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Pseudo-Carleson Measures for Fock Spaces

Yongqing Liu and Shengzhao Hou*

Abstract. In this paper, we introduce pseudo-Carleson measure, a generalization of Carleson measure, for Fock spaces, and give sufficient and necessary conditions for a complex Borel measure to be pseudo-Carleson. We then give integral representations of functions in a Fock space, and use the measures to characterize boundedness and compactness of small Hankel operators on Fock spaces.

1. Introduction

Carleson measures were introduced in the early 1960s by L. Carleson [9] to characterize the interpolating sequences in the algebra H^∞ of bounded analytic functions and give a solution to the corona problem. As we all know, Carleson [9] also proved that for $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with $T = \partial\mathbb{D}$, $1 \le p < \infty$, and a positive measure μ on $\mathbb{D}$, the Poisson operator $P\colon H^p(T) \to L^p(\mathbb{D},\mu)$ is bounded if and only if the measure μ is Carleson. Since then, the Carleson measures and their extensions have found numerous applications in the study of various spaces of functions, complex analysis, and the theory of integral operators (see [2, 7, 11, 12, 14, 15, 18, 22, 26]).

As a generalization of Carleson measures, Xiao [29] studied pseudo-Carleson measures for weighted Bergman spaces. Arcozzi, Rochberg, Sawyer, and Wick [1] studied independently pseudo-Carleson measures for Dirichlet spaces (see Theorem 5 in [1]). Xiao [28] and Bao, Ye, Zhu [3] studied pseudo-Carleson measures (or Hankel measures) for Hardy spaces.

Recently, the study on Fock spaces (or Segal–Bargmann spaces) has become a popular topic in operator theory and has attracted the attention of many mathematicians (see [4, 5, 8, 10, 13, 21, 23, 27]). Fock spaces are deeply connected with quantum mechanics. For instance, creation and annihilation operators in quantum mechanics are multiplication and, respectively, differential operators on Fock spaces. Characterizing the zero sequence of Fock space is equivalent to studying the completeness of coherent state system in quantum mechanics (see [16, 17, 24]).

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*Corresponding author.

Now, the purpose of this article is to study pseudo-Carleson measures for Fock spaces. Let α be positive number, and $d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2}\, dA(z)$ be the Gaussian measure on the complex plane $\mathbb{C}$, where dA is the Lebesgue area measure on $\mathbb{C}$. For $0 < p < \infty$, the Fock space F^p_α consists of all entire functions f on $\mathbb{C}$ such that

$$\|f\|_{p,\alpha} = \left(\frac{p\alpha}{2\pi}\int_{\mathbb{C}} \left|f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^p dA(z)\right)^{1/p} < \infty.$$

For $p = \infty$, Fock space F^∞_α consists of all entire functions f on $\mathbb{C}$ such that

$$\|f\|_{\infty,\alpha} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

When $1 \le p \le \infty$, Fock space F^p_α is a Banach space. In particular, F^2_α is a Hilbert space with the following inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{C}} f(z)\overline{g(z)}\, d\lambda_\alpha(z).$$

The reproducing kernel of F^2_α is given by $K_\alpha(z,w) = e^{\alpha z\overline{w}}$, and, when normalized, it becomes $k_w(z) = e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2}$.

A complex Borel measure ν on $\mathbb{C}$ is called a *Fock–Carleson measure* if there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{C}} \left|f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^2 d|\nu|(z)\right)^{1/2} \le C\|f\|_{2,\alpha}, \quad \forall\, f \in F^2_\alpha;$$

ν is a *vanishing Fock–Carleson measure* if

$$\lim_{n\to\infty} \int_{\mathbb{C}} \left|f_n(z)e^{-\frac{\alpha}{2}|z|^2}\right|^2 d|\nu|(z) = 0$$

whenever $\{f_n\}$ is a bounded sequence in F^2_α that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$ (see [19]).

For $z \in \mathbb{C}$ and $r > 0$, let $D(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$. Isralowitz and Zhu proved that ν is a Fock–Carleson measure if and only if there exists a constant $C > 0$ such that

$$|\nu|(D(z,r)) \le C$$

for all $z \in \mathbb{C}$ (Theorem 3 in [19], see also [25]). And ν is a vanishing Fock–Carleson measure if and only if $|\nu|(D(z,r)) \to 0$ as $z \to \infty$ (see Theorem 4 in [19]).

Defining the Toeplitz operator by $T_\nu f(z) = \int_{\mathbb{C}} e^{\alpha z\overline{w}} f(w)e^{-\alpha|w|^2}\, d|\nu|(w)$, Isralowitz and Zhu [19] characterized the boundedness, compactness, and Schatten class membership of T_ν on the Fock space F^2_α in terms of Fock–Carleson measures.

Let $\mathbb{D}$ be the unit disk and $\gamma > -1$. The weighted Bergman space A^2_γ consists of analytic function f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\gamma \, dA(z) < \infty.$$

Xiao [29] introduced pseudo-Carleson measures (or called Hankel measures) for A^2_γ as follows. A complex Borel measure μ on $\mathbb{D}$ is called a pseudo-Carleson measure for A^2_γ if there exists a constant $C > 0$ such that

$$\left| \int_{\mathbb{D}} f(z)^2 \, d\mu(z) \right| \le C \|f\|^2_{A^2_\gamma}, \quad f \in A^2_\gamma.$$

Xiao [29] proved that μ is a pseudo-Carleson measure for A^2_γ if and only if

$$\sup_{w \in \mathbb{D}} \left| \int_{\mathbb{D}} \left[\frac{1-|w|^2}{(1-\overline{w}z)} \right]^{2+\gamma} d\mu(z) \right| < \infty. \tag{1.1}$$

He also obtained a Carleson measure type equivalent characterization for the pseudo-Carleson measure, namely, μ is a pseudo-Carleson measure for A^2_γ if and only if

$$\sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^{\gamma+2}} \int_{S(I)} \left| \int_{\mathbb{D}} \frac{\overline{w} \, d\overline{\mu}(w)}{(1-\overline{w}z)^{\gamma+3}} \right|^2 (1-|z|^2)^{\gamma+2} \, dA(z) < \infty,$$

where $S(I)$ is the Carleson square.

