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The Least Squares Solution with the Minimal Norm to a System of Mixed Generalized Sylvester Reduced Biquaternion Tensor Equations

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Abstract. In this paper, we investigate the least squares solution with the minimal norm to the system (1.1) over reduced biquaternion via complex representation of reduced biquaternion tensors and the Moore–Penrose inverse of tensors. Besides, we establish some necessary and sufficient conditions for the solvability to the above system and give an expression of the general solution to the system when the solvability conditions are met. Moreover, the algorithm and numerical example are presented to verify the main results of this paper.

1. Introduction

In this paper, we prescribe the following notations. $\mathbb{C}^{m\times n}$ represents the set of all $m\times n$ complex matrices, $\mathbb{H}_r^{m\times n}$ represents the set of all $m\times n$ reduced biquaternion matrices. For a positive integer N, let $[N]=\{1,\ldots,N\}$, an order N tensor $\mathcal{A}=(a_{i_1\cdots i_N})_{1\le i_j\le I_j}$ $(j=1,\ldots,N)$ is a multidimensional array with $I=I_1I_2\cdots I_N$ entries. $\mathbb{R}^{I_1\times\cdots\times I_N}$, $\mathbb{C}^{I_1\times\cdots\times I_N}$, $\mathbb{H}_r^{I_1\times\cdots\times I_N}$ stand for the sets of all order N and dimension $I_1\times\cdots\times I_N$ tensors over the real number field $\mathbb{R}$, complex number field $\mathbb{C}$, and real reduced biquaternion algebra $\mathbb{H}_r$, respectively. Given $\mathcal{A}=(a_{i_1\cdots i_N j_1\cdots j_M})\in\mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$, define $\overline{a}_{i_1\cdots i_N j_1\cdots j_M}$ to be conjugate of $a_{i_1\cdots i_N j_1\cdots j_M}$, and let $\mathcal{B}=(b_{j_1\cdots j_M i_1\cdots i_N})\in\mathbb{C}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N}$ be the conjugate transpose of $\mathcal{A}$, where $b_{j_1\cdots j_M i_1\cdots i_N}=\overline{a}_{i_1\cdots i_N j_1\cdots j_M}$, denoted by $\mathcal{A}^*$. When $b_{j_1\cdots j_M i_1\cdots i_N}=a_{i_1\cdots i_N j_1\cdots j_M}$, $\mathcal{B}$ is called the transpose of $\mathcal{A}$, denoted by $\mathcal{A}^T$. A tensor $\mathcal{D}=(d_{i_1\cdots i_N j_1\cdots j_N})\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is called a diagonal tensor if all its entries are zero except for $d_{i_1\cdots i_N i_1\cdots i_N}$. For a diagonal tensor, if all the diagonal entries $d_{i_1\cdots i_N i_1\cdots i_N}=1$, then $\mathcal{D}$ is a unit tensor, denoted by $\mathcal{I}$. The zero tensor with suitable order is denoted by $\mathcal{O}$. For $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, the Frobenius norm $\|\cdot\|$ of $\mathcal{A}$ is defined as $\|\mathcal{A}\|=\big(\sum_{i_1\cdots i_N j_1\cdots j_N}|a_{i_1\cdots i_N j_1\cdots j_N}|^2\big)^{1/2}$. $\operatorname{Re}\mathcal{A}$ and $\operatorname{Im}\mathcal{A}$ represent the real and imaginary parts

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of the complex tensor $\mathcal{A}$, respectively. $\otimes$ stands for the Kronecker product. $*_N$ stands for the Einstein product.

Reduced biquaternions were introduced by Schütte and Wenzel in 1990, and have great applications in signal and image processing, control and system theory, neural network, etc. Sylvester equation is a very important kind of matrix equation in matrix theory. It is widely applied in characteristic structure configuration, aerospace control technology, numerical solution of differential equations, pattern recognition and so on. At present, there have been a huge amount of papers to discuss the standard Sylvester equation and its various generalized forms [3,4,6,8,17,18,20,22]. In recent decades, tensor conceived by Tullio Levi-Civita [10], has attracted a lot of scholars to study. Tensor equations can be used to model many problems in quantum physics, engineering and science, general relativity, data mining and so on [1,5,14,15,19]. More and more people are getting interested in the Sylvester tensor equation and its generalization. Some results on tensor equations and related problems can be found in [2,7,9,11–13,16,23–26]. In particular, [2] proposed a projection method based on the tensor format and considered the preconditioned iterative solvers of Sylvester tensor equations; [9] was concerned with the conjugate gradient least squares algorithm to solve a class of tensor equations via the Einstein product and proved that the solution of the tensor equation can be obtained within a finite number of iterative steps in the absence of round-off errors; [26] focused on solving high order Sylvester tensor equation arising in control theory and proposed some effective iterative algorithms for solving Sylvester tensor equation; [25] investigated the solution to the least squares problem for the quaternion Sylvester tensor equation and studied the convergence properties of the proposed iterative method; [7] established some necessary and sufficient solvability conditions for a system of quaternary-coupled Sylvester-type quaternion tensor equations and gave an expression of the general solution to this system when it is solvable; [24] gave some necessary and sufficient conditions for the solvability to a system of a pair of coupled two-sided Sylvester-type tensor equations over the quaternion algebra and derived some solvability conditions and expressions of the η-Hermitian solutions to some systems of coupled two-sided Sylvester-type quaternion tensor equations as applications, etc.

To our knowledge, there has been little information on the system of mixed generalized Sylvester reduced biquaternion tensor equations

$$\begin{gathered}\mathcal{A}_1 *_N \mathcal{X} = \mathcal{B}_1, \quad \mathcal{Y} *_N \mathcal{A}_2 = \mathcal{B}_2, \quad \mathcal{A}_3 *_N \mathcal{Z} = \mathcal{B}_3, \\ \mathcal{C}_1 *_N \mathcal{X} - \mathcal{Y} *_N \mathcal{D}_1 = \mathcal{E}_1, \quad \mathcal{C}_2 *_N \mathcal{Z} - \mathcal{Y} *_N \mathcal{D}_2 = \mathcal{E}_2,\end{gathered} \tag{1.1}$$

where $\mathcal{A}_\sigma$, $\mathcal{B}_\sigma$, $\mathcal{C}_\tau$, $\mathcal{D}_\tau$ and $\mathcal{E}_\tau$ ($\sigma = 1, 2, 3$, $\tau = 1, 2$) are given reduced biquaternion tensors and $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are unknown reduced biquaternion tensors.

Motivated by above mentioned, as well as the wide applications of Sylvester-type tensor equations, we in this paper discuss the least squares solution with the minimal norm of the system of mixed generalized Sylvester tensor equations (1.1) over the reduced biquaternion algebra based on complex representation of reduced biquaternion tensors together with the Moore–Penrose inverse of tensors. The least squares solution with the minimal norm of (1.1) can be stated as follows.

Problem 1.1. Given the tensors in (1.1): $\mathcal{A}_1, \mathcal{B}_1 \in \mathbb{H}_r^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{A}_2, \mathcal{B}_2 \in \mathbb{H}_r^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_N}$, $\mathcal{A}_3, \mathcal{B}_3 \in \mathbb{H}_r^{L_1\times\cdots\times L_N\times J_1\times\cdots\times J_N}$, $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2, \mathcal{E}_1, \mathcal{E}_2 \in \mathbb{H}_r^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ and

$$\begin{aligned}\mathbb{H}_L = \big\{(\mathcal{X},\mathcal{Y},\mathcal{Z}) \mid \mathcal{X},\mathcal{Y},\mathcal{Z} \in \mathbb{H}_r^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N},\\ \|\mathcal{A}_1 *_N \mathcal{X} - \mathcal{B}_1\|^2 + \|\mathcal{Y} *_N \mathcal{A}_2 - \mathcal{B}_2\|^2 + \|\mathcal{A}_3 *_N \mathcal{Z} - \mathcal{B}_3\|^2\\ + \|\mathcal{C}_1 *_N \mathcal{X} - \mathcal{Y} *_N \mathcal{D}_1 - \mathcal{E}_1\|^2 + \|\mathcal{C}_2 *_N \mathcal{Z} - \mathcal{Y} *_N \mathcal{D}_2 - \mathcal{E}_2\|^2 = \min\big\}.\end{aligned}$$

Find out $(\mathcal{X}_l, \mathcal{Y}_l, \mathcal{Z}_l) \in \mathbb{H}_L$ such that

$$\|(\mathcal{X}_l, \mathcal{Y}_l, \mathcal{Z}_l)\|^2 = \min_{(\mathcal{X},\mathcal{Y},\mathcal{Z})\in\mathbb{H}_L} \|(\mathcal{X},\mathcal{Y},\mathcal{Z})\|^2.$$

The solution $(\mathcal{X}_l, \mathcal{Y}_l, \mathcal{Z}_l)$ in Problem 1.1 is called the minimal norm least squares solution.

The remainder of this paper is organized as follows. In Section 2, we first overview some basic definitions and related properties with regard to a complex tensor. Then we give its complex representation for a reduced biquaternion tensor, and on this basis we deduce some important properties. In Section 3, we establish some necessary and sufficient conditions for the existence of a solution to the system (1.1), and give the general solution to this system when it is solvable. In Section 4, we give an algorithm and numerical example to prove that our results are feasible. In Section 5, we put some conclusions.

