dt dx \right| = \left| \int_I \int_k^{\xi} (-it) e^{-itx}\, dx\, g(t)\, dt \right| \\
&= \left| \int_I (e^{-it\xi} - e^{-itk}) g(t)\, dt \right| = |\widehat{g}(\xi) - \widehat{g}(k)|,
\end{aligned}$$

which implies $|\widehat{g}(\xi)| \le |\widehat{g}(k)| + \|\widehat{g}'\|_{L^2(k,k+1)}$.

For $g \in L^2(I)$, recall that

$$\|g\|_{L^2(I)}^2 = \|\widehat{g}\|_{L^2}^2 = \sum_{k \in \mathbb{Z}} |\widehat{g}(k)|^2. \tag{2.4}$$

Since $\sum_{n \le x} \omega_n \approx x (\log x)^{2^\alpha - 1}$, inequalities (2.3) and (2.4) show that

$$\begin{aligned}
\sum_{n=2}^{N} \frac{|\widehat{g}(\log n)|^2 \omega_n}{(\log n)^{2^\alpha - 1} n} &\le \frac{|\widehat{g}(\log 2)|^2 \omega_2}{(\log 2)^{2^\alpha - 1}} + \sum_{k=1}^{\infty} \sum_{n \in (e^k, e^{k+1})} \frac{|\widehat{g}(\log n)|^2 \omega_n}{(\log n)^{2^\alpha - 1} n} \\
&\le \frac{|\widehat{g}(\log 2)|^2 \omega_2}{(\log 2)^{2^\alpha - 1}} + \sum_{k=1}^{\infty} \frac{2\left(|\widehat{g}(k)|^2 + \|\widehat{g}'\|_{L^2(k,k+1)}^2\right)}{k^{2^\alpha - 1} e^k} \sum_{n \le e^{k+1}} \omega_n \\
&\lesssim \|g\|_{L^2(I)}^2 + \sum_{k=1}^{\infty} \frac{2\left(|\widehat{g}(k)|^2 + \|\widehat{g}'\|_{L^2(k,k+1)}^2\right)}{k^{2^\alpha - 1} e^k} e^{k+1} (k+1)^{2^\alpha - 1} \\
&\le \|g\|_{L^2(I)}^2 + 2^{2^\alpha} e \left(\sum_{k=1}^{\infty} |\widehat{g}(k)|^2 + \int_1^{\infty} |\widehat{g}'(x)|^2\, dx \right) \lesssim \|g\|_{L^2(I)}^2.
\end{aligned} \tag{2.5}$$

Inequalities (2.2) and (2.5) yield

$$\int_{1/2}^{\theta}\int_I |f'(\delta+it)|^2(\delta-1/2)^{2^\alpha}\,dtd\delta \lesssim \sum_{n=2}^{N}\frac{|a_n|^2}{\omega_n}\int_{1/2}^{\theta}(\log n)^{2^\alpha+1}n^{-2(\delta-1/2)}(\delta-1/2)^{2^\alpha}\,d\delta$$
$$\leq \frac{\Gamma(2^\alpha+1)}{2^{2^\alpha+1}}\sum_{n=1}^{N}\frac{|a_n|^2}{\omega_n},$$

where Γ denotes the gamma function. Using the density of Dirichlet polynomials on $\mathscr{H}_\omega$, we complete the proof. □

We immediately observe the following corollary by Lemma 2.4.

Corollary 2.5. *Let $\alpha > 0$, $\sigma > 1$ and $\{\omega_n\}_{n\geq 1}$ be a sequence of nonnegative numbers with $\sum_{n\leq x}\omega_n \approx x(\log x)^{2^\alpha-1}$. Then there exists a constant $C > 0$ such that*

$$\int_{1/2}^{\infty}\int_{\mathbb{R}} |f'(\delta+it)|^2\frac{(\delta-1/2)^{2^\alpha}}{|\delta+it+1/2|^\sigma}\,dtd\delta \leq C\|f\|_{\mathscr{H}_\omega}^2$$

for every $f \in \mathscr{H}_\omega$.