Xiao [28] asked whether a similar condition (1.1) holds for pseudo-Carleson measure on Hardy space. Recently, Bao, Ye, and Zhu [3] answered Xiao's question negatively in the Hardy space setting. Bao, Ye, and Zhu [3] further explored the notion of pseudo-Carleson measures for Hardy spaces and obtained some new characterizations of pseudo-Carleson measures.

In this paper, we introduce pseudo-Carleson measures on Fock spaces (see Definition 2.1) and give several conditions to characterize pseudo-Carleson measures (see Theorem 2.5) and vanishing pseudo-Carleson measures (see Theorem 2.6). We then give an integral representations of functions in Fock spaces (see Theorem 3.1). Finally, we characterize boundedness and compactness of small Hankel operators on Fock spaces in terms of pseudo-Carleson measures and vanishing pseudo-Carleson measures, respectively (see Theorems 4.2 and 4.5).

Besides this introduction, this article consists of three sections. Section 2 is devoted to the study of pseudo-Carleson measures for Fock spaces. Section 3 presents integral representations of functions in Fock spaces. Finally, in Section 4, we study boundedness and compactness of small Hankel operators on Fock spaces.

2. (Vanishing) Pseudo-Carleson measures

In this section, we introduce (vanishing) pseudo-Carleson measures for Fock spaces, and give sufficient and necessary conditions for a measure to be pseudo-Carleson and to be vanishing pseudo-Carleson, respectively.

Definition 2.1. A complex Borel measure μ on $\mathbb{C}$ is called a pseudo-Carleson measure if there exists a constant $C > 0$ such that

$$\left| \int_{\mathbb{C}} f(z)^2 e^{-\alpha|z|^2} \, d\mu(z) \right| \leq C \int_{\mathbb{C}} |f(z)|^2 \, d\lambda_\alpha(z)$$

for $f \in F^2_\alpha$. With the same spirit as vanishing Carleson measures, we call a complex Borel measure μ on $\mathbb{C}$ a vanishing pseudo-Carleson measure for the Fock space F^2_α if

$$\lim_{n\to\infty} \left| \int_{\mathbb{C}} f_n(z)^2 e^{-\alpha|z|^2} \, d\mu(z) \right| = 0$$

whenever $\{f_n\}$ is a bounded sequence in F^2_α that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$.

Remark 2.2. By definition, every Fock–Carleson measure is a pseudo-Carleson measure. But a pseudo-Carleson measure may not be a Fock–Carleson measure. For example, for any fixed $\xi > 0$, let

$$d\mu(z) = e^{-4\alpha\xi^2} e^{\alpha\xi\overline{z} + \frac{\alpha}{2}|z|^2} \, dA(z).$$

Then the measure μ is pseudo-Carleson measure, not Carleson measure. In fact, the reproducing kernel formula gives that

$$\begin{aligned} \left| \int_{\mathbb{C}} k_w(z)^2 e^{-\alpha|z|^2} \, d\mu(z) \right| &= \left| \int_{\mathbb{C}} k_w(z)^2 e^{-4\alpha\xi^2} e^{\alpha\xi\overline{z} - \frac{\alpha}{2}|z|^2} \, dA(z) \right| \\ &= \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}} e^{-4\alpha\xi^2 - \alpha|w|^2} e^{\alpha\xi\overline{z} - \frac{\alpha}{2}|z|^2} \, dA(z) \right| \\ &= \left| e^{4\alpha\xi\overline{w}} \right| e^{-4\alpha\xi^2 - \alpha|w|^2} \\ &= e^{-\alpha|2\xi - w|^2} < \infty. \end{aligned}$$

By Theorem 2.5, we see that μ is pseudo-Carleson measure. However,

$$\begin{aligned} \int_{\mathbb{C}} |k_w(z)|^2 e^{-\alpha|z|^2} \, d|\mu|(z) &= \int_{\mathbb{C}} |k_w(z)|^2 e^{-4\alpha\xi^2} \left| e^{\alpha\xi\overline{z}} \right| e^{-\frac{\alpha}{2}|z|^2} \, dA(z) \\ &= e^{\frac{\alpha}{2}|2w+\xi|^2} e^{-\alpha|w|^2 - 4\alpha\xi^2} \\ &= Ce^{\alpha|w|^2 + 2\alpha \operatorname{Re}(\xi w)} \to \infty \end{aligned}$$

as $|w| \to \infty$.

The following lemma provides optimal pointwise estimates for functions in Fock spaces.

Lemma 2.3. [19, Lemma 2.1] *For any $r > 0$ and $p > 0$, there exists a constant $C > 0$ such that*

$$\left|f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^p \leq C \int_{D(z,r)} \left|f(w)e^{-\frac{\alpha}{2}|w|^2}\right|^p dA(w)$$

for all entire function f and all $z \in \mathbb{C}$.

We also need the following atomic decomposition theorem for Fock spaces.

Lemma 2.4. [32, Theorem 2.34] *Let $0 < p \leq \infty$. There exists a positive constant r_0 such that for any $0 < r < r_0$, the Fock space F^p_α consists exactly of the following functions*

$$f(z) = \sum_{w \in r\mathbb{Z}^2} c_w e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2}, \tag{2.1}$$

where $\{c_w : w \in \mathbb{Z}^2\} \in l^p$. Moreover, there exists a positive constant C such that

$$C^{-1}\|f\|_{p,\alpha} \leq \inf \|\{c_w\}\|_{l^p} \leq C\|f\|_{p,\alpha}$$

for all $f \in F^p_\alpha$, where the infimum is taken over all sequences $\{c_w\}$ that give rise to the decomposition (not unique) in (2.1).

We have the following characterization of pseudo-Carleson measures for Fock spaces.