2. Preliminaries

2.1. An introduction to tensors

Definition 2.1. [1] Suppose $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_N}$, $\mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_N\times J_1\times\cdots\times J_M}$, the Einstein product of tensors $\mathcal{A}$ and $\mathcal{B}$ is defined by the operation $*_N$ via

$$(\mathcal{A} *_N \mathcal{B})_{i_1\cdots i_N j_1\cdots j_M} = \sum_{k_1\cdots k_N} a_{i_1\cdots i_N k_1\cdots k_N} b_{k_1\cdots k_N j_1\cdots j_M},$$

where $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$.

In the following, we start by introducing the Kronecker product of tensors and block complex tensors.

Definition 2.2. [23] Suppose $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_M\times L_1\times\cdots\times L_M}$, the Kronecker product of the tensors $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A} \otimes \mathcal{B} := (a_{i_1\cdots i_N j_1\cdots j_N}\mathcal{B}).$$

Remark 2.3. (1) Note that it is a 'Kr-block tensor' whose (s,t)-subblock is $a_{i_1\cdots i_N j_1\cdots j_N}\mathcal{B}$ obtained via multiplied all the entries of $\mathcal{B}$ by a constant $a_{i_1\cdots i_N j_1\cdots j_N}$, where $s = i_N + \sum_{K=1}^{N-1}\left[(i_K-1)\prod_{L=K+1}^{N} I_L\right]$ and $t = j_N + \sum_{K=1}^{N-1}\left[(j_K-1)\prod_{L=K+1}^{N} J_L\right]$.

(2) Obviously, this Kronecker product is non-commutative, that is, $\mathcal{A}\otimes\mathcal{B} \neq \mathcal{B}\otimes\mathcal{A}$ in general.

Furthermore, the following properties of this Kronecker product can be found in [26].

Lemma 2.4. *Suppose* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$ *and* $\mathcal{B},\mathcal{C} \in \mathbb{C}^{K_1\times\cdots\times K_M\times L_1\times\cdots\times L_M}$. *Then*

(1) $(\mathcal{A}\otimes\mathcal{B})^* = \mathcal{A}^*\otimes\mathcal{B}^*$;

(2) $\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) = (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}$;

(3) $\mathcal{A}\otimes(\mathcal{B}+\mathcal{C}) = \mathcal{A}\otimes\mathcal{B}+\mathcal{A}\otimes\mathcal{C}$ *and* $(\mathcal{B}+\mathcal{C})\otimes\mathcal{A} = \mathcal{B}\otimes\mathcal{A}+\mathcal{C}\otimes\mathcal{A}$.

For a tensor $\mathcal{D} = (d_{j_1\cdots j_N l_1\cdots l_M}) \in \mathbb{C}^{J_1\times\cdots\times J_N\times L_1\times\cdots\times L_M}$, $\mathcal{D}_{(j_1\cdots j_N|:)} = (d_{j_1\cdots j_N:\cdots:}) \in \mathbb{C}^{L_1\times\cdots\times L_M}$ is a subblock of $\mathcal{D}$. $V_c(\mathcal{D})$ is obtained by lining up all the subtensors in a column. The t-th subblock of $V_c(\mathcal{D})$ is $\mathcal{D}_{(j_1\cdots j_N|:)}$, where $t = j_N + \sum_{K=1}^{N-1}\left[(j_K - 1)\prod_{P=K+1}^{N} J_P\right]$. For instance, if $\mathcal{D} = (d_{j_1j_2l_1}) \in \mathbb{C}^{3\times2\times3}$, then

$$V_c(\mathcal{D}) = \begin{pmatrix} \mathcal{D}_{(11|:)} \\ \mathcal{D}_{(12|:)} \\ \mathcal{D}_{(21|:)} \\ \mathcal{D}_{(22|:)} \\ \mathcal{D}_{(31|:)} \\ \mathcal{D}_{(32|:)} \end{pmatrix}.$$

Lemma 2.5. *Suppose* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_M\times L_1\times\cdots\times L_M}$ *and* $\mathcal{D} \in \mathbb{C}^{J_1\times\cdots\times J_N\times L_1\times\cdots\times L_M}$, *then we have*

$$(\mathcal{A}\otimes\mathcal{B}) *_M V_c(\mathcal{D}) = V_c(\mathcal{A} *_N \mathcal{D} *_M \mathcal{B}^T).$$

Example 2.6. Let $\mathcal{A} = (a_{i_1i_2j_1j_2}) \in \mathbb{C}^{2\times 3\times 2\times 2}$ and $\mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_M\times L_1\times\cdots\times L_M}$, then we can obtain

$$\mathcal{A}\otimes\mathcal{B} = \begin{pmatrix} a_{1111}\mathcal{B} & a_{1112}\mathcal{B} & a_{1121}\mathcal{B} & a_{1122}\mathcal{B} \\ a_{1211}\mathcal{B} & a_{1212}\mathcal{B} & a_{1221}\mathcal{B} & a_{1222}\mathcal{B} \\ a_{1311}\mathcal{B} & a_{1312}\mathcal{B} & a_{1321}\mathcal{B} & a_{1322}\mathcal{B} \\ a_{2111}\mathcal{B} & a_{2112}\mathcal{B} & a_{2121}\mathcal{B} & a_{2122}\mathcal{B} \\ a_{2211}\mathcal{B} & a_{2212}\mathcal{B} & a_{2221}\mathcal{B} & a_{2222}\mathcal{B} \\ a_{2311}\mathcal{B} & a_{2312}\mathcal{B} & a_{2321}\mathcal{B} & a_{2322}\mathcal{B} \end{pmatrix}.$$

For $\mathcal{D} \in \mathbb{C}^{2\times 2\times L_1\times\cdots\times L_M}$, we can further get

$$\begin{aligned} &(\mathcal{A}\otimes\mathcal{B}) *_M V_c(\mathcal{D}) \\ &= \begin{pmatrix} a_{1111}\mathcal{B} & a_{1112}\mathcal{B} & a_{1121}\mathcal{B} & a_{1122}\mathcal{B} \\ a_{1211}\mathcal{B} & a_{1212}\mathcal{B} & a_{1221}\mathcal{B} & a_{1222}\mathcal{B} \\ a_{1311}\mathcal{B} & a_{1312}\mathcal{B} & a_{1321}\mathcal{B} & a_{1322}\mathcal{B} \\ a_{2111}\mathcal{B} & a_{2112}\mathcal{B} & a_{2121}\mathcal{B} & a_{2122}\mathcal{B} \\ a_{2211}\mathcal{B} & a_{2212}\mathcal{B} & a_{2221}\mathcal{B} & a_{2222}\mathcal{B} \\ a_{2311}\mathcal{B} & a_{2312}\mathcal{B} & a_{2321}\mathcal{B} & a_{2322}\mathcal{B} \end{pmatrix} *_M \begin{pmatrix} \mathcal{D}_{(11|:)} \\ \mathcal{D}_{(12|:)} \\ \mathcal{D}_{(21|:)} \\ \mathcal{D}_{(22|:)} \end{pmatrix} = \begin{pmatrix} \sum_{j_1j_2} a_{11j_1j_2}\mathcal{D}_{(j_1j_2|:)} *_M \mathcal{B}^T \\ \sum_{j_1j_2} a_{12j_1j_2}\mathcal{D}_{(j_1j_2|:)} *_M \mathcal{B}^T \\ \sum_{j_1j_2} a_{13j_1j_2}\mathcal{D}_{(j_1j_2|:)} *_M \mathcal{B}^T \\ \sum_{j_1j_2} a_{21j_1j_2}\mathcal{D}_{(j_1j_2|:)} *_M \mathcal{B}^T \\ \sum_{j_1j_2} a_{22j_1j_2}\mathcal{D}_{(j_1j_2|:)} *_M \mathcal{B}^T \\ \sum_{j_1j_2} a_{23j_1j_2}\mathcal{D}_{(j_1j_2|:)} *_M \mathcal{B}^T \end{pmatrix} \\ &= V_c(\mathcal{A} *_2 \mathcal{D} *_M \mathcal{B}^T). \end{aligned}$$

Definition 2.7. [23] Let $\mathcal{A} = (a_{i_1\cdots i_N j_1\cdots j_M}) \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$, $\mathcal{B} = (a_{i_1\cdots i_N k_1\cdots k_M}) \in \mathbb{C}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_M}$. Then the 'row block tensor' consisting of $\mathcal{A}$ and $\mathcal{B}$ is denoted by

$$(\mathcal{A}\ \ \mathcal{B}) \in \mathbb{C}^{\alpha^N\times L_1\times\cdots\times L_M},$$

where $\alpha^N = I_1\times\cdots\times I_N$, $L_i = J_i + K_i$, $i = 1,\ldots,M$ and

$$\begin{aligned} &(\mathcal{A}\ \ \mathcal{B})_{i_1\cdots i_N l_1\cdots l_M} \\ &= \begin{cases} a_{i_1\cdots i_N l_1\cdots l_M}, & i_1\cdots i_N \in [I_1]\times\cdots\times[I_N],\ l_1\cdots l_M \in [J_1]\times\cdots\times[J_M], \\ b_{i_1\cdots i_N l_1\cdots l_M}, & i_1\cdots i_N \in [I_1]\times\cdots\times[I_N],\ l_1\cdots l_M \in \Gamma_1\times\cdots\times\Gamma_M, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

where $\Gamma_i = \{J_i+1,\ldots,J_i+K_i\}$, $i = 1,\ldots,M$.