Proof. Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ belong to $\mathscr{H}_\omega$. One readily observes that $|a_n| \leq \sqrt{\omega_n}\|f\|_{\mathscr{H}_\omega}$. Since $\omega_n \leq \sum_{k\leq n}\omega_k \leq n(\log n)^{2^\alpha-1}$ and $\log n = \mathcal{O}(n^\epsilon)$ for every $\epsilon > 0$, we deduce

$$\sqrt{\omega_n}(\log n) \leq n^{1/2}(\log n)^{2^{\alpha-1}+1/2} \lesssim n^{3/4}.$$

Then, for $\operatorname{Re} s > 2$, we have

$$|f'(s)| = \left|\sum_{n=2}^{\infty} a_n(\log n)n^{-s}\right| \leq \sum_{n=2}^{\infty}\frac{|a_n|(\log n)}{n^{\operatorname{Re} s}} \leq \|f\|_{\mathscr{H}_\omega}\sum_{n=2}^{\infty}\frac{\sqrt{\omega_n}(\log n)}{n^{\operatorname{Re} s}}$$
$$\leq \|f\|_{\mathscr{H}_\omega}\sum_{n=2}^{\infty} n^{-\operatorname{Re} s+3/4}$$
$$\leq \|f\|_{\mathscr{H}_\omega}\left(2^{-\operatorname{Re} s+3/4} + \int_2^\infty x^{-\operatorname{Re} s+3/4}\,dx\right) \lesssim 2^{-\operatorname{Re} s}\|f\|_{\mathscr{H}_\omega}.$$

On account of $\sigma > 1$, Lemma 2.4 shows that

$$\int_{1/2}^{\infty}\int_{\mathbb{R}} |f'(\delta+it)|^2\frac{(\delta-1/2)^{2^\alpha}}{|\delta+it+1/2|^\sigma}\,dtd\delta$$
$$= \int_{1/2}^{2}\int_{\mathbb{R}} |f'(\delta+it)|^2\frac{(\delta-1/2)^{2^\alpha}}{|\delta+it+1/2|^\sigma}\,dtd\delta + \int_{2}^{\infty}\int_{\mathbb{R}} |f'(\delta+it)|^2\frac{(\delta-1/2)^{2^\alpha}}{|\delta+it+1/2|^\sigma}\,dtd\delta$$
$$\lesssim \sum_k \frac{1}{(1+k^2)^{\sigma/2}}\int_{1/2}^{2}\int_k^{k+1} |f'(\delta+it)|^2(\delta-1/2)^{2^\alpha}\,dtd\delta$$
$$+ \|f\|_{\mathscr{H}_\omega}^2\int_2^\infty\int_{\mathbb{R}}\frac{2^{-2\delta}(\delta-1/2)^{2^\alpha}}{(1+t^2)^{\sigma/2}}\,dtd\delta$$
$$\lesssim \|f\|_{\mathscr{H}_\omega}^2.$$

This completes the proof. $\square$

Proof of Lemma 2.3. (1) $\Rightarrow$ (2). Let f belong to $\mathscr{D}_\alpha$. Since μ has bounded support, it is easy to see that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}\int_{\mathbb{C}_{1/2}} |f'(s)|^2\, d\mu(s) &= \int_{\mathbb{C}_{1/2}} \left|\frac{f'(s)}{(s+1/2)^{2^{\alpha-1}+2}}\right|^2 |s+1/2|^{2^\alpha+4}\, d\mu(s) \\ &\le C_1 \int_{\mathbb{C}_{1/2}} \left|\frac{f'(s)}{(s+1/2)^{2^{\alpha-1}+2}}\right|^2 d\mu(s).\end{aligned}$$

Put $F(s) = \frac{f'(s)}{(s+1/2)^{2^{\alpha-1}+2}}$. It follows from Lemma 2.2 that condition (1) implies μ is a Carleson measure for $D_{2^\alpha+1}(\mathbb{C}_{1/2})$. By the following asymptotic formula (refer to [19])

$$\sum_{n\le x} [d(n)]^\alpha \approx x(\log x)^{2^\alpha-1},$$

Corollary 2.5 shows that

$$\int_{\mathbb{C}_{1/2}} |f'(s)|^2\, d\mu(s) \lesssim \int_{\mathbb{C}_{1/2}} |F(s)|^2\, d\mu(s) \lesssim \|F\|^2_{D_{2^\alpha+1}} \lesssim \|f\|^2_{\mathscr{D}_\alpha}.$$

