TAIWANESE JOURNAL OF MATHEMATICS
Vol. xx, No. x, pp. 1–14, xx 20xx
DOI: 10.11650/tjm/220903

Canonical Comultiplication and Double Centraliser Property

Nan Gao*, Jing Ma and Juxia Zhang

Abstract. In this paper, we show the existence of the attached comultiplication structure on $\mathrm{Hom}_{eAe}(eA, D(Ae))$ if an (eAe, A)-bimodule eA has the double centraliser property over an algebra A with the idempotent e. Then we apply it on gendo-Gorenstein algebras. As applications, we give a sufficient and necessary condition for a gendo-Gorenstein algebra to be Gorenstein, and give a bocs-theoretic characterisation of the double centraliser property.

1. Introduction and main results

The double centraliser property is a frequent phenomenon in mathematics. It means that the endomorphism algebra of A-B-bimodule M as a left A-module M is B, while the endomorphism algebra of M as a right B-module M is A. The famous Schur–Weyl duality is a special case of the double centraliser property. It refers in particular to the double centraliser property of bimodules of classical Schur algebras and group algebras of symmetric groups [7]. Soergel [9] gave Schur–Weyl duality of BGG-categories on complex semisimple Lie algebras. The general double centraliser property is also called generalized Schur–Weyl duality in the literature. With this property, there is a close connection and comparison method. König–Slungard–Xi [8] exhibited algebraic structures behind the double centraliser properties, which can be used to prove the classical Schur–Weyl duality between Schur algebras and group algebras of symmetric groups or Soergel's structure theorem for blocks of category $\mathcal{O}$. What's more, they studied double centraliser properties on projective-injective modules and tilting modules respectively. Fang–König [5] constructed from the double centraliser property a comultiplication on gendo-symmetric algebras. Brundan–Kleshchev [2] gave the applications of the double centraliser property in higher Schur–Weyl duality, and there are examples of the double centraliser property in quantum affine Schur–Weyl duality in [3].

Received May 10, 2022; Accepted September 4, 2022.
Communicated by Kunio Yamagata.
2020 *Mathematics Subject Classification.* 16G10, 16E65, 13E10.
Key words and phrases. comultiplication, double centraliser property, gendo-Gorenstein algebra, dominant dimension.
The first author is supported by the National Natural Science Foundation of China (Grant Nos. 11771272 and 11871326).
*Corresponding author.

Gao and König [6] introduced gendo-d-Gorenstein algebras, which are as correspondents of Gorenstein algebras under a Morita–Tachikawa correspondence. It is known that the double centraliser property plays a key role in defining gendo-d-Gorenstein algebras. They described the double centraliser property in terms of grade, which is also described via dominant dimension.

In this paper, we will show the existence of the comultiplication structure attached to the double centraliser property, and based on it, we will give a sufficient and necessary condition for a gendo-Gorenstein algebra to be Gorenstein. We will give a bocs-theoretic characterisation of the double centraliser property. Inspired by the comultiplication, we will characterise the dominant dimension.

Our main results are the following.

Definition 1.1. (see Definition 2.5) There is a canonical comultiplication Δ on $\mathrm{Hom}_{eAe}(eA, D(Ae))$ attached to the (eAe, A)-bimodule eA with the double centraliser property up to an invertible central element of A.

In particular, if moreover (A, e) is gendo-Gorenstein, we call Δ the associated comultiplication on (A, e).

Theorem 1.2. (see Theorem 3.4) *Let A be a finite-dimensional k-algebra and e an idempotent of A such that $\mathrm{Hom}_{eAe}(eA, eAe) \cong Ae$. The canonical bimodule associated to A is defined to be the A-bimodule*

$$V := \mathrm{Hom}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)), A).$$

Then the following are equivalent.

(i) *The (eAe, A)-bimodule eA has the double centraliser property.*

(ii) *There is a comultiplication and counit such that (A, V) is a bocs.*

The paper is organised as follows. In Section 2, we show the existence of the attached comultiplication structure for a gendo-Gorenstein algebra, and using it, a sufficient and necessary condition is given for a gendo-Gorenstein algebra to be Gorenstein. We also prove that gendo-Gorenstein algebras with the attached comultiplication are preserved under Morita equivalences. In Section 3, we give a bocs-theoretic characterisation of the double centraliser property.

2. Canonical comultiplications on gendo-Gorenstein algebras

In the section, we show the existence of the comultiplication structure for a gendo-Gorenstein algebra, and moreover, give a sufficient and necessary condition for a gendo-Gorenstein algebra to be Gorenstein. We also show that gendo-Gorenstein algebras are preserved under Morita equivalences.

2.1. Canonical comultiplication attached to the double centraliser property

In this subsection, we aim to exhibit a closed structure with double centraliser property. We will derive the existence of a comultiplication from homological properties of finite-dimensional algebras.

Let A be a finite-dimensional k-algebra over a field k and e an idempotent of A. Let $D\colon A\text{-mod} \to A^{\mathrm{op}}\text{-mod}$ be the duality. Then we have an A-bimodule isomorphism

$$D(Ae \otimes_{eAe} eA) \cong \mathrm{Hom}_{eAe}(eA, D(Ae)).$$

So there is the induced A-bimodule isomorphism

$$\gamma\colon Ae \otimes_{eAe} eA \cong D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$$

such that

$$\gamma(ae \otimes eb)(f) = f(eb)(ae)$$

for $a, b \in A$ and $f \in \mathrm{Hom}_{eAe}(eA, D(Ae))$.

