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# Analysis and Approximation of Hemivariational Inequality for a Frictional Thermo-electro-visco-elastic Contact Problem with Damage

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Abstract. The aim of this paper is to investigate a contact problem involving thermo-electro-visco-elastic body with damage and a rigid foundation. The friction is modelled with a subgradient of a locally Lipschitz mapping, and the contact is described by the Signorini's unilateral contact condition. A parabolic differential inclusion for the damage function is used to include the damaging effect in the model. We establish the model's variational formulation using four systems of three hemivariational inequalities and a parabolic equation, we prove an existence and uniqueness result of this problem. The proof is based on a fixed point argument and a recent finding from hemivariational inequality theory. Finally, by employing the finite element approach, we investigate a fully discrete approximation of the model and we derive error estimates on the approximate solution.

## 1. Introduction

Hemivariational inequalities theory has recently played an important role in the study of nonlinear problems arising in Contact Mechanics, Physics, Economics, and Engineering [17, 19]. Panagiotopouls [21, 22] introduced the notion of hemivariational inequalities as a useful generalization of variational inequalities in the 1980s. It is based on Clarke's subdifferential of locally Lipschitz function, see for example [3, 14]. The damage function was first introduced in [10, 11] to quantify the material's damage. Here, we also consider the damage of the material. This is described by an interval variable, which is modelled by a parabolic differential inclusion. Moreover, for contact problems involving damage phenomena, we refer to [13, 15, 16, 24] and the references therein.

The contact problem between a nonlinear thermo-electro-visco-elastic body and a stiff foundation is studied in this paper using a mathematical model in the form of a hemivariational inequality, describing both visco-piezoelectric and thermal effects. We can see [12, 20, 23, 26, 28] for piezoelectric and visco-piezolectric contact problems. On the

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other hand, we use the references [1, 6, 7, 18] to describe thermo-visco-piezolectric materials. The friction law is given in the form of a subdifferential condition, and the material's damage is factored in. Here, $T > 0$ and $[0, T]$ is the time interval of interest, and dots above a variable represent time derivatives, that is $\dot{u} = \partial u/\partial t$. Then, we shall deal with a thermo-electro-visco-elastic materials for which the constitutive laws are given, without indicating explicitly the dependence of various functions on the independent variables $x \in \Omega \cup \Gamma$, as follows:

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{B}(\varepsilon(u(t)), \zeta(t)) - \mathcal{P}^T E(\varphi(t)) - \mathcal{C}\theta(t), \tag{1.1}$$

$$D(t) = \mathcal{P}\varepsilon(u(t)) + \beta E(\varphi(t)) + \mathcal{G}\theta(t), \tag{1.2}$$

$$\dot{\theta}(t) - \operatorname{div} \mathcal{K}(\nabla\theta(t)) = \mathcal{M}\varepsilon(u)(t) - \mathcal{N}E(\varphi(t)) + h_0(t), \tag{1.3}$$

in which $\sigma$ denotes the stress tensor, $u$ is the displacement field, $\zeta$ is the damage field, $\varphi$ is the electric potential field and $\theta$ is the temperature. The extension of the thermo-visco-elastic constitutive laws with damage employed in [24] is represented by a constitutive equation of the type (1.1)–(1.3). They generalize the thermo-piezoelectric and thermo-electro-visco-elastic constitutive equations employed in [2, 25]. As in [10], we suppose that the damage evolution is governed by the following differential inclusion

$$\dot{\zeta}(t) - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta(t)) \ni \phi(\varepsilon(u(t)), \zeta(t)),$$

where $\kappa > 0$ is the micro-crack diffusion coefficient and $\phi$ is the mechanical cause of damage growth. The damage function $\zeta$ takes its values in $[0, 1]$, the value $\zeta = 1$ indicates that the material is undamaged, the value $\zeta = 0$ indicates that the material is totally damaged, and when $0 < \zeta < 1$, there is partial damage and the system's load carrying capability is lowered.

From a mathematical point of view, models characterizing thermo-electro-visco-elastic are recent, see for example, [2, 25], and the first novelty of the current paper is to extend such models to thermo-electro-visco-elastic contact problems with damage and to the hemivariational case. We shall provide an existence and uniqueness of a fractional solution for the Signorini's contact problem with non-monotone boundary conditions described by the Clarke subdifferential. Moreover, up to date, no work has dealt with numerical analysis of a hemivariational inequalities arising in thermo-electro-visco-elastic contact problems with damage, and that represents the second novelty of this paper. In the study of a fully discrete scheme for the numerical solutions, a finite element approach is used to approximate the spatial variable and finite differences are used for the time derivatives, and as result, we obtain some error estimates on the approximate solutions. For further information on numerical aspects of elastic and electro-elastic contact problems, we refer [4, 5, 8, 9, 27].

The paper is organized as follows. Section 2 is devoted to study the existence and uniqueness of the unique solution of a quasi-static frictional unilateral contact problem with damage between a thermo-electro-visco-elastic material's body and a conductive foundation. Moreover, we derive the variational formulation of this problem, as a coupled system of three hemivariational inequalities and a parabolic equation. Finally, in Section 3, we investigate a fully discrete approximation of the relate model, and we derive error estimates and convergence results.

## 2. Contact problem for thermo-electro-visco-elastic with damage

In this section, we discuss a static contact problem for a nonlinear thermo-electro-visco-elastic with damage body which is described by an unilateral constraints with multi-valued normal compliance function, and non-monotone multi-valued friction condition with slip dependent coefficient. We describe the physical setting of the problem and we provide its classical variational-hemivariational formulation using four systems of three hemivariational inequalities and a parabolic equation. Next, we study the existence and uniqueness of solution for this coupled system.

We consider a thermo-visco-piezoelectric body that occupies the domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$, which is supposed to be an open, bounded and connected subset of $\mathbb{R}^d$, and with a Lipschitz boundary $\Gamma = \partial\Omega$. The body is acted upon by body forces $f_0$, a volume free electric charge $q_0$, a surface electric charge $q_b$ and heat source $h_n$. It is also constrained mechanically and electrically on $\Gamma$, and to describe these constraints, we consider three open and measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ such that $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3 = \Gamma$ and $\text{meas}(\Gamma_1) > 0$, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open measurable parts $\Gamma_a$ and $\Gamma_b$ such that $\text{meas}(\Gamma_a) > 0$. Let $\nu$ be the outer normal to $\Gamma$ and throughout this paper $i$, $j$, $k$ run from 1 to $d$. The summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the variable.

Let $\mathbb{S}^d$ denote the space of second order symmetric tensors on $\mathbb{R}^d$ while $\cdot$ and $\|\cdot\|$ represent both the inner product and the associated Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, defined by

$$u \cdot v = u_i v_i, \qquad \|v\| = (v \cdot v)^{1/2}, \qquad \forall\, u, v \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\| = (\tau \cdot \tau)^{1/2}, \qquad \forall\, \sigma, \tau \in \mathbb{S}^d.$$

The normal and tangential components of the displacement vector $v \in \mathbb{R}^d$ and the stress tensor $\sigma \in \mathbb{S}^d$ on the boundary $\Gamma$ are given by

$$v_\nu = v \cdot \nu, \qquad v_\tau = v - v_\nu \nu,$$
$$\sigma_\nu = (\sigma\nu) \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

From the two orthogonality relations $v_\tau \cdot \nu = 0$ and $\sigma_\tau \cdot \nu = 0$, we derive the following important result

$$\sigma\nu \cdot v = \sigma_\nu v_\nu + \sigma_\tau \cdot v_\tau.$$

Then, the classical formulation of our frictional thermo-electro-visco-elastic contact problem with damage, is as follows.

**Problem 2.1.** Find a displacement $u\colon \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma\colon \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi\colon \Omega \times [0,T] \to \mathbb{R}$, an electric displacement $D\colon \Omega \times [0,T] \to \mathbb{R}^d$, a temperature $\theta\colon \Omega \times [0,T] \to \mathbb{R}$ and a damage field such that $\zeta\colon \Omega \times [0,T] \to \mathbb{R}$:

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{B}(\varepsilon(u(t)), \zeta(t)) - \mathcal{P}^T E(\varphi)(t) - \mathcal{C}\theta(t) \quad \text{in } \Omega \times [0,T], \tag{2.1}$$

$$D(t) = \beta E(\varphi(t)) + \mathcal{P}\varepsilon(u(t)) + \mathcal{G}\theta(t) \quad \text{in } \Omega \times [0,T], \tag{2.2}$$

$$\dot{\theta}(t) - \operatorname{div} \mathcal{K}(\nabla\theta(t)) = \mathcal{M}\varepsilon(u(t)) - \mathcal{N}E(\varphi(t)) + h_0(t) \quad \text{in } \Omega \times [0,T], \tag{2.3}$$

$$\dot{\zeta}(t) - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta(t)) \ni \phi(\varepsilon(u(t)), \zeta(t)) \quad \text{in } \Omega \times [0,T], \tag{2.4}$$

$$\operatorname{Div} \sigma(t) + f_0(t) = 0 \quad \text{in } \Omega \times [0,T], \tag{2.5}$$

$$\operatorname{div} D(t) - q_0(t) = 0 \quad \text{in } \Omega \times [0,T], \tag{2.6}$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \times [0,T], \tag{2.7}$$

$$\varphi(t) = 0 \quad \text{on } \Gamma_a \times [0,T], \tag{2.8}$$

$$\theta(t) = 0 \quad \text{on } \Gamma_1 \times [0,T], \tag{2.9}$$

$$\frac{\partial\zeta}{\partial\nu} = 0 \quad \text{on } \Gamma_1 \times [0,T], \tag{2.10}$$

$$\sigma(t)\nu = f_2(t) \quad \text{on } \Gamma_2 \times [0,T], \tag{2.11}$$

$$D(t) \cdot \nu = q_b(t) \quad \text{on } \Gamma_b \times [0,T], \tag{2.12}$$

$$q(t) \cdot \nu = h_n(t) \quad \text{on } \Gamma_2 \times [0,T], \tag{2.13}$$

$$-\sigma_\nu(t) \in \partial j_\nu(\dot{u}_\nu(t)), \quad -\sigma_\tau(t) \in \partial j_\tau(\dot{u}_\tau(t)) \quad \text{on } \Gamma_3 \times [0,T], \tag{2.14}$$

$$D(t) \cdot \nu \in h_e(u_\nu(t))\partial j_e(\varphi(t) - \varphi_0) \quad \text{on } \Gamma_3 \times [0,T], \tag{2.15}$$

$$-\mathcal{K}(\nabla\theta(t)) \cdot \nu \in \partial j_\theta(\theta(t)) \quad \text{on } \Gamma_3 \times [0,T], \tag{2.16}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \tag{2.17}$$

Here, conditions (2.1)–(2.4) represent the thermo-electro-visco-elastic constitutive laws with damage, see [2, 9, 13, 24] for more details, where $\mathcal{A} \in L^\infty(\Omega)$ and $\mathcal{B} \in L^\infty(\Omega)$ are the viscous and the elastic tensors, $\mathcal{P} = (e_{ijk}) \in L^\infty(\Omega)$ is the piezoelectric tensor, $\beta = (\beta_{ij})$ is the symmetric and coercive electric permittivity tensors, $\mathcal{G}$ is the pyroelectric tensor, $\mathcal{M} = (m_{ij})$ is the thermal expansion tensor, $\mathcal{N} = (n_i)$ is the pyroelectric tensor, $\mathcal{K} = (k_{ij})$ is the thermal conductivity tensor and $\phi$ is the mechanical source of damage growth. In addition, $\varepsilon(u) = (\nabla u + (\nabla u)^T)/2$ is the linearized strain tensor, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{P}^T = (\mathcal{P}_{kij})$ is the transpose tensor of $\mathcal{P}$, $I_{[0,1]}$ is the indicator function of the interval $[0,1]$ and $\partial I_{[0,1]}$ denotes its subdifferential. Relations (2.5), (2.6) are the equilibrium equations for the stress and the electric displacement fields where Div and div denote the divergence operator for tensors and vector valued functions. Moreover, (2.7)–(2.13) are the mechanical, electrical, thermal and damage boundary conditions. The relations (2.14) represent the normal stress and normal velocity satisfying the non-monotone damped response condition and the friction law in which $j_\nu$, $j_\tau$ are locally Lipschitz functions and $\partial j_\nu$, $\partial j_\tau$ denotes the Clarke generalized gradient of the functions $j_\nu$ and $j_\tau$. The condition (2.15) represents a regularized condition for the electrical contact on $\Gamma_3$ in which $\varphi_0$ is the electric potential of the foundation and $h_e$, $j_e$ are given functions. The relation (2.16) represents the heat exchange between $\Gamma_3$ and the foundation. Finally (2.17) denotes the initial displacement, temperature and damage conditions.