Theorem 2.5. *Let μ be a complex Borel measure on $\mathbb{C}$ and $r > 0$. Then the following conditions are equivalent.*

(a) *The measure μ is a pseudo-Carleson measure.*

(b) *The measure μ satisfies*

$$\sup_{w \in \mathbb{C}} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w} - \alpha|w|^2 - \alpha|z|^2} \, d\mu(z) \right| < \infty.$$

(c) *There exists a constant $C > 0$ such that*

$$\int_{D(z,r)} \left| \int_{\mathbb{C}} e^{2\alpha\xi\overline{w} - \alpha|\xi|^2} \, d\mu(\xi) \right| d\lambda_\alpha(w) \leq C$$

for all $z \in \mathbb{C}$.

Proof. (a) $\Rightarrow$ (b). Let $f(z) = k_w(z)$, the normalized reproducing kernel of F^2_α (introduced in Section 1). Then

$$\left| \int_{\mathbb{C}} e^{2\alpha z\overline{w} - \alpha|w|^2 - \alpha|z|^2} \, d\mu(z) \right| \leq C\|k_w\|^2_{2,\alpha} = C$$

for all $w \in \mathbb{C}$.

(b) $\Rightarrow$ (a). Suppose that (b) holds. We only need to prove that there exists a constant $C > 0$ such that

$$\left| \int_{\mathbb{C}} f(z) e^{-\alpha|z|^2} \, d\mu(z) \right| \leq C \|f\|_{1,2\alpha}, \quad \forall f \in F^1_{2\alpha}. \tag{2.2}$$

In fact, for any $f \in F^2_\alpha$, then $f^2 \in F^1_{2\alpha}$ and $\|f\|_{2,\alpha} = \|f^2\|_{1,2\alpha}$. Applying (2.2), we have

$$\left| \int_{\mathbb{C}} f(z)^2 e^{-\alpha|z|^2} \, d\mu(z) \right| \leq C \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} \, dA(z).$$

So μ is a pseudo-Carleson measure, that is, condition (a) holds. Next, we prove that (2.2) holds. For any $f \in F^1_{2\alpha}$, by Lemma 2.4, there exists a sequence $\{c_w\} \in l^1$ such that

$$f(z) = \sum_{w \in r\mathbb{Z}^2} c_w e^{2\alpha z \overline{w} - \alpha|w|^2},$$

and $\inf \|\{c_w\}\|_{l^1} \leq C \|f\|_{1,2\alpha}$. We thus have

$$\begin{aligned} \int_{\mathbb{C}} f(z) e^{-\alpha|z|^2} \, d\mu(z) &= \int_{\mathbb{C}} \sum_{w \in r\mathbb{Z}^2} c_w e^{2\alpha z \overline{w} - \alpha|w|^2} e^{-\alpha|z|^2} \, d\mu(z) \\ &= \sum_{w \in r\mathbb{Z}^2} c_w \int_{\mathbb{C}} e^{2\alpha z \overline{w} - \alpha|w|^2} e^{-\alpha|z|^2} \, d\mu(z). \end{aligned}$$

It follows that

$$\left| \int_{\mathbb{C}} f(z) e^{-\alpha|z|^2} \, d\mu(z) \right| \leq C \|f\|_{1,2\alpha} \sup_{w \in \mathbb{C}} \left| \int_{\mathbb{C}} e^{2\alpha z \overline{w} - \alpha|w|^2} e^{-\alpha|z|^2} \, d\mu(z) \right|.$$

This shows that condition (b) implies condition (a).

(b) $\Leftrightarrow$ (c). By Lemma 2.3, we have

$$\left| \int_{\mathbb{C}} e^{2\alpha z \overline{w}} e^{-\alpha|z|^2} \, d\mu(z) \right| e^{-\alpha|w|^2} \leq C \int_{D(w,r)} \left| \int_{\mathbb{C}} e^{2\alpha \xi \overline{u}} e^{-\alpha|\xi|^2} \, d\mu(\xi) \right| e^{-\alpha|u|^2} \, dA(u).$$

Taking supremum, we get (c) $\Rightarrow$ (b). On the other hand, since

$$\begin{aligned} \int_{D(z,r)} \left| \int_{\mathbb{C}} e^{2\alpha \xi \overline{w}} e^{-\alpha|\xi|^2} \, d\mu(\xi) \right| d\lambda_\alpha(w) &= \frac{\alpha}{\pi} \int_{D(z,r)} \left| \int_{\mathbb{C}} e^{2\alpha \xi \overline{w} - \alpha|w|^2} e^{-\alpha|\xi|^2} \, d\mu(\xi) \right| dA(w) \\ &\leq \frac{\alpha}{\pi} \int_{D(z,r)} \sup_{w \in \mathbb{C}} \left| \int_{\mathbb{C}} e^{2\alpha \xi \overline{w} - \alpha|w|^2} e^{-\alpha|\xi|^2} \, d\mu(\xi) \right| dA(w) \\ &\leq C \sup_{w \in \mathbb{C}} \left| \int_{\mathbb{C}} e^{2\alpha z \overline{w} - \alpha|w|^2 - \alpha|z|^2} \, d\mu(z) \right| < \infty, \end{aligned}$$

condition (b) implies condition (c). $\square$

Similarly, we give the following characterization of vanishing pseudo-Carleson measures for Fock spaces.