Lemma 2.1. *The following are equivalent.*

(i) *The (eAe, A)-bimodule eA has the double centraliser property.*

(ii) *There is an A-bimodule isomorphism*

$$\mathrm{End}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae))) \cong A. \tag{2.1}$$

(iii) *There is an A-bimodule isomorphism*

$$\mathrm{Hom}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)), A) \cong A.$$

Proof. (i) $\Leftrightarrow$ (ii) Note that there is a series of A-bimodule isomorphisms induced by the isomorphism γ:

$$\begin{aligned}&\mathrm{End}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)))\\ \cong\ &\mathrm{End}_A(Ae \otimes_{eAe} eA) \cong \mathrm{Hom}_{eAe}(eA, \mathrm{Hom}_A(Ae, Ae \otimes_{eAe} eA))\\ \cong\ &\mathrm{Hom}_{eAe}(eA, eAe \otimes_{eAe} eA) \cong \mathrm{End}_{eAe}(eA).\end{aligned}$$

This implies the desired result (i).

(i) $\Leftrightarrow$ (iii) Since there are the following A-bimodule isomorphisms induced by the isomorphism γ:

$$\begin{aligned}&\mathrm{Hom}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)), A)\\ \cong\ &\mathrm{Hom}_A(Ae \otimes_{eAe} eA, A) \cong \mathrm{Hom}_{eAe}(eA, \mathrm{Hom}_A(Ae, A)) \cong \mathrm{End}_{eAe}(eA),\end{aligned}$$

it follows that the (eAe, A)-bimodule eA has the double centraliser property if and only if $\mathrm{Hom}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)), A) \cong A$. □

Now we are going to construct the comultiplication on $\mathrm{Hom}_{eAe}(eA, D(Ae))$ hidden in the double centraliser property.

Notice that there is an isomorphism

$$D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_A D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$$
$$\xrightarrow{\gamma^{-1}\otimes_A\gamma^{-1}} (Ae \otimes_{eAe} eA) \otimes_A (Ae \otimes_{eAe} eA) \cong Ae \otimes_{eAe} eA \xrightarrow{\gamma} D\,\mathrm{Hom}_{eAe}(eA, D(Ae)).$$

Let m be the composition of the canonical morphism $D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_k D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \to D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_A D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$ with the above isomorphism, i.e.,

$$m\colon D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_k D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \to D\,\mathrm{Hom}_{eAe}(eA, D(Ae)).$$

Dualising m yields the map

$$\Delta\colon \mathrm{Hom}_{eAe}(eA, D(Ae)) \to \mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_k \mathrm{Hom}_{eAe}(eA, D(Ae))$$

such that $(\phi \otimes \varphi)\Delta(f) = m(\varphi \otimes \phi)(f)$ for any $\phi, \varphi \in D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$ and $f \in \mathrm{Hom}_{eAe}(eA, D(Ae))$.

Now we assume that the (eAe, A)-bimodule eA has the double centraliser property. Then from Lemma 2.1(iii) we consider the isomorphisms

$$\mathrm{Hom}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)), A) \xrightarrow{\mathrm{Hom}_A(\gamma, A)} \mathrm{Hom}_A(Ae \otimes_{eAe} eA, A) \cong A. \tag{2.2}$$

Let $\theta\colon D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \to A$ be the inverse image of $1 \in A$ under the isomorphism (2.2). Then θ is an A-bimodule morphism with $(\theta \circ \gamma)(ae \otimes eb) = aeb$ for $a, b \in A$, and $e\theta = \mathrm{Id}_{eA}$.

In the following, we represent an element ϕ in $D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$ by $\gamma(ae \otimes eb)$ for some $a, b \in A$ as γ is an isomorphism for simplicity. Then by definition of m we have the equalities

$$m(\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed)) = \gamma(ae \otimes eb \otimes ce \otimes ed) = \gamma(aebce \otimes ed)$$

for any $a, b, c, d, x, y \in A$.

Lemma 2.2. *The map m satisfies*

$$m(1 \otimes m) = m(m \otimes 1)$$

as k-morphisms from $D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_k D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_k D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \to D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$, and for any $\phi, \varphi \in D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$,

$$\theta(m(\phi \otimes \varphi)) = \theta(\phi)\theta(\varphi).$$

Proof.

$$\begin{aligned} &m(1\otimes m)(\gamma(ae\otimes eb)\otimes\gamma(ce\otimes ed)\otimes\gamma(xe\otimes ey))\\ &=m(\gamma(ae\otimes eb)\otimes\gamma(cedxe\otimes ey))=\gamma(aebcedxe\otimes ey),\\ &m(m\otimes 1)(\gamma(ae\otimes eb)\otimes\gamma(ce\otimes ed)\otimes\gamma(xe\otimes ey))\\ &=m(\gamma(aebce\otimes ed)\otimes\gamma(xe\otimes ey))=\gamma(aebcedxe\otimes ey)\end{aligned}$$

and

$$(\theta\circ m)(\gamma(ae\otimes eb)\otimes\gamma(ce\otimes ed))=\theta(\gamma(aebce\otimes ed))=aebced=(aeb)(ced).$$ □

Lemma 2.3. *Let* $\Delta\colon \operatorname{Hom}_{eAe}(eA,D(Ae))\to\operatorname{Hom}_{eAe}(eA,D(Ae))\otimes_k\operatorname{Hom}_{eAe}(eA,D(Ae))$ *be as above. Then we have the following.*