Next, to derive the variational-hemivariational formulation of Problem 2.1, we have to recall some necessary definitions and notations that we will use later. We consider the following spaces

$$H = L^2(\Omega)^d, \quad Z = H^1(\Omega)^d, \quad Z_0 = L^2(\Omega),$$
$$\mathcal{H} = \big\{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\big\}, \quad \mathcal{H}_1 = \big\{\sigma \in \mathcal{H} \mid \operatorname{Div}\sigma \in \mathcal{H}\big\},$$

which are real Hilbert spaces for the following inner products and their associated norms

$$(u,v)_H = \int_\Omega u_i v_i\, dx, \qquad (u,v)_Z = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$
$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx, \qquad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_{\mathcal{H}}.$$

Furthermore, according to the boundary conditions, we introduce the following variational subspaces

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_1\},$$
$$W = \{\psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\},$$
$$Q = \{\theta \in H^1(\Omega) \mid \theta = 0 \text{ on } \Gamma_1\},$$

and the following set of admissible damage functions

$$K = \{\xi \in Z \mid \xi \in [0,1] \text{ a.e. in } \Omega\}.$$

The spaces $V$, $W$ and $Q$ are Hilbert spaces for the following Euclidean norms

$$\|u\|_V = (u,u)_V^{1/2}, \quad (u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$
$$\|\varphi\|_W = (\varphi,\varphi)_W^{1/2}, \quad (\varphi,\psi)_W = (\nabla\varphi, \nabla\psi)_H, \tag{2.18}$$
$$\|\theta\|_Q = (\theta,\theta)_Q^{1/2}, \quad (\theta,\eta)_Q = (\nabla\theta, \nabla\eta)_{\mathcal{H}}. \tag{2.19}$$

Since $V$ is a closed subspace of $Z$ and $\mathrm{meas}(\Gamma_1) > 0$, the Korn's inequality holds and there exists a constant $c_k > 0$ depending only on $\Omega$ and $\Gamma_1$ such that

$$\|v\|_Z \leq c_k \|\varepsilon(v)\|_{\mathcal{H}}, \quad \forall\, v \in V.$$

Hence, the norms $\|\cdot\|_Z$ and $\|\cdot\|_V$ are equivalent on $V$ and then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists $c_0 > 0$ depending only on $\Omega$, $\Gamma_3$ and $\Gamma_1$ such that

$$\|v\|_{L^2(\Gamma)^d} \leq c_0 \|v\|_V, \quad \forall\, v \in V.$$

Since $\mathrm{meas}(\Gamma_a) > 0$ the Friedrichs–Poincaré inequality holds and thus we have

$$\|\psi\|_{H^1(\Omega)} \leq c_F \|\nabla\psi\|_{\mathcal{H}}, \quad \forall\, \psi \in W, \tag{2.20}$$

where $c_F > 0$ is a constant which depends only on $\Omega$ and $\Gamma_a$. It follows from (2.18) and (2.20) that the norms $\|\cdot\|_W$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $W$, and thus $(W, \|\cdot\|_W)$ is a real Hilbert space. In addition, the Sobolev trace theorem implies that there exists $c_1 > 0$ depending on $\Omega$, $\Gamma_a$ and $\Gamma_3$ such that

$$\|\xi\|_{L^2(\Gamma_3)} \leq c_1 \|\xi\|_W, \quad \forall\, \xi \in W.$$

Moreover, since $\mathrm{meas}(\Gamma_1) > 0$, the Friedrichs–Poincaré inequality holds and thus we have

$$\|\theta\|_{H^1(\Omega)} \leq c_R \|\nabla\theta\|_{\mathcal{H}}, \quad \forall\, \theta \in Q, \tag{2.21}$$

where $c_R > 0$ is a constant which depends only on $\Omega$ and $\Gamma_1$. It follows from (2.19) and (2.21), that the norms $\|\cdot\|_Q$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $Q$, and thus $(Q, \|\cdot\|_Q)$ is a real Hilbert space. In addition, the Sobolev trace theorem implies that there exists $c_2 > 0$ depending on $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\|\eta\|_{L^2(\Gamma_3)} \leq c_2 \|\eta\|_Q, \quad \forall\, \eta \in Q.$$

Next, in the study of the solvability of Problem 2.1, we need the following hypotheses.

$(\mathcal{H}_1)$ The tensor $\mathcal{A}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(a) $\mathcal{A}(\,\cdot\,, \varepsilon)$ is measurable on $\Omega$ for all $\varepsilon \in \mathbb{S}^d$,

(b) there exist $L_{\mathcal{A}} > 0$ such that for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$ and $x \in \Omega$, we have

$$\|\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)\| \leq L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|, \tag{2.22}$$

(c) there exist $\alpha_{\mathcal{A}} > 0$ such that for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$ and $x \in \Omega$, we have

$$(\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq \alpha_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2, \tag{2.23}$$

(d) $\mathcal{A}(x,0)=0$ for all $x\in\Omega$.

$(\mathcal{H}_2)$ The tensor $\mathcal{B}\colon \Omega\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{S}^d$ is such that

(a) $\mathcal{B}(\,\cdot\,,\varepsilon,r)$ is measurable on $\Omega$ for all $\varepsilon\in\mathbb{S}^d$ and $r\in\mathbb{R}$,

(b) there exist $L_{\mathcal{B}}>0$ such that for all $\varepsilon_1,\varepsilon_2\in\mathbb{S}^d$, $r_1,r_2\in\mathbb{R}$ and $x\in\Omega$, we have

$$\|\mathcal{B}(x,\varepsilon_1,r_1)-\mathcal{B}(x,\varepsilon_2,r_2)\|\le L_{\mathcal{B}}(\|\varepsilon_1-\varepsilon_2\|+|r_1-r_2|), \tag{2.24}$$

(c) $\mathcal{B}(x,\varepsilon,0)=0$ for all $x\in\Omega$.

$(\mathcal{H}_3)$ The tensor of piezoelectric $\mathcal{P}=(e_{ijk})\colon \Omega\times\mathbb{S}^d\to\mathbb{R}^d$ is such that

(a) $e_{ijk}\in L^\infty(\Omega)$,

(b) there exist $L_{\mathcal{P}}>0$ such that for all $\varepsilon_1,\varepsilon_2\in\mathbb{S}^d$ and $x\in\Omega$, we have

$$\|\mathcal{P}(x,\varepsilon_1)-\mathcal{P}(x,p_2,\varepsilon_2)\|\le L_{\mathcal{P}}\|\varepsilon_1-\varepsilon_2\|, \tag{2.25}$$

(c) $\mathcal{P}(x,0)=0$ for all $x\in\Omega$.

$(\mathcal{H}_4)$ The permittivity tensor $\beta=(\beta_{ij})\colon \Omega\times\mathbb{R}^d\to\mathbb{R}^d$ is such that

(a) $\beta_{ij}=\beta_{ji}\in L^\infty(\Omega)$,

(b) there exist $L_\beta>0$ such that for all $\xi_1,\xi_2\in\mathbb{R}^d$ and $x\in\Omega$, we have

$$\|\beta(x,\xi_1)-\beta(x,\xi_2)\|\le L_\beta\|\xi_1-\xi_2\|, \tag{2.26}$$

(c) there exist $\alpha_\beta>0$ such that for all $\xi_1,\xi_2\in\mathbb{R}^d$ and $x\in\Omega$, we have

$$(\beta(x,\xi_1)-\beta(x,\xi_2))\cdot(\xi_1-\xi_2)\ge\alpha_\beta\|\xi_1-\xi_2\|^2, \tag{2.27}$$

(d) $\beta(x,0)=0$ for all $x\in\Omega$.

$(\mathcal{H}_5)$ The functions $j_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}$, $j_\tau\colon \Gamma_3\times\mathbb{R}^d\to\mathbb{R}$ and $j_e,j_\theta\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}$ satisfy

(I)(a) $j_\nu(\,\cdot\,,r)$ is measurable on $\Gamma_3$ for all $r\in\mathbb{R}$,

(b) $j_\nu(x,\cdot\,)$ is locally Lipschitz on $\mathbb{R}$ for all $x\in\Gamma_3$,

(c) there exist $c_{0\nu},c_{1\nu}\ge0$ such that for all $r\in\mathbb{R}$ and $x\in\Gamma_3$, we have

$$|\partial j_\nu(x,r)|\le c_{0\nu}+c_{1\nu}|r|,$$

(d) there exist $\alpha_{j\nu}\ge0$ such that for all $r_1,r_2\in\mathbb{R}$ and $x\in\Gamma_3$, we have

$$j_\nu^0(x,r_1;r_2-r_1)+j_\nu^0(x,r_2;r_1-r_2)\le\alpha_{j\nu}|r_1-r_2|^2. \tag{2.28}$$

(II)(a) $j_\tau(\cdot,\xi)$ is measurable on $\Gamma_3$ for all $\xi \in \mathbb{R}^d$,

(b) $j_\tau(x,\cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for all $x \in \Gamma_3$,

(c) there exist $c_{0\tau}, c_{1\tau} \geq 0$ such that for all $\xi \in \mathbb{R}^d$ and $x \in \Gamma_3$, we have

$$\|\partial j_\tau(x,\xi)\| \leq c_{0\tau} + c_{1\tau}\|\xi\|_{\mathbb{R}^d},$$

(d) there exist $\alpha_{j\tau} \geq 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ and $x \in \Gamma_3$, we have

$$j_\tau^0(x,\xi_1;\xi_2-\xi_1) + j_\tau^0(x,\xi_2;\xi_1-\xi_2) \leq \alpha_{j\tau}\|\xi_1-\xi_2\|^2_{\mathbb{R}^d}. \tag{2.29}$$

(III)(a) $j_e(\cdot,r)$ is measurable on $\Gamma_3$ for all $r \in \mathbb{R}$,

(b) $j_e(x,\cdot)$ is locally Lipschitz on $\mathbb{R}$ for all $x \in \Gamma_3$,

(c) there exist $c_{0e}, c_{1e} \geq 0$ such that for all $r \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$|\partial j_e(x,r)| \leq c_{0e} + c_{1e}|r|,$$

(d) there exist $\alpha_{je} \geq 0$ such that for all $r_1, r_2 \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$j_e^0(x,r_1;r_2-r_1) + j_e^0(x,r_2;r_1-r_2) \leq \alpha_{je}|r_1-r_2|^2. \tag{2.30}$$

(IV)(a) $j_\theta(\cdot,r)$ is measurable on $\Gamma_3$ for all $r \in \mathbb{R}$,

(b) $j_\theta(x,\cdot)$ is locally Lipschitz on $\mathbb{R}$ for all $x \in \Gamma_3$,

(c) there exist $c_{0\theta}, c_{1\theta} \geq 0$ such that for all $r \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$|\partial j_\theta(x,r)| \leq c_{0\theta} + c_{1\theta}|r|,$$

(d) there exist $\alpha_{j\theta} \geq 0$ such that for all $r_1, r_2 \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$j_\theta^0(x,r_1;r_2-r_1) + j_\theta^0(x,r_2;r_1-r_2) \leq \alpha_{j\theta}|r_1-r_2|^2. \tag{2.31}$$

$(\mathcal{H}_6)$ The function $h_e\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfy

(a) $h_e(\cdot,r)$ is measurable on $\Gamma_3$ for all $r \in \mathbb{R}$,

(b) $h_e(x,\cdot)$ is continuous on $\mathbb{R}$ for all $x \in \Gamma_3$,

(c) there exists $\overline{h}_e > 0$ for all $r \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$0 \leq h_e(x,r) \leq \overline{h}_e.$$

$(\mathcal{H}_7)$ The thermal operator $\mathcal{C}\colon \Omega \times \mathbb{R} \to \mathbb{S}^d$ is such that

(a) $\mathcal{C}(\cdot,r)$ is measurable on $\Omega$ for all $r \in \mathbb{R}$,

(b) there exist $L_{\mathcal{C}} > 0$ such that for all $r_1, r_2 \in \mathbb{R}$ and $x \in \Omega$, we have

$$\|\mathcal{C}(x, r_1) - \mathcal{C}(x, r_2)\| \le L_{\mathcal{C}} |r_1 - r_2|, \tag{2.32}$$

(c) $\mathcal{C}(x, 0) = 0$ for all $x \in \Omega$.

($\mathcal{H}_8$) The function $\mathcal{M}\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ is such that

(a) $\mathcal{M}(\,\cdot\,, \xi) \in L^\infty(\Omega)$,

(b) there exist $L_{\mathcal{M}} > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\mathcal{M}(x, \xi_1) - \mathcal{M}(x, \xi_2)\| \le L_{\mathcal{M}} \|\xi_1 - \xi_2\|. \tag{2.33}$$

($\mathcal{H}_9$) The piezoelectric tensor $\mathcal{N}\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ is such that

(a) $\mathcal{N}(\,\cdot\,, \xi) \in L^\infty(\Omega)$,

(b) there exist $L_{\mathcal{N}} > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\mathcal{N}(x, \xi_1) - \mathcal{N}(x, \xi_2)\| \le L_{\mathcal{N}} \|\xi_1 - \xi_2\|, \tag{2.34}$$

(c) $\mathcal{N}(x, 0) = 0$ for all $x \in \Omega$.

($\mathcal{H}_{10}$) The pyroelectric tensor $\mathcal{G}\colon \Omega \times \mathbb{R} \to \mathbb{R}^d$ is such that

(a) $\mathcal{G}(\,\cdot\,, r) \in L^\infty(\Omega)$ for all $r \in \mathbb{R}$,

(b) there exist $L_{\mathcal{G}} > 0$ such that for all $r_1, r_2 \in \mathbb{R}$ and $x \in \Omega$, we have

$$\|\mathcal{G}(x, r_1) - \mathcal{G}(x, r_2)\| \le L_{\mathcal{G}} |r_1 - r_2|. \tag{2.35}$$

($\mathcal{H}_{11}$) The thermal conductivity operator $\mathcal{K}\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is such that

(a) $\mathcal{K}(\,\cdot\,, \xi)$ is measurable on $\Omega$ for all $\xi \in \mathbb{R}^d$,

(b) there exist $L_{\mathcal{K}} > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\mathcal{K}(x, \xi_1) - \mathcal{K}(x, \xi_2)\| \le L_{\mathcal{K}} \|\xi_1 - \xi_2\|,$$

(c) there exist $\alpha_{\mathcal{K}} > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$(\mathcal{K}(x, \xi_1) - \mathcal{K}(x, \xi_2)) \cdot (\xi_1 - \xi_2) \ge \alpha_{\mathcal{K}} \|\xi_1 - \xi_2\|^2, \tag{2.36}$$

(d) $\mathcal{K}(x, 0) = 0$ for all $x \in \Omega$.

($\mathcal{H}_{12}$) The damage source function $\phi\colon \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ is such that

(a) $\phi(\,\cdot\,,\varepsilon,r)$ is measurable on $\Omega$ for all $\varepsilon\in\mathbb{S}^d$ and $r\in\mathbb{R}$,

(b) there exist $L_\phi>0$ such that for all $\varepsilon_1,\varepsilon_2\in\mathbb{S}^d$, $r_1,r_2\in\mathbb{R}$ and $x\in\Omega$, we have

$$\|\phi(x,\varepsilon_1,\zeta_1)-\phi(x,\varepsilon_2,\zeta_2)\|\le L_\phi(\|\varepsilon_1-\varepsilon_2\|+|r_1-r_2|), \tag{2.37}$$

(c) $\phi(x,0,0)=0$, for all $x\in\Omega$.