Theorem 2.6. *Let μ be a complex Borel measure on $\mathbb{C}$ and $r > 0$. Then the following conditions are equivalent.*

(a) *The measure μ is a vanishing pseudo-Carleson measure.*

(b) *The measure μ satisfies*

$$\lim_{|w|\to\infty} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2-\alpha|z|^2}\, d\mu(z) \right| = 0.$$

(c) *The measure μ satisfies*

$$\lim_{|z|\to\infty} \int_{D(z,r)} \left| \int_{\mathbb{C}} e^{2\alpha\xi\overline{w}-\alpha|\xi|^2}\, d\mu(\xi) \right| d\lambda_\alpha(w) = 0.$$

Proof. (a) $\Rightarrow$ (b). Let $f_w(z) = e^{\alpha z\overline{w}-\frac{\alpha}{2}|w|^2}$. Then we have $\|f_w\|_{2,\alpha} = 1$ and $f_w(z) \to 0$ uniformly on compact sets of $\mathbb{C}$ as $|w| \to \infty$, and

$$\lim_{|w|\to\infty} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2-\alpha|z|^2}\, d\mu(z) \right| = 0.$$

(b) $\Rightarrow$ (a). Notice that if (b) holds, we only need to show that

$$\lim_{n\to\infty} \left| \int_{\mathbb{C}} f_n(z) e^{-\alpha|z|^2}\, d\mu(z) \right| = 0,$$

whenever $\{f_n\}$ is a bounded sequence in $F^1_{2\alpha}$ that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. Now for any bounded sequence $\{f_n\}$ in $F^1_{2\alpha}$ that converges to 0 uniformly on compact sets of $\mathbb{C}$, by the reproducing formula, we have

$$\begin{aligned}
\int_{\mathbb{C}} f_n(z) e^{-\alpha|z|^2}\, d\mu(z) &= \int_{\mathbb{C}} \int_{\mathbb{C}} e^{2\alpha z\overline{w}} f_n(w)\, d\lambda_{2\alpha}(w)\, e^{-\alpha|z|^2}\, d\mu(z) \\
&= \int_{\mathbb{C}} f_n(w) \int_{\mathbb{C}} e^{2\alpha z\overline{w}} e^{-\alpha|z|^2}\, d\mu(z)\, d\lambda_{2\alpha}(w) \\
&= \frac{2\alpha}{\pi} \int_{\mathbb{C}} f_n(w) e^{-\alpha|w|^2} \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z)\, dA(w).
\end{aligned}$$

Therefore, for $0 < t < \infty$, we have

$$\begin{aligned}
\left| \int_{\mathbb{C}} f_n(z) e^{-\alpha|z|^2}\, d\mu(z) \right| &\le \frac{2\alpha}{\pi} \int_{\mathbb{C}} |f_n(w)| e^{-\alpha|w|^2} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z) \right| dA(w) \\
&= \frac{2\alpha}{\pi} \int_{|w|\le t} |f_n(w)| e^{-\alpha|w|^2} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z) \right| dA(w) \\
&\quad + \frac{2\alpha}{\pi} \int_{t<|w|} |f_n(w)| e^{-\alpha|w|^2} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z) \right| dA(w) \\
&= I_1 + I_2.
\end{aligned}$$

We estimate I_1 and I_2 respectively. We first estimate I_1. Since $f_n \to 0$ uniformly on compact sets in $\mathbb{C}$ and $\sup_{w\in\mathbb{C}} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z)\right| < \infty$, we have

$$\begin{aligned} I_1 &= \frac{2\alpha}{\pi} \int_{|w|\le t} |f_n(w)| e^{-\alpha|w|^2} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z) \right| dA(w) \\ &\le C \sup_{|w|\le t} |f_n(w)| \to 0 \end{aligned}$$

as $n \to \infty$. We next estimate I_2. It follows from $\sup_n \|f_n\|_{1,2\alpha} < \infty$ that

$$\begin{aligned} I_2 &= \frac{2\alpha}{\pi} \int_{t<|w|} |f_n(w)| e^{-\alpha|w|^2} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z) \right| dA(w) \\ &\le C \sup_{t<|w|} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z) \right|, \end{aligned}$$

which implies that $I_2 \to 0$ as $t \to \infty$. Therefore, we obtain

$$\lim_{n\to\infty} \left| \int_{\mathbb{C}} f_n(z) e^{-\alpha|z|^2}\, d\mu(z) \right| = 0.$$

(b) $\Leftrightarrow$ (c). Lemma 2.3 gives that (c) $\Rightarrow$ (b). Now suppose that

$$\lim_{|w|\to\infty} \left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z) \right| = 0.$$

Then for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\left| \int_{\mathbb{C}} e^{2\alpha z\overline{w}-\alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z) \right| < \varepsilon,$$

whenever $|w| > R$. Therefore, when $|z|$ is large enough, we have

$$\int_{D(z,r)} \left| \int_{\mathbb{C}} e^{2\alpha\xi\overline{w}-\alpha|\xi|^2}\, d\mu(\xi) \right| d\lambda_\alpha(w) < C\varepsilon.$$

So condition (b) implies condition (c). □

3. The integral representations of Fock spaces

Inspired by the study of the previous section, in this section, we give integral representations of functions in Fock spaces.

Theorem 3.1. *Let f be an entire function, $\alpha > 0$, $r > 0$ and $1 \le p \le \infty$. Then $f \in F^p_\alpha$ if and only if there exists a complex Borel measure μ such that*

$$f(z) = \int_{\mathbb{C}} e^{\alpha z\overline{w}}\, d\mu(w)$$

and

$$\int_{D(z,r)} \left| \int_{\mathbb{C}} e^{\alpha\xi\overline{w}}\, d\mu(w) \right| e^{-\frac{\alpha}{2}|\xi|^2}\, dA(\xi) \in L^p(\mathbb{C}, dA). \tag{3.1}$$

Proof. If

$$f(z) = \int_{\mathbb{C}} e^{\alpha z\overline{w}}\, d\mu(w),$$

then by Lemma 2.3 we have

$$\begin{aligned}|f(z)|e^{-\frac{\alpha}{2}|z|^2} &= \left|\int_{\mathbb{C}} e^{\alpha z\overline{w}}\, d\mu(w)\right| e^{-\frac{\alpha}{2}|z|^2} \\ &\le C\int_{D(z,r)} \left|\int_{\mathbb{C}} e^{\alpha u\overline{w}}\, d\mu(w)\right| e^{-\frac{\alpha}{2}|u|^2}\, dA(u).\end{aligned}$$

Therefore, condition (3.1) gives that $f \in F^p_\alpha$.