(i) Δ *is an* A*-bimodule morphism.*

(ii) *As* k*-morphisms from* $\operatorname{Hom}_{eAe}(eA,D(Ae))\to\operatorname{Hom}_{eAe}(eA,D(Ae))\otimes_k\operatorname{Hom}_{eAe}(eA,D(Ae))\otimes_k\operatorname{Hom}_{eAe}(eA,D(Ae))$,

$$(1\otimes\Delta)\Delta=(\Delta\otimes 1)\Delta.$$

(iii) $\operatorname{Im}\Delta=\big\{\sum_i f'_i\otimes f''_i \;\big|\; \sum_i f'_i a\otimes f''_i=\sum_i f'_i\otimes af''_i, \forall\, a\in A\big\}$.

Consequently, the map Δ *is a coassociative comultiplication on the* A*-bimodule* $\operatorname{Hom}_{eAe}(eA,D(Ae))$.

Proof. (i) For $a,b,c,d,x,y\in A$ and $f\in\operatorname{Hom}_{eAe}(eA,D(Ae))$, we have the equalities

$$\begin{aligned}(\gamma(ae\otimes eb)\otimes\gamma(ce\otimes ed))\Delta(xfy)&=m(\gamma(ce\otimes ed)\otimes\gamma(ae\otimes eb))(xfy)\\ &=\gamma(cedae\otimes eb)(xfy)\\ &=f(ebx)(ycedae),\\ (\gamma(ae\otimes eb)\otimes\gamma(ce\otimes ed))((x\Delta(f))y)&=(\gamma(ae\otimes ebx)\otimes\gamma(yce\otimes ed))\Delta(f)\\ &=m(\gamma(yce\otimes ed)\otimes\gamma(ae\otimes ebx))(f)\\ &=\gamma(ycedae\otimes ebx)(f)\\ &=f(ebx)(ycedae),\\ (\gamma(ae\otimes eb)\otimes\gamma(ce\otimes ed))(x(\Delta(f)y))&=(\gamma(ae\otimes ebx)\otimes\gamma(yce\otimes ed))\Delta(f)\\ &=\gamma(ycedae\otimes ebx)(f)\\ &=f(ebx)(ycedae).\end{aligned}$$

So $\Delta(xfy)=x\Delta(f)y$, i.e., Δ is an A-bimodule morphism.

(ii) For any $u \in \mathrm{Hom}_{eAe}(eA, D(Ae))$ and $\phi, \varphi, \psi \in D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$, let $\Delta(u) = \sum_i u_i' \otimes u_i''$, where $u_i', u_i'' \in \mathrm{Hom}_{eAe}(eA, D(Ae))$. Then

$$\begin{aligned}
&(\phi \otimes \varphi \otimes \psi)(1 \otimes \Delta)\Delta(u)\\
&= (\phi \otimes \varphi \otimes \psi)(1 \otimes \Delta)\left(\sum_i u_i' \otimes u_i''\right) = \sum_i \phi(u_i')(\varphi \otimes \psi)\Delta(u_i'')\\
&= \sum_i \phi(u_i')m(\psi \otimes \varphi)(u_i'') = (\phi \otimes m(\psi \otimes \varphi))\Delta(u) = m(m(\psi \otimes \varphi) \otimes \phi)(u),\\
&(\phi \otimes \varphi \otimes \psi)(\Delta \otimes 1)\Delta(u)\\
&= (\phi \otimes \varphi \otimes \psi)(\Delta \otimes 1)\left(\sum_i u_i' \otimes u_i''\right) = \sum_i (\phi \otimes \varphi)\Delta(u_i')\psi(u_i'')\\
&= \sum_i m(\varphi \otimes \phi)(u_i')\psi(u_i'') = (m(\varphi \otimes \phi) \otimes \psi)\Delta(u) = m(\psi \otimes m(\varphi \otimes \phi))(u).
\end{aligned}$$

Thus it follows from Lemma 2.2 that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

(iii) Let $\Sigma = \left\{ \sum_i f_i' \otimes f_i'' \;\middle|\; \sum_i f_i'a \otimes f_i'' = \sum_i f_i' \otimes af_i'', \forall\, a \in A \right\}$. Then $\mathrm{Im}\,\Delta \subseteq \Sigma$. Indeed, for any $f \in \mathrm{Hom}_{eAe}(eA, D(Ae))$, let $\Delta(f) = \sum_i f_i' \otimes f_i''$. Then for any $\varphi, \psi \in D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$ and $a \in A$,

$$\begin{aligned}
(\varphi \otimes \psi)\left(\sum_i f_i'a \otimes f_i''\right) &= (a\varphi \otimes \psi)\left(\sum_i f_i' \otimes f_i''\right) = (a\varphi \otimes \psi)\Delta(f) = m(\psi \otimes a\varphi)(f),\\
(\varphi \otimes \psi)\left(\sum_i f_i' \otimes af_i''\right) &= (\varphi \otimes \psi a)\left(\sum_i f_i' \otimes f_i''\right) = (\varphi \otimes \psi a)\Delta(f) = m(\psi a \otimes \varphi)(f).
\end{aligned}$$

By definition of m, there is an equality $m(\psi \otimes a\varphi) = m(\psi a \otimes \varphi)$. Thus $\Delta(f) \in \Sigma$.

