TAIWANESE JOURNAL OF MATHEMATICS
Vol. 26, No. 6, pp. 1133–1144, December 2022
DOI: 10.11650/tjm/220605

Log Canonical Thresholds on Burniat Surfaces with $K^2 = 6$ via Pluricanonical Divisors

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Abstract. Let S be a Burniat surface with $K_S^2 = 6$ and φ be the bicanonical map of S. In this paper we show optimal lower bounds of log canonical thresholds of members of pluricanonical sublinear systems of S via Klein group G induced by φ. Indeed, for a positive even integer m, the log canonical threshold of members of an invariant (resp. anti-invariant) part of $|mK_S|$ is greater than or equal to $1/(2m)$ (resp. $1/(2m-2)$). For a positive odd integer m, the log canonical threshold of members of an invariant (resp. anti-invariant) part of $|mK_S|$ is greater than or equal to $1/(2m-5)$ (resp. $1/(2m)$). The inequalities are all optimal.

1. Introduction

Let X be a variety and $\mathsf{p} \in X$ be a smooth point. And let D be an effective Cartier divisor on X. The log canonical threshold or the complex singularity exponent of D at p is the number

$$\operatorname{lct}_{\mathsf{p}}(X, D) := \sup \left\{ c \in \mathbb{Q} \mid |f|^{-c} \text{ is locally } L^2 \text{ near } \mathsf{p} \right\},$$

where f is a local defining equation of D at p. In [7] we have the following inequalities

$$\frac{1}{\operatorname{mult}_{\mathsf{p}}(D)} \leq \operatorname{lct}_{\mathsf{p}}(X, D) \leq \frac{\dim X}{\operatorname{mult}_{\mathsf{p}}(D)},$$

and the log canonical threshold of D at p is equal to the absolute value of the largest root of the Bernstein–Sato polynomial of f.

The log canonical threshold can be formally defined for log pairs (cf. [7, 8.2 Proposition]). Let X be a normal variety with at worst log canonical singularities, Z be a closed subvariety of X and D be an effective $\mathbb{Q}$-Cartier divisor on X. The log canonical threshold of D along Z on X is the number

$$\operatorname{lct}_Z(X, D) := \sup\{c \in \mathbb{Q} \mid (X, cD) \text{ is log canonical in an open neighborhood of } Z\}.$$

For simplicity, we put $\operatorname{lct}(X, D) = \operatorname{lct}_X(X, D)$.

We have the following invariant for every polarised pair $(X, \mathcal{L})$.

Received December 29, 2021; Accepted June 27, 2022.
Communicated by Jungkai Alfred Chen.
2020 *Mathematics Subject Classification*. 14J17, 14J29.
Key words and phrases. Burniat surface, log canonical threshold, surface of general type, pluricanonical divisor.
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Definition 1.1. Let X be a normal variety with at worst log canonical singularities, and $\mathcal{L}$ be an ample $\mathbb{Q}$-Cartier divisor on X. The global log canonical threshold of a pair $(X, \mathcal{L})$ is the number

$$\begin{aligned}&\operatorname{glct}(X, \mathcal{L})\\ &:= \inf\{\operatorname{lct}(X, D) \mid D \text{ is an effective } \mathbb{Q}\text{-Cartier divisor on } X,\ \mathbb{Q}\text{-linearly equivalent to } \mathcal{L}\}.\end{aligned}$$

Chen, Chen and Jiang [5] proved the Noether inequality for projective 3-folds of general type. They use the global log canonical threshold of a surface of general type with $p_g = 2$ and $K^2 = 1$ via its ample canonical divisor (see the appendix by Kollár in [5]).

The authors in [6] showed that the global log canonical threshold of a Burniat surface with $K^2 = 6$ via its ample canonical divisor is $1/2$, where the Burniat surface is a minimal surface of general surface with $p_g = 0$ and $K^2 = 6$.

In this paper, we give optimal lower bounds of log canonical thresholds of members of pluricanonical sublinear systems via Klein group induced by the bicanonical map of a Burniat surface with $K^2 = 6$.

Let S be a Burniat surface with $K_S^2 = 6$ (see [1, 2, 8–10]). The bicanonical map φ of S has an image, a del Pezzo surface Σ of degree 6 in $\mathbb{P}^6$ which is a blow-up $\rho\colon \Sigma \to \mathbb{P}^2$ at three point p_1, p_2, p_3 in general position. Denote by e_i the (-1)-curve corresponding to p_i, by e_i' the strict transform of the line passing through the two points p_j and p_k by ρ, and by m_l^i the strict transform of a general line passing through the point p_i by ρ for each $\{i, j, k\} = \{1, 2, 3\}$ and $l = 1, 2$. Then φ is a bidouble covering map over Σ with a branch divisor $B := B_1 + B_2 + B_3$ satisfying $2L_i \sim B_j + B_k$ for a line bundle L_i on Σ and $\{i, j, k\} = \{1, 2, 3\}$, where

$$\begin{aligned} B_1 &= e_1 + e_1' + m_1^2 + m_2^2,\\ B_2 &= e_2 + e_2' + m_1^3 + m_2^3,\\ B_3 &= e_3 + e_3' + m_1^1 + m_2^1,\end{aligned}$$

and $\sim$ means the linearly equivalent relation between divisors.