($\mathcal{H}_{13}$) The forces, tractions, volume and surface charge densities gap and initial functions satisfy

(i) $f_0\in L^2(\Omega)^d$, $f_2\in L^2(\Gamma_2)^d$, $q_0\in L^2(\Omega)$, $q_b\in L^2(\Gamma_b)$, $h_0\in L^2(\Omega)$, $h_n\in L^2(\Gamma_2)$,

(ii) $\varphi_0\in L^\infty(\Gamma_3)$, $\kappa>0$, $u_0\in V$, $\theta_0\in Q$, $\zeta_0\in K$.

Using Riesz's representation theorem, we consider the elements $f\in V$, $q\in W$ and $h\in Q$ defined by

$$\begin{aligned}
\langle f,v\rangle_V &= \langle f_0,v\rangle_{L^2(\Omega)^d}+\langle f_2,v\rangle_{L^2(\Gamma_2)^d} && \text{for all } v\in V,\\
\langle q,\psi\rangle_W &= \langle q_0,\psi\rangle_{L^2(\Omega)}-\langle q_b,\psi\rangle_{L^2(\Gamma_b)} && \text{for all } \psi\in W,\\
\langle h,\xi\rangle_Q &= \langle h_0,\xi\rangle_{L^2(\Omega)}-\langle h_n,\xi\rangle_{L^2(\Gamma_2)} && \text{for all } \xi\in Q.
\end{aligned}$$

We consider the following bilinear form $a\colon H_1\times H_1\to\mathbb{R}$ given by

$$a(\xi,\eta)=\kappa\int_\Omega \nabla\xi\cdot\nabla\eta\,dx,\quad \forall\,\xi,\eta\in H_1.$$

Then, by standard arguments, the variational formulation of Problem 2.1 is as follows.

**Problem 2.2.** Find a displacement $u\colon[0,T]\to V$, an electric potential $\varphi\colon[0,T]\to W$, a temperature $\theta\colon[0,T]\to W$ and a damage field $\zeta\colon[0,T]\to Z$ such that for all $t\in[0,T]$, we have

$$\begin{aligned}
&\langle\mathcal{A}\varepsilon(\dot u(t)),\varepsilon(v-\dot u(t))\rangle_{\mathcal{H}}+\langle\mathcal{B}(\varepsilon(u(t)),\zeta(t))+\mathcal{P}^T\nabla\varphi(t)-\mathcal{C}\theta(t),\varepsilon(v-\dot u(t))\rangle_{\mathcal{H}}\\
&\quad+\int_{\Gamma_3}\big(j_\nu^0(\dot u_\nu(t);v_\nu-\dot u_\nu(t))+j_\tau^0(\dot u_\tau(t);v_\tau-\dot u_\tau(t))\big)\,da\\
&\ge\langle f(t),v-\dot u(t)\rangle_V,\quad\forall\,v\in V,
\end{aligned}\tag{2.38}$$

$$\begin{aligned}
&\langle\beta\nabla(\varphi(t))-\mathcal{P}\varepsilon(u(t))-\mathcal{G}\theta(t),\nabla(\psi-\varphi(t))\rangle_{\mathcal{H}}\\
&\quad+\int_{\Gamma_3}h_e(u(t))j_e^0(\varphi(t)-\varphi_0;\psi-\varphi(t))\,da\\
&\ge\langle q(t),\psi-\varphi(t)\rangle_W,\quad\forall\,\psi\in W,
\end{aligned}\tag{2.39}$$

$$\begin{aligned}
&\langle\dot\theta(t),\lambda-\theta(t)\rangle_{\mathcal{H}}+\langle\mathcal{K}\nabla\theta(t),\nabla(\lambda-\theta(t))\rangle_{\mathcal{H}}\\
&\quad-\langle\mathcal{M}\varepsilon(u(t))-\mathcal{N}\nabla\varphi(t),\lambda-\theta(t)\rangle_{\mathcal{H}}+\int_{\Gamma_3}j_\theta^0(\theta(t);\lambda-\theta(t))\,da\\
&\ge\langle h(t),\lambda-\theta(t)\rangle_Q,\quad\forall\,\eta\in Q,
\end{aligned}\tag{2.40}$$

$$\langle\dot\zeta(t),\xi-\zeta(t)\rangle_{Z_0}+a(\zeta(t),\xi-\zeta(t))\ge\langle\phi(\varepsilon(u(t)),\zeta(t)),\xi-\zeta(t)\rangle,\quad\forall\,\xi\in K.$$

Under these considerations, we have the following existence and uniqueness result.

**Theorem 2.3.** *Assume hypotheses* $(\mathcal{H}_1)$–$(\mathcal{H}_{13})$ *and the following smallness condition are satisfied.*

$$\begin{aligned} \alpha_{\mathcal{A}} &> c_0^2(\alpha_{j_\nu} + \alpha_{j_\tau})\sqrt{\operatorname{meas}(\Gamma_3)}, \\ \alpha_{\mathcal{K}} &> c_0^2\alpha_{j_\theta}\sqrt{\operatorname{meas}(\Gamma_3)}, \\ \alpha_\beta &> \overline{h}_e\alpha_{j_e}c_0^2\sqrt{\operatorname{meas}(\Gamma_3)}, \\ \alpha_{\mathcal{K}} - c_0^2\alpha_{j_\theta}\sqrt{\operatorname{meas}(\Gamma_3)} &> L_{\mathcal{M}}T/2. \end{aligned} \tag{2.41}$$

*Then, Problem* 2.2 *has a unique solution* $(u, \varphi, \theta, \zeta)$ *such that*

$$u \in L^2(0,T;V), \quad \varphi \in L^2(0,T;W), \quad \theta \in L^2(0,T;Q),$$
$$\zeta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

The proof will be done in several claims, and it is based on fixed point and hemivariational inequalities arguments.

**Claim 1.** Let $\eta \in L^2(0,T;\mathcal{H})$, $z \in L^2(0,T;Q)$ and $\omega \in L^2(0,T;L^2(\Omega))$ supposed given. Then, we consider the following auxiliary problems.

**Problem 2.4.** Find $u_\eta\colon [0,T] \to V$ such that for all $t \in [0,T]$, we have

$$\begin{aligned} &\langle \mathcal{A}\varepsilon(\dot{u}_\eta)(t), \varepsilon(v - \dot{u}_\eta(t))\rangle_{\mathcal{H}} + \langle \eta(t), \varepsilon(v - \dot{u}_\eta(t))\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} j_\nu^0(\dot{u}_{\eta\nu}(t); v_\nu - \dot{u}_{\eta\nu}(t))\,da + \int_{\Gamma_3} j_\tau^0(\dot{u}_{\eta\tau}(t); v_\tau - \dot{u}_{\eta\tau}(t)))\,da \\ &\geq \langle f(t), v - \dot{u}_\eta(t)\rangle_V, \quad \forall\, v \in V. \end{aligned}$$

**Problem 2.5.** Find $\varphi_{\eta,z}\colon [0,T] \to W$ such that for all $t \in [0,T]$ and all $\psi \in W$, we have

$$\begin{aligned} &\langle \beta\nabla(\varphi_{\eta,z}(t)), \nabla(\psi - \varphi_{\eta,z}(t))\rangle_{\mathcal{H}} - \langle \mathcal{P}(\varepsilon(u_\eta(t))) + \mathcal{G}\theta_{\eta,z}(t), \nabla(\psi - \varphi_{\eta,z}(t))\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} h_e(u_\eta(t)) j_e^0(\varphi_{\eta,z}(t) - \varphi_0; \psi - \varphi_{\eta,z}(t))\,da \\ &\geq \langle q(t), \psi - \varphi_{\eta,z}(t)\rangle_W. \end{aligned}$$

**Problem 2.6.** Find $\theta_{\eta,z}\colon [0,T] \to W$ such that for all $t \in [0,T]$ and all $\lambda \in Q$, we have

$$\begin{aligned} &\langle \dot{\theta}_{\eta,z}(t), \lambda - \theta_{\eta,z}(t)\rangle_{\mathcal{H}} + \langle \mathcal{K}\nabla\theta_{\eta,z}(t), \nabla(\lambda - \theta_{\eta,z}(t))\rangle_{\mathcal{H}} - \langle \mathcal{M}\varepsilon(u_\eta(t)), \lambda - \theta_{\eta,z}(t)\rangle_{\mathcal{H}} \\ &\quad + \langle z(t), \lambda - \theta_{\eta,z}(t)\rangle + \int_{\Gamma_3} j_\theta^0(\theta_{\eta,z}(t); \lambda - \theta_{\eta,z}(t))\,da \\ &\geq \langle h(t), \lambda - \theta_{\eta,z}(t)\rangle_Q. \end{aligned}$$

**Problem 2.7.** Find $\zeta\colon [0,T] \to Z$ such that for all $t \in [0,T]$ and all $\xi \in K$, we have

$$\langle \dot{\zeta}_\omega(t), \xi - \zeta_\omega(t)\rangle_{L^2(\Omega)} + a(\zeta_\omega(t), \xi - \zeta_\omega(t)) \geq \langle \omega(t), \xi - \zeta_\omega(t)\rangle.$$

**Lemma 2.8.** *Problem 2.4 has a unique solution. Moreover, there exists a constant $c > 0$ such that*

$$\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*}^2 \, ds, \tag{2.42}$$

*where $u_{\eta_i}$ denotes the solution of Problem 2.4 corresponding to $\eta_i$ with $i = 1, 2$.*

*Proof.* Using similar techniques as in [25, Lemma 4.5], we prove the existence part of Lemma 2.8. For the estimate (2.42), let $u_{\eta_i}$ be the solution of Problem 2.4 corresponding to $\eta_i \in L^2(0,T;\mathcal{H})$ with $i = 1, 2$. Then, for all $t \in [0,T]$ and all $v \in V$, we write

$$\begin{aligned} &\langle \mathcal{A}\varepsilon(\dot{u}_{\eta_1})(t), \varepsilon(v - \dot{u}_{\eta_1}(t))\rangle_{\mathcal{H}} + \langle \eta_1(t), \varepsilon(v - \dot{u}_{\eta_1}(t))\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} j_\nu^0(\dot{u}_{\eta_1\nu}(t); v_\nu - \dot{u}_{\eta_1\nu}(t))\, da + \int_{\Gamma_3} j_\tau^0(\dot{u}_{\eta_1\tau}(t); v_\tau - \dot{u}_{\eta_1\tau}(t))\, da \\ &\geq \langle f(t), v - \dot{u}_{\eta_1}(t)\rangle_V, \end{aligned} \tag{2.43}$$

$$\begin{aligned} &\langle \mathcal{A}\varepsilon(\dot{u}_{\eta_2})(t), \varepsilon(v - \dot{u}_{\eta_2}(t))\rangle_{\mathcal{H}} + \langle \eta_2(t), \varepsilon(v - \dot{u}_{\eta_2}(t))\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} j_\nu^0(\dot{u}_{\eta_2\nu}(t); v_\nu - \dot{u}_{\eta_2\nu}(t))\, da + \int_{\Gamma_3} j_\tau^0(\dot{u}_{\eta_2\tau}(t); v_\tau - \dot{u}_{\eta_2\tau}(t))\, da \\ &\geq \langle f(t), v - \dot{u}_{\eta_2}(t)\rangle_V. \end{aligned} \tag{2.44}$$

Taking $v = \dot{u}_{\eta_2}(t)$ in (2.43) and $v = \dot{u}_{\eta_1}(t)$ in (2.44), we add the obtained inequalities to obtain

$$\begin{aligned} &\langle \mathcal{A}\varepsilon(\dot{u}_{\eta_1})(t) - \mathcal{A}\varepsilon(\dot{u}_{\eta_2})(t), \varepsilon(\dot{u}_{\eta_1}(t) - \dot{u}_{\eta_2}(t))\rangle_{\mathcal{H}} \\ &\leq \langle \eta_1(t) - \eta_2(t), \varepsilon(\dot{u}_{\eta_2}(t) - \dot{u}_{\eta_1}(t))\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} \big[j_\nu^0(\dot{u}_{\eta_1\nu}(t); \dot{u}_{\eta_2\nu}(t) - \dot{u}_{\eta_1\nu}(t)) + j_\nu^0(\dot{u}_{\eta_2\nu}(t); \dot{u}_{\eta_1\nu}(t) - \dot{u}_{\eta_2\nu}(t))\big]\, da \\ &\quad + \int_{\Gamma_3} \big[j_\tau^0(\dot{u}_{\eta_1\tau}(t); \dot{u}_{\eta_2\tau}(t) - \dot{u}_{\eta_1\tau}(t)) + j_\tau^0(\dot{u}_{\eta_2\tau}(t); \dot{u}_{\eta_1\tau}(t) - \dot{u}_{\eta_2\tau}(t))\big]\, da. \end{aligned}$$

Then, we combine the inequalities (2.23), (2.28) and (2.29) to deduce

$$\big(\alpha_{\mathcal{A}} - c_0^2(\alpha_{j_\nu} + \alpha_{j_\tau})\sqrt{\mathrm{meas}(\Gamma_3)}\big)\|\dot{u}_{\eta_1}(t) - \dot{u}_{\eta_2}(t)\|_V^2 \leq \langle \eta_1(t) - \eta_2(t), \varepsilon(\dot{u}_{\eta_2}(t) - \dot{u}_{\eta_1}(t))\rangle_{\mathcal{H}}.$$

Remembering $u_{\eta_1}(0) = u_{\eta_2}(0) = u_0$, we integrate by part the previous inequality over $(0,T)$ to find

$$\begin{aligned} &\big(\alpha_{\mathcal{A}} - c_0^2(\alpha_{j_\nu} + \alpha_{j_\tau})\sqrt{\mathrm{meas}(\Gamma_3)}\big) \int_0^t \|\dot{u}_{\eta_1}(s) - \dot{u}_{\eta_2}(s)\|_V^2\, ds \\ &\leq c\int_0^t \|\dot{u}_{\eta_1}(s) - \dot{u}_{\eta_2}(s)\|_V^2\, ds + \frac{1}{4c}\int_0^t \|\eta_1(s) - \eta_2(s)\|\, ds. \end{aligned}$$

Thus, from the previous inequality, we conclude

$$\big(\alpha_{\mathcal{A}} - c_0^2(\alpha_{j_\nu} + \alpha_{j_\tau})\sqrt{\text{meas}(\Gamma_3)} - c\big) \int_0^t \|\dot{u}_{\eta_1}(s) - \dot{u}_{\eta_2}(s)\|_V^2\, ds$$
$$\le \frac{1}{4c}\int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*}^2\, ds.$$

Finally, we use the condition (2.41) and the Cauchy inequality to get the wanted estimation (2.42). □

**Lemma 2.9.** *Problem* 2.6 *has a unique solution. Moreover there exists a constant* $c > 0$ *such that*

$$\|\theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\|_Q^2 \le c\int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*}^2 + \|z_1(s) - z_2(s)\|_{Q^*}^2\, ds, \tag{2.45}$$

*where* $\theta_{\eta_i,z_i}(t)$ *denotes the solution of Problem* 2.6 *corresponding to* $(\eta_i, z_i)$ *with* $i = 1, 2$.