Conversely, on the one hand, for each $f = \sum_i f_i' \otimes f_i'' \in \Sigma$, there is a k-linear map $D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \to \mathrm{Hom}_{eAe}(eA, D(Ae))$, denoted by $\widehat{f}$, such that $\widehat{f}(\varphi) = \sum_i \varphi(f_i')f_i''$ for any $\varphi \in D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$. Since $\sum_i f_i'a \otimes f_i'' = \sum_i f_i' \otimes af_i''$ for any $a \in A$, it follows that

$$\widehat{f}(a\varphi) = \sum_i (a\varphi)(f_i')f_i'' = \sum_i \varphi(f_i'a)f_i'' = \sum_i \varphi(f_i')af_i'' = a\widehat{f}(\varphi),$$

$\widehat{f}$ is a left A-module morphism, i.e., $\widehat{f} \in \mathrm{Hom}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)), \mathrm{Hom}_{eAe}(eA, D(Ae)))$. This means that we define an injective map

$$\Sigma \to \mathrm{Hom}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)), \mathrm{Hom}_{eAe}(eA, D(Ae)))$$

by sending f to $\widehat{f}$. On the other hand, we have a series of isomorphisms

$$\mathrm{Hom}_{eAe}(eA, D(Ae)) \cong \mathrm{Hom}_{eAe}(eA, \mathrm{Hom}_{eAe}(eAe, D(Ae)))$$

$$\begin{aligned}&\cong \mathrm{Hom}_{eAe}(eA, \mathrm{Hom}_A(Ae, \mathrm{Hom}_{eAe}(eA, D(Ae))))\\&\cong \mathrm{Hom}_A(Ae \otimes_{eAe} eA, \mathrm{Hom}_{eAe}(eA, D(Ae)))\\&\cong \mathrm{Hom}_A(D\,\mathrm{Hom}_{eAe}(eA, D(Ae)), \mathrm{Hom}_{eAe}(eA, D(Ae))).\end{aligned}$$

Thus we obtain from Δ being injective that $\Sigma \subseteq \mathrm{Im}\,\Delta$. This completes the proof. □

Lemma 2.4. *Let*

$$\widetilde{\Delta}\colon \mathrm{Hom}_{eAe}(eA, D(Ae)) \to \mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_k \mathrm{Hom}_{eAe}(eA, D(Ae))$$

be a k-linear map satisfying all conditions in Lemma 2.3. *Then there exists an invertible central element z of A such that*

$$\widetilde{\Delta} = \Delta \circ z.$$

Proof. Note that $\widetilde{\Delta}$ induces a map

$$\mu\colon D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_k D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \to D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$$

such that for any $\phi, \varphi \in D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$ and $f \in \mathrm{Hom}_{eAe}(eA, D(Ae))$,

$$\mu(\phi \otimes \varphi)(f) = (\varphi \otimes \phi)\widetilde{\Delta}(f).$$

By assumption μ induces an A-bimodule isomorphism

$$D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \otimes_A D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \to D\,\mathrm{Hom}_{eAe}(eA, D(Ae)).$$

This means that there exists an A-bimodule isomorphism $\omega \in \mathrm{End}_A(\mathrm{Hom}_{eAe}(eA, D(Ae)))$ such that $\widetilde{\Delta}(f) = \Delta(\omega(f))$ for all $f \in \mathrm{Hom}_{eAe}(eA, D(Ae))$. Therefore there is an invertible central element $z \in A$ from (2.1), so that $\widetilde{\Delta}(f) = \Delta(fz)$ for all $f \in \mathrm{Hom}_{eAe}(eA, D(Ae))$. □

Definition 2.5. We call Δ the canonical comultiplication on $\mathrm{Hom}_{eAe}(eA, D(Ae))$ attached to the (eAe, A)-bimodule eA with the double centraliser property up to an invertible central element of A. In particular, if moreover (A, e) is gendo-Gorenstein, we call Δ the associated comultiplication on (A, e).

Remark 2.6. Let (A, e) be a gendo-symmetric k-algebra. Then $\mathrm{Hom}_{eAe}(eA, D(Ae)) \cong A$ as A-bimodules, and so the above map $\Delta\colon A \to A \otimes_k A$ is exactly the comultiplication on A in the sense of Fang and König [5].

Now let Δ be the canonical comultiplication on $\mathrm{Hom}_{eAe}(eA, D(Ae))$. If B is Morita equivalent to A, then there is a projective generator P in A-mod such that $B \cong \mathrm{End}_A(P)^{\mathrm{op}}$, and

$$M \mapsto \mathrm{Hom}_A(P, A) \otimes_A M \otimes_A P$$

defines an equivalence F from A-bimod to B-bimod. It follows that $F(Ae \otimes_{eAe} eA) = \mathrm{Hom}_A(P, A) \otimes_A Ae \otimes_{eAe} eA \otimes_A P = \mathrm{Hom}_A(P, Ae) \otimes_{eAe} eP$.