For $i = 1, 2, 3$, we note $\varphi^*(B_i) = 2R_i$ for some divisor R_i ramified by φ, and denote by G the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{\mathrm{Id}_S, \sigma_1, \sigma_2, \sigma_3\}$ induced by φ such that R_i is the divisorial fixed part of σ_i.

For a positive integer m, the natural action of the group G splits the set of global sections of the pluricanonical divisor mK_S of S into eigen spaces via the characters of G:

$$H^0(S, mK_S) = H^0(S, mK_S)^{\mathrm{inv}} \oplus \bigoplus_{i=1}^{3} H^0(S, mK_S)^{\chi_i},$$

where χ_i is a character of G such that $\chi_i(\sigma_j) = \delta_{ij}$ for $i, j \in \{1, 2, 3\}$. Then the pluricanonical linear system $|mK_S|$ for a positive integer m contains an *invariant part* $|mK_S|_0$

(resp. an *anti-invariant part* $|mK_S|_i$) that consists of zeros of sections of $H^0(S, mK_S)^{\mathrm{inv}}$ (resp. $H^0(S, mK_S)^{\chi_i}$) for $i = 1, 2, 3$, that is,

$$|mK_S| \supseteq |mK_S|_0 \cup \bigcup_{i=1}^{3} |mK_S|_i.$$

We consider the log canonical threshold of members of the invariant and anti-invariant parts of the complete linear system $|mK_S|$, where m is a positive integer. To calculate the log canonical threshold, we use the following representation of pluricanonical linear systems for a bidouble covering map $\varphi\colon S \to \Sigma$. Denote by R the ramification divisor $R_1 + R_2 + R_3$ of φ.

Proposition 1.2. (cf. [10, Proposition 1.6]) *For a positive integer n and each $i = 1, 2, 3$ with $\{i, j, k\} = \{1, 2, 3\}$,*

(i) $|2nK_S|_0 = \varphi^*|n(2K_\Sigma + B)|$ *and* $|2nK_S|_i = R_j + R_k + |\varphi^*(n(2K_\Sigma + B) - L_i)|$;

(ii) $|(2n+1)K_S|_0 = R + |\varphi^*((2n+1)K_\Sigma + nB)|$ *and* $|(2n+1)K_S|_i = R_i + |\varphi^*((2n+1)K_\Sigma + nB + L_i)|$.

We apply

$$B \sim -3K_\Sigma$$

to Proposition 1.2 and obtain log canonical thresholds of members of the pluricanonical sublinear systems of Burniat surfaces S with $K_S^2 = 6$ via the Klein group induced by the bicanonical map of φ as follows.

Theorem 1.3 (Main theorem)**.** *Let S be a Burniat surface with $K_S^2 = 6$. Then for a positive integer n and each $i = 1, 2, 3$,*

(i) *if $D_0 \in |2nK_S|_0$ and $D_i \in |2nK_S|_i$,*

$$\operatorname{lct}(S, D_0) \geq \frac{1}{4n} \quad \textit{and} \quad \operatorname{lct}(S, D_i) \geq \frac{1}{4n-2};$$

(ii) *if $D'_0 \in |(2n+1)K_S|_0$ and $D'_i \in |(2n+1)K_S|_i$,*

$$\operatorname{lct}(S, D'_0) \geq \frac{1}{4n-3} \quad \textit{and} \quad \operatorname{lct}(S, D'_i) \geq \frac{1}{4n+2}.$$

Moreover the inequalities are optimal.

Remark 1.4. Since $|2K_S|_i = \emptyset$ for all $i = 1, 2, 3$ (see [9, Proposition 3.1]), we actually have $\operatorname{lct}(S, D_i) \geq 1/(4n-2)$ for any $D_i \in |2nK_S|_i$ when an integer $n \geq 2$ in Theorem 1.3(i).

Corollary 1.5. *Let S be a Burniat surface with $K_S^2 = 6$. Then for a positive integer n and each $i = 1, 2, 3$,*

(i) *if $D_i \in |2nK_S|_i$,*

$$\mathrm{lct}(S, D_i) > \frac{1}{4n};$$

(ii) *if $D_0' \in |(2n+1)K_S|_0$,*

$$\mathrm{lct}(S, D_0') > \frac{1}{4n+2}.$$

Remark 1.6. Corollary 1.5(i) is [6, Proposition 5.2].