*Proof.* For the existence part, we follow the same steps as in [25, Lemma 4.6]. For the estimate (2.45), let us denotes $\theta_{\eta_i,z_i}$ the solution of Problem 2.6 corresponding to $\eta_i, z_i \in L^2(0,T;\mathcal{H}\times Q)$ with $i = 1, 2$. Hence, for all $t \in (0,T)$ and all $\lambda \in Q$, we find that

$$\langle \dot{\theta}_{\eta_1,z_1}(t), \lambda - \theta_{\eta_1,z_1}(t)\rangle_{\mathcal{H}} + \langle \mathcal{K}\nabla\theta_{\eta_1,z_1}(t), \nabla(\lambda - \theta_{\eta_1,z_1}(t))\rangle_{\mathcal{H}}$$
$$- \langle \mathcal{M}\varepsilon(u_{\eta_1}(t)), \lambda - \theta_{\eta_1,z_1}(t)\rangle_{\mathcal{H}} + \langle z_1(t), \lambda - \theta_{\eta_1,z_1}(t)\rangle + \int_{\Gamma_3} j_\theta^0(\theta_{\eta_1,z_1}(t); \lambda - \theta_{\eta_1,z_1}(t))\, da$$
$$\ge \langle h(t), \lambda - \theta_{\eta_1,z_1}(t)\rangle_Q,$$
$$\langle \dot{\theta}_{\eta_2,z_2}(t), \lambda - \theta_{\eta_2,z_2}(t)\rangle_{\mathcal{H}} + \langle \mathcal{K}\nabla\theta_{\eta_2,z_2}(t), \nabla(\lambda - \theta_{\eta_2,z_2}(t))\rangle_{\mathcal{H}}$$
$$- \langle \mathcal{M}\varepsilon(u_{\eta_2}(t)), \lambda - \theta_{\eta_2,z_2}(t)\rangle_{\mathcal{H}} + \langle z_2(t), \lambda - \theta_{\eta_2,z_2}(t)\rangle + \int_{\Gamma_3} j_\theta^0(\theta_{\eta_2,z_2}(t); \lambda - \theta_{\eta_2,z_2}(t))\, da$$
$$\ge \langle h(t), \lambda - \theta_{\eta_2,z_2}(t)\rangle_Q.$$

After taking $\lambda = \theta_{\eta_2,z_2}(t)$ in the first inequality, $\lambda = \theta_{\eta,z_1}(t)$ in the second inequality, we add the two obtained inequalities to obtain that for all $t \in [0,T]$, the following inequality holds:

$$\langle \dot{\theta}_{\eta_1,z_1}(t) - \dot{\theta}_{\eta_2,z_2}(t), \theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\rangle_{\mathcal{H}}$$
$$+ \langle \mathcal{K}\nabla\theta_{\eta_1,z_1}(t) - \mathcal{K}\nabla\theta_{\eta_2,z_2}(t), \nabla(\theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t))\rangle_{\mathcal{H}}$$
$$\le \langle \mathcal{M}\varepsilon(u_{\eta_1}(t)) - \mathcal{M}\varepsilon(u_{\eta_2}(t)), \theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\rangle_{\mathcal{H}} + \langle z_1(t) - z_2(t), \theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\rangle$$
$$+ \int_{\Gamma_3} \big(j_\theta^0(\theta_{\eta_1,z_1}(t); \theta_{\eta_2,z_2}(t) - \theta_{\eta_1,z_1}(t)) + j_\theta^0(\theta_{\eta_2,z_2}(t); \theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t))\big)\, da.$$

Proceeding in the same way as in the proof of Lemma 2.8, we exploit (2.31), (2.33) and (2.36) to get

$$\begin{aligned}
&\langle \dot{\theta}_{\eta_1,z_1}(t) - \dot{\theta}_{\eta_2,z_2}(t), \theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\rangle_{\mathcal{H}} + \alpha_{\mathcal{K}}\|\theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\|_Q^2 \\
&\le L_{\mathcal{M}}\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V \|\theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\|_Q \\
&+ \langle z_1(t) - z_2(t), \theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\rangle + c_0^2\alpha_{j_\theta}\sqrt{\text{meas}(\Gamma_3)}\|\theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\|_Q^2.
\end{aligned} \tag{2.46}$$

Remembering $\theta_{\eta_1,z_1}(0) = \theta_{\eta_2,z_2}(0) = \theta_0$, we integrate by part the above inequality to deduce

$$\begin{aligned}
&\int_0^t \langle \dot{\theta}_{\eta_1,z_1}(s) - \dot{\theta}_{\eta_2,z_2}(s), \theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\rangle_{\mathcal{H}} \\
&+ \left(\alpha_{\mathcal{K}} - c_0^2\alpha_{j_\theta}\sqrt{\text{meas}(\Gamma_3)}\right)\int_0^t \|\theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\|_Q^2 \\
\le\ & L_{\mathcal{M}}\int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V \|\theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\|_Q \\
&+ \int_0^t \langle z_1(s) - z_2(s), \theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\rangle.
\end{aligned}$$

By using the Cauchy inequality, there exists $c > 0$ such that for all $t \in (0, T)$, we have

$$\begin{aligned}
&\frac{1}{2}\|\theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\|_Q^2 + \left(\alpha_{\mathcal{K}} - c_0^2\alpha_{j_\theta}\sqrt{\text{meas}(\Gamma_3)}\right)\int_0^t \|\theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\|_Q^2 \\
&\le c\int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V^2 + \frac{L_{\mathcal{M}}}{4c}\int_0^t \|\theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\|_Q^2 \\
&+ c\int_0^t \|z_1(s) - z_2(s)\|_{Q^*}^2 + \frac{1}{4c}\int_0^t \|\theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\|_Q^2.
\end{aligned} \tag{2.47}$$

Next, we keep in mind the inequality (2.42) to deduce the following estimate

$$\begin{aligned}
&\frac{1}{2}\|\theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\|_Q^2 \\
&+ \left(\alpha_{\mathcal{K}} - c_0^2\alpha_{j_\theta}\sqrt{\text{meas}(\Gamma_3)} - \frac{L_{\mathcal{M}}}{4c} - \frac{1}{4c}\right)\int_0^t \|\theta_{\eta_1,z_1}(s) - \theta_{\eta_2,z_2}(s)\|_Q^2 \\
\le\ & c\int_0^t \left(\|\eta_{\eta_1}(s) - \eta_{\eta_2}(s)\|_{V^*}^2 + \|z_1(s) - z_2(s)\|_{Q^*}^2\right) ds.
\end{aligned} \tag{2.48}$$

Finally, we conclude that the estimation (2.45) is verified. □

**Lemma 2.10.** *Problem* 2.5 *has a unique solution. Moreover there exists a constant* $c > 0$ *such that*

$$\|\varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t)\|_W^2 \le c\int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*}^2 + \|z_1(s) - z_2(s)\|_{Q^*}^2\, ds, \tag{2.49}$$

*where* $\varphi_{\eta_i,z_i}(t)$ *denotes the solution of Problem* 2.5 *corresponding to* $(\eta_i, z_i)$ *with* $i = 1, 2$.

*Proof.* We use the same techniques as in [25, Lemma 4.8] to prove the existence part of Problem 2.5. To prove the previous estimate (2.49), let $\varphi_{\eta_i,z_i}$ denotes solutions to Problem 2.5 corresponding to the elements $\eta_i, z_i \in L^2(0,T;\mathcal{H}\times Q)$ with $i = 1, 2$. Then, for all $\psi \in W$, we have

(2.50)
$$\begin{aligned}&\langle \beta E(\varphi_{\eta_1,z_1}(t)), \nabla(\psi - \varphi_{\eta_1,z_1}(t))\rangle_{\mathcal{H}} - \langle \mathcal{P}(\varepsilon(u_{\eta_1}(t))) + \mathcal{G}\theta_{\eta_1,z_1}(t), \nabla(\psi - \varphi_{\eta_1,z_1}(t))\rangle_{\mathcal{H}}\\ &\quad + \int_{\Gamma_3} h_e(u_{\eta_1}(t)) j_e^0(\varphi_{\eta_1,z_1}(t) - \varphi_0; \psi - \varphi_{\eta_1,z_1}(t))\, da\\ &\geq \langle q(t), \psi - \varphi_{\eta_1,z_1}(t)\rangle_W,\end{aligned}$$

(2.51)
$$\begin{aligned}&\langle \beta E(\varphi_{\eta_2,z_2}(t)), \nabla(\psi - \varphi_{\eta_2,z_2}(t))\rangle_{\mathcal{H}} - \langle \mathcal{P}(\varepsilon(u_{\eta_2}(t))) + \mathcal{G}\theta_{\eta_2,z_2}(t), \nabla(\psi - \varphi_{\eta_2,z_2}(t))\rangle_{\mathcal{H}}\\ &\quad + \int_{\Gamma_3} h_e(u_{\eta_2}(t)) j_e^0(\varphi_{\eta_2,z_2}(t) - \varphi_0; \psi - \varphi_{\eta_2,z_2}(t))\, da\\ &\geq \langle q(t), \psi - \varphi_{\eta_2,z_2}(t)\rangle_W.\end{aligned}$$

We take $\psi = \varphi_{\eta_2,z_2}(t)$ in (2.50), $\psi = \varphi_{\eta_1,z_1}(t)$ in (2.51). Then we add the obtained inequalities to get

$$\begin{aligned}&\langle \beta E(\varphi_{\eta_1,z_1}(t)) - \beta E(\varphi_{\eta_2,z_2}(t)), \nabla(\varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t))\rangle_{\mathcal{H}}\\ &\leq \langle \mathcal{P}(\varepsilon(u_{\eta_1}(t))) - \mathcal{P}(\varepsilon(u_{\eta_2}(t))), \nabla(\varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t))\rangle_{\mathcal{H}}\\ &\quad + \langle \mathcal{G}\theta_{\eta_1,z_1}(t) - \mathcal{G}\theta_{\eta_2,z_2}(t), \nabla(\varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t))\rangle_{\mathcal{H}}\\ &\quad + \overline{h}_e \int_{\Gamma_3} j_e^0(\varphi_{\eta_1,z_1}(t) - \varphi_0; \varphi_{\eta_2,z_2}(t) - \varphi_{\eta_1,z_1}(t))\, da\\ &\quad + \overline{h}_e \int_{\Gamma_3} j_e^0(\varphi_{\eta_2,z_2}(t) - \varphi_0; \varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t))\, da, \quad \forall\, t \in [0,T].\end{aligned}$$

Remembering the estimations (2.25), (2.27), (2.30) and (2.35), we obtain

$$\begin{aligned}&\big(\alpha_\beta - \overline{h}_e \alpha_{j_e} c_0^2 \sqrt{\operatorname{meas}(\Gamma_3)}\big)\|\varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t)\|_W^2\\ &\leq L_{\mathcal{P}}\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V \|\varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t)\|_W\\ &\quad + L_{\mathcal{G}}\|\theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\|_Q \|\varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t)\|_W.\end{aligned}$$

Using the Cauchy inequality, there exists $c > 0$ such that for all $t \in [0,T]$, we have

$$\begin{aligned}&\big(\alpha_\beta - \overline{h}_e \alpha_{j_e} c_0^2 \sqrt{\operatorname{meas}(\Gamma_3)} - 2c\big)\|\varphi_{\eta_1,z_1}(t) - \varphi_{\eta_2,z_2}(t)\|_W^2\\ &\leq \frac{L_{\mathcal{P}}}{4c}\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V^2 + \frac{L_{\mathcal{G}}}{4c}\|\theta_{\eta_1,z_1}(t) - \theta_{\eta_2,z_2}(t)\|_Q^2.\end{aligned} \tag{2.52}$$

Finally, applying the inequalities (2.42) and (2.45)–(2.52), we deduce (2.49). □

**Lemma 2.11.** *Problem 2.7 has a unique solution. Moreover, there exists $c > 0$ such that*

$$\|\zeta_{\omega_1}(t) - \zeta_{\omega_2}(t)\|^2_{L^2(\Omega)} \leq c \int_0^t \|\omega_1(s) - \omega_2(s)\|^2_{L^2(\Omega)}\, ds, \tag{2.53}$$

*where $\zeta_{\omega_i}$ is a solution of Problem 2.7 corresponding to $\omega_i$ with $i = 1, 2$.*

*Proof.* Using the same arguments as in [24, Theorem 3] to prove the unique solvability of Problem 2.7. It remains to show the estimate (2.53). To this end, let $\zeta_{\omega_i}$ denotes the solution to Problem 2.7 corresponding to $w_i \in L^2(0, T; L^2(\Omega))$ with $i = 1, 2$. Then, for all $t \in (0, T)$, we have

$$\langle \dot{\zeta}_{\omega_1}(t), \xi - \zeta_{\omega_1}(t)\rangle_{L^2(\Omega)} + a(\zeta_{\omega_1}(t), \xi - \zeta_{\omega_1}(t)) \geq \langle \omega_1(t), \xi - \zeta_{\omega_1}(t)\rangle, \tag{2.54}$$

$$\langle \dot{\zeta}_{\omega_2}(t), \xi - \zeta_{\omega_2}(t)\rangle_{L^2(\Omega)} + a(\zeta_{\omega_2}(t), \xi - \zeta_{\omega_2}(t)) \geq \langle \omega_2(t), \xi - \zeta_{\omega_2}(t)\rangle. \tag{2.55}$$