Similarly, for given tensors $\mathcal{C} = (c_{j_1\cdots j_M i_1\cdots i_N}) \in \mathbb{C}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N}$ and $\mathcal{D} = (d_{k_1\cdots k_M i_1\cdots i_N}) \in \mathbb{C}^{K_1\times\cdots\times K_M\times I_1\times\cdots\times I_N}$, the 'column block tensor' consisting of $\mathcal{C}$ and $\mathcal{D}$ is denoted by

$$\begin{pmatrix} \mathcal{C} \\ \mathcal{D} \end{pmatrix} = (\mathcal{C}^T\ \ \mathcal{D}^T)^T \in \mathbb{C}^{L_1\times\cdots\times L_M\times\alpha^N}.$$

Suppose $\rho_1 = (\mathcal{A}_1 \quad \mathcal{B}_1)$ and $\rho_2 = (\mathcal{A}_2 \quad \mathcal{B}_2)$ are two 'row block tensors', where $\mathcal{A}_1 \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$, $\mathcal{B}_1 \in \mathbb{C}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_M}$, $\mathcal{A}_2 \in \mathbb{C}^{T_1\times\cdots\times T_N\times J_1\times\cdots\times J_M}$, $\mathcal{B}_2 \in \mathbb{C}^{T_1\times\cdots\times T_N\times K_1\times\cdots\times K_M}$. The 'column block tensor' $\left(\begin{smallmatrix}\rho_1\\ \rho_2\end{smallmatrix}\right)$ can be written as

$$\begin{pmatrix}\mathcal{A}_1 & \mathcal{B}_1\\ \mathcal{A}_2 & \mathcal{B}_2\end{pmatrix} \in \mathbb{C}^{\beta_1\times\cdots\times\beta_N\times L_1\times\cdots\times L_M},$$

where $\beta_i = I_i + T_i$, $i = 1,\dots,N$ and $L_j = J_j + K_j$, $j = 1,\dots,M$.

Next, we present some properties with regard to the product of some 'block tensors'.

Lemma 2.8. [23] *Suppose* $(\mathcal{A}\ \ \mathcal{B})$, $\left(\begin{smallmatrix}\mathcal{C}\\ \mathcal{D}\end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix}\mathcal{A}_1 & \mathcal{B}_1\\ \mathcal{A}_2 & \mathcal{B}_2\end{smallmatrix}\right)$ *are the forms as in Definition* 2.7*. Then*

(1) $\mathcal{F} *_N (\mathcal{A}\ \ \mathcal{B}) = (\mathcal{F} *_N \mathcal{A}\ \ \mathcal{F} *_N \mathcal{B}) \in \mathbb{C}^{\alpha^N\times L_1\times\cdots\times L_M}$;

(2) $\left(\begin{smallmatrix}\mathcal{C}\\ \mathcal{D}\end{smallmatrix}\right) *_N \mathcal{F} = \left(\begin{smallmatrix}\mathcal{C}*_N\mathcal{F}\\ \mathcal{D}*_N\mathcal{F}\end{smallmatrix}\right) \in \mathbb{C}^{L_1\times\cdots\times L_M\times\alpha^N}$;

(3) $(\mathcal{A}\ \ \mathcal{B}) *_M \left(\begin{smallmatrix}\mathcal{C}\\ \mathcal{D}\end{smallmatrix}\right) = \mathcal{A} *_M \mathcal{C} + \mathcal{B} *_M \mathcal{D} \in \mathbb{C}^{\alpha^N\times\alpha^N}$;

(4) $\left(\begin{smallmatrix}\mathcal{C}\\ \mathcal{D}\end{smallmatrix}\right) *_N (\mathcal{A}\ \ \mathcal{B}) = \left(\begin{smallmatrix}\mathcal{C}*_N\mathcal{A} & \mathcal{C}*_N\mathcal{B}\\ \mathcal{D}*_N\mathcal{A} & \mathcal{D}*_N\mathcal{B}\end{smallmatrix}\right) \in \mathbb{C}^{L_1\times\cdots\times L_M\times L_1\times\cdots\times L_M}$;

(5) $\left(\begin{smallmatrix}\mathcal{A}_1 & \mathcal{B}_1\\ \mathcal{A}_2 & \mathcal{B}_2\end{smallmatrix}\right) *_M \left(\begin{smallmatrix}\mathcal{C}\\ \mathcal{D}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\mathcal{A}_1*_M\mathcal{C}+\mathcal{B}_1*_M\mathcal{D}\\ \mathcal{A}_2*_M\mathcal{C}+\mathcal{B}_2*_M\mathcal{D}\end{smallmatrix}\right) \in \mathbb{C}^{\beta_1\times\cdots\times\beta_N\times\alpha^N}$;

(6) $(\mathcal{G}\ \ \mathcal{H}) *_N \left(\begin{smallmatrix}\mathcal{A}_1 & \mathcal{B}_1\\ \mathcal{A}_2 & \mathcal{B}_2\end{smallmatrix}\right) = (\mathcal{G}*_N\mathcal{A}_1+\mathcal{H}*_N\mathcal{A}_2\ \ \mathcal{G}*_N\mathcal{B}_1+\mathcal{H}*_N\mathcal{B}_2) \in \mathbb{C}^{S_1\times\cdots\times S_N\times L_1\times\cdots\times L_M}$,

where $\mathcal{F} \in \mathbb{C}^{\alpha^N\times\alpha^N}$, $\mathcal{G} \in \mathbb{C}^{S_1\times\cdots\times S_N\times I_1\times\cdots\times I_N}$ *and* $\mathcal{H} \in \mathbb{C}^{S_1\times\cdots\times S_N\times T_1\times\cdots\times T_N}$.

Now we introduce the definition of the Moore–Penrose inverse of a tensor over $\mathbb{C}$ via the Einstein product, which is a generalization of the Moore–Penrose inverse of a matrix.

Definition 2.9. [21] Suppose $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$. The tensor $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_N}$ satisfying the following four complex tensor equalities:

(1) $\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}$,

(2) $\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}$,

(3) $(\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}$,

(4) $(\mathcal{X} *_N \mathcal{A})^* = \mathcal{X} *_N \mathcal{A}$

is called the Moore–Penrose inverse of $\mathcal{A}$, denoted by $\mathcal{A}^\dagger$.

The following lemma provides the solvability conditions and general solution to the multilinear system $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$, which is used to prove our main results thereinafter.

Lemma 2.10. [21] *Suppose* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_N}$ *and* $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_N}$. *Then we have the following statements:*

(1) *The least squares solutions of the multilinear system* $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$ *can be represented as*

$$\mathcal{X} = \mathcal{A}^{\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) *_N \mathcal{W},$$

where $\mathcal{W} \in \mathbb{R}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_N}$ *is an arbitrary tensor. The minimal norm least squares solution is* $\mathcal{X} = \mathcal{A}^{\dagger} *_N \mathcal{B}$.

(2) *The multilinear system* $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$ *has a solution* $\mathcal{X}^{\star}$ *if and only if* $\mathcal{A} *_N \mathcal{A}^{\dagger} *_N \mathcal{B} = \mathcal{B}$. *In that case,* $\mathcal{X}^{\star}$ *can be represented as*

$$\mathcal{X}^{\star} = \mathcal{A}^{\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) *_N \mathcal{W},$$

where $\mathcal{W} \in \mathbb{R}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_N}$ *is an arbitrary tensor.*

2.2. The complex representation of reduced biquaternion tensors

We know that a matrix $A \in \mathbb{H}_r^{m\times n}$ can be written as $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in \mathbb{C}^{m\times n}$. Similarly, a tensor $\mathcal{A} \in \mathbb{H}_r^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$ can also be expressed as $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2\mathbf{j}$, where $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$. Thus, the complex representation tensor of $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2\mathbf{j}$ is given by $f(\mathcal{A}) = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_2 & \mathcal{A}_1 \end{pmatrix} \in \mathbb{C}^{2I_1\times\cdots\times 2I_N\times 2J_1\times\cdots\times 2J_M}$. Notice that $f(\mathcal{A})$ is uniquely determined by $\mathcal{A}$.

Theorem 2.11. *Suppose* $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2\mathbf{j} \in \mathbb{H}_r^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$ *and* $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2\mathbf{j} \in \mathbb{H}_r^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_N}$. *Then we have*

$$f(\mathcal{A} *_N \mathcal{B}) = f(\mathcal{A}) *_N f(\mathcal{B}).$$

Proof. By the complex representation tensor of $\mathcal{A} *_N \mathcal{B}$ and Lemma 2.8, we have

$$\begin{aligned} f(\mathcal{A} *_N \mathcal{B}) &= f(\mathcal{A}_1 *_N \mathcal{B}_1 + \mathcal{A}_2 *_N \mathcal{B}_2 + (\mathcal{A}_1 *_N \mathcal{B}_2 + \mathcal{A}_2 *_N \mathcal{B}_1)\mathbf{j}) \\ &= \begin{pmatrix} \mathcal{A}_1 *_N \mathcal{B}_1 + \mathcal{A}_2 *_N \mathcal{B}_2 & \mathcal{A}_1 *_N \mathcal{B}_2 + \mathcal{A}_2 *_N \mathcal{B}_1 \\ \mathcal{A}_1 *_N \mathcal{B}_2 + \mathcal{A}_2 *_N \mathcal{B}_1 & \mathcal{A}_1 *_N \mathcal{B}_1 + \mathcal{A}_2 *_N \mathcal{B}_2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} f(\mathcal{A}) *_N f(\mathcal{B}) &= \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_2 & \mathcal{A}_1 \end{pmatrix} *_N \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_2 & \mathcal{B}_1 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{A}_1 *_N \mathcal{B}_1 + \mathcal{A}_2 *_N \mathcal{B}_2 & \mathcal{A}_1 *_N \mathcal{B}_2 + \mathcal{A}_2 *_N \mathcal{B}_1 \\ \mathcal{A}_1 *_N \mathcal{B}_2 + \mathcal{A}_2 *_N \mathcal{B}_1 & \mathcal{A}_1 *_N \mathcal{B}_1 + \mathcal{A}_2 *_N \mathcal{B}_2 \end{pmatrix}. \end{aligned}$$