(2) $\Rightarrow$ (1). Consider a sequence of Carleson squares $Q_k = Q(\tau_k, \epsilon_k)$ with $\epsilon_k \to 0^+$. Let $s_k = 1/2 + \epsilon_k + i\tau_k$ and define the following functions

$$f_k(s) = \frac{K_\alpha(s, s_k)}{\|K_\alpha(s, s_k)\|_{\mathscr{D}_\alpha}}.$$

By equality (2.1), we have

(2.6)

$$\begin{aligned}|f_k'(s)|^2 = \frac{|\zeta_\alpha'(s+\overline{s_k})|^2}{\zeta_\alpha(2\operatorname{Re} s_k)} &= \frac{\left|([\zeta(s+\overline{s_k})]^{2^\alpha}\cdot \Phi_\alpha(s+\overline{s_k}))'\right|^2}{[\zeta(2\operatorname{Re} s_k)]^{2^\alpha}\Phi_\alpha(2\operatorname{Re} s_k)} \\ &= \frac{\left|2^\alpha[\zeta(s+\overline{s_k})]^{2^\alpha-1}\cdot\zeta'(s+\overline{s_k})\cdot\Phi_\alpha(s+\overline{s_k}) + [\zeta(s+\overline{s_k})]^{2^\alpha}\cdot\Phi_\alpha'(s+\overline{s_k})\right|^2}{[\zeta(2\operatorname{Re} s_k)]^{2^\alpha}\Phi_\alpha(2\operatorname{Re} s_k)} \\ &\ge \frac{\left(2^\alpha|\zeta(s+\overline{s_k})|^{2^\alpha-1}\cdot|\zeta'(s+\overline{s_k})|\cdot|\Phi_\alpha(s+\overline{s_k})| - |\zeta(s+\overline{s_k})|^{2^\alpha}\cdot|\Phi_\alpha'(s+\overline{s_k})|\right)^2}{[\zeta(2\operatorname{Re} s_k)]^{2^\alpha}\Phi_\alpha(2\operatorname{Re} s_k)}.\end{aligned}$$

Let $\Phi_\alpha(s) := \sum_{m=1}^\infty c_m m^{-s}$. Since Φ_α converges absolutely in $\mathbb{C}_{1/2}$, we have for $\operatorname{Re} w > 1/2$,

$$\sum_{m=1}^\infty |c_m| m^{-\operatorname{Re} w} \lesssim 1.$$

Furthermore the fact $\log m = \mathcal{O}(m^\epsilon)$ for every $\epsilon > 0$ yields that

$$\begin{aligned}|\Phi_\alpha'(s+\overline{s_k})| = \left|\sum_{m=2}^\infty c_m (\log m) m^{-s-\overline{s_k}}\right| &\le \sum_{m=2}^\infty |c_m|(\log m) m^{-\operatorname{Re} s - \operatorname{Re} s_k} \\ &\lesssim \sum_{m=2}^\infty |c_m| m^{-\operatorname{Re} s-\operatorname{Re} s_k+1/4}.\end{aligned}$$

Since $\operatorname{Re} s + \operatorname{Re} s_k > 1$, we have $\operatorname{Re} s + \operatorname{Re} s_k - 1/4 > 3/4$. Therefore,

$$|\Phi'_\alpha(s + \overline{s_k})| \lesssim \sum_{m=2}^{\infty} |c_m| m^{-\operatorname{Re} s - \operatorname{Re} s_k + 1/4} \lesssim 1. \tag{2.7}$$