Taking $\xi = \zeta_{\omega_2}(t)$ in (2.54), $\xi = \zeta_{\omega_1}(t)$ in (2.55), and then adding the obtained inequalities to obtain

$$\begin{aligned} &\langle \dot{\zeta}_{\omega_1}(t) - \dot{\zeta}_{\omega_2}(t), \zeta_{\omega_1}(t) - \zeta_{\omega_2}(t)\rangle_{L^2(\Omega)} + a(\zeta_{\omega_1}(t) - \zeta_{\omega_2}(t), \zeta_{\omega_1}(t) - \zeta_{\omega_2}(t)) \\ &\leq \langle \omega_1(t) - \omega_2(t), \zeta_{\omega_1}(t) - \zeta_{\omega_2}(t)\rangle. \end{aligned}$$

Remembering $\zeta_{\omega_1}(0) = \zeta_{\omega_1}(0) = \zeta_0$, we integrate by part the previous inequality over $(0, T)$ to get

$$\frac{1}{2}\|\zeta_{\omega_1}(t) - \zeta_{\omega_2}(t)\|^2_{L^2(\Omega)} \leq \int_0^t (\omega_1(s) - \omega_2(s), \zeta_{\omega_1}(s) - \zeta_{\omega_2}(s))\, ds.$$

Then, using the Hölder inequality, we conclude that

$$\|\zeta_{\omega_1}(t) - \zeta_{\omega_2}(t)\|^2_{L^2(\Omega)} \leq c \int_0^t \|\omega_1(s) - \omega_2(s)\|^2_{L^2(\Omega)} + \|\zeta_{\omega_1}(s) - \zeta_{\omega_2}(s)\|^2_{L^2(\Omega)}\, ds.$$

Moreover, the Gronwall's inequality leads to the following estimate

$$\|\zeta_{\omega_1}(t) - \zeta_{\omega_2}(t)\|^2_{L^2(\Omega)} \leq c \int_0^t \|\omega_1(s) - \omega_2(s)\|^2_{L^2(\Omega)}\, ds.$$

Finally, the desired estimate (2.53) holds. □

**Claim 2.** To complete the proof of Theorem 2.3, we consider the following operator

$$\begin{aligned} &\Lambda\colon L^2(0, T; \mathcal{H} \times Q^* \times L^2(\Omega)) \to L^2(0, T; \mathcal{H} \times Q^* \times L^2(\Omega)) \\ &\Lambda(\eta, z, \omega) = (\Lambda_1(\eta, z, \omega), \Lambda_2(\eta, z), \Lambda_3(\eta, \omega)), \end{aligned} \tag{2.56}$$

where $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ are given for all $(\eta, z, \omega) \in L^2(0,T;\mathcal{H}\times Q^*\times L^2(\Omega))$ and $t\in[0,T]$ by

$$\langle \Lambda_1(\eta,z,\omega),\varepsilon(v)\rangle_{\mathcal{H}} = \langle \mathcal{B}(t,\varepsilon(u_\eta(t)),\zeta_\omega(t)) + \mathcal{P}^T E\varphi_{\eta,z}(t) - \mathcal{C}\theta_{\eta,z}(t),\varepsilon(v)\rangle_{\mathcal{H}}, \tag{2.57}$$

$$\langle \Lambda_2(\eta,\omega),\xi\rangle_{Q^*\times Q} = \langle \mathcal{N}\nabla\varphi_{\eta,z}(t),\xi\rangle_{\mathcal{H}}, \tag{2.58}$$

$$\Lambda_3(\eta,\omega) = \phi(\varepsilon(u_\eta(t)),\zeta_\omega(t)), \tag{2.59}$$

where $u_\eta$, $\varphi_{\eta,z}$, $\theta_{\eta,z}$ and $\xi_w$ are, respectively, the solution of Problems 2.4, 2.5, 2.6 and 2.7. Next, we state the following lemma which shows that the operator $\Lambda$ has a fixed point.

**Lemma 2.12.** *The operator* $\Lambda$ *defined by* (2.56)–(2.59) *has a unique fixed point.*

*Proof.* Consider $(\eta_1,z_1,\omega_1),(\eta_2,z_2,\omega_2)\in L^2(0,T;\mathcal{H}\times Q^*\times L^2(\Omega))$, from the definition of $\Lambda$, we get

$$\begin{aligned}
&\|\Lambda(\eta_1,z_1,\omega_1)(t)-\Lambda(\eta_2,z_2,\omega_2)(t)\|^2_{\mathcal{H}\times Q^*\times L^2(\Omega)}\\
&= \|\Lambda_1(\eta_1,z_1,\omega_1)(t)-\Lambda_1(\eta_2,z_2,\omega_2)(t)\|^2_{\mathcal{H}\times Q^*\times L^2(\Omega)}\\
&\quad + \|\Lambda_2(\eta_1,z_1)(t)-\Lambda_2(\eta_2,z_2)(t)\|^2_{\mathcal{H}\times Q^*} + \|\Lambda_3(\eta_1,\omega_1)(t)-\Lambda(\eta_2,\omega_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\\
&\le \|\mathcal{B}(t,\varepsilon(u_{\eta_1}(t)),\zeta_{\omega_1}(t))-\mathcal{B}(t,\varepsilon(u_{\eta_2}(t)),\zeta_{\omega_2}(t))\|^2_V + \|\mathcal{P}^T E\varphi_{\eta_1,z_1}(t)-\mathcal{P}^T E\varphi_{\eta_2,z_2}(t)\|^2_W\\
&\quad + \|\mathcal{C}\theta_{\eta_1,z_1}(t)-\mathcal{C}\theta_{\eta_2,z_2}(t)\|^2_Q + \|\mathcal{N}E\varphi_{\eta_1,z_1}(t)-\mathcal{N}E\varphi_{\eta_2,z_2}(t)\|^2_W\\
&\quad + \|\phi(\varepsilon(u_{\eta_1}(t)),\zeta_{\omega_1}(t))-\phi(\varepsilon(u_{\eta_2}(t)),\zeta_{\omega_2}(t))\|^2.
\end{aligned}$$

Using the relations (2.24), (2.25), (2.32), (2.34) and (2.37), we find that for all $t\in(0,T)$, we have

$$\begin{aligned}
&\|\Lambda(\eta_1,z_1,\omega_1)(t)-\Lambda(\eta_2,z_2,\omega_2)(t)\|^2_{\mathcal{H}\times Q^*\times L^2(\Omega)}\\
&\le \left(L^2_{\mathcal{B}}+L^2_{\phi}\right)\left(\|u_{\eta_1}(t)-u_{\eta_2}(t)\|^2_V + \|\zeta_{\omega_1}(t)-\zeta_{\omega_2}(t)\|^2_{L^2(\Omega)}\right)\\
&\quad + \left(L^2_{\mathcal{P}}+L^2_{\mathcal{N}}\right)\|\varphi_{\eta_1,z_1}(t)-\varphi_{\eta_2,z_2}(t)\|^2_W + L^2_{\mathcal{C}}\|\theta_{\eta_1,z_1}(t)-\theta_{\eta_2,z_2}(t)\|^2_Q.
\end{aligned}$$

Hence, by applying the previous lemmas, we deduce that

$$\begin{aligned}
&\|\Lambda(\eta_1,z_1,\omega_1)(t)-\Lambda(\eta_2,z_2,\omega_2)(t)\|^2_{\mathcal{H}\times Q^*\times L^2(\Omega)}\\
&\le c\int_0^t \|\eta_1(s)-\eta_2(s)\|^2 + \|z_1(s)-z_2(s)\|^2 + \|\omega_1(s)-\omega_2(s)\|^2\,ds.
\end{aligned}$$

Thus, for a.e. $t\in[0,T]$, there exists a constant $c>0$ such that

$$\begin{aligned}
&\|\Lambda(\eta_1,z_1,\omega_1)(t)-\Lambda(\eta_2,z_2,\omega_2)(t)\|_{\mathcal{H}\times Q^*\times L^2(\Omega)}\\
&\le c\int_0^t \|(\eta_1,z_1,\omega_1)-(\eta_2,z_2,\omega_2)\|^2_{V^*\times Q^*\times L^2(\Omega)}\,ds.
\end{aligned}$$

Finally, it comes from [25, Lemma 2.1] that operator $\Lambda$ has a unique fixed point. □

**Claim 3.** Let $(\eta^*, z^*, \omega^*) \in L^2(0,T;\mathcal{H}\times Q^*\times L^2(\Omega))$ be the unique fixed point of operator $\Lambda$.

*Proof.* We move to prove Theorem 2.3. For that, let $u = u_{\eta^*}$, $\varphi = \varphi_{\eta^*,z^*}$, $\theta = \theta_{\eta^*,z^*}$ and $\zeta = \zeta_{\omega^*}$ be solutions of Problems 2.4, 2.5, 2.6 and 2.7, respectively. Hence $(u,\varphi,\theta,\zeta)$ is a solution of Problem 2.2, and the uniqueness of the fixed point of $\Lambda$ leads to the uniqueness part of Theorem. □

## 3. Numerical analysis of Problem 2.1

In this section, we present a fully discrete approach for Problem 2.2 and we derive an approximate solution error estimate. To start, we use the finite-difference method to approximate the derivative of function. We consider the uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$ with a time step-size $k = T/N$, and for each continuous function $v$, we denote

$$v(t_n) = v_n, \quad \delta v_n = \frac{v_n - v_{n-1}}{k}.$$

Moreover, we apply the finite element method for the spatial discretization. Let $\Omega$ be a polygonal domain, then we consider a regular family of partitions $\{\mathcal{T}^h\}$ of $\overline{\Omega}$ into triangles that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3$ and $\Gamma_1\cup\Gamma_2 = \Gamma_a\cup\Gamma_b$. Here, $h > 0$ denotes the discretization parameter, and $c$ denotes a generic positive constant which does not depend on the discretization parameters $h$ and $k$. To approximate the spaces $V$, $W$ and $Q$, respectively, we introduce the following linear finite element spaces corresponding to $\mathcal{T}^h$.

$$\begin{aligned}
V^h &= \big\{v^h \in C(\overline{\Omega}) \mid v^h_{|_T} \in \mathbb{P}_1(T)^d \text{ for } T\in\mathcal{T}^h, v^h = 0 \text{ on } \Gamma_1\big\},\\
W^h &= \big\{\psi^h \in C(\overline{\Omega}) \mid \psi^h_{|_T} \in \mathbb{P}_1(T)^d \text{ for } T\in\mathcal{T}^h, \psi^h = 0 \text{ on } \Gamma_1\big\},\\
Q^h &= \big\{\theta^h \in C(\overline{\Omega}) \mid \theta^h_{|_T} \in \mathbb{P}_1(T)^d \text{ for } T\in\mathcal{T}^h, \theta^h = 0 \text{ on } \Gamma_1\big\}.
\end{aligned}$$

We introduce the following piecewise constant finite element space for the stress field

$$\mathcal{H}^h = \big\{\tau^h \in \mathcal{H} \mid \tau^h_{|_T} \in \mathbb{R}^{d\times d} \text{ for } T\in\mathcal{T}^h\big\},$$

and the following linear finite element space for the damage field

$$Z^h = \big\{\xi^h \in C(\overline{\Omega}) \mid \xi^h_{|_T} \in \mathbb{P}_1(T)^d \text{ for } T\in\mathcal{T}^h\big\}.$$

Then, we define the following constrained subset of $Z^h$:

$$K^h = \big\{\xi^h \in Z^h \mid \xi^h_{|_T} \in [0,T] \text{ for } T\in\mathcal{T}^h\big\}.$$

Let $u_0^{hk} = u_0^h \in V^h$, $\theta_0^{hk} = \theta_0^h \in Q^h$ and $\zeta_0^{hk} = \zeta_0^h \in Z^h$ be an appropriate approximation of the initial conditions $u_0$, $\theta_0$ and $\zeta_0$, respectively, i.e., such that

$$\|u_0 - u_0^h\|_V \le ch, \quad \|\theta_0 - \theta_0^h\|_Q \le ch, \quad \|\zeta_0 - \zeta_0^h\|_Z \le ch. \tag{3.1}$$

Hence, the discrete scheme for Problem 2.2 is given as follows.