Thus, $f(\mathcal{A} *_N \mathcal{B}) = f(\mathcal{A}) *_N f(\mathcal{B})$. $\square$

For $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2\mathbf{j} \in \mathbb{H}_r^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, we denote such an identification by the symbol $\mathrel{\hat{=}}$, that is,

$$\mathcal{A}_1 + \mathcal{A}_2\mathbf{j} = \mathcal{A} \mathrel{\hat{=}} \Phi_{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2).$$

We also denote

$$\overrightarrow{\mathcal{A}} = (\operatorname{Re}\mathcal{A}_1, \operatorname{Im}\mathcal{A}_1, \operatorname{Re}\mathcal{A}_2, \operatorname{Im}\mathcal{A}_2).$$

Note that $\|\Phi_{\mathcal{A}}\| = \|\overrightarrow{\mathcal{A}}\|$. Besides, we have $V_c(\mathcal{A}) = V_c(\mathcal{A}_1 + \mathcal{A}_2\mathbf{j}) = V_c(\mathcal{A}_1) + V_c(\mathcal{A}_2)\mathbf{j}$,

$$\begin{aligned} V_c(\mathcal{A}) \mathrel{\hat{=}} V_c(\Phi_{\mathcal{A}}) &= \begin{pmatrix} V_c(\mathcal{A}_1) \\ V_c(\mathcal{A}_2) \end{pmatrix} = \begin{pmatrix} V_c(\operatorname{Re}\mathcal{A}_1) + V_c(\operatorname{Im}\mathcal{A}_1)\mathbf{i} \\ V_c(\operatorname{Re}\mathcal{A}_2) + V_c(\operatorname{Im}\mathcal{A}_2)\mathbf{i} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{I} & \mathbf{i}\mathcal{I} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} & \mathbf{i}\mathcal{I} \end{pmatrix} *_N \begin{pmatrix} V_c(\operatorname{Re}\mathcal{A}_1) \\ V_c(\operatorname{Im}\mathcal{A}_1) \\ V_c(\operatorname{Re}\mathcal{A}_2) \\ V_c(\operatorname{Im}\mathcal{A}_2) \end{pmatrix} \\ &= \Omega_{J_N} *_N V_c(\overrightarrow{\mathcal{A}}), \end{aligned}$$

where $\mathcal{I} \in \mathcal{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ and $\Omega_{J_N} = \left(\begin{smallmatrix} \mathcal{I} & \mathbf{i}\mathcal{I} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} & \mathbf{i}\mathcal{I} \end{smallmatrix}\right)$.

Addition of two reduced biquaternion tensors $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2\mathbf{j}$ and $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2\mathbf{j}$ is defined by

$$(\mathcal{A}_1 + \mathcal{B}_1) + (\mathcal{A}_2 + \mathcal{B}_2)\mathbf{j} = (\mathcal{A} + \mathcal{B}) \mathrel{\hat{=}} \Phi_{\mathcal{A}+\mathcal{B}} = (\mathcal{A}_1 + \mathcal{B}_1, \mathcal{A}_2 + \mathcal{B}_2).$$

Whereas multiplication of two reduced biquaternion tensors $\mathcal{A}$, $\mathcal{C}$ is defined as

$$\mathcal{A} *_N \mathcal{C} = (\mathcal{A}_1 + \mathcal{A}_2\mathbf{j}) *_N (\mathcal{C}_1 + \mathcal{C}_2\mathbf{j}) = (\mathcal{A}_1 *_N \mathcal{C}_1 + \mathcal{A}_2 *_N \mathcal{C}_2) + (\mathcal{A}_1 *_N \mathcal{C}_2 + \mathcal{A}_2 *_N \mathcal{C}_1)\mathbf{j}.$$

So $\mathcal{A} *_N \mathcal{C} \mathrel{\hat{=}} \Phi_{\mathcal{A}*_N\mathcal{C}}$. We derive some properties of $\Phi_{\mathcal{A}}$ as follows.

Theorem 2.12. *Suppose k is a real number and $\mathcal{A}, \mathcal{B} \in \mathbb{H}_r^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{C} \in \mathbb{H}_r^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_N}$. Then*

(1) $\mathcal{A} = \mathcal{B} \iff \Phi_{\mathcal{A}} = \Phi_{\mathcal{B}}$;

(2) $\Phi_{\mathcal{A}+\mathcal{B}} = \Phi_{\mathcal{A}} + \Phi_{\mathcal{B}}$, $\Phi_{k\mathcal{A}} = k\Phi_{\mathcal{A}}$;

(3) $\Phi_{\mathcal{A}*_N\mathcal{C}} = \Phi_{\mathcal{A}} *_N f(\mathcal{C})$.

Proof. Since the proofs of (1) and (2) are easy, we omit them. We only prove (3). $\Phi_{\mathcal{A}*_N\mathcal{C}}$ can be expressed as

$$\begin{aligned} \Phi_{\mathcal{A}*_N\mathcal{C}} &= (\mathcal{A}_1 *_N \mathcal{C}_1 + \mathcal{A}_2 *_N \mathcal{C}_2, \mathcal{A}_1 *_N \mathcal{C}_2 + \mathcal{A}_2 *_N \mathcal{C}_1) \\ &= (\mathcal{A}_1, \mathcal{A}_2) *_N \begin{pmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ \mathcal{C}_2 & \mathcal{C}_1 \end{pmatrix} = \Phi_{\mathcal{A}} *_N f(\mathcal{C}). \end{aligned}$$

□

Theorem 2.13. *Suppose* $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2\mathbf{j} \in \mathbb{H}_r^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2\mathbf{j} \in \mathbb{H}_r^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_M}$ *and* $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2\mathbf{j} \in \mathbb{H}_r^{K_1\times\cdots\times K_M\times J_1\times\cdots\times J_N}$. *Then*

$$V_c(\Phi_{\mathcal{A}*_N\mathcal{B}*_M\mathcal{C}}) = f\big[(\mathcal{A}_1\otimes\mathcal{C}_1^T + \mathcal{A}_2\otimes\mathcal{C}_2^T) + (\mathcal{A}_2\otimes\mathcal{C}_1^T + \mathcal{A}_1\otimes\mathcal{C}_2^T)\mathbf{j}\big] *_M \Omega_{K_M} *_M V_c(\overrightarrow{\mathcal{B}}),$$

where Ω_{K_M} *has the same structure as* Ω_{J_N}, *except for dimension.*

Proof. By Theorems 2.11 and 2.12, we have

$$\begin{aligned}
\Phi_{\mathcal{A}*_N\mathcal{B}*_M\mathcal{C}} &= \Phi_{\mathcal{A}} *_N f(\mathcal{B} *_M \mathcal{C}) = \Phi_{\mathcal{A}} *_N f(\mathcal{B}) *_M f(\mathcal{C}) \\
&= (\mathcal{A}_1, \mathcal{A}_2) *_N \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_2 & \mathcal{B}_1 \end{pmatrix} *_M \begin{pmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ \mathcal{C}_2 & \mathcal{C}_1 \end{pmatrix} \\
&= (\mathcal{A}_1 *_N \mathcal{B}_1 *_M \mathcal{C}_1 + \mathcal{A}_2 *_N \mathcal{B}_2 *_M \mathcal{C}_1 + \mathcal{A}_1 *_N \mathcal{B}_2 *_M \mathcal{C}_2 + \mathcal{A}_2 *_N \mathcal{B}_1 *_M \mathcal{C}_2, \\
&\quad \mathcal{A}_1 *_N \mathcal{B}_1 *_M \mathcal{C}_2 + \mathcal{A}_2 *_N \mathcal{B}_2 *_M \mathcal{C}_2 + \mathcal{A}_1 *_N \mathcal{B}_2 *_M \mathcal{C}_1 + \mathcal{A}_2 *_N \mathcal{B}_1 *_M \mathcal{C}_1).
\end{aligned}$$