Conversely, for $f \in F^p_\alpha$, by Lemma 2.4, f can be represented as

$$f(z) = \sum_{w\in r\mathbb{Z}^2} c_w e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2},$$

where $\{c_w : w \in r\mathbb{Z}^2\} \in l^p$. We define a measure μ on $\mathbb{C}$ as follows:

$$\mu = \sum_{w\in r\mathbb{Z}^2} c_w e^{-\frac{\alpha}{2}|w|^2}\delta_w,$$

where every δ_a is the unit atomic measure at $a \in \mathbb{C}$. It follows that

$$f(z) = \int_{\mathbb{C}} e^{\alpha z\overline{u}}\, d\mu(u).$$

Next we prove that μ satisfies the condition (3.1). For $p = \infty$, we have

$$\int_{D(a,r)} \left|\int_{\mathbb{C}} e^{\alpha\xi\overline{u}}\, d\mu(u)\right| e^{-\frac{\alpha}{2}|\xi|^2}\, dA(\xi) \le C\|\{c_w\}\|_\infty.$$

For $p = 1$, we have

$$\int_{D(a,r)} \left|\int_{\mathbb{C}} e^{\alpha\xi\overline{u}}\, d\mu(u)\right| e^{-\frac{\alpha}{2}|\xi|^2}\, dA(u) = \int_{D(a,r)} \left|\sum_{w\in r\mathbb{Z}^2} c_w e^{\alpha\xi\overline{w} - \frac{\alpha}{2}|w|^2}\right| e^{-\frac{\alpha}{2}|\xi|^2}\, dA(u).$$

Let $g(u) = \sum_{w\in r\mathbb{Z}^2} c_w e^{\alpha u\overline{w} - \frac{\alpha}{2}|w|^2}$. Since $\{c_w : w \in r\mathbb{Z}^2\} \in l^1$, by Lemma 2.4, we see that $g \in F^1_\alpha$. By Fubini's theorem and the fact $\chi_{D(z,r)}(w) = \chi_{D(w,r)}(z)$ for any $z, w \in \mathbb{C}$, we get

$$\begin{aligned}\int_{\mathbb{C}}\int_{D(a,r)} |g(u)|e^{-\frac{\alpha}{2}|u|^2}\, dA(u)\, dA(a) &= \int_{\mathbb{C}}\int_{\mathbb{C}} |g(u)|e^{-\frac{\alpha}{2}|u|^2}\chi_{D(a,r)}(u)\, dA(u)\, dA(a) \\ &= \int_{\mathbb{C}}\int_{\mathbb{C}} |g(u)|e^{-\frac{\alpha}{2}|u|^2}\chi_{D(u,r)}(a)\, dA(u)\, dA(a) \\ &= \int_{\mathbb{C}} |g(u)|e^{-\frac{\alpha}{2}|u|^2}\int_{\mathbb{C}} \chi_{D(u,r)}(a)\, dA(a)\, dA(u) \\ &\le C\int_{\mathbb{C}} |g(u)|e^{-\frac{\alpha}{2}|u|^2}\, dA(u) \\ &\le C\|\{c_w\}\|_1.\end{aligned}$$

The condition (3.1) for $1 < p < \infty$ comes from the complex interpolation. □

4. Small Hankel operators

In this section, for certain measures μ, we prove that small Hankel operator H_μ on F^2_α is bounded (compact) if and only if μ is pseudo-Carleson (vanishing pseudo-Carleson).

Let

$$\mathbf{D} = \left\{ f(z) = \sum_{k=1}^{n} c_k K_\alpha(z, w_k) : n \in \mathbb{N}, c_k \in \mathbb{C} \text{ and } w_k \in \mathbb{C} \right\}.$$

By Lemma 2.1 in [32], $\mathbf{D}$ is dense in F^p_α. We say that a complex Borel measure μ on $\mathbb{C}$ satisfies condition (M) if

$$\int_{\mathbb{C}} |K_\alpha(z, w)| e^{-\alpha|w|^2} \, d|\mu|(w) < \infty$$

for all $z \in \mathbb{C}$. Let $\overline{F^2_\alpha}$ be the space of conjugate analytic functions in F^2_α. For $\varphi \in L^2(\mathbb{C}, d\lambda_\alpha)$, the small Hankel operator h_φ with symbol φ is defined on a dense subset of F^2_α by

$$h_\varphi f(z) = \int_{\mathbb{C}} e^{\alpha \overline{z} w} f(w) \varphi(w) \, d\lambda_\alpha(w).$$

In fact, whenever $f \in \mathbf{D}$, $h_\varphi \colon F^2_\alpha \to \overline{F^2_\alpha}$ is densely well-defined (see [32]).

The integral representation for the small Hankel operator motivates us to define integral operators on Fock spaces with much more general symbols.

Definition 4.1. Let μ be a measure satisfying condition (M). We define the integral operator H_μ with symbol μ on Fock space F^2_α by

$$H_\mu f(z) = \int_{\mathbb{C}} e^{\alpha z w} f(w) e^{-\alpha|w|^2} \, d\mu(w).$$

Here the definition of the operator H_μ uses the kernel $e^{\alpha z w}$, which is different from the kernel $e^{\alpha \overline{z} w}$ of small Hankle operator h_φ. However, this difference is not essential. In fact, if take $d\mu(w) = \varphi(w) \, dA(w)$, then $H_\mu f(z) = h_\varphi f(\overline{z})$ for any $f \in \mathbf{D}$.

If μ satisfies condition (M), then the operator H_μ is densely well-defined on Fock spaces. In fact, for any $f \in \mathbf{D}$, since μ satisfies condition (M),

$$\begin{aligned}
\int_{\mathbb{C}} \left| e^{\alpha z w} f(w) \right| e^{-\alpha|w|^2} \, d|\mu|(w) &= \int_{\mathbb{C}} \left| e^{\alpha z w} \sum_{k=1}^{n} c_k K_\alpha(w, w_k) \right| e^{-\alpha|w|^2} \, d|\mu|(w) \\
&\le \sum_{k=1}^{n} |c_k| \int_{\mathbb{C}} \left| e^{\alpha z w} K_\alpha(w, w_k) \right| e^{-\alpha|w|^2} \, d|\mu|(w) \\
&= \sum_{k=1}^{n} |c_k| \int_{\mathbb{C}} |K_\alpha(w, w_k + \overline{z})| e^{-\alpha|w|^2} \, d|\mu|(w) < \infty.
\end{aligned}$$

We also call H_μ a small Hankel operator with the symbol μ. We say that H_μ is bounded on F^2_α if there exists a positive constant C such that $\|H_\mu f\|_{2,\alpha} \le C\|f\|_{2,\alpha}$ for any $f \in \mathbf{D}$.