Lemma 2.7. *Let f be an idempotent of B corresponding to B-module $\mathrm{Hom}_A(P, Ae)$. Then $F(\Delta)$ is the canonical comultiplication on (fBf, B)-bimodule fB.*

Proof. By assumption $\mathrm{Hom}_A(P, Ae) \otimes_{eAe} eP = Bf \otimes_{fBf} fB$. Then by Lemma 2.1(iii) there are the isomorphisms

$$B \cong F(A) \cong \mathrm{Hom}_B(F(Ae \otimes_{eAe} eA), B) = \mathrm{Hom}_B(Bf \otimes_{fBf} fB, B) \cong \mathrm{End}_{fBf}(fB).$$

Thus we obtain that the (fBf, B)-bimodule fB has the double centraliser property, and moreover, $F(\Delta)$ is the canonical comultiplication attached to the (fBf, B)-bimodule fB. □

Proposition 2.8. *Let A be a finite-dimensional k-algebra and e an idempotent of A such that the (eAe, A)-bimodule eA has the double centraliser property. Then the following hold.*

(i) *Let E/k be any field extension. If (A, e) is gendo-Gorenstein, then $A_E = A \otimes_k E$ is gendo-Gorenstein with the associated idempotent $e \otimes 1_E$.*

(ii) *Let B be Morita equivalent to A via the functor F. Use the notation in Corollary 2.7. If (A, e) is gendo-Gorenstein, then (B, f) is gendo-Gorenstein with the associated comultiplication $F(\Delta)$.*

Proof. (i) Since E/k is a field extension, we have that $eA_Ee = eAe \otimes_k E$. Since eAe is a Gorenstein algebra, it follows that eA_Ee is a Gorenstein algebra.

Note that each A_E-module is of the form $M \otimes_k E$, where M is an A-module, and also for any A-modules M, N,

$$\mathrm{Hom}_{A_E}(M \otimes_k E, N \otimes_k E) \cong \mathrm{Hom}_A(M, N) \otimes_k E.$$

From this view,

$$\mathrm{End}_{eA_Ee}(eA_E) \cong \mathrm{End}_A(eA) \otimes_k E \cong A \otimes_k E = A_E,$$

and for all $n \geq 1$,

$$\mathrm{Ext}^n_{eA_Ee}(eA_E, eA_Ee) \cong \mathrm{Ext}^n_{eAe}(eA, eAe) = 0.$$

It follows that the (eA_Ee, A_E)-bimodule eA_E has the double centraliser property, and eA_E is a Gorenstein projective left eA_Ee-module. Thus A_E is gendo-Gorenstein.

(ii) By Lemma 2.7 the (fBf, B)-bimodule fB has the double centraliser property and $F(\Delta)$ is the canonical comultiplication on fB. Since there are the isomorphisms

$$fBf = \mathrm{End}_B(Bf) = \mathrm{End}_B(\mathrm{Hom}_A(P, Ae)) \cong \mathrm{End}_A(Ae) \cong eAe,$$

and

$$\mathrm{Ext}^n_{fBf}(fB, fBf) \cong \mathrm{Ext}^n_{eAe}(eA, eAe) = 0,$$

we get that fBf is a Gorenstein algebra and fB is a Gorenstein projective left fBf-module. Thus (B, f) is gendo-Gorenstein with the associated comultiplication $F(\Delta)$. □

In the last of this subsection, we give a sufficient and necessary condition for a gendo-Gorenstein algebra to be Gorenstein by the above comultiplication.

Proposition 2.9. *Let* $\widetilde{\Delta}$ *be the coassociative comultiplication on* $\mathrm{Hom}_{eAe}(eA, D(Ae))$ *such that* $\widetilde{\Delta} = \Delta \circ z$ *for an invertible central element* z*.*

(i) $(\mathrm{Hom}_{eAe}(eA, D(Ae)), \widetilde{\Delta})$ *has a counit if and only if there is an* A*-bimodule isomorphism*

$$D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \cong {}_1A_\sigma,$$

where $\sigma\colon A \to A$ *is an* A*-bimodule isomorphism induced by* z*.*

(ii) *If* (A, e) *is a gendo-Gorenstein* k*-algebra, then* $(\mathrm{Hom}_{eAe}(eA, D(Ae)), \Delta)$ *has a counit if and only if* A *is Gorenstein.*

Proof. (i) If $\varepsilon \in D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$ is a counit of $(\mathrm{Hom}_{eAe}(eA, D(Ae)), \widetilde{\Delta})$, then for any $\gamma(ae \otimes eb) \in D\,\mathrm{Hom}_{eAe}(eA, D(Ae))$ and $f \in \mathrm{Hom}_{eAe}(eA, D(Ae))$,

$$m(\gamma(ae \otimes eb) \otimes \varepsilon)(f) = (\varepsilon \otimes \gamma(ae \otimes eb))\Delta(f) = \gamma(ae \otimes ebz^-)(f)$$

and

$$m(\varepsilon \otimes \gamma(ae \otimes eb))(f) = (\gamma(ae \otimes eb) \otimes \varepsilon)\Delta(f) = \gamma(z^- ae \otimes eb)(f).$$

Let $\theta(\varepsilon) = \nu$. Then $\theta(m(\gamma(ae \otimes eb) \otimes \varepsilon)) = \theta(\gamma(ae \otimes ebz^-))$ implies

$$aeb\nu = aebz^-$$

for any $a, b \in A$ by Lemma 2.2. Since the (eAe, A)-bimodule eA has the double centraliser property, we get that AeA is a faithful right A-module. Hence θ is surjective as an A-bimodule morphism and thus an isomorphism by Lemma 2.1. Therefore

$$D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \cong {}_1A_\sigma.$$

Conversely, if $D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \cong {}_1A_\sigma$, we write this isomorphism by $\beta\colon$ $D\,\mathrm{Hom}_{eAe}(eA, D(Ae)) \to {}_1A_\sigma$, then

$$\beta^{-1}(\beta(\varphi)\beta(\psi)) = z^-\beta^{-1}(\beta(m(\varphi \otimes \psi))) = z^- m(\varphi \otimes \psi)$$

for all $\varphi, \psi \in D\operatorname{Hom}_{eAe}(eA, D(Ae))$. Let $\varepsilon = z^{-}\beta^{-1}(1)$. Then

$$\beta(m(\varphi \otimes \varepsilon)) = \beta(\varphi),$$

that is,

$$m(\varphi \otimes \varepsilon) = \varphi \circ z^{-}, \quad m(\varepsilon \otimes \varphi) = \varphi \circ z^{-}.$$