Further, we can obtain

$$\begin{aligned}
&V_c(\Phi_{\mathcal{A}*_N\mathcal{B}*_M\mathcal{C}}) \\
&= \begin{bmatrix} V_c(\mathcal{A}_1 *_N \mathcal{B}_1 *_M \mathcal{C}_1 + \mathcal{A}_2 *_N \mathcal{B}_2 *_M \mathcal{C}_1 + \mathcal{A}_1 *_N \mathcal{B}_2 *_M \mathcal{C}_2 + \mathcal{A}_2 *_N \mathcal{B}_1 *_M \mathcal{C}_2) \\ V_c(\mathcal{A}_1 *_N \mathcal{B}_1 *_M \mathcal{C}_2 + \mathcal{A}_2 *_N \mathcal{B}_2 *_M \mathcal{C}_2 + \mathcal{A}_1 *_N \mathcal{B}_2 *_M \mathcal{C}_1 + \mathcal{A}_2 *_N \mathcal{B}_1 *_M \mathcal{C}_1) \end{bmatrix} \\
&= \begin{bmatrix} (\mathcal{A}_1\otimes\mathcal{C}_1^T) *_M V_c(\mathcal{B}_1) + (\mathcal{A}_2\otimes\mathcal{C}_1^T) *_M V_c(\mathcal{B}_2) + (\mathcal{A}_1\otimes\mathcal{C}_2^T) *_M V_c(\mathcal{B}_2) + (\mathcal{A}_2\otimes\mathcal{C}_2^T) *_M V_c(\mathcal{B}_1) \\ (\mathcal{A}_1\otimes\mathcal{C}_2^T) *_M V_c(\mathcal{B}_1) + (\mathcal{A}_2\otimes\mathcal{C}_2^T) *_M V_c(\mathcal{B}_2) + (\mathcal{A}_1\otimes\mathcal{C}_1^T) *_M V_c(\mathcal{B}_2) + (\mathcal{A}_2\otimes\mathcal{C}_1^T) *_M V_c(\mathcal{B}_1) \end{bmatrix} \\
&= \begin{bmatrix} (\mathcal{A}_1\otimes\mathcal{C}_1^T + \mathcal{A}_2\otimes\mathcal{C}_2^T) *_M V_c(\mathcal{B}_1) + (\mathcal{A}_2\otimes\mathcal{C}_1^T + \mathcal{A}_1\otimes\mathcal{C}_2^T) *_M V_c(\mathcal{B}_2) \\ (\mathcal{A}_1\otimes\mathcal{C}_2^T + \mathcal{A}_2\otimes\mathcal{C}_1^T) *_M V_c(\mathcal{B}_1) + (\mathcal{A}_2\otimes\mathcal{C}_2^T + \mathcal{A}_1\otimes\mathcal{C}_1^T) *_M V_c(\mathcal{B}_2) \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}_1\otimes\mathcal{C}_1^T + \mathcal{A}_2\otimes\mathcal{C}_2^T & \mathcal{A}_2\otimes\mathcal{C}_1^T + \mathcal{A}_1\otimes\mathcal{C}_2^T \\ \mathcal{A}_1\otimes\mathcal{C}_2^T + \mathcal{A}_2\otimes\mathcal{C}_1^T & \mathcal{A}_2\otimes\mathcal{C}_2^T + \mathcal{A}_1\otimes\mathcal{C}_1^T \end{bmatrix} *_M \begin{pmatrix} V_c(\mathcal{B}_1) \\ V_c(\mathcal{B}_2) \end{pmatrix} \\
&= f\big[(\mathcal{A}_1\otimes\mathcal{C}_1^T + \mathcal{A}_2\otimes\mathcal{C}_2^T) + (\mathcal{A}_2\otimes\mathcal{C}_1^T + \mathcal{A}_1\otimes\mathcal{C}_2^T)\mathbf{j}\big] *_M \begin{pmatrix} V_c(\mathcal{B}_1) \\ V_c(\mathcal{B}_2) \end{pmatrix} \\
&= f\big[(\mathcal{A}_1\otimes\mathcal{C}_1^T + \mathcal{A}_2\otimes\mathcal{C}_2^T) + (\mathcal{A}_2\otimes\mathcal{C}_1^T + \mathcal{A}_1\otimes\mathcal{C}_2^T)\mathbf{j}\big] *_M \Omega_{K_M} *_M V_c(\overrightarrow{\mathcal{B}}). \qquad \square
\end{aligned}$$

3. The solution for Problem 1.1

In this section, we are now in a position to solve Problem 1.1. For convenience, we set $\mathcal{A}_\sigma = \mathcal{A}_1^\sigma + \mathcal{A}_2^\sigma\mathbf{j}$ $(\sigma = 1,2,3)$, $\mathcal{C}_\tau = \mathcal{C}_1^\tau + \mathcal{C}_2^\tau\mathbf{j}$, $\mathcal{D}_\tau = \mathcal{D}_1^\tau + \mathcal{D}_2^\tau\mathbf{j}$ $(\tau = 1,2)$, $\mathcal{A}_1, \mathcal{B}_1 \in \mathbb{H}_r^{I_1\times\cdots I_N\times J_1\times\cdots J_N}$, $\mathcal{A}_2, \mathcal{B}_2 \in \mathbb{H}_r^{J_1\times\cdots J_N\times K_1\times\cdots K_N}$, $\mathcal{A}_3, \mathcal{B}_3 \in \mathbb{H}_r^{L_1\times\cdots L_N\times J_1\times\cdots J_N}$, $\mathcal{C}_\tau, \mathcal{D}_\tau, \mathcal{E}_\tau \in \mathbb{H}_r^{J_1\times\cdots J_N\times J_1\times\cdots J_N}$,

$$\begin{aligned}
&\mathcal{P}_1 = f[(\mathcal{A}_1^1\otimes\mathcal{I}) + (\mathcal{A}_2^1\otimes\mathcal{I})\mathbf{j}] *_N \Omega_{J_N}, && \mathcal{P}_2 = f[(\mathcal{I}\otimes(\mathcal{A}_1^2)^T) + (\mathcal{I}\otimes(\mathcal{A}_2^2)^T)\mathbf{j}] *_N \Omega_{J_N}, \\
&\mathcal{P}_3 = f[(\mathcal{A}_1^3\otimes\mathcal{I}) + (\mathcal{A}_2^3\otimes\mathcal{I})\mathbf{j}] *_N \Omega_{J_N}, && \mathcal{P}_4 = f[(\mathcal{C}_1^1\otimes\mathcal{I}) + (\mathcal{C}_2^1\otimes\mathcal{I})\mathbf{j}] *_N \Omega_{J_N}, \\
&\mathcal{P}_5 = f[(\mathcal{I}\otimes(\mathcal{D}_1^1)^T) + (\mathcal{I}\otimes(\mathcal{D}_2^1)^T)\mathbf{j}] *_N \Omega_{J_N}, && \mathcal{P}_6 = f[(\mathcal{C}_1^2\otimes\mathcal{I}) + (\mathcal{C}_2^2\otimes\mathcal{I})\mathbf{j}] *_N \Omega_{J_N}, \\
&\mathcal{P}_7 = f[(\mathcal{I}\otimes(\mathcal{D}_1^2)^T) + (\mathcal{I}\otimes(\mathcal{D}_2^2)^T)\mathbf{j}] *_N \Omega_{J_N},
\end{aligned}$$

and

$$\mathcal{P}=\begin{pmatrix}\mathcal{P}_1 & \mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{P}_2 & \mathcal{O}\\ \mathcal{O} & \mathcal{O} & \mathcal{P}_3\\ \mathcal{P}_4 & -\mathcal{P}_5 & \mathcal{O}\\ \mathcal{O} & -\mathcal{P}_7 & \mathcal{P}_6\end{pmatrix},\quad \mathcal{G}=\begin{pmatrix}\operatorname{Re}\mathcal{P}\\ \operatorname{Im}\mathcal{P}\end{pmatrix},\quad \mathcal{H}=\begin{pmatrix}V_c(\operatorname{Re}\Phi_{\mathcal{B}_1})\\ V_c(\operatorname{Re}\Phi_{\mathcal{B}_2})\\ V_c(\operatorname{Re}\Phi_{\mathcal{B}_3})\\ V_c(\operatorname{Re}\Phi_{\mathcal{E}_1})\\ V_c(\operatorname{Re}\Phi_{\mathcal{E}_2})\\ V_c(\operatorname{Im}\Phi_{\mathcal{B}_1})\\ V_c(\operatorname{Im}\Phi_{\mathcal{B}_2})\\ V_c(\operatorname{Im}\Phi_{\mathcal{B}_3})\\ V_c(\operatorname{Im}\Phi_{\mathcal{E}_1})\\ V_c(\operatorname{Im}\Phi_{\mathcal{E}_2})\end{pmatrix}. \tag{3.1}$$

By the complex representation of reduced biquaternion tensors mentioned above, we can turn least squares problem of the system of mixed generalized Sylvester reduced biquaternion tensor equations (1.1) into a corresponding problem of complex tensor equations. Next we give the main results of this paper.

Theorem 3.1. *Suppose $\mathcal{A}_\sigma=\mathcal{A}_1^\sigma+\mathcal{A}_2^\sigma\mathbf{j}$, $\mathcal{B}_\sigma=\mathcal{B}_1^\sigma+\mathcal{B}_2^\sigma\mathbf{j}$ ($\sigma=1,2,3$), $\mathcal{C}_\tau=\mathcal{C}_1^\tau+\mathcal{C}_2^\tau\mathbf{j}$, $\mathcal{D}_\tau=\mathcal{D}_1^\tau+\mathcal{D}_2^\tau\mathbf{j}$, $\mathcal{E}_\tau=\mathcal{E}_1^\tau+\mathcal{E}_2^\tau\mathbf{j}$ ($\tau=1,2$), $\mathcal{A}_1,\mathcal{B}_1\in\mathbb{H}_r^{I_1\times\cdots I_N\times J_1\times\cdots J_N}$, $\mathcal{A}_2,\mathcal{B}_2\in\mathbb{H}_r^{J_1\times\cdots J_N\times K_1\times\cdots K_N}$, $\mathcal{A}_3,\mathcal{B}_3\in\mathbb{H}_r^{L_1\times\cdots L_N\times J_1\times\cdots J_N}$, $\mathcal{C}_\tau,\mathcal{D}_\tau,\mathcal{E}_\tau\in\mathbb{H}_r^{J_1\times\cdots J_N\times J_1\times\cdots J_N}$ and let $\mathcal{P}$, $\mathcal{G}$, $\mathcal{H}$ be as in* (3.1). *Hence the set $\mathbb{H}_L$ of Problem* 1.1 *can be expressed as*

$$\mathbb{H}_L=\left\{(\mathcal{X},\mathcal{Y},\mathcal{Z})\in\mathbb{H}_r^{J_1\times\cdots J_N\times J_1\times\cdots J_N}\;\middle|\;\begin{pmatrix}V_c(\overrightarrow{\mathcal{X}})\\ V_c(\overrightarrow{\mathcal{Y}})\\ V_c(\overrightarrow{\mathcal{Z}})\end{pmatrix}=\mathcal{G}^\dagger *_N\mathcal{H}+(\mathcal{I}-\mathcal{G}^\dagger *_N\mathcal{G}) *_N\mho\right\}, \tag{3.2}$$

where $\mho$ is an arbitrary tensor vector of appropriate order. And then, the minimal norm least squares solution $(\mathcal{X}_l,\mathcal{Y}_l,\mathcal{Z}_l)$ of Problem 1.1 *satisfies*

$$\begin{pmatrix}V_c(\overrightarrow{\mathcal{X}_l})\\ V_c(\overrightarrow{\mathcal{Y}_l})\\ V_c(\overrightarrow{\mathcal{Z}_l})\end{pmatrix}=\mathcal{G}^\dagger *_N\mathcal{H}. \tag{3.3}$$