Our first result in this section is about boundedness of small Hankel operators.

Theorem 4.2. *Suppose μ is a complex Borel measure on $\mathbb{C}$ and satisfies condition* (M). *Then H_μ is bounded on F^2_α if and only if μ is a pseudo-Carleson measure.*

Proof. Assume that H_μ is bounded on F^2_α. For any $f, g \in F^2_\alpha$, the reproducing formula gives that

$$\begin{aligned}\langle H_\mu f, g\rangle_\alpha &= \int_{\mathbb{C}} H_\mu f(z)\overline{g(z)}\, d\lambda_\alpha(z) \\ &= \int_{\mathbb{C}}\int_{\mathbb{C}} e^{\alpha z w} f(w) e^{-\alpha|w|^2}\, d\mu(w)\overline{g(z)}\, d\lambda_\alpha(z) \\ &= \int_{\mathbb{C}} f(w)\overline{\left[\int_{\mathbb{C}} e^{\alpha\overline{z w}} g(z)\, d\lambda_\alpha(z)\right]} e^{-\alpha|w|^2}\, d\mu(w) \\ &= \int_{\mathbb{C}} f(w)\overline{g(\overline{w})} e^{-\alpha|w|^2}\, d\mu(w).\end{aligned}$$

In particular, choosing $f(z) = k_w(z) = e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2}$ and $g(z) = k_{\overline{w}}(z) = e^{\alpha z w - \frac{\alpha}{2}|w|^2}$, we get

$$\begin{aligned}\langle H_\mu k_w, k_{\overline{w}}\rangle_\alpha &= \int_{\mathbb{C}} f(z)\overline{g(\overline{z})} e^{-\alpha|z|^2}\, d\mu(z) \\ &= \int_{\mathbb{C}} e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2}\overline{e^{\alpha\overline{z} w - \frac{\alpha}{2}|w|^2}} e^{-\alpha|z|^2}\, d\mu(z) \\ &= \int_{\mathbb{C}} e^{2\alpha z\overline{w} - \alpha|w|^2} e^{-\alpha|z|^2}\, d\mu(z).\end{aligned}$$

Since H_μ on Fock space F^2_α is bounded, by the Cauchy–Schwartz inequality, we obtain

$$\left|\int_{\mathbb{C}} e^{2\alpha z\overline{w} - \alpha|w|^2 - \alpha|z|^2}\, d\mu(z)\right| \le \|H_\mu\|$$

for all $w \in \mathbb{C}$. It follows by Theorem 2.5 that μ is a pseudo-Carleson measure.

Now suppose that μ is a pseudo-Carleson measure for Fock space. We first claim that $H_\mu f$ is well-defined for $f \in \mathbf{D}$. In fact, for any $z, u \in \mathbb{C}$, the reproducing formula yields that

$$\begin{aligned}\left|\int_{\mathbb{C}} e^{\alpha z w} e^{\alpha w\overline{u}} e^{-\alpha|w|^2}\, d\mu(w)\right| &= \left|\int_{\mathbb{C}} e^{\alpha w(z+\overline{u})} e^{-\alpha|w|^2}\, d\mu(w)\right| \\ &\le C\int_{\mathbb{C}} \left|e^{\alpha w\frac{(z+\overline{u})}{2}}\right|^2 d\lambda_\alpha(w) = Ce^{\frac{\alpha}{4}|z+\overline{u}|^2} < \infty.\end{aligned}$$

This also shows that $H_\mu f \in F^2_\alpha$. On the other hand, for any $f, g \in \mathbf{D}$, we have

$$\langle H_\mu f, g\rangle_\alpha = \int_{\mathbb{C}} f(w)\overline{g(\overline{w})} e^{-\alpha|w|^2}\, d\mu(w). \tag{4.1}$$

In fact, by definition of $\mathbf{D}$, to prove the conclusion (4.1), we only need to show

$$\langle H_\mu K_\alpha(\,\cdot\,,u_1), K_\alpha(\,\cdot\,,u_2)\rangle_\alpha = \int_{\mathbb{C}} K_\alpha(z,u_1)\overline{K_\alpha(\overline{z},u_2)}e^{-\alpha|z|^2}\,d\mu(z)$$

for $u_1, u_2 \in \mathbb{C}$. Since

$$\begin{aligned}
&\int_{\mathbb{C}} \left|\int_{\mathbb{C}} e^{\alpha zw} e^{\alpha w\overline{u}_1} e^{-\alpha|w|^2}\,d\mu(w)\right| \left|e^{\alpha z\overline{u}_2}\right| d\lambda_\alpha(z) \\
&\leq C\int_{\mathbb{C}} e^{\frac{\alpha}{4}|z+\overline{u}_1|^2}\left|e^{\alpha z\overline{u}_2}\right| d\lambda_\alpha(z) \leq Ce^{\frac{\alpha}{4}|u_1|^2+\frac{\alpha}{12}|u_1+2\overline{u}_2|} < \infty,
\end{aligned}$$

with the help of Fubini's theorem, we obtain the desirable conclusion. Therefore,

$$\begin{aligned}
\left|\langle H_\mu f, g\rangle_\alpha\right| &= \left|\int_{\mathbb{C}} f(w)\overline{g(\overline{w})}e^{-\alpha|w|^2} w\,d\mu(w)\right| \\
&\leq C\int_{\mathbb{C}} \left|f(w)\overline{g(\overline{w})}\right| d\lambda_\alpha(w) \leq C\|f\|_{2,\alpha}\|g\|_{2,\alpha}.
\end{aligned}$$

Since $\mathbf{D}$ is dense in Fock space F^2_α, the duality argument shows that H_μ is bounded on F^2_α. □