**Problem 3.1.** Find a displacement $\{u_n^{hk}\}_{n=0}^N \subset V^h$, an electric potential $\{\varphi_n^{hk}\}_{n=0}^N \subset W^h$, a temperature $\{\theta_n^{hk}\}_{n=0}^N \subset Q^h$ and a damage field $\{\zeta_n^{hk}\}_{n=0}^N \subset K^h$ such that for $n = 1, \dots, N$, we have

$$\begin{aligned}
&\langle \mathcal{A}\varepsilon(w_n^{hk}), \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} + \langle \mathcal{B}(\varepsilon(u_n^{hk}), \zeta_n^{hk}) + \mathcal{P}^T\nabla\varphi_n^{hk} - \mathcal{C}\theta_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} \\
&+ \int_{\Gamma_3} \big(j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\tau}^h - w_{n\tau}^{hk})\big)\, da \\
&\ge \langle f_n, v_n^h - w_n^{hk}\rangle, \quad \forall\, v_n^h \in V^h,
\end{aligned}\tag{3.2}$$

$$\begin{aligned}
&\langle \beta\nabla\varphi_n^{hk} - \mathcal{P}\varepsilon(u_n^{hk}) - \mathcal{G}\theta_n^{hk}, \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}} \\
&+ \int_{\Gamma_3} h_e(u_{n\nu}^{hk}) j_e^0(\varphi_n^{hk} - \varphi_0; \psi_n^h - \varphi_n^{hk})\, da \\
&\ge \langle q(t), \psi_n^h - \varphi_n^{hk}\rangle_W, \quad \forall\, w_n^h \in W^h,
\end{aligned}\tag{3.3}$$

$$\begin{aligned}
&\langle \delta\theta_n^{hk}, \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}} + \langle \mathcal{K}\nabla\theta_n^{hk}, \nabla(\lambda_n^h - \theta_n^{hk})\rangle_{\mathcal{H}} \\
&- \langle \mathcal{M}\varepsilon(u_n^{hk}) + \mathcal{N}\nabla\varphi_n^{hk}, \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}} + \int_{\Gamma_3} j_\theta^0(\theta_n^{hk}; \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}})\, da \\
&\ge \langle h_n, \lambda_n^h - \theta_n^{hk}\rangle_Q, \quad \forall\, \lambda_n^h \in Q^h,
\end{aligned}\tag{3.4}$$

$$\langle \delta\zeta_n^{hk}, \xi_n^h - \zeta_n^{hk}\rangle_{\mathcal{H}} + a(\zeta_n^{hk}, \xi_n^h - \zeta_n^{hk}) \ge \langle \phi(\varepsilon(u_n^{hk}), \zeta_n^{hk}), \xi_n^h - \zeta_n^{hk}\rangle_Z, \quad \forall\, \xi_n^h \in K^h,$$

$$u_0^{hk} = u_0^h, \quad \theta_0^{hk} = \theta_0^h \quad \text{and} \quad \zeta_0^{hk} = \zeta_0^h.$$

Here the sequences $\{u_n^{hk}\}_{n=0}^N$ and $\{w_n^{hk}\}_{n=0}^N$ are related by the following equalities

$$w_n^{hk} = \delta u_n^{hk} \quad \text{and} \quad u_n^{hk} = u_0^h + k\sum_{j=1}^n w_j^{hk}, \quad n = 1, \dots, N.$$

From assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_{13})$, we derive by using the same arguments as for Problem 2.2, that Problem 3.1 has a unique solution $(u_n^{hk}, \varphi_n^{hk}, \theta_n^{hk}, \zeta_n^{hk}) \subset V^* \times W^h \times Q^* \times K^h$. For error estimations, it will be derived using the Céa inequalities.

**Theorem 3.2.** *Let assumption of Theorem* 2.3 *still hold and the condition* (3.1) *satisfied. Consider* $(u, \varphi, \theta, \zeta)$ *the solution to Problem* 2.2 *and* $(u_n^{hk}, w_n^{hk}, \varphi_n^{hk}, \theta_n^{hk}, \zeta_n^{hk})$ *the solution* *to Problem* 3.1. *Then for* $n = 1, \dots, N$, *the following error estimate hold:*

(3.5)
$$\begin{aligned}
&\max_{1\le n\le N}\big\{\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2\|\theta_n - \theta_n^{hk}\|_Q^2 + \|\zeta_n - \zeta_n^{hk}\|_Z^2\big\}\\
&+ k\sum_{n=1}^{N}|\zeta_n - \zeta_n^{hk}|_Z^2\\
&\le c\max_{1\le n\le N}\big\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \psi_n^h\|_W^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)}\big\}\\
&+ c\sum_{n=1}^{N}\|\theta_n - \lambda_n^h\|_Q^2 + \|\theta_n - \lambda_n^h\|_{L^2(\Gamma_3)} + c\sum_{n=1}^{N-1}\|(\theta_n - \lambda_n^h) - (\theta_{n+1} - \lambda_{n+1}^h)\|\\
&+ c\big(\|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2\big) + c(h^2 + k^2).
\end{aligned}$$

*Proof.* First, the following equality holds:

$$\begin{aligned}
&\langle \mathcal{A}\varepsilon(w_n) - \mathcal{A}\varepsilon(w_n^{hk}), \varepsilon(w_n - w_n^{hk})\rangle_{\mathcal{H}}\\
&= \langle \mathcal{A}\varepsilon(w_n) - \mathcal{A}\varepsilon(w_n^{hk}), \varepsilon(w_n - v_n^h)\rangle_{\mathcal{H}} + \langle \mathcal{A}\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}}\\
&\quad + \langle \mathcal{A}\varepsilon(w_n), \varepsilon(w_n - w_n^{hk})\rangle_{\mathcal{H}} + \langle \mathcal{A}\varepsilon(w_n^{hk}), \varepsilon(w_n^{hk}) - \varepsilon(v_n^h)\rangle_{\mathcal{H}}.
\end{aligned} \tag{3.6}$$

Furthermore, by taking $t = t_n$ and $v = w_n^{hk}$ in the inequality (2.38), we get

$$\begin{aligned}
&\langle \mathcal{A}\varepsilon(w_n), \varepsilon(w_n - w_n^{hk})\rangle_{\mathcal{H}}\\
&\le \langle \mathcal{B}(\varepsilon(u_n), \zeta_n), \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} + \langle \mathcal{P}^T\nabla\varphi_n, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} - \langle \mathcal{C}\theta_n, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}}\\
&\quad + \int_{\Gamma_3} j_\nu^0(w_{n\nu}; w_{n\nu}^{hk} - w_{n\nu}) + j_\tau^0(w_{n\tau}; w_{n\nu}^{hk} - w_{n\nu})\,da + \langle f_n, w_n - w_n^{hk}\rangle_V.
\end{aligned} \tag{3.7}$$

Remembering the inequality (3.2), it follows from the previous inequality that

(3.8)
$$\begin{aligned}
&\langle \mathcal{A}\varepsilon(w_n^{hk}), \varepsilon(w_n^{hk} - v_n^h)\rangle_{\mathcal{H}}\\
&\le \langle \mathcal{B}(\varepsilon(u_n^{hk}), \zeta_n^{hk}), \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} + \langle \mathcal{P}^T\nabla\varphi_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} - \langle \mathcal{C}\theta_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}}\\
&+ \int_{\Gamma_3} j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\,da + \langle f_n, w_n^{hk} - v_n^h\rangle_V.
\end{aligned}$$

We now combine hypothesis (2.23) and inequalities (3.6)–(3.8) to deduce

$$\begin{aligned}
&\alpha_{\mathcal{A}}\|w_n - w_n^{hk}\|_V^2\\
&\le \langle \mathcal{A}\varepsilon(w_n) - \mathcal{A}\varepsilon(w_n^{hk}), \varepsilon(w_n - v_n^h)\rangle_{\mathcal{H}} + \langle \mathcal{A}\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} + \langle f_n, w_n - v_n^h\rangle_V\\
&\quad + \langle \mathcal{B}(\varepsilon(u_n), \zeta_n), \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} + \langle \mathcal{B}(\varepsilon(u_n^{hk}), \zeta_n^{hk}), \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}}\\
&\quad + \langle \mathcal{P}^T\nabla\varphi_n, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} + \langle \mathcal{P}^T\nabla\varphi_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} - \langle \mathcal{C}\theta_n, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}}\\
&\quad - \langle \mathcal{C}\theta_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} + \int_{\Gamma_3} j_\nu^0(w_{n\nu}; w_{n\nu}^{hk} - w_{n\nu}) + j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\,da
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Gamma_3} j_\tau^0(w_{n\tau}; w_{n\nu}^{hk}-w_{n\nu}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\,da\\
&\le \langle \mathcal{A}\varepsilon(w_n)-\mathcal{A}\varepsilon(w_n^{hk}), \varepsilon(w_n-v_n^h)\rangle_{\mathcal{H}} + \langle \mathcal{A}\varepsilon(w_n), \varepsilon(v_n^h-w_n)\rangle_{\mathcal{H}}\\
&\quad+\langle \mathcal{B}(\varepsilon(u_n^{hk}),\zeta_n^{hk}), \varepsilon(v_n^h-w_n)\rangle_{\mathcal{H}} + \langle \mathcal{B}(\varepsilon(u_n),\zeta_n)-\mathcal{B}(\varepsilon(u_n^{hk}),\zeta_n^{hk}), \varepsilon(w_n^{hk}-w_n)\rangle_{\mathcal{H}}\\
&\quad+\langle \mathcal{P}^T\nabla\varphi_n-\mathcal{P}^T\nabla\varphi_n^{hk}, \varepsilon(w_n^{hk}-w_n)\rangle_{\mathcal{H}} + \langle \mathcal{P}^T\nabla\varphi_n^{hk}, \varepsilon(v_n^h-w_n)\rangle_{\mathcal{H}}\\
&\quad-\langle \mathcal{C}\theta_n-\mathcal{C}\theta_n^{hk}, \varepsilon(w_n^{hk}-w_n)\rangle_{\mathcal{H}} - \langle \mathcal{C}\theta_n^{hk}, \varepsilon(v_n^h-w_n)\rangle_{\mathcal{H}} + \langle f_n, w_n-v_n^h\rangle_V\\
&\quad+\int_{\Gamma_3} j_\nu^0(w_{n\nu}; w_{n\nu}^{hk}-w_{n\nu}) + j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\,da\\
&\quad+\int_{\Gamma_3} j_\tau^0(w_{n\tau}; w_{n\nu}^{hk}-w_{n\nu}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\,da\\
&\quad+\int_{\Gamma_3} j_\nu^0(w_{n\nu}^{hk}; v_n^h-w_n)\,da + \int_{\Gamma_3} j_\tau^0(w_{n\tau}^{hk}; v_n^h-w_{n\nu})\,da.
\end{aligned}
$$

Next, we use hypotheses (2.22), (2.24), (2.25), (2.28), (2.29) and (2.32) to find

$$
\begin{aligned}
&\alpha_{\mathcal{A}}\|w_n-w_n^{hk}\|_V^2\\
&\le L_{\mathcal{A}}\|w_n-w_n^{hk}\|_V\|w_n-v_n^h\|_V + L_{\mathcal{P}}\|\varphi_n-\varphi_n^{hk}\|_W\big(\|w_n-w_n^{hk}\|_V+\|w_n-v_n^h\|_V\big)\\
&\quad+L_{\mathcal{B}}\big(\|u_n-u_n^{hk}\|_V+\|\zeta_n-\zeta_n^{hk}\|_Z\big)\big(\|w_n-w_n^{hk}\|_V+\|w_n-v_n^h\|_V\big)+S_1(u_n,\varphi_n,\theta_n)\\
&\quad+I_1(\varphi_n^{hk},\varphi_n,\psi_n)+L_{\mathcal{M}}\|\theta_n-\theta_n^{hk}\|_Q\big(\|w_n-w_n^{hk}\|_V+\|w_n-v_n^h\|_V\big)\\
&\quad+c_0^2\sqrt{\text{meas}(\Gamma_3)}(\alpha_{j\nu}+\alpha_{j\tau})\|w_n-w_n^{hk}\|_V^2,
\end{aligned}
$$

where the quantities $S_1$ and $I_1$ are given by

$$
\begin{aligned}
S_1(u_n,\varphi_n,\theta_n) &= \langle \mathcal{A}\varepsilon(w_n), \varepsilon(v_n^h-w_n)\rangle_{\mathcal{H}} + \langle \mathcal{B}(\varepsilon(u_n),\zeta_n), \varepsilon(v_n^h-w_n)\rangle_{\mathcal{H}}\\
&\quad+\langle \mathcal{P}^T\nabla\varphi_n, \varepsilon(v_n^h-w_n)\rangle_{\mathcal{H}} - \langle \mathcal{C}\theta_n, \varepsilon(v_n^h-w_n)\rangle_{\mathcal{H}} + \langle f_n, w_n-v_n^h\rangle_V
\end{aligned}
$$

and

$$
I_1(w_n^{hk},w_n,v_n^h) = \int_{\Gamma_3} j_\nu^0(w_{n\nu}^{hk}; v_n^h-w_n)\,da + \int_{\Gamma_3} j_\tau^0(w_{n\tau}^{hk}; v_n^h-w_{n\nu})\,da.
$$

We further assume that $j_\nu(x,\cdot)$ and $j_\tau(x,\cdot)$ are $c$-Locally Lipschtiz on $\mathbb{R}$ and $\mathbb{R}^d$ respectively for a.e. $x\in\Gamma_3$, where the Lipschitiz constant $c>0$ is independent of $x$. Hence, we have

$$
j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h-w_{n\nu}) \le c\|w_n-v_n^h\|_{L^2(\Gamma_3)} \quad\text{and}\quad j_\tau^0(w_{n\tau}^{hk}; v_{n\tau}^h-w_{n\tau}) \le c\|w_n-v_n^h\|_{L^2(\Gamma_3)}.
$$

Then, it should be concluded that

$$
I_1(w_n^{hk},w_n,v_n^h) \le c\|w_n-v_n^h\|_{L^2(\Gamma_3)}.
$$

We next multiply (2.5) by an arbitrary element $v\in V$ to find that

$$
\int_\Omega \sigma\nu\cdot\varepsilon(v)\,da - \int_\Omega f\cdot v\,da = \int_{\Gamma_3} \sigma\cdot v\,da \quad\text{and}\quad \sigma\nu\in L^2(\Gamma_3;\mathbb{R}^d).
$$

Thus, we conclude that the following inequalities hold:

$$S_1(u_n,\varphi_n,\theta_n)=\int_{\Gamma_3}\sigma\nu\cdot(v_n^h-w_n)\,da\le c\|\sigma\|\|w_n-v_n^h\|_{L^2(\Gamma_3)}\le c\|w_n-v_n^h\|_{L^2(\Gamma_3)}. \tag{3.9}$$

Moreover, by applying Cauchy inequality with $\epsilon>0$, we find

$$\begin{aligned}&\big(\alpha_{\mathcal{A}}-c_0^2\sqrt{\text{meas}(\Gamma_3)}(\alpha_{j\nu}+\alpha_{j\tau})-5\epsilon\big)\|w_n-w_n^{hk}\|_V^2\\ &\le c\big\{\|w_n-v_n^h\|_V^2+\|u_n-u_n^{hk}\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_W^2\\ &\quad+\|\theta_n-\theta_n^{hk}\|_Q^2+\|\zeta_n-\zeta_n^{hk}\|_{Z_0}^2+\|w_n-v_n^h\|_{L^2(\Gamma_3)}\big\}.\end{aligned} \tag{3.10}$$