Proof. For $\mathcal{X},\mathcal{Y},\mathcal{Z}\in\mathbb{H}_r^{J_1\times\cdots J_N\times J_1\times\cdots J_N}$, it follows from Theorem 2.13 that

$$\|\mathcal{A}_1 *_N\mathcal{X}-\mathcal{B}_1\|^2+\|\mathcal{Y} *_N\mathcal{A}_2-\mathcal{B}_2\|^2+\|\mathcal{A}_3 *_N\mathcal{Z}-\mathcal{B}_3\|^2+\|\mathcal{C}_1 *_N\mathcal{X}-\mathcal{Y} *_N\mathcal{D}_1-\mathcal{E}_1\|^2$$
$$+\|\mathcal{C}_2 *_N\mathcal{Z}-\mathcal{Y} *_N\mathcal{D}_2-\mathcal{E}_2\|^2$$

$$
\begin{aligned}
&= \|\Phi_{\mathcal{A}_1 *_N \mathcal{X}} - \Phi_{\mathcal{B}_1}\|^2 + \|\Phi_{\mathcal{Y} *_N \mathcal{A}_2} - \Phi_{\mathcal{B}_2}\|^2 + \|\Phi_{\mathcal{A}_3 *_N \mathcal{Z}} - \Phi_{\mathcal{B}_3}\|^2 + \|\Phi_{\mathcal{C}_1 *_N \mathcal{X}} - \Phi_{\mathcal{Y} *_N \mathcal{D}_1} - \Phi_{\mathcal{E}_1}\|^2 \\
&\quad + \|\Phi_{\mathcal{C}_2 *_N \mathcal{Z}} - \Phi_{\mathcal{Y} *_N \mathcal{D}_2} - \Phi_{\mathcal{E}_2}\|^2 \\
&= \|V_c(\Phi_{\mathcal{A}_1 *_N \mathcal{X}}) - V_c(\Phi_{\mathcal{B}_1})\|^2 + \|V_c(\Phi_{\mathcal{Y} *_N \mathcal{A}_2}) - V_c(\Phi_{\mathcal{B}_2})\|^2 + \|V_c(\Phi_{\mathcal{A}_3 *_N \mathcal{Z}}) - V_c(\Phi_{\mathcal{B}_3})\|^2 \\
&\quad + \|V_c(\Phi_{\mathcal{C}_1 *_N \mathcal{X}}) - V_c(\Phi_{\mathcal{Y} *_N \mathcal{D}_1}) - V_c(\Phi_{\mathcal{E}_1})\|^2 + \|V_c(\Phi_{\mathcal{C}_2 *_N \mathcal{Z}}) - V_c(\Phi_{\mathcal{Y} *_N \mathcal{D}_2}) - V_c(\Phi_{\mathcal{E}_2})\|^2 \\
&= \left\| f[(\mathcal{A}_1^1 \otimes \mathcal{I}) + (\mathcal{A}_2^1 \otimes \mathcal{I})\mathbf{j}] *_N \Omega_{J_N} *_N V_c(\overrightarrow{\mathcal{X}}) - V_c(\Phi_{\mathcal{B}_1}) \right\|^2 \\
&\quad + \left\| f[(\mathcal{I} \otimes (\mathcal{A}_1^2)^T) + (\mathcal{I} \otimes (\mathcal{A}_2^2)^T)\mathbf{j}] *_N \Omega_{J_N} *_N V_c(\overrightarrow{\mathcal{Y}}) - V_c(\Phi_{\mathcal{B}_2}) \right\|^2 \\
&\quad + \left\| f[(\mathcal{A}_1^3 \otimes \mathcal{I}) + (\mathcal{A}_2^3 \otimes \mathcal{I})\mathbf{j}] *_N \Omega_{J_N} *_N V_c(\overrightarrow{\mathcal{Z}}) - V_c(\Phi_{\mathcal{B}_3}) \right\|^2 \\
&\quad + \Big\| f[(\mathcal{C}_1^1 \otimes \mathcal{I}) + (\mathcal{C}_2^1 \otimes \mathcal{I})\mathbf{j}] *_N \Omega_{J_N} *_N V_c(\overrightarrow{\mathcal{X}}) \\
&\qquad - f[(\mathcal{I} \otimes (\mathcal{D}_1^1)^T) + (\mathcal{I} \otimes (\mathcal{D}_2^1)^T)\mathbf{j}] *_N \Omega_{J_N} *_N V_c(\overrightarrow{\mathcal{Y}}) - V_c(\Phi_{\mathcal{E}_1}) \Big\|^2 \\
&\quad + \Big\| f[(\mathcal{C}_1^2 \otimes \mathcal{I}) + (\mathcal{C}_2^2 \otimes \mathcal{I})\mathbf{j}] *_N \Omega_{J_N} *_N V_c(\overrightarrow{\mathcal{Z}}) \\
&\qquad - f[(\mathcal{I} \otimes (\mathcal{D}_1^2)^T) + (\mathcal{I} \otimes (\mathcal{D}_2^2)^T)\mathbf{j}] *_N \Omega_{J_N} *_N V_c(\overrightarrow{\mathcal{Y}}) - V_c(\Phi_{\mathcal{E}_2}) \Big\|^2 \\
&= \left\| \mathcal{P}_1 *_N V_c(\overrightarrow{\mathcal{X}}) - V_c(\Phi_{\mathcal{B}_1}) \right\|^2 + \left\| \mathcal{P}_2 *_N V_c(\overrightarrow{\mathcal{Y}}) - V_c(\Phi_{\mathcal{B}_2}) \right\|^2 + \left\| \mathcal{P}_3 *_N V_c(\overrightarrow{\mathcal{Z}}) - V_c(\Phi_{\mathcal{B}_3}) \right\|^2 \\
&\quad + \left\| \mathcal{P}_4 *_N V_c(\overrightarrow{\mathcal{X}}) - \mathcal{P}_5 *_N V_c(\overrightarrow{\mathcal{Y}}) - V_c(\Phi_{\mathcal{E}_1}) \right\|^2 + \left\| \mathcal{P}_6 *_N V_c(\overrightarrow{\mathcal{Z}}) - \mathcal{P}_7 *_N V_c(\overrightarrow{\mathcal{Y}}) - V_c(\Phi_{\mathcal{E}_2}) \right\|^2 \\
&= \left\| \begin{pmatrix} \mathcal{P}_1 & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{P}_2 & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{P}_3 \\ \mathcal{P}_4 & -\mathcal{P}_5 & \mathcal{O} \\ \mathcal{O} & -\mathcal{P}_7 & \mathcal{P}_6 \end{pmatrix} *_N \begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} - \begin{pmatrix} V_c(\Phi_{\mathcal{B}_1}) \\ V_c(\Phi_{\mathcal{B}_2}) \\ V_c(\Phi_{\mathcal{B}_3}) \\ V_c(\Phi_{\mathcal{E}_1}) \\ V_c(\Phi_{\mathcal{E}_2}) \end{pmatrix} \right\|^2 = \left\| \mathcal{P} *_N \begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} - \begin{pmatrix} V_c(\Phi_{\mathcal{B}_1}) \\ V_c(\Phi_{\mathcal{B}_2}) \\ V_c(\Phi_{\mathcal{B}_3}) \\ V_c(\Phi_{\mathcal{E}_1}) \\ V_c(\Phi_{\mathcal{E}_2}) \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} \operatorname{Re}\mathcal{P} \\ \operatorname{Im}\mathcal{P} \end{pmatrix} *_N \begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} - \begin{pmatrix} V_c(\operatorname{Re}\Phi_{\mathcal{B}_1}) \\ V_c(\operatorname{Re}\Phi_{\mathcal{B}_2}) \\ V_c(\operatorname{Re}\Phi_{\mathcal{B}_3}) \\ V_c(\operatorname{Re}\Phi_{\mathcal{E}_1}) \\ V_c(\operatorname{Re}\Phi_{\mathcal{E}_2}) \\ V_c(\operatorname{Im}\Phi_{\mathcal{B}_1}) \\ V_c(\operatorname{Im}\Phi_{\mathcal{B}_2}) \\ V_c(\operatorname{Im}\Phi_{\mathcal{B}_3}) \\ V_c(\operatorname{Im}\Phi_{\mathcal{E}_1}) \\ V_c(\operatorname{Im}\Phi_{\mathcal{E}_2}) \end{pmatrix} \right\|^2 = \left\| \mathcal{G} *_N \begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} - \mathcal{H} \right\|^2 .
\end{aligned}
$$

By Lemma 2.10, we get

$$
\begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} = \mathcal{G}^\dagger *_N \mathcal{H} + (\mathcal{I} - \mathcal{G}^\dagger *_N \mathcal{G}) *_N \mho.
$$