Moreover, for every $s \in Q_k$, it follows from [14, Theorems 8.3 and 8.4] that $|\zeta(s + \overline{s_k})| \approx |\zeta(2 \operatorname{Re} s_k)| \approx (\epsilon_k)^{-1}$ and $|\zeta'(s+\overline{s_k})| \approx (\epsilon_k)^{-2}$. Since Φ_α converges absolutely in $\mathbb{C}_{1/2}$ with $\Phi_\alpha(1) \neq 0$, these results together with inequality (2.7) show that

$$2^\alpha |\zeta(s + \overline{s_k})|^{2^\alpha - 1} \cdot |\zeta'(s + \overline{s_k})| \cdot |\Phi_\alpha(s + \overline{s_k})| \approx (\epsilon_k)^{-2^\alpha - 1},$$

and

$$|\zeta(s + \overline{s_k})|^{2^\alpha} \cdot |\Phi'_\alpha(s + \overline{s_k})| \lesssim (\epsilon_k)^{-2^\alpha},$$

which implies

$$2^\alpha |\zeta(s + \overline{s_k})|^{2^\alpha - 1} \cdot |\zeta'(s + \overline{s_k})| \cdot |\Phi_\alpha(s + \overline{s_k})| - |\zeta(s + \overline{s_k})|^{2^\alpha} \cdot |\Phi'_\alpha(s + \overline{s_k})| \gtrsim (\epsilon_k)^{-2^\alpha - 1}.$$

Then, by inequality (2.6), we deduce

$$|f'_k(s)|^2 \gtrsim \epsilon_k^{-2^\alpha - 2}. \tag{2.8}$$

Then

$$\int_{\mathbb{C}_{1/2}} |f'_k(s)|^2 \, d\mu(s) \geq \int_{Q_k} |f'_k(s)|^2 \, d\mu(s) \gtrsim \frac{\mu(Q_k)}{\epsilon_k^{2^\alpha + 2}}.$$

Hence

$$\mu(Q_k) \lesssim \epsilon_k^{2^\alpha + 2} \int_{\mathbb{C}_{1/2}} |f'_k(s)|^2 \, d\mu(s) \lesssim \epsilon_k^{2^\alpha + 2} \|f_k\|^2_{\mathscr{D}_\alpha} = \epsilon_k^{2^\alpha + 2}.$$

□

2.3. Equivalent norm and a mean counting function

We will need the following non-injective change of variable formula concerning $\mathscr{H}^2$ norm, which can be found in [9, Theorem 1.3].

Lemma 2.6. *Let $\varphi \in \mathscr{G}_0$. Then for every $f \in \mathscr{H}^2$,*

$$\|C_\varphi f\|^2_{\mathscr{H}^2} = |f(\varphi(+\infty))|^2 + \frac{2}{\pi} \int_{1/2}^{\infty} \int_{\mathbb{R}} |f'(\delta + it)|^2 \mathscr{M}_\varphi(\delta + it) \, dt d\delta.$$

To study the property concerning the mean counting function $\mathscr{M}_\varphi(w)$, we introduce the Nevanlinna class $\mathscr{N}_u$ of all Dirichlet series f which converge uniformly in $\mathbb{C}_\epsilon$ with $\epsilon > 0$ and satisfy

$$\overline{\lim_{\delta \to 0^+}} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log^+ |f(\delta + it)| \, dt < \infty,$$

where $\log^+ |f(\delta + it)| = \max\{0, \log |f(\delta + it)|\}$. We recall the following result which plays a significant role in our arguments. The proof can be found in [9, Lemma 6.5].

Lemma 2.7. *Let $f \in \mathscr{N}_u$. Then for every open disc $\mathbb{D}(w,r)$ which does not contain $f(+\infty)$,*

$$\mathscr{M}_f(w) \leq \frac{1}{\pi r^2} \int_{\mathbb{D}(w,r)} \mathscr{M}_f(\delta + it)\, dtd\delta.$$

Lemma 2.8. *Let $\alpha > 0$ and $\varphi \in \mathscr{G}_0$. Then the following statements are equivalent:*

(1) $\mathscr{M}_\varphi(s) = \mathcal{O}(\operatorname{Re} s - 1/2)^{2^\alpha}$ *for* $\operatorname{Re} s \to (1/2)^+$,

(2) $\int_{Q(\tau,\epsilon)} \mathscr{M}_\varphi(\delta + it)\, dtd\delta = \mathcal{O}(\epsilon^{2^\alpha+2})$ *for every Carleson square $Q(\tau,\epsilon)$, where the implied constant is independent of τ.*