Hence, using same arguments as in [13], we deduce

$$\|u_n-u_n^{hk}\|_V^2\le c(h^2+k^2)+ck\sum_{i=1}^{n}\|w_i-w_i^{hk}\|_V^2, \tag{3.11}$$

$$\|\zeta_n-\zeta_n^{hk}\|_{Z_0}^2+k\sum_{i=1}^{n}|\zeta_i-\zeta_i^{hk}|_Z^2\le c(h^2+k^2)+ck\sum_{i=1}^{n-1}\|u_i-u_i^{hk}\|_V^2. \tag{3.12}$$

So, it follows from the two previous inequalities (3.11) and (3.12) that

$$\|\zeta_n-\zeta_n^{hk}\|_{Z-0}^2\le c(h^2+k^2)+ck\sum_{i=1}^{n-1}\|w_i-w_i^{hk}\|_V^2. \tag{3.13}$$

Next, we combine (3.10), (3.11) and (3.13) to find

$$\begin{aligned}\|w_n-w_n^{hk}\|_V^2&\le c\big\{\|w_n-v_n^h\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_W^2+\|\theta_n-\theta_n^{hk}\|_Q^2+\|w_n-v_n^h\|_{L^2(\Gamma_3)}\big\}\\ &\quad+c(h^2+k^2)+ck\sum_{i=1}^{n-1}\|w_i-w_i^{hk}\|_V^2.\end{aligned} \tag{3.14}$$

Then, by applying the Gronwall inequality, the inequality (3.14) leads to

$$\begin{aligned}\|w_n-w_n^{hk}\|_V^2&\le c\big\{\|w_n-v_n^h\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_W^2+\|\theta_n-\theta_n^{hk}\|_Q^2+\|w_n-v_n^h\|_{L^2(\Gamma_3)}\big\}\\ &\quad+c(h^2+k^2).\end{aligned} \tag{3.15}$$

We now combine (3.11), (3.12) and (3.15) to get that there exist a constant $c>0$ such that

$$\begin{aligned}&\|w_n-w_n^{hk}\|_V^2+\|u_n-u_n^{hk}\|_V^2+\|\zeta_n-\zeta_n^{hk}\|_{Z_0}^2+k\sum_{i=1}^{n}|\zeta_i-\zeta_i^{hk}|_Z^2\\ &\le c\big\{\|w_n-v_n^h\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_W^2+\|\theta_n-\theta_n^{hk}\|_Q^2+\|w_n-v_n^h\|_{L^2(\Gamma_3)}\big\}\\ &\quad+c(h^2+k^2)+ck\sum_{i=1}^{n}\big(\|w_i-w_i^{hk}\|_V^2+\|u_i-u_i^{hk}\|_V^2+|\zeta_i-\zeta_i^{hk}|_Z^2\big).\end{aligned}$$

To simplify the notations, let us consider

$$e_n = \|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_{Z_0}^2 + k\sum_{i=1}^{n}|\zeta_i - \zeta_i^{hk}|_Z^2,$$

$$g_n = \|w_n - v_n^h\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + h^2 + k^2.$$

Then, we find that there exists a constant $c > 0$ such that

$$e_n \le c g_n + c\sum_{j=}^{n} e_j \quad \text{with } c > 0.$$

Hence, using again Gronwall inequality, we get

$$\begin{aligned}
&\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_{Z_0}^2 + k\sum_{i=1}^{n}|\zeta_i - \zeta_i^{hk}|_Z^2 \\
&\le c\big\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}\big\} \\
&\quad + c(h^2 + k^2).
\end{aligned} \tag{3.16}$$

Moreover, it comes from the assumption (2.27) that

$$\begin{aligned}
\alpha_\beta\|\varphi_n - \varphi_n^{hk}\|_W^2 \le{}& \langle \beta\nabla\varphi_n - \beta\nabla\varphi_n^{hk}, \nabla(\varphi_n - \psi_n^h)\rangle_{\mathcal{H}} + \langle \beta\nabla\varphi_n, \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} \\
&+ \langle \beta\nabla\varphi_n, \nabla(\varphi_n - \varphi_n^{hk})\rangle_{\mathcal{H}} + \langle \beta\nabla\varphi_n^{hk}, \nabla(\varphi_n^{hk} - \psi_n^h)\rangle_{\mathcal{H}}.
\end{aligned} \tag{3.17}$$

Taking $t = t_n$ and $\psi = \varphi_n^{hk}$ in the relation (2.39) to get

$$\begin{aligned}
\langle \beta\nabla\varphi_n, \nabla(\varphi_n - \varphi_n^{hk})\rangle_{\mathcal{H}} \le{}& \langle \mathcal{P}\varepsilon(u_n), \nabla(\varphi_n - \varphi_n^{hk})\rangle_{\mathcal{H}} + \langle \mathcal{G}\theta_n, \nabla(\varphi_n - \varphi_n^{hk})\rangle_{\mathcal{H}} \\
&+ \int_{\Gamma_3} h_e(u_{n\nu}) j_e^0(\varphi_n - \varphi_0; \varphi_n^{hk} - \varphi_n)\, da \\
&+ \langle q_n, \varphi_n - \varphi_n^{hk}\rangle_W.
\end{aligned} \tag{3.18}$$

Remembering the inequality (3.3), we obtain

$$\begin{aligned}
\langle \beta\nabla\varphi_n^{hk}, \nabla(\varphi_n^{hk} - \psi_n^h)\rangle_{\mathcal{H}} \le{}& \langle \mathcal{P}\varepsilon(u_n^{hk}), \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}} + \langle \mathcal{G}\theta_n^{hk}, \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}} \\
&+ \int_{\Gamma_3} h_e(u_{n\nu}^{hk}) j_e^0(\varphi_n^{hk} - \varphi_0; \psi_n^h - \varphi_n^{hk})\, da \\
&+ \langle q_n, \varphi_n^{hk} - \psi_n^h\rangle_W.
\end{aligned} \tag{3.19}$$

Then, we combine the previous inequalities (3.17)–(3.19) to find

$$\begin{aligned}
&\alpha_\beta\|\varphi_n - \varphi_n^{hk}\|_W^2 \\
&\le \langle \beta\nabla\varphi_n - \beta\nabla\varphi_n^{hk}, \nabla(\varphi_n - \psi_n^h)\rangle_{\mathcal{H}} + \langle \beta\nabla\varphi_n, \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} - \langle \mathcal{P}\varepsilon(u_n), \nabla(\varphi_n^{hk} - \varphi_n)\rangle_{\mathcal{H}} \\
&\quad - \langle \mathcal{P}\varepsilon(u_n^{hk}), \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}} - \langle \mathcal{G}\theta_n, \nabla(\varphi_n^{hk} - \varphi_n)\rangle_{\mathcal{H}} + \langle \mathcal{G}\theta_n^{hk}, \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}}
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Gamma_3}\overline{h}_e j_e^0(\varphi_n-\varphi_0;\varphi_n^{hk}-\varphi_n)\,da+\int_{\Gamma_3}\overline{h}_e j_e^0(\varphi_n^{hk}-\varphi_0;\psi_n^h-\varphi_n^{hk})\,da+\langle q_n,\varphi_n-\psi_n^h\rangle_W\\
&\le\langle\beta\nabla\varphi_n-\beta\nabla\varphi_n^{hk},\nabla(\varphi_n-\psi_n^h)\rangle_{\mathcal{H}}+\langle\beta\nabla\varphi_n,\nabla(\psi_n^h-\varphi_n)\rangle_{\mathcal{H}}\\
&+\langle\mathcal{P}\varepsilon(u_n)-\mathcal{P}\varepsilon(u_n^{hk}),\nabla(\varphi_n^{hk}-\varphi_n)\rangle_{\mathcal{H}}+\langle\mathcal{P}\varepsilon(u_n^{hk}),\nabla(\psi_n^h-\varphi_n)\rangle_{\mathcal{H}}\\
&+\langle\mathcal{G}\theta_n-\mathcal{G}\theta_n^{hk},\nabla(\varphi_n^{hk}-\varphi_n)\rangle_{\mathcal{H}}+\langle\mathcal{G}\theta_n^{hk},\nabla(\psi_n^h-\varphi_n)\rangle_{\mathcal{H}}+\langle q_n,\varphi_n-\psi_n^h\rangle_W\\
&+\int_{\Gamma_3}\overline{h}_e j_e^0(\varphi_n-\varphi_0;\varphi_n^{hk}-\varphi_n)+\overline{h}_e j_e^0(\varphi_n^{hk}-\varphi_0;\varphi_n-\varphi_n^{hk})+\overline{h}_e j_e^0(\varphi_n^{hk}-\varphi_0;\psi_n^h-\varphi_n)\,da.
\end{aligned}
$$

We now use (2.25), (2.26), (2.30) and (2.35) to deduce

$$
\begin{aligned}
\alpha_\beta\|\varphi_n-\varphi_n^{hk}\|_W^2\le{}& L_\beta\|\varphi_n-\varphi_n^{hk}\|_W\|\varphi_n-\psi_n^h\|_W+L_{\mathcal{P}}\|u_n-u_n^{hk}\|_V\|\varphi_n-\varphi_n^{hk}\|_W\\
&+L_{\mathcal{P}}\|u_n-u_n^{hk}\|_V\|\psi_n^h-\varphi_n\|_W+L_{\mathcal{G}}\|\theta_n-\theta_n^{hk}\|_Q\|\varphi_n-\varphi_n^{hk}\|_W\\
&+L_{\mathcal{G}}\|\theta_n-\theta_n^{hk}\|_Q\|\psi_n^h-\varphi_n\|_W+\overline{h}_e\alpha_{j_e}c_0^2\|\varphi_n-\varphi_n^{hk}\|_W^2\\
&+S_2(u_n,\varphi_n,\theta_n)+I_2(\varphi_n^{hk},\varphi_n,\psi_n),
\end{aligned}
$$

where the quantities $S_2$ and $I_2$ are defined as follows:

$$
\begin{aligned}
S_2(u_n,\varphi_n,\theta_n)={}&\langle\beta\nabla\varphi_n,\nabla(\psi_n^h-\varphi_n)\rangle_{\mathcal{H}}-\langle\mathcal{P}\varepsilon(u_n),\nabla(\psi_n^h-\varphi_n)\rangle_{\mathcal{H}}\\
&-\langle\mathcal{G}\theta_n,\nabla(\psi_n^h-\varphi_n)\rangle_{\mathcal{H}}+\langle q_n,\varphi_n-\psi_n^h\rangle_W
\end{aligned}
$$

and

$$
I_2(\varphi_n^{hk},\varphi_n,\psi_n)=\int_{\Gamma_3}\overline{h}_e j_e^0(\varphi_n^{hk}-\varphi_0;\psi_n^h-\varphi_n)\,da.
$$

We now multiply the equation (2.6) by an arbitrary element $\psi_n^h\in W$ to get

$$
\int_\Omega D\cdot\nu(\psi_n^h-\varphi_n)\,dx-\int_\Omega q_0(\psi_n^h-\varphi_n)\,dx+\int_{\Gamma_b}(\psi_n^h-\varphi_n)\,d\Gamma=\int_{\Gamma_3}D\cdot\nu(\psi_n^h-\varphi_n)\,d\Gamma.
$$

We have $D\cdot\nu\in L^2(0,T;L^2(\Gamma_3))$, then we find

$$
S_2(u_n,\varphi_n,\theta_n)\le c\|\varphi_n-\psi_n^h\|_{L^2(\Gamma_3)}.
$$

We assume that $j_e(x,\cdot)$ is locally Lipschitz on $\mathbb{R}$ with Lipschitz constant $c>0$ independent of $x$. Thus,

$$
I_2(\varphi_n^{hk},\varphi_n,\psi_n)\le c\|\varphi_n-\psi_n^h\|_{L^2(\Gamma_3)}.
$$

So, applying the Cauchy inequality with $c>0$, we obtain the following estimation

$$
\|\varphi_n-\varphi_n^{hk}\|_W^2\le c\big\{\|\varphi_n-\psi_n^h\|_W^2+\|u_n-u_n^{hk}\|_V^2+\|\theta_n-\theta_n^{hk}\|_Q^2+\|\varphi_n-\psi_n^h\|_{L^2(\Gamma_3)}\big\}.
$$

On the other hand, it comes from assumption (2.36) that

$$
\begin{aligned}
\alpha_{\mathcal{K}}\|\theta_n-\theta_n^{hk}\|_Q^2\le{}&\langle\mathcal{K}\nabla\theta_n-\mathcal{K}\nabla\theta_n^{hk},\nabla(\theta_n-\lambda_n^h)\rangle_{\mathcal{H}}+\langle\mathcal{K}\nabla\theta_n,\nabla(\lambda_n^h-\theta_n)\rangle_{\mathcal{H}}\\
&+\langle\mathcal{K}\nabla\theta_n,\nabla(\theta_n-\theta_n^{hk})\rangle_{\mathcal{H}}+\langle\mathcal{K}\nabla\theta_n^{hk},\nabla(\theta_n^{hk}-\lambda_n^h)\rangle_{\mathcal{H}}.
\end{aligned}
$$

Taking $t = t_n$ and $\lambda = \theta_n^{hk}$ in the inequality (2.40), we get

$$\langle \mathcal{K}\nabla\theta_n, \nabla(\theta_n - \theta_n^{hk})\rangle_{\mathcal{H}} \le \langle \dot{\theta}_n, \theta_n^{hk} - \theta_n\rangle_{\mathcal{H}} - \langle \mathcal{M}\varepsilon(u_n), \theta_n^{hk} - \theta_n\rangle_{\mathcal{H}} + \langle \mathcal{N}\nabla\varphi_n, \theta_n^{hk} - \theta_n\rangle_{\mathcal{H}} + \int_{\Gamma_3} j_\theta^0(\theta_n; \theta_n^{hk} - \theta_n)\, da + \langle h_n, \theta_n - \theta^{hk}\rangle_Q.$$