Moreover, from (3.2), we know that the solution set $\mathbb{H}_L$ is nonempty and is a closed convex set. Therefore, Problem 1.1 has a unique solution $(\mathcal{X}_l, \mathcal{Y}_l, \mathcal{Z}_l) \in \mathbb{H}_L$. Now, we prove that this unique solution $(\mathcal{X}_l, \mathcal{Y}_l, \mathcal{Z}_l)$ can be expressed as (3.3). It follows from the above that

$$\begin{aligned}\min_{(\mathcal{X},\mathcal{Y},\mathcal{Z})\in\mathbb{H}_L}\left(\|(\mathcal{X},\mathcal{Y},\mathcal{Z})\|^2\right) &= \min_{(\mathcal{X},\mathcal{Y},\mathcal{Z})\in\mathbb{H}_L}\left(\|\mathcal{X}\|^2+\|\mathcal{Y}\|^2+\|\mathcal{Z}\|^2\right)\\ &= \min_{(\mathcal{X},\mathcal{Y},\mathcal{Z})\in\mathbb{H}_L}\left(\left\|V_c(\overrightarrow{\mathcal{X}})\right\|^2+\left\|V_c(\overrightarrow{\mathcal{Y}})\right\|^2+\left\|V_c(\overrightarrow{\mathcal{Z}})\right\|^2\right)\\ &= \min_{(\mathcal{X},\mathcal{Y},\mathcal{Z})\in\mathbb{H}_L}\left\|\begin{pmatrix}V_c(\overrightarrow{\mathcal{X}})\\ V_c(\overrightarrow{\mathcal{Y}})\\ V_c(\overrightarrow{\mathcal{Z}})\end{pmatrix}\right\|^2.\end{aligned}$$

Further, using Lemma 2.10 and (3.2), we can derive

$$\begin{pmatrix}V_c(\overrightarrow{\mathcal{X}})\\ V_c(\overrightarrow{\mathcal{Y}})\\ V_c(\overrightarrow{\mathcal{Z}})\end{pmatrix} = \mathcal{G}^\dagger *_N \mathcal{H}.$$

Thus we can get (3.2) and (3.3). The proof is completed. □

By virtue of Theorem 3.1 and Lemma 2.10, we derive the following conclusion.

Corollary 3.2. *The system of mixed generalized Sylvester reduced biquaternion tensor equations* (1.1) *has a solution* $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ *if and only if*

$$\mathcal{G} *_N \mathcal{G}^\dagger *_N \mathcal{H} = \mathcal{H}. \tag{3.4}$$

In this case, the solution set of system (1.1) *can be represented as*

$$\mathbb{H}_S = \left\{(\mathcal{X},\mathcal{Y},\mathcal{Z})\ \middle|\ \begin{pmatrix}V_c(\overrightarrow{\mathcal{X}})\\ V_c(\overrightarrow{\mathcal{Y}})\\ V_c(\overrightarrow{\mathcal{Z}})\end{pmatrix} = \mathcal{G}^\dagger *_N \mathcal{H} + (\mathcal{I} - \mathcal{G}^\dagger *_N \mathcal{G}) *_N \mho\right\},$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathbb{H}_r^{J_1\times\cdots J_N\times J_1\times\cdots J_N}$ *and* $\mho$ *is an arbitrary tensor vector of appropriate order. Furthermore,* (1.1) *has a unique solution* $(\mathcal{X}_l', \mathcal{Y}_l', \mathcal{Z}_l') \in \mathbb{H}_S$ *if and only if* $\mathcal{G}$ *has full column rank, and the unique solution* $(\mathcal{X}_l', \mathcal{Y}_l', \mathcal{Z}_l')$ *satisfies*

$$\begin{pmatrix}V_c(\overrightarrow{\mathcal{X}_l'})\\ V_c(\overrightarrow{\mathcal{Y}_l'})\\ V_c(\overrightarrow{\mathcal{Z}_l'})\end{pmatrix} = \mathcal{G}^\dagger *_N \mathcal{H}. \tag{3.5}$$

Proof. According to the proof of Theorem 3.1 and Definition 2.9, we have

$$\begin{aligned}
&\|\mathcal{A}_1 *_N \mathcal{X} - \mathcal{B}_1\|^2 + \|\mathcal{Y} *_N \mathcal{A}_2 - \mathcal{B}_2\|^2 + \|\mathcal{A}_3 *_N \mathcal{Z} - \mathcal{B}_3\|^2 \\
&+ \|\mathcal{C}_1 *_N \mathcal{X} - \mathcal{Y} *_N \mathcal{D}_1 - \mathcal{E}_1\|^2 + \|\mathcal{C}_2 *_N \mathcal{Z} - \mathcal{Y} *_N \mathcal{D}_2 - \mathcal{E}_2\|^2 \\
=& \left\| \mathcal{G} *_N \begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} - \mathcal{H} \right\|^2 = \left\| \mathcal{G} *_N \mathcal{G}^\dagger *_N \mathcal{G} *_N \begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} - \mathcal{H} \right\|^2 \\
=& \|\mathcal{G} *_N \mathcal{G}^\dagger *_N \mathcal{H} - \mathcal{H}\|^2,
\end{aligned}$$

thus we can obtain

$$\begin{aligned}
&\|\mathcal{A}_1 *_N \mathcal{X} - \mathcal{B}_1\|^2 + \|\mathcal{Y} *_N \mathcal{A}_2 - \mathcal{B}_2\|^2 + \|\mathcal{A}_3 *_N \mathcal{Z} - \mathcal{B}_3\|^2 \\
&+ \|\mathcal{C}_1 *_N \mathcal{X} - \mathcal{Y} *_N \mathcal{D}_1 - \mathcal{E}_1\|^2 + \|\mathcal{C}_2 *_N \mathcal{Z} - \mathcal{Y} *_N \mathcal{D}_2 - \mathcal{E}_2\|^2 = 0 \\
\Longleftrightarrow \quad & \|\mathcal{G} *_N \mathcal{G}^\dagger *_N \mathcal{H} - \mathcal{H}\|^2 = 0 \\
\Longleftrightarrow \quad & \mathcal{G} *_N \mathcal{G}^\dagger *_N \mathcal{H} = \mathcal{H}.
\end{aligned}$$

So we get the formula in (3.4). Under the condition that (3.4) is established, the solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of (1.1) satisfies

$$\mathcal{G} *_N \begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} = \mathcal{H}.$$

Also, in the light of Lemma 2.10, the solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of (1.1) satisfies

$$\begin{pmatrix} V_c(\overrightarrow{\mathcal{X}}) \\ V_c(\overrightarrow{\mathcal{Y}}) \\ V_c(\overrightarrow{\mathcal{Z}}) \end{pmatrix} = \mathcal{G}^\dagger *_N \mathcal{H} + (\mathcal{I} - \mathcal{G}^\dagger *_N \mathcal{G}) *_N \mho.$$

At the same time, the unique solution (3.5) can also be obtained. □

4. Algorithm and numerical experiment

In this section, we first present an algorithm for solving Problem 1.1, which is based on the discussions in Section 3. And then we use an example to illustrate our main results.

Algorithm 4.1. (Problem 1.1)

(1) *Input the given tensors:* $\mathcal{A}_\sigma = \mathcal{A}_1^\sigma + \mathcal{A}_2^\sigma \mathbf{j}$ *(*$\sigma = 1, 2, 3$*),* $\mathcal{C}_\tau = \mathcal{C}_1^\tau + \mathcal{C}_2^\tau \mathbf{j}$*,* $\mathcal{D}_\tau = \mathcal{D}_1^\tau + \mathcal{D}_2^\tau \mathbf{j}$*,* $\mathcal{A}_1, \mathcal{B}_1 \in \mathbb{H}_r^{I_1 \times \cdots I_N \times J_1 \times \cdots J_N}$*,* $\mathcal{A}_2, \mathcal{B}_2 \in \mathbb{H}_r^{J_1 \times \cdots J_N \times K_1 \times \cdots K_N}$*,* $\mathcal{A}_3, \mathcal{B}_3 \in \mathbb{H}_r^{L_1 \times \cdots L_N \times J_1 \times \cdots J_N}$*,* $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2, \mathcal{E}_1, \mathcal{E}_2 \in \mathbb{H}_r^{J_1 \times \cdots J_N \times J_1 \times \cdots J_N}$*.*

(2) *Compute* Ω_{J_N}*,* $\mathcal{P}_1$*,* $\mathcal{P}_2$*,* $\mathcal{P}_3$*,* $\mathcal{P}_4$*,* $\mathcal{P}_5$*,* $\mathcal{P}_6$*,* $\mathcal{P}_7$*,* $\mathcal{P}$*,* $\mathcal{G}$ *and* $\mathcal{H}$*, which are defined in Section* 3*.*

(3) *According to* (3.3)*, calculate the minimal norm least squares solution* $(\mathcal{X}_l, \mathcal{Y}_l, \mathcal{Z}_l)$ *of Problem* 1.1*.*

Now, we give the following numerical example to explain Algorithm 4.1. On the basis of the discussions in Section 3, it is not hard to find that the expression of the minimal norm least squares solution is the same as that of the solution. To make sure that Problem 1.1 has a solution, we suppose that the system (1.1) is consistent.