So ε is a counit of $(\operatorname{Hom}_{eAe}(eA, D(Ae)), \widetilde{\Delta})$. The second part is verified if we take $z = 1_A$.

(ii) Since A is gendo-Gorenstein, we get from the definition that eAe is Gorenstein. Since $D\operatorname{Hom}_{eAe}(eA, D(Ae)) \cong A$ as A-bimodules if and only if $A/AeA = 0$ if and only if A and eAe are Morita equivalent. It follows that A is Gorenstein if and only if $D\operatorname{Hom}_{eAe}(eA, D(Ae)) \cong A$ as A-bimodules. Thus the desired result follows from (i) in which $z = 1_A$. □

3. Bocs-theoretic characterisation

In the section, we will give a bocs-theoretic characterisation of the double centraliser property.

As we have seen, given an algebra A and an idempotent e, the A-bimodule $\operatorname{Hom}_A(D\operatorname{Hom}_{eAe}(eA, D(Ae)), A)$ is crucial in characterisations of the double centraliser property.

3.1. Associated canonical bimodule

In this subsection, we study the properties of this bimodule, and utilize them to provide new characterisations of the (eAe, A)-bimodule eA having the double centraliser property and dominant dimension of A-modules.

Definition 3.1. Let A be an algebra and e an idempotent of A. The canonical bimodule associated to A is defined to be the A-bimodule

$$V := \operatorname{Hom}_A(D\operatorname{Hom}_{eAe}(eA, D(Ae)), A).$$

The following result collects some properties of the canonical bimodule.

Lemma 3.2. (i) $\operatorname{Hom}_A(V, D(eA)) \cong D(eA)$ *as a left A-module.*

(ii) $eV \cong eA$ *as a left eAe-module.*

(iii) $D\operatorname{Hom}_{eAe}(eA, D(Ae)) \otimes_A V \otimes_A D\operatorname{Hom}_{eAe}(eA, D(Ae)) \cong D\operatorname{Hom}_{eAe}(eA, D(Ae))$ *as an A-bimodule.*

(iv) *There is the A-bimodule isomorphism $V \cong \operatorname{End}_A(V)$.*

(v) *If* $F\colon A\text{-mod}\to B\text{-mod}$ *is a Morita equivalence between algebras* A *and* B*, then the induced equivalence from* A-bimod *to* B-bimod *preserves the canonical bimodules.*

Proof. (i) We have the isomorphisms of left A-modules

$$\begin{aligned}\operatorname{Hom}_A(V, D(eA)) &= \operatorname{Hom}_A(\operatorname{Hom}_A(D\operatorname{Hom}_{eAe}(eA, D(Ae)), A), D(eA))\\ &\cong \operatorname{Hom}_A(\operatorname{Hom}_A(Ae\otimes_{eAe} eA, A), D(eA))\\ &\cong \operatorname{Hom}_{A^{\mathrm{op}}}(eA, D\operatorname{Hom}_A(Ae\otimes_{eAe} eA, A))\\ &\cong D(eA\otimes_A \operatorname{Hom}_A(Ae\otimes_{eAe} eA, A))\\ &\cong D\operatorname{Hom}_A(Ae, A)\cong D(eA).\end{aligned}$$

(ii) Note that as a left eAe-module,

$$\begin{aligned}eV &= e\operatorname{Hom}_A(D\operatorname{Hom}_{eAe}(eA, D(Ae)), A)\cong \operatorname{Hom}_A(D\operatorname{Hom}_{eAe}(eA, D(Ae))e, A)\\ &\cong \operatorname{Hom}_A(D\operatorname{Hom}_{eAe}(eAe, D(Ae)), A)\cong \operatorname{Hom}_A(Ae, A)\cong eA.\end{aligned}$$

(iii) We have the following A-bimodule isomorphisms by (ii)

$$\begin{aligned}&D\operatorname{Hom}_{eAe}(eA, D(Ae))\otimes_A V\otimes_A D\operatorname{Hom}_{eAe}(eA, D(Ae))\\ \cong\ & Ae\otimes_{eAe} eV\otimes_A Ae\otimes_{eAe} eA\cong Ae\otimes_{eAe} eA\otimes_A Ae\otimes_{eAe} eA\cong Ae\otimes_{eAe} eA.\end{aligned}$$

(iv) We have the following A-bimodule isomorphisms by (ii)

$$\begin{aligned}\operatorname{End}_A(V) &\cong \operatorname{Hom}_A(D\operatorname{Hom}_{eAe}(eA, D(Ae))\otimes_A V, A)\\ &\cong \operatorname{Hom}_A(Ae\otimes_{eAe} eA\otimes_A V, A)\\ &\cong \operatorname{Hom}_A(Ae\otimes_{eAe} eA, A) = V.\end{aligned}$$