Furthermore, it follows from (3.4) that

$$\begin{aligned} &\langle \mathcal{K}\nabla\theta_n^{hk}, \nabla(\theta_n^{hk} - \lambda_n^h)\rangle_{\mathcal{H}} \\ &\le \langle \delta\theta_n^{hk}, \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}} - \langle \mathcal{M}\varepsilon(u_n^{hk}), \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}} + \langle \mathcal{N}\nabla\varphi_n^{hk}, \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} j_\theta^0(\theta_n^{hk}; \lambda_n^h - \theta_n^{hk})\, da + \langle h_n, \theta_n^{hk} - \lambda_n^h\rangle_Q. \end{aligned}$$

Thus, from the previous inequalities, we conclude

$$\begin{aligned} &\alpha_{\mathcal{K}}\|\theta_n - \theta_n^{hk}\|_Q^2 \\ &\le \langle \mathcal{K}\nabla\theta_n - \mathcal{K}\nabla\theta_n^{hk}, \nabla(\theta_n - \lambda_n^h)\rangle_{\mathcal{H}} + \langle \mathcal{K}\nabla\theta_n, \nabla(\lambda_n^h - \theta_n)\rangle_{\mathcal{H}} + \langle \delta\theta_n - \delta\theta_n^{hk}, \theta_n^{hk} - \theta_n\rangle_{\mathcal{H}} \\ &\quad + \langle \delta\theta_n^{hk}, \lambda_n^h - \theta_n\rangle_{\mathcal{H}} - \langle \mathcal{M}\varepsilon(u_n) - \mathcal{M}\varepsilon(u_n^{hk}), \theta_n^{hk} - \theta_n\rangle_{\mathcal{H}} + \langle \mathcal{M}\varepsilon(u_n^{hk}), \lambda_n^h - \theta_n\rangle_{\mathcal{H}} \\ &\quad - \langle \mathcal{N}\nabla\varphi_n - \mathcal{N}\nabla\varphi_n^{hk}, \theta_n^{hk} - \theta_n\rangle_{\mathcal{H}} + \langle \mathcal{N}\nabla\varphi_n, \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}} + \langle h_n, \theta_n - \lambda_n^h\rangle_Q \\ &\quad + \int_{\Gamma_3} j_\theta^0(\theta_n; \theta_n^{hk} - \theta_n) + j_\theta^0(\theta_n^{hk}; \theta_n - \theta_n^{hk})\, da + \int_{\Gamma_3} j_\theta^0(\theta_n^{hk}; \lambda_n^h - \theta_n)\, da. \end{aligned}$$

So, we can deduce the following estimation

$$\begin{aligned} &\alpha_{\mathcal{K}}\|\theta_n - \theta_n^{hk}\|_Q^2 + \langle \delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\rangle \\ &\le c\{\|\theta_n - \lambda_n^h\|_Q^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_Q^2\} \\ &\quad + \langle \delta\theta_n^{hk} - \delta\theta_n, \lambda_n^h - \theta_n\rangle_{\mathcal{H}} + S_3(u_n, \varphi_n, \theta_n) + I_3(\theta_n^{hk}, \theta_n, \lambda_n^h), \end{aligned} \tag{3.20}$$

where the quantities $S_3$ and $I_3$ are given by the expressions below

$$S_3(u_n, \varphi_n, \theta_n) = \langle \dot{\theta}_n, \lambda_n^h - \theta_n\rangle_{\mathcal{H}} + \langle \mathcal{K}\nabla\theta_n, \nabla(\lambda_n^h - \theta_n)\rangle_{\mathcal{H}} - \langle \mathcal{M}\varepsilon(u_n), \lambda_n^h - \theta_n\rangle_{\mathcal{H}} + \langle \mathcal{N}\nabla\varphi_n, \lambda_n^h - \theta_n\rangle_{\mathcal{H}} + \langle h_n, \theta_n - \lambda_n^h\rangle_{\mathcal{H}}$$

and

$$I_3(\theta_n^{hk}, \theta, \lambda_n^h) = \int_{\Gamma_3} j_\theta^0(\theta_n^{hk}; \lambda_n^h - \theta_n)\, da.$$

In the same way as for (3.9), we get that for $\lambda_n^h \in Q^h$, we have

$$\begin{aligned} &\int_\Omega \mathcal{K}\nabla\theta \cdot \nu(\lambda_n^h - \theta_n)\, dx - \int_\Omega h_0(\lambda_n^h - \theta_n)\, dx + \int_{\Gamma_2} (\lambda_n^h - \theta_n)\, d\Gamma \\ &= \int_{\Gamma_3} \mathcal{K}\nabla\theta \cdot \nu \cdot \nu(\lambda_n^h - \theta_n)\, d\Gamma. \end{aligned}$$

Then, since we have $\mathcal{K}\nabla\theta\cdot\nu \in L^2(0,T;L^2(\Gamma_3))$, we can deduce that

$$S_3(u_n,\varphi_n,\theta_n) \le c\|\theta_n-\lambda_n^h\|_{L^2(\Gamma_3)}. \tag{3.21}$$

We assume additionally that $j_\theta(x,\cdot\,)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x\in\Gamma_3$ for Lipschitz-constant $c>0$ independent of $x$. Then, the following estimations holds:

$$I_3(\theta_n^{hk},\theta,\lambda_n^h) \le c\|\theta_n-\lambda_n^h\|_{L^2(\Gamma_3)}. \tag{3.22}$$

We next use the inequalities (3.20), (3.21) and (3.22) to find

$$\begin{aligned}
&\alpha_{\mathcal{K}}\|\theta_n-\theta_n^{hk}\|_Q^2 + \langle\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\rangle_{L(\Omega)}\\
&\le c\big\{\|\theta_n-\lambda_n^h\|_Q^2+\|u_n-u_n^{hk}\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_Q^2+\|\theta_n-\lambda_n^h\|_{L^2(\Gamma_3)}\big\}\\
&\quad+\langle\delta\theta_n^{hk}-\delta\theta_n,\lambda_n^h-\theta_n\rangle_{\mathcal{H}}.
\end{aligned} \tag{3.23}$$

Using formula $2\langle a-b,a\rangle=\|a-b\|^2+\|a\|^2-\|b\|^2$ for $a=\theta_n-\theta_n^{hk}$ and $b=\theta_{n-1}-\theta_{n-1}^{hk}$, we get

$$\frac{1}{2k}\big(\|\theta_n-\theta_n^{hk}\|_Q^2-\|\theta_{n-1}-\theta_{n-1}^{hk}\|_Q^2\big)\le\langle\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\rangle_{L(\Omega)}. \tag{3.24}$$

Then, by combining (3.23) and (3.24), we deduce the following majoration

$$\begin{aligned}
&\alpha_{\mathcal{K}}\|\theta_n-\theta_n^{hk}\|_Q^2+\frac{1}{2k}\big(\|\theta_n-\theta_n^{hk}\|_Q^2-\|\theta_{n-1}-\theta_{n-1}^{hk}\|_Q^2\big)\\
&\le c\big\{\|\theta_n-\lambda_n^h\|_Q^2+\|u_n-u_n^{hk}\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_Q^2+\|\theta_n-\lambda_n^h\|_{L^2(\Gamma_3)}\big\}\\
&\quad+\langle\delta\theta_n^{hk}-\delta\theta_n,\lambda_n^h-\theta_n\rangle_{\mathcal{H}}.
\end{aligned}$$

Replacing $n$ by $j$ in the above relation, and summing up from $j=1$ to $n$, we deduce that

$$\begin{aligned}
&\|\theta_n-\theta_n^{hk}\|_Q^2+2k\alpha_{\mathcal{K}}\sum_{j=1}^n\|\theta_j-\theta_j^{hk}\|_Q^2\\
&\le ck\sum_{j=1}^n\big\{\|\theta_j-\lambda_j^h\|_Q^2+\|u_j-u_j^{hk}\|_V^2+\|\varphi_j-\varphi_j^{hk}\|_Q^2+\|\theta_j-\lambda_j^h\|_{L^2(\Gamma_3)}\big\}\\
&\quad+2k\sum_{j=1}^n\langle\delta\theta_j^{hk}-\delta\theta_j,\lambda_j^h-\theta_j\rangle_{\mathcal{H}}+\|\theta_0-\theta_0^h\|_Q^2.
\end{aligned}$$

Now, as done [5], we derive the following majoration

$$\begin{aligned}
&2k\sum_{j=1}^n\langle\delta\theta_j^{hk}-\delta\theta_j,\lambda_j^h-\theta_j\rangle_{\mathcal{H}}\\
&\le c\|\theta_n-\theta_n^{hk}\|_Q^2+c\|\theta_n-\lambda_n^h\|_Q^2+c\|\theta_0-\theta_0^h\|_Q^2+c\|\theta_1-\lambda_1^h\|_Q^2\\
&\quad+\frac{k}{2}\sum_{j=1}^{n-1}\|\theta_j-\theta_j^{hk}\|_Q^2+\frac{2}{k}\sum_{j=1}^{n-1}\|(\theta_j-\lambda_j^h)-(\theta_{j+1}-\lambda_{j+1}^h)\|_{L^2(\Omega)}^2.
\end{aligned}$$

For the sake of simplification, we note

$$e_n = \|\theta_n - \theta_n^{hk}\|_Q^2 + 2k\alpha_{\mathcal{K}} \sum_{j=1}^{n} \|\theta_j - \theta_j^{hk}\|_Q^2$$

and

$$\begin{aligned} g_n = k\sum_{j=1}^{n} \big\{ &\|\theta_j - \lambda_j^h\|_Q^2 + \|u_j - u_j^{hk}\|_V^2 + \|\varphi_j - \varphi_j^{hk}\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)} \big\} \\ &+ \frac{1}{k}\sum_{j=1}^{n-1} \|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 + \|\theta_0 - \theta_0^h\|_Q^2 \\ &+ \|\theta_1 - \lambda_1^h\|_Q^2 + \|\theta_n - \lambda_n^h\|_Q^2. \end{aligned}$$

Then, the previous majorations imply that there exists a constant $c > 0$ such that

$$e_n \le c g_n + c\sum_{j=}^{n} e_j.$$

We now use the Gronwall inequality to establish that

$$\begin{aligned} &\|\theta_n - \theta_n^{hk}\|_Q^2 + \sum_{j=1}^{n} \|\theta_j - \theta_j^{hk}\|_Q^2 \\ &\le ck\sum_{j=1}^{n} \big\{ \|\theta_j - \lambda_j^h\|_Q^2 + \|u_j - u_j^{hk}\|_V^2 + \|\varphi_j - \varphi_j^{hk}\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)} \big\} \\ &\quad + c\sum_{j=1}^{n-1} \|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 + \|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2. \end{aligned} \tag{3.25}$$

From the previous estimations (3.11), (3.12), (3.15) and (3.25), we deduce

$$\begin{aligned} &\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 \\ &\le c\Big\{ \|w_n - v_n^h\|_V^2 + \|\varphi_n - \psi_n^h\|_W^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)} \\ &\quad + \sum_{j=1}^{n} \big( \|\theta_j - \lambda_j^h\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)} \big) + \sum_{j=1}^{n-1} \|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 \\ &\quad + \sum_{j=1}^{n} \big( \|w_j - w_j^{hk}\|_V^2 + \|u_j - u_j^{hk}\|_V^2 + \|\varphi_j - \varphi_j^{hk}\|_Q^2 + \|\theta_j - \theta_j^{hk}\|_Q^2 \big) \\ &\quad + \|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2 \Big\} + c(h^2 + k^2). \end{aligned} \tag{3.26}$$

Let us consider the two following quantities

$$e_n = \|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2$$

and

$$\begin{aligned} g_n = &\|w_n - v_n^h\|_V^2 + \|\varphi_n - \psi_n^h\|_W^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)} \\ &+ \sum_{j=1}^{n} \left(\|\theta_j - \lambda_j^h\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)}\right) + \sum_{j=1}^{n-1} \|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 \\ &+ \|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2 + h^2 + k^2. \end{aligned}$$

Then, according to these new notations, the inequality (3.26) can be written as follows:

$$e_n \leq c g_n + c \sum_{j=}^{n} e_j.$$

Hence, by applying the Gronwall inequality, it comes from the previous inequality that

$$\begin{aligned} &\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 \\ &\leq c\Big\{ \|w_n - v_n^h\|_V^2 + \|\varphi_n - \psi_n^h\|_W^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)} \\ &\quad + \sum_{j=1}^{n} \left(\|\theta_j - \lambda_j^h\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)}\right) + \sum_{j=1}^{n-1} \|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 \\ &\quad + \|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2 \Big\} + c(h^2 + k^2). \end{aligned} \tag{3.27}$$

Finally, we combine (3.16) and (3.27) to derive the estimation (3.5), and this establishes Theorem 3.2. □

Keeping now in mind the standard finite element approximation theory (see [2,5,13]), we derive from Theorem 2.3, the following error estimate result.

**Corollary 3.3.** *Assume the assumptions of Theorem* 2.3 *hold, as well as the following conditions*

$$\begin{gathered} u \in L^2([0,T];V) \cap C^1([0,T];H^2(\Omega)), \quad w \in C([0,T];H^2(\Omega)) \cap L^2([0,T];H^2(\Gamma_3)), \\ \varphi \in C([0,T];H^2(\Omega)), \quad \theta \in C([0,T];H^2(\Omega)) \cap L^2(0,T;H^2(\Gamma_3)), \quad \dot{\theta} \in L^2(0,T;H^2(\Omega)), \\ \zeta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \end{gathered}$$

*we have the following order error estimate*

$$\begin{aligned} &\max_{1\leq n\leq N} \|w_n - w_n^{hk}\|_V + \max_{1\leq n\leq N} \|u_n - u_n^{hk}\|_V + \max_{1\leq n\leq N} \|\varphi_n - \varphi_n^{hk}\|_W \\ &\quad + \max_{1\leq n\leq N} \|\theta_n - \theta_n^{hk}\|_Q + \max_{1\leq n\leq N} \|\zeta_n - \zeta_n^{hk}\|_Z + \left[k \sum_{n=1}^{N} \|\zeta_n - \zeta_n^{hk}\|_Z\right]^{1/2} \\ &\leq c(h+k). \end{aligned}$$

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