Example 4.2. Let $I_1 = J_1 = K_1 = L_1 = 2$, $I_2 = J_2 = K_2 = L_2 = 2$ and

$$\mathcal{A}_1(:,:,1,1) = \begin{pmatrix} \mathbf{k} & \mathbf{j} \\ \mathbf{i}+\mathbf{j}+\mathbf{k} & 1+\mathbf{i}+\mathbf{j} \end{pmatrix}, \qquad \mathcal{A}_1(:,:,2,1) = \begin{pmatrix} \mathbf{k} & \mathbf{k} \\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & 1+\mathbf{k} \end{pmatrix},$$

$$\mathcal{A}_1(:,:,1,2) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & 1 \\ 1+\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j} \end{pmatrix}, \qquad \mathcal{A}_1(:,:,2,2) = \begin{pmatrix} 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & 1+\mathbf{i}+\mathbf{k} \\ 0 & \mathbf{i}+\mathbf{k} \end{pmatrix},$$

$$\mathcal{A}_2(:,:,1,1) = \begin{pmatrix} -1+\mathbf{i}+\mathbf{j}+\mathbf{k} & -\mathbf{i}+\mathbf{k} \\ 0 & \mathbf{i} \end{pmatrix}, \qquad \mathcal{A}_2(:,:,2,1) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & \mathbf{i}-\mathbf{j} \\ -\mathbf{i} & 1 \end{pmatrix},$$

$$\mathcal{A}_2(:,:,1,2) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & 0 \\ 0 & 1+\mathbf{j} \end{pmatrix}, \qquad \mathcal{A}_2(:,:,2,2) = \begin{pmatrix} \mathbf{k} & 1 \\ 1-\mathbf{k} & 1+\mathbf{i}-\mathbf{j}-\mathbf{k} \end{pmatrix},$$

$$\mathcal{A}_3(:,:,1,1) = \begin{pmatrix} 0 & \mathbf{i}+\mathbf{j} \\ 1+\mathbf{j} & 1+\mathbf{i}+\mathbf{k} \end{pmatrix}, \qquad \mathcal{A}_3(:,:,2,1) = \begin{pmatrix} 1+\mathbf{j}+\mathbf{k} & 1+\mathbf{i}+\mathbf{j} \\ 1+\mathbf{i}+\mathbf{j} & 1+\mathbf{j} \end{pmatrix},$$

$$\mathcal{A}_3(:,:,1,2) = \begin{pmatrix} \mathbf{j} & 0 \\ \mathbf{k} & -1-\mathbf{j} \end{pmatrix}, \qquad \mathcal{A}_3(:,:,2,2) = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ 1+\mathbf{k} & 1+\mathbf{k} \end{pmatrix},$$

$$\mathcal{C}_1(:,:,1,1) = \begin{pmatrix} -1 & -1-\mathbf{i}+\mathbf{j}+\mathbf{k} \\ -\mathbf{i} & -\mathbf{i}+\mathbf{j} \end{pmatrix}, \qquad \mathcal{C}_1(:,:,2,1) = \begin{pmatrix} 1+\mathbf{j}+\mathbf{k} & 1+\mathbf{i}+\mathbf{j}+\mathbf{k} \\ 1+\mathbf{k} & \mathbf{k} \end{pmatrix},$$

$$\mathcal{C}_1(:,:,1,2) = \begin{pmatrix} \mathbf{i}+\mathbf{j}+\mathbf{k} & -\mathbf{j} \\ -1+\mathbf{i}+\mathbf{j} & \mathbf{i}+\mathbf{j}+\mathbf{k} \end{pmatrix}, \qquad \mathcal{C}_1(:,:,2,2) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & \mathbf{k} \\ \mathbf{i}+\mathbf{k} & 1+\mathbf{j}+\mathbf{k} \end{pmatrix},$$

$$\mathcal{C}_2(:,:,1,1) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & -1+\mathbf{k} \\ -1+\mathbf{i}+\mathbf{k} & 1 \end{pmatrix}, \qquad \mathcal{C}_2(:,:,2,1) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & 0 \\ 1 & -1-\mathbf{k} \end{pmatrix},$$

$$\mathcal{C}_2(:,:,1,2) = \begin{pmatrix} 1+\mathbf{j}+\mathbf{k} & 1+\mathbf{i}+\mathbf{j}+\mathbf{k} \\ 1+\mathbf{j} & \mathbf{k} \end{pmatrix}, \qquad \mathcal{C}_2(:,:,2,2) = \begin{pmatrix} \mathbf{i}+\mathbf{k} & \mathbf{i}+\mathbf{j} \\ 1+\mathbf{i}+\mathbf{k} & 0 \end{pmatrix},$$

$$\mathcal{D}_1(:,:,1,1)=\begin{pmatrix}\mathbf{k} & \mathbf{i}\\ \mathbf{j}+\mathbf{k} & \mathbf{j}+\mathbf{k}\end{pmatrix},\qquad \mathcal{D}_1(:,:,2,1)=\begin{pmatrix}1 & -1\\ -1+\mathbf{i}-\mathbf{k} & -\mathbf{k}\end{pmatrix},$$

$$\mathcal{D}_1(:,:,1,2)=\begin{pmatrix}1+\mathbf{j} & \mathbf{i}+\mathbf{j}+\mathbf{k}\\ 1+\mathbf{i}+\mathbf{j} & \mathbf{j}+\mathbf{k}\end{pmatrix},\qquad \mathcal{D}_1(:,:,2,2)=\begin{pmatrix}\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k}\\ \mathbf{j} & \mathbf{j}\end{pmatrix},$$

$$\mathcal{D}_2(:,:,1,1)=\begin{pmatrix}-1+\mathbf{j} & \mathbf{i}\\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{j}\end{pmatrix},\qquad \mathcal{D}_2(:,:,2,1)=\begin{pmatrix}\mathbf{i}+\mathbf{j}+\mathbf{k} & 1+\mathbf{k}\\ 0 & 0\end{pmatrix},$$

$$\mathcal{D}_2(:,:,1,2)=\begin{pmatrix}1-\mathbf{j}+\mathbf{k} & 1+\mathbf{i}+\mathbf{j}+\mathbf{k}\\ -\mathbf{k} & \mathbf{i}-\mathbf{j}\end{pmatrix},\quad \mathcal{D}_2(:,:,2,2)=\begin{pmatrix}-1+\mathbf{k} & -1+\mathbf{i}+\mathbf{k}\\ \mathbf{j} & -\mathbf{i}+\mathbf{j}-\mathbf{k}\end{pmatrix}.$$

Let

$$\mathcal{X}(:,:,1,1)=\begin{pmatrix}-1-\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k}\\ -1+\mathbf{i}+\mathbf{j}-\mathbf{k} & \mathbf{j}\end{pmatrix},\qquad \mathcal{X}(:,:,2,1)=\begin{pmatrix}-\mathbf{k} & \mathbf{i}+\mathbf{k}\\ \mathbf{i} & \mathbf{j}+\mathbf{k}\end{pmatrix},$$

$$\mathcal{X}(:,:,1,2)=\begin{pmatrix}1+\mathbf{i}+\mathbf{k} & \mathbf{i}\\ 1+\mathbf{i}+\mathbf{j} & \mathbf{k}\end{pmatrix},\qquad \mathcal{X}(:,:,2,2)=\begin{pmatrix}-\mathbf{i} & 0\\ 1-\mathbf{k} & -1-\mathbf{j}-\mathbf{k}\end{pmatrix},$$

$$\mathcal{Y}(:,:,1,1)=\begin{pmatrix}-\mathbf{k} & -1-\mathbf{i}-\mathbf{j}-\mathbf{k}\\ -1-\mathbf{i}-\mathbf{j} & \mathbf{j}+\mathbf{k}\end{pmatrix},\qquad \mathcal{Y}(:,:,2,1)=\begin{pmatrix}1 & \mathbf{i}\\ 1+\mathbf{i}+\mathbf{j} & \mathbf{i}+\mathbf{k}\end{pmatrix},$$

$$\mathcal{Y}(:,:,1,2)=\begin{pmatrix}1 & \mathbf{i}\\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & 0\end{pmatrix},\qquad \mathcal{Y}(:,:,2,2)=\begin{pmatrix}\mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}-\mathbf{k}\\ \mathbf{i}+\mathbf{j} & 1+\mathbf{i}+\mathbf{k}\end{pmatrix},$$

$$\mathcal{Z}(:,:,1,1)=\begin{pmatrix}-\mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k}\\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & -\mathbf{i}+\mathbf{k}\end{pmatrix},\qquad \mathcal{Z}(:,:,2,1)=\begin{pmatrix}1-\mathbf{i}-\mathbf{k} & 1+\mathbf{k}\\ 1+\mathbf{j} & 1-\mathbf{i}+\mathbf{j}\end{pmatrix},$$

$$\mathcal{Z}(:,:,1,2)=\begin{pmatrix}1+\mathbf{j}+\mathbf{k} & \mathbf{i}-\mathbf{j}\\ -1-\mathbf{i}-\mathbf{j}-\mathbf{k} & 1+\mathbf{i}-\mathbf{j}\end{pmatrix},\qquad \mathcal{Z}(:,:,2,2)=\begin{pmatrix}0 & 1+\mathbf{i}+\mathbf{k}\\ 1+\mathbf{i}+\mathbf{j} & \mathbf{k}\end{pmatrix}.$$

We can obtain the solution $(\mathcal{X}_m,\mathcal{Y}_m,\mathcal{Z}_m)$ with the minimal norm of the system of mixed generalized Sylvester reduced biquaternion tensor equations (1.1) by MATLAB. In this way, we can compute $\log_{10}\|(\mathcal{X}_l,\mathcal{Y}_l,\mathcal{Z}_l)-(\mathcal{X}_m,\mathcal{Y}_m,\mathcal{Z}_m)\|=-13.4292$, which can demonstrate the feasibility of Algorithm 4.1.

5. Conclusion

Based on complex representation of reduced biquaternion tensors and the Moore–Penrose inverse of tensors, we have derived some necessary and sufficient conditions for the existence of the general solution to the system (1.1) and provided an expression of the general solution to the system when it is solvable. Moreover, an example has been furnished to illustrate the main results.

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