Now we study compactness of small Hankel operators. It is well-known that an operator T is compact if and only if its essential norm $\|T\|_e = 0$. Let X and Y be two Banach spaces, and $\mathcal{K}(X,Y)$ be the set of all compact operators from X to Y. The essential norm of a bounded linear operator T from X to Y, denoted as $\|T\|_e$, is defined by

$$\|T\|_e = \inf\{\|T-Q\| : Q \in \mathcal{K}(X,Y)\}.$$

Let small Hankel operator H_μ with symbol μ be bounded on F^2_α. By the reproducing formula, we have

$$H_\mu f(z) = \int_{\mathbb{C}} e^{\alpha zw} f(w)\overline{\psi(w)}\,d\lambda_\alpha(w),$$

where

$$\psi(w) = \int_{\mathbb{C}} e^{\alpha w\overline{\xi}-\alpha|\xi|^2}\,d\overline{\mu}(\xi).$$

Define $Jf(z) = f(\overline{z})$, $f \in F^2_\alpha$. Then $J\colon F^2_\alpha \to \overline{F^2_\alpha}$ is a unitary operator, and

$$J \circ H_\mu = h_{\overline{\psi}}, \tag{4.2}$$

where $h_{\overline{\psi}}$ is a small Hankel operator. We thus turn to give essential norm estimates for small Hankel operators.

Let V_α be the integral transform defined by

$$V_\alpha f(z) = 2e^{-\alpha|z|^2}\int_{\mathbb{C}} e^{2\alpha z\overline{w}} f(w)\,d\lambda_\alpha(w).$$

For a given function $\varphi \in L^2(\mathbb{C}, d\lambda_\alpha)$, one has $h_{\overline{\varphi}} = h_{\overline{V_\alpha \varphi}}$ in the sense that $h_{\overline{\varphi}} f = h_{\overline{V_\alpha \varphi}} f$ for all $f \in \mathbf{D}$ (see [31] for a proof). The properties of $V_\alpha \varphi$ can be used to obtain characterizations of the boundedness, compactness, and Schatten classes of small Hankel operator $h_{\overline{\varphi}}$ (see [32]). We now estimate the essential norms of small Hankel operators with symbols in $L^2(\mathbb{C}, d\lambda_\alpha)$, in terms of the integral transform V_α, on Fock spaces.

Theorem 4.3. *Let $\varphi \in L^2(\mathbb{C}, d\lambda_\alpha)$. If $h_{\overline{\varphi}}$ is bounded from F^2_α to $\overline{F^2_\alpha}$, then there exists a positive constant $C > 0$ such that*

$$\limsup_{|z|\to\infty} |V_\alpha \varphi(z)| \le \|h_{\overline{\varphi}}\|_e \le C \limsup_{|z|\to\infty} |V_\alpha \varphi(z)|.$$

So, $h_{\overline{\varphi}}$ is a compact from F^2_α to $\overline{F^2_\alpha}$ if and only if

$$\lim_{|z|\to\infty} |V_\alpha \varphi(z)| = 0.$$

Proof. We first prove the upper estimate. For a positive number r, we define

$$I_r f(z) = \int_{\mathbb{C}} \chi_r(w) K_\alpha(w, z) f(w) \overline{V_\alpha \varphi(w)}\, d\lambda_\alpha(w),$$

where χ_r is the characteristic function of $\{w : |w| \le r\}$. Since

$$\begin{aligned}
&\int_{\mathbb{C}} \int_{\mathbb{C}} \left|\chi_r(w) K_\alpha(w, z)\overline{V_\alpha \varphi(w)}\right|^2 d\lambda_\alpha(z)\, d\lambda_\alpha(w) \\
&= \int_{|w|\le r} \int_{\mathbb{C}} \left|K_\alpha(w, z)\overline{V_\alpha \varphi(w)}\right|^2 d\lambda_\alpha(z)\, d\lambda_\alpha(w) \\
&\le \|V_\alpha \varphi\|^2_\infty \int_{|w|\le r} \int_{\mathbb{C}} |K_\alpha(w, z)|^2\, d\lambda_\alpha(z)\, d\lambda_\alpha(w) \\
&= \|V_\alpha \varphi\|^2_\infty \int_{|w|\le r} K_\alpha(w, w)\, d\lambda_\alpha(w) \\
&= \alpha r^2 \|V_\alpha \varphi\|^2_\infty < \infty,
\end{aligned}$$

the operator I_r is Hilbert–Schmidt. In particular, I_r is a compact operator from F^2_α to $\overline{F^2_\alpha}$. It follows from

$$h_{\overline{\varphi}} = h_{\overline{V_\alpha \varphi}}$$

that

$$\|h_{\overline{\varphi}}\|_e = \|h_{\overline{V_\alpha \varphi}}\|_e \le \|h_{\overline{V_\alpha \varphi}} - I_r\|. \tag{4.3}$$

Write

$$(h_{\overline{V_\alpha \varphi}} - I_r) f(z) = \int_{\mathbb{C}} H(z, w) f(w)\, d\lambda_\alpha(w), \quad f \in F^2_\alpha,$$

where

$$H(z,w) = (1-\chi_r(w))K_\alpha(w,z)\overline{V_\alpha\varphi(w)}.$$

We shall use Schur's test (see Theorem 3.6 in [31]) to obtain an estimate on the norm of $h_{\overline{V_\alpha\varphi}} - I_r$. To this end, let $h(z) = e^{\frac{\alpha}{2}|z|^2}$. Since

$$\begin{aligned} I(z) &= \int_{\mathbb{C}} \left|(1-\chi_r(w))K_\alpha(w,z)\overline{V_\alpha\varphi(w)}\right| h(w)\, d\lambda_\alpha(w) \\ &= \int_{\mathbb{C}-\{w:|w|\le r\}} \left|K_\alpha(w,z)\overline{V_\alpha\varphi(w)}\right| h(w)\, d\lambda_\alpha(w) \\ &\le \sup_{|w|>r} |V_\alpha\varphi(w)| \int_{\mathbb{C}} |K_\alpha(w,z)| h(w)\, d\lambda_\alpha(w) \\ &= 2\sup_{|w|>r} |V_\alpha\varphi(w)| h(z) \end{aligned}$$