(v) By Lemma 2.7 we know that $F\colon A\text{-bimod}\to B\text{-bimod}$ is an equivalence given by $M\mapsto \operatorname{Hom}_A(P, A)\otimes_A M\otimes_A P$, where $B=\operatorname{End}_A^{\mathrm{op}}(P)$. It follows that $F(Ae\otimes_{eAe} eA) = \operatorname{Hom}_A(P, A)\otimes_A Ae\otimes_{eAe} eA\otimes_A P = \operatorname{Hom}_A(P, Ae)\otimes_{eAe} eP$. Let f be an idempotent of B corresponding to B-module $\operatorname{Hom}_A(P, Ae)$. This implies that $\operatorname{Hom}_A(P, Ae)\otimes_{eAe} eP = Bf\otimes_{fBf} fB$. Thus the result follows from the isomorphisms

$$\begin{aligned}F(V(A)) &\cong \operatorname{Hom}_B(F(Ae\otimes_{eAe} eA), B) = \operatorname{Hom}_B(Bf\otimes_{fBf} fB, B)\\ &\cong \operatorname{Hom}_B(D\operatorname{Hom}_{fBf}(fB, D(Bf)), B) = V(B).\end{aligned}$$ □

In the following, we give a bocs-theoretic characterisation of the double centraliser property. Before doing this, we recall the notion of the bocs.

Definition 3.3. Let A be a finite dimensional algebra and W an A-bimodule. Then the pair (A, W) is called a bocs if there are A-bimodule maps $\mu\colon W\to W\otimes_A W$ (the comultiplication) and $\varepsilon\colon W\to A$ (the counit) with the following properties: $(1_W\otimes_A\varepsilon)\mu = 1_W$, $(\varepsilon\otimes_A 1_W)\mu = 1_W$ and $(1\otimes_A\mu)\mu = (\mu\otimes_A 1)\mu$.

Theorem 3.4. *Let A be a finite-dimensional k-algebra and e an idempotent of A such that* $\mathrm{Hom}_{eAe}(eA, eAe) \cong Ae$. *Then the following are equivalent.*

(i) *The (eAe, A)-bimodule eA has the double centraliser property.*

(ii) *There is a comultiplication and counit such that (A, V) is a bocs.*

Proof. If the (eAe, A)-bimodule eA has the double centraliser property, then from Lemma 2.1 that $V \cong A$, and thus (A, V) is a bocs.

Now assume that (A, V) is a bocs with comultiplication μ and counit ε. By definition, the comultiplication $\mu\colon V \to V \otimes_A V$ is injective. Since there is the A-bimodule isomorphism $V \cong \mathrm{End}_A(V) \cong D(V \otimes_A D(V))$ by Lemma 3.2, comparing the dimension, we obtain that μ is an isomorphism, and furthermore, $V \cong A$. Thus the (eAe, A)-bimodule eA has the double centraliser property. □

Notice that Lemma 3.2 exhibits the use of the canonical bimodule V in characterising the $D(eA)$-dominant dimension of A-modules.

Lemma 3.5. *Let A be a finite-dimensional k-algebra over a field k and e an idempotent of A. Let F and G be the endofunctors on A*-mod *such that* $F(M) = \mathrm{Hom}_{(eAe)^{\mathrm{op}}}(\mathrm{Hom}_A(M, D(eA)), D(eA))$ *and* $G(M) = \mathrm{Hom}_{eAe}(eA, eM)$ *for* $M \in A$-mod. *Then there exists a natural equivalence* $\eta\colon F \xrightarrow{\cong} G$ *on* A-mod. *In particular, if the (eAe, A)-bimodule eA has the double centraliser property, then $F(V) \cong A$.*

Proof. For each $M \in A$-mod, by the isomorphisms

$$\begin{aligned} F(M) &= \mathrm{Hom}_{(eAe)^{\mathrm{op}}}(\mathrm{Hom}_A(M, D(eA)), D(eA)) \\ &\cong \mathrm{Hom}_{(eAe)^{\mathrm{op}}}(D(eM), D(eA)) \cong \mathrm{Hom}_{(eAe)}(eA, eM) = G(M), \end{aligned}$$

we obtain a natural isomorphism $\eta\colon F \xrightarrow{\cong} G$.

If the (eAe, A)-bimodule eA has the double centraliser property, then by Lemma 3.2

$$F(V) \cong \mathrm{Hom}_{eAe}(eA, eV) \cong \mathrm{Hom}_{eAe}(eA) \cong A.$$ □

Theorem 3.6. *Let A and F be as in Lemma* 3.5. *Let M be a finitely generated A-module and $n \geq 2$ an integer. Then $D(eA)$-domdim $M \geq n$ if and only if $F(M) \cong M$ canonically and $R^iF(M) = 0$ for $1 \leq i \leq n-1$, where R^iF denotes the i-th right derived functor of F.*

Proof. If $D(eA)$-domdim $M \geq n$, then by definition, there is an exact sequence

$$0 \to M \to D(eA)^1 \to D(eA)^2 \to \cdots \to D(eA)^n$$

with all $D(eA)^i \in \operatorname{add} D(eA)$. Since $n \geq 2$, it follows from [1, Proposition 2.1] that $F(M) \cong M$ canonically. Applying F to the exact sequence, we get from the isomorphism $F(D(eA)) \cong D(eA)$ that $R^iF(M) = 0$ for $1 \leq i \leq n-1$.