Proof. (1) $\Rightarrow$ (2). Consider a Carleson square $Q = Q(\tau,\epsilon)$ with $\epsilon \to 0^+$. From (1), we deduce that there exists an absolute constant C such that

$$\int_{Q(\tau,\epsilon)} \mathscr{M}_\varphi(\delta + it)\, dtd\delta \leq C \int_{Q(\tau,\epsilon)} (\delta - 1/2)^{2^\alpha}\, dtd\delta \leq C\epsilon^{2^\alpha+2}.$$

(2) $\Rightarrow$ (1). Recall that if $\varphi \in \mathscr{G}_0$, then $\varphi \in \mathscr{N}_u$ (see [9, p. 35]). Set $\epsilon = \operatorname{Re} s - 1/2$. By Lemma 2.7, we obtain

$$\mathscr{M}_\varphi(s) \lesssim \frac{1}{\epsilon^2} \int_{\mathbb{D}(s,\epsilon)} \mathscr{M}_\varphi(\delta + it)\, dtd\delta \leq \frac{1}{\epsilon^2} \int_{Q(\operatorname{Im} s, 2\epsilon)} \mathscr{M}_\varphi(\delta + it)\, dtd\delta \lesssim \epsilon^{2^\alpha}.$$

This completes the proof. □

3. The boundedness of composition operators

The goal of this section is devoted to giving some sufficient and necessary conditions for the boundedeness of composition operators from Bergman space to Hardy space of Dirichlet series. We begin with the following lemma.

Lemma 3.1. *Let $\alpha > 0$, $\theta > 1/2$ and $\varphi \in \mathscr{G}_0$. Then there exists a constant $C := C(\theta)$ such that*

$$\int_\theta^\infty \int_{\mathbb{R}} |P'(\delta + it)|^2 \mathscr{M}_\varphi(\delta + it)\, dtd\delta \leq C\|P\|^2_{\mathscr{D}_\alpha}$$

for every Dirichlet polynomial $P(s) = \sum_{n=1}^N a_n n^{-s}$.

Proof. Let $e_2(s) = 2^{-s}$ and $\epsilon = \theta - 1/2$. Since $d(n) = \mathcal{O}(n^\epsilon)$ and $\log n = \mathcal{O}(n^\epsilon)$ for every $\epsilon > 0$, we have

$$[d(n)]^\alpha (\log n)^2 \lesssim [n^{\epsilon/(2\alpha)}]^\alpha n^{\epsilon/2} = n^\epsilon.$$

The Cauchy–Schwarz inequality and $\operatorname{Re} s - \epsilon/2 \geq \theta/2 + 1/4 > 1/2$ show that

$$\begin{aligned} |P'(s)|^2 = \left|\sum_{n=2}^N a_n (\log n) n^{-s}\right|^2 &\leq \sum_{n=2}^N \frac{|a_n|^2}{[d(n)]^\alpha} \sum_{n=2}^N \frac{[d(n)]^\alpha (\log n)^2}{n^{2\operatorname{Re} s}} \\ &\lesssim \|P\|^2_{\mathscr{D}_\alpha} \sum_{n=2}^N n^{-2\operatorname{Re} s + \epsilon} \lesssim (\log 2)^2 2^{-2\operatorname{Re} s} \|P\|^2_{\mathscr{D}_\alpha} = |e_2'(s)|^2 \cdot \|P\|^2_{\mathscr{D}_\alpha}. \end{aligned}$$

Then, by $\varphi \in \mathscr{G}_0$ and Lemma 2.6, we deduce

$$\begin{aligned}\int_\theta^\infty \int_{\mathbb{R}} |P'(\delta+it)|^2 \mathscr{M}_\varphi(\delta+it)\,dtd\delta &\lesssim \int_\theta^\infty \int_{\mathbb{R}} |e_2'(s)|^2 \mathscr{M}_\varphi(\delta+it)\,dtd\delta \cdot \|P\|_{\mathscr{D}_\alpha}^2 \\ &\le \|C_\varphi e_2\|_{\mathscr{H}^2}^2 \cdot \|P\|_{\mathscr{D}_\alpha}^2 \lesssim \|e_2\|_{\mathscr{H}^2}^2 \cdot \|P\|_{\mathscr{D}_\alpha}^2 = \|P\|_{\mathscr{D}_\alpha}^2.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.2. Let $P(s) = \sum_{n=1}^N a_n n^{-s}$ be a Dirichlet polynomial, then $P \circ \varphi$ is a Dirichlet series which is bounded in $\mathbb{C}_0$. Hence $P \circ \varphi$ is in $\mathscr{H}^\infty$ and consequently belongs to $\mathscr{H}^2$. Let A be a constant such that $|\operatorname{Im}\varphi| \le A$. Put $w = \delta + it$, then