and

$$\begin{aligned} J(w) &= \int_{\mathbb{C}} \left|(1-\chi_r(w))K_\alpha(w,z)\overline{V_\alpha\varphi(w)}\right| h(z)\, d\lambda_\alpha(z) \\ &\le \sup_{|w|>r} |V_\alpha\varphi(w)| \int_{\mathbb{C}} |K_\alpha(w,z)| h(z)\, d\lambda_\alpha(z) \\ &= 2\sup_{|w|>r} |V_\alpha\varphi(w)| h(w), \end{aligned}$$

we obtain

$$\int_{\mathbb{C}} |H(z,w)| h(w)\, d\lambda_\alpha(w) \le 2\sup_{|w|>r} |V_\alpha\varphi(w)| h(z), \quad z\in\mathbb{C},$$

and

$$\int_{\mathbb{C}} |H(z,w)| h(z)\, d\lambda_\alpha(z) \le 2\sup_{|w|>r} |V_\alpha\varphi(w)| h(w), \quad w\in\mathbb{C}.$$

Therefore,

$$\|h_{\overline{V_\alpha\varphi}} - I_r\| \le 2\sup_{|w|>r} |V_\alpha\varphi(w)|.$$

Letting $r\to\infty$, from (4.3), we get

$$\|h_{\overline{\varphi}}\|_e = \|h_{\overline{V_\alpha\varphi}}\|_e \le 2\limsup_{|z|\to\infty} |V_\alpha\varphi(z)|.$$

We next prove the lower estimate. For any compact operator Q from F^2_α to $\overline{F^2_\alpha}$, we have $\|Qk_z\|_{2,\alpha}\to 0$ as $|z|\to\infty$. Hence,

$$\|h_{\overline{\varphi}}-Q\| \ge \limsup_{|z|\to\infty} \left\|(h_{\overline{\varphi}}-Q)k_z\right\|_{2,\alpha} \ge \limsup_{|z|\to\infty} \left(\|h_{\overline{\varphi}}k_z\|_{2,\alpha} - \|Qk_z\|_{2,\alpha}\right) = \limsup_{|z|\to\infty} \|h_{\overline{\varphi}}k_z\|_{2,\alpha}.$$

Therefore,

$$\|h_{\overline{\varphi}}\|_e \ge \limsup_{|z|\to\infty} \|h_{\overline{\varphi}}k_z\|_{2,\alpha} \ge \limsup_{|z|\to\infty} |V_\alpha\varphi(z)|. \qquad \square$$

Remark 4.4. Compactness characterization of small Hankel operators $h_{\overline{\varphi}}$ with bounded symbols have been shown previously in Corollary 3.12 of [6]. Theorem 4.3 also shows that, when φ is an entire function, $h_{\overline{\varphi}}$ is compact from F^2_α to $\overline{F^2_\alpha}$ if and only if $\lim_{|z|\to\infty} |\varphi(z)|e^{-\frac{\alpha}{4}|z|^2} = 0$, and the result has been proved in [20, 32]. Here we used a different method to prove it and gave the essential norm estimates for $h_{\overline{\varphi}}$.

Next we give a sufficient and necessary condition for the small Hankel operator H_μ to be compact.

Theorem 4.5. *Suppose μ is a complex Borel measure on $\mathbb{C}$ and satisfies condition* (M). *Then H_μ is compact on F^2_α if and only if μ is a vanishing pseudo-Carleson measure.*

Proof. Assume that H_μ is a compact operator on F^2_α. Using the argument in Theorem 4.2, we get

$$\langle H_\mu k_z, k_{\overline{z}}\rangle_\alpha = \int_{\mathbb{C}} e^{2\alpha w\overline{z}-\alpha|z|^2} e^{-\alpha|w|^2}\, d\mu(w).$$

Since $k_z \to 0$ weakly in F^2_α as $z \to \infty$, the Cauchy–Schwartz inequality and compactness of H_μ imply that

$$\left|\int_{\mathbb{C}} e^{2\alpha w\overline{z}-\alpha|z|^2-\alpha|w|^2}\, d\mu(w)\right| \le \|H_\mu k_z\|_{2,\alpha} \to 0.$$

By Theorem 2.6, μ is a vanishing pseudo-Carleson measure.

Conversely, let μ be a vanishing pseudo-Carleson measure for F^2_α. We show that H_μ is a compact operator. In fact, since

$$\begin{aligned} V_\alpha\psi(z) &= 2e^{-\alpha|z|^2}\int_{\mathbb{C}}\int_{\mathbb{C}} e^{\alpha w\overline{\xi}-\alpha|\xi|^2}\, d\overline{\mu}(\xi)\, e^{2\alpha z\overline{w}}\, d\lambda_\alpha(w) \\ &= 2\int_{\mathbb{C}} e^{2\alpha z\overline{\xi}-\alpha|z|^2-\alpha|\xi|^2}\, d\overline{\mu}(\xi), \end{aligned}$$

we have

$$\lim_{|z|\to\infty} |V_\alpha\psi(z)| = 0.$$

By Theorem 4.3 and (4.2), H_μ is compact on F^2_α. □

Remark 4.6. Let $p \ge 1$ and μ be a complex Borel measure on $\mathbb{C}$ and satisfy condition (M). By the method of the proof of Theorem 15 in [30], we can get that $H_\mu \in S_p(F^2_\alpha)$ (the Schatten-p class) if and only if

$$\int_{\mathbb{C}} \left|\int_{\mathbb{C}} e^{2\alpha\overline{z}w-\alpha|z|^2-\alpha|w|^2}\, d\mu(w)\right|^p dA(z) < \infty.$$

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Yongqing Liu and Shengzhao Hou
School of Mathematical Sciences, Soochow University, Suzhou, Jiangsu 215006, China
E-mail addresses: `yqliu0808@stu.suda.edu.cn`, `shou@suda.edu.cn`