Conversely, $F(M) \cong M$ canonically implies that $D(eA)$-domdim $M \geq 2$ again by [1, Proposition 2.1]. Take a minimal injective resolution of M,

$$0 \to M \to I^1 \to I^2 \to I^3 \to \cdots \to I^{n-1} \to I^n \to \cdots .$$

To show that $D(eA)$-domdim $M \geq n$, we prove by induction that $I^1, \ldots, I^n$ above are in $\operatorname{add} D(eA)$. Assume that $I^1, \ldots, I^t$ are in $\operatorname{add} D(eA)$ for $t \leq n-1$. Since $R^iF(M) = 0$ for $1 \leq i \leq n-1$, it follows that $0 \to F(M) \to F(I^1) \to F(I^2) \to \cdots \to F(I^n)$ is exact. This means that $I^{t+1} \to F(I^{t+1})$ is a split monomorphism.

On the other hand, the eAe-module $\operatorname{Hom}_A(I^{t+1}, D(eA))$ is finitely generated. So we have an eAe-projective presentation

$$\operatorname{Hom}_A(D(eA)^m, D(eA)) \to \operatorname{Hom}_A(D(eA)^n, D(eA)) \to \operatorname{Hom}_A(I^{t+1}, D(eA)) \to 0.$$

Applying the functor $\operatorname{Hom}_{eAe}(-, D(eA))$ to this sequence, we get the following exact sequence

$$0 \to F(I^{t+1}) \to D(eA)^n \to D(eA)^m.$$

It follows that $I^{t+1} \in \operatorname{add} D(eA)$. □

Proposition 3.7. *Let A be a finite-dimensional k-algebra over a field k and e an idempotent of A. Let M be a finitely generated A-module and $n \geq 2$ an integer. If $D(eA)$-domdim $M \geq n$, then we have the (restricted) first quadrant Grothendieck spectral sequence:*

$$E_2^{p,0} = \operatorname{Ext}^p_{eAe}(eA, eM) \quad \Longrightarrow \quad R^pF(M), \quad \forall\, 0 \leq p \leq n-1.$$

Proof. Since $D(eA)$-domdim $M \geq n$, there is an exact sequence

$$0 \to M \to D(eA)^1 \to D(eA)^2 \to \cdots \to D(eA)^n$$

with all $D(eA)^i \in \operatorname{add} D(eA)$. Since $eA \otimes_A D(eA)$ is an injective eAe-module, and moreover, $eA \otimes_A D(eA)$ is $\operatorname{Hom}_{eAe}(eA, -)$-acyclic for $1 \leq i \leq n-1$ by Theorem 3.6, we get from [4, Lemma 4.3] and Lemma 3.5 the (restricted) first quadrant Grothendieck spectral sequence:

$$E_2^{p,q} = \operatorname{Ext}^p_{eAe}(eA, \operatorname{Tor}^q_A(eA, M)) \quad \Longrightarrow \quad R^{p+q}F(M), \quad \forall\, 0 \leq p+q \leq n-1.$$

Since $\operatorname{Tor}^q_A(eA, M) = 0$ for all $q \geq 1$, we get the (restricted) first quadrant Grothendieck spectral sequence:

$$E_2^{p,0} = \operatorname{Ext}^p_{eAe}(eA, eM) \quad \Longrightarrow \quad R^pF(M), \quad \forall\, 0 \leq p \leq n-1.$$ □

Acknowledgments

The part of this paper was written during the first author's visit to Stuttgart in 2019. She would like to express their gratitude to Steffen Koenig for hospitality and many useful discussions.

References

[1] M. Auslander and Ø. Solberg, *Relative homology and representation theory III: Cotilting modules and Wedderburn correspondence*, Comm. Algebra **21** (1993), no. 9, 3081–3097.

[2] J. Brundan and A. Kleshchev, *Schur–Weyl duality for higher levels*, Selecta Math. (N.S.) **14** (2008), no. 1, 1–57.

[3] V. Chari and A. Pressley, *Quantum affine algebras and affine Hecke algebras*, Pacific. J. Math. **174** (1996), no. 2, 295–326.

[4] M. Fang, O. Kerner and K. Yamagata, *Canonical bimodules and dominant dimension*, Tran. Amer. Math. Soc. **370** (2018), no. 2, 847–872.

[5] M. Fang and S. Koenig, *Gendo-symmetric algebras, canonical comultiplication, bar cocomplex and dominant dimension*, Tran. Amer. Math. Soc. **368** (2016), no. 7, 5037–5055.

[6] N. Gao and S. Koenig, *Grade, dominant dimension and Gorenstein algebras*, J. Algebra **427** (2015), 118–141.

[7] J. A. Green, *Polynomial Representations of* GL_n, Lecture Notes in Mathematics **830**, Springer-Verlag, Berlin, 1980.

[8] S. König, I. H. Slungård and C. Xi, *Double centralizer properties, dominant dimension, and tilting modules*, J. Algebra **240** (2001), no. 1, 393–412.

[9] W. Soergel, *Kategorie $\mathscr{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. **3** (1990), no. 2, 421–445.

Nan Gao, Jing Ma, Juxia Zhang
Department of Mathematics, Shanghai University, Shanghai 200444, China
E-mail addresses: nangao@shu.edu.cn, majingmmmm@shu.edu.cn,
zhangjuxia@shu.edu.cn