$$\|C_\varphi P\|_{\mathscr{H}^2}^2 = |P(\varphi(+\infty))|^2 + \frac{2}{\pi} \int_{1/2}^\infty \int_{-A}^A |P'(\delta+it)|^2 \mathscr{M}_\varphi(w)\,dtd\delta.$$

Since $\operatorname{Re}\varphi(+\infty) > 1/2$ (see [11, p. 319]), we have

$$|P(\varphi(+\infty))| \lesssim \|P\|_{\mathscr{D}_\alpha}.$$

Hence, it suffices to prove that

$$\int_{1/2}^\infty \int_{-A}^A |P'(\delta+it)|^2 \mathscr{M}_\varphi(w)\,dtd\delta \lesssim \|P\|_{\mathscr{D}_\alpha}^2.$$

For $\operatorname{Re} w \to (1/2)^+$, $\mathscr{M}_\varphi(w) = \mathcal{O}(\operatorname{Re} w - 1/2)^{2^\alpha}$, we see that there exists $\theta > 1/2$ such that for all $1/2 < \delta = \operatorname{Re} w < \theta$,

$$\mathscr{M}_\varphi(w) \lesssim (\delta - 1/2)^{2^\alpha}. \tag{3.1}$$

We split the integral above into pieces I_1 and I_2 as follows:

$$I_1 := \int_{1/2}^\theta \int_{-A}^A |P'(\delta+it)|^2 \mathscr{M}_\varphi(w)\,dtd\delta$$

and

$$I_2 := \int_\theta^\infty \int_{-A}^A |P'(\delta+it)|^2 \mathscr{M}_\varphi(w)\,dtd\delta.$$

Since $\sum_{n\le x}[d(n)]^\alpha \approx x(\log x)^{2^\alpha-1}$ (see [19]), applying Lemma 2.4 with $I = [-A, A]$ and inequality (3.1), we have

$$I_1 \lesssim \int_{1/2}^\theta \int_{-A}^A |P'(\delta+it)|^2 (\delta-1/2)^{2^\alpha}\,dtd\delta \lesssim \|P\|_{\mathscr{D}_\alpha}^2.$$

On the other hand, Lemma 3.1 yields

$$I_2 \le \int_\theta^\infty \int_{\mathbb{R}} |P'(w)|^2 \mathscr{M}_\varphi(w)\,dtd\delta \lesssim \|P\|_{\mathscr{D}_\alpha}^2.$$

From I_1 and I_2, we have $\|C_\varphi P\|_{\mathscr{H}^2} \lesssim \|P\|_{\mathscr{D}_\alpha}$.

Assume that $\mathcal{P}$ denotes the class of all Dirichlet polynomials. By the density of $\mathcal{P}$ in $\mathscr{D}_\alpha$, we can extend C_φ to a bounded operator $T\colon \mathscr{D}_\alpha \to \mathscr{H}^2$ such that $T(P) = C_\varphi(P)$ for every $P \in \mathcal{P}$. For any $f \in \mathscr{D}_\alpha$, there exists a sequence of Dirichlet polynomials $\{P_n\}$ converging to f in norm. Theorem 3 in [4] shows that for every $s \in \mathbb{C}_{1/2}$,

$$|T(f)(s) - T(P_n)(s)| \lesssim \|T(f) - T(P_n)\|_{\mathscr{H}^2} \lesssim \|T\| \cdot \|f - P_n\|_{\mathscr{D}_\alpha},$$

which implies $T(P_n(s))$ converges to $T(f)(s)$ as $n \to +\infty$. Since point evaluation at $\varphi(s) \in \mathbb{C}_{1/2}$ is bounded by Lemma 2.1, one observes that $P_n(\varphi)(s)$ converges to $f(\varphi)(s)$ when $n \to +\infty$. Furthermore, $T(P_n(s)) = P_n(\varphi)(s)$ yields $T(f)(s) = f(\varphi(s)) = C_\varphi(f)(s)$. Then $\|C_\varphi f\|_{\mathscr{H}^2} \lesssim \|f\|_{\mathscr{D}_\alpha}$. □

Proof of Theorem 1.3. Let $\{w_n\}_{n\geq 1}$ be any sequence in $\mathbb{C}_{1/2}$ such that $\operatorname{Re} w_n \to (1/2)^+$. Without loss of generality, we assume that

$$\operatorname{Re} w_n \leq \frac{1/2 + \operatorname{Re}\varphi(+\infty)}{2}.$$

Put $r_n = \frac{\operatorname{Re} w_n - 1/2}{2}$. For any $s \in \mathbb{D}(w_n, r_n)$, the fact $\operatorname{Re}\varphi(+\infty) > 1/2$ shows that

$$\operatorname{Re} s \leq \operatorname{Re} w_n + r_n \leq \frac{\operatorname{Re}\varphi(+\infty) + 1/2}{2} + \frac{\operatorname{Re} w_n - 1/2}{2} < \operatorname{Re}\varphi(+\infty).$$

This means $\varphi(+\infty)$ is uniformly bounded away from the disc $\mathbb{D}(w_n, r_n)$. Recall that if $\varphi \in \mathscr{G}_0$, then $\varphi \in \mathscr{N}_u$ (see [9, p. 35]). Hence Lemma 2.7 yields

$$\mathscr{M}_\varphi(w_n) \leq \frac{1}{\pi r_n^2} \int_{\mathbb{D}(w_n, r_n)} \mathscr{M}_\varphi(\delta + it)\, dt d\delta.$$

Let us consider

$$f_n(s) = \frac{K_\alpha(s, w_n)}{\|K_\alpha(s, w_n)\|_{\mathscr{D}_\alpha}}.$$

Similar to the proof of inequality (2.8), we have

$$|f_n'(s)|^2 \gtrsim \frac{1}{r_n^{2\alpha+2}}.$$

Therefore, in light of Lemma 2.6, we have

$$\begin{aligned}\frac{\mathscr{M}_\varphi(w_n)}{(\operatorname{Re} w_n - 1/2)^{2\alpha}} &\leq \frac{1}{2^{2\alpha}\pi} \int_{\mathbb{D}(w_n, r_n)} \frac{1}{r_n^{2\alpha+2}} \mathscr{M}_\varphi(\delta + it)\, dt d\delta \\ &\lesssim \int_{\mathbb{D}(w_n, r_n)} |f_k'(\delta + it)|^2 \mathscr{M}_\varphi(\delta + it)\, dt d\delta \leq \|C_\varphi f_k\|_{\mathscr{H}^2}^2 \lesssim \|f_k\|_{\mathscr{D}_\alpha}^2 = 1,\end{aligned}$$

which completes the proof. □

On the basis of Lemma 2.8, Theorems 1.2 and 1.3, we obtain the following results.

Corollary 3.2. *Let $\alpha > 0$ and $\varphi \in \mathscr{G}_0$. If $\operatorname{Im}\varphi$ is bounded on $\mathbb{C}_0$ and $\int_{Q(\tau,\epsilon)} \mathscr{M}_\varphi(\delta + it)\, dtd\delta = \mathcal{O}(\epsilon^{2^\alpha+2})$ for every Carleson square $Q(\tau,\epsilon)$, where the implied constant is independent of τ, then C_φ is bounded from $\mathscr{D}_\alpha$ to $\mathscr{H}^2$.*

Corollary 3.3. *Let $\alpha > 0$ and $\varphi \in \mathscr{G}_0$. If C_φ is bounded from $\mathscr{D}_\alpha$ to $\mathscr{H}^2$, then $\int_{Q(\tau,\epsilon)} \mathscr{M}_\varphi(\delta + it)\, dtd\delta = \mathcal{O}(\epsilon^{2^\alpha+2})$ for every Carleson square $Q(\tau,\epsilon)$, where the implied constant is independent of τ.*

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