Since

$$\mathrm{glct}(S, K_S) = \frac{1}{2}$$

(see [6, Theorem 1.3],) we obtain

Corollary 1.7. *Let S be a Burniat surface with $K_S^2 = 6$. For any positive even (resp. odd) integer m, if a divisor D is in the linear system $|mK_S|$ such that $\mathrm{glct}(S, K_S) = \mathrm{lct}\left(S, \frac{1}{m}D\right)$, then the divisor D is not in the anti-invariant parts $|mK_S|_i$ (resp. the invariant part $|mK_S|_0$) for $i = 1, 2, 3$.*

Proof. We get the result by Corollary 1.5. □

2. Preliminaries

Let X be a normal variety with at worst log canonical singularities. Note that $\sim_{\mathbb{Q}}$ means the $\mathbb{Q}$-linearly equivalent relation.

Lemma 2.1. *Let $\mathcal{N}_0 \sim_{\mathbb{Q}} A$ be an effective $\mathbb{Q}$-Cartier divisor on X such that the log pair $(X, \mathcal{N}_0)$ is not log canonical at a point p. And let $\mathcal{N} \sim_{\mathbb{Q}} A$ be an effective $\mathbb{Q}$-Cartier divisor on X such that the log pair $(X, \mathcal{N})$ is log canonical at the point p. Then there is an effective $\mathbb{Q}$-Cartier divisor $\mathcal{N}' \sim_{\mathbb{Q}} A$ on X such that at least one component of $\mathcal{N}$ is not contained in the support of $\mathcal{N}'$ and the log pair $(X, \mathcal{N}')$ is not log canonical at the point p.*

Proof. See [4, Remark 2.22]. □

The following is used for a non log canonical pair at some smooth point.

Lemma 2.2. (cf. [7, 8.10 Lemma]) *Let D be an effective $\mathbb{Q}$-Cartier divisor on X. If the log pair (X, D) is not log canonical at some smooth point p, then the inequality*

$$\mathrm{mult}_{\mathsf{p}}(D) > 1$$

holds.

3. Proof of the main theorem

We remark that for $i = 1, 2, 3$ and $j = 1, 2$,

$$E_i^2 = E_i'^2 = -1, \quad K_S \cdot E_i = K_S \cdot E_i' = 1, \quad M_j^{i2} = 0 \quad \text{and} \quad K_S \cdot M_j^i = 2,$$

where $\varphi^*(e_i) = 2E_i$, $\varphi^*(e_i') = 2E_i'$ and $\varphi^*(m_j^i) = 2M_j^i$.

3.1. Even pluricanonical linear system

For a positive integer n, the complete linear system $|2nK_S|$ contains the invariant part $|2nK_S|_0$ and the anti-invariant parts $|2nK_S|_i$ with $i = 1, 2, 3$, that is,

$$|2nK_S| \supseteq \bigcup_{i=0}^{3} |2nK_S|_i.$$

3.1.1. Invariant part

In [6] we have

$$\operatorname{glct}(S, 2K_S) = \operatorname{lct}(S, \overline{D}_0) = \frac{1}{4}$$

for some divisor $\overline{D}_0 \in |2K_S|$. For example, $\overline{D}_0 := 2E_1 + 4E_3 + 2E_1' + 4E_2'$, then

$$\operatorname{lct}(S, D_0) \geq \frac{1}{4n}$$

for any $D_0 \in |2nK_S|_0$ and the inequality is optimal.

3.1.2. Anti-invariant parts

To show

$$\operatorname{lct}(S, D_i) \geq \frac{1}{4n-2}$$

for any $D_i \in |2nK_S|_i$, we need the following lemma.

Lemma 3.1. [6, Lemma 4.1] *Let $\psi\colon X \to Y$ be a bidouble covering map between a normal variety X and a smooth variety Y branched along an effective divisor $\mathcal{B}$ on Y, and $\mathcal{D}$ be an effective $\mathbb{Q}$-Cartier divisor on X. Then*

$$(X, \mathcal{D}) \ \textit{is log canonical if} \ \left(Y, \psi(\mathcal{D}) + \frac{1}{2}\mathcal{B}\right) \ \textit{is log canonical.}$$

We deal with an integer $n \geq 2$ by Remark 1.4. Suppose that $\operatorname{lct}(S, D_i) < 1/(4n-2)$. Then the log pair $\left(S, \frac{1}{4n-2} D_i\right)$ is not log canonical at some point p. By Lemma 2.2,

$$\operatorname{mult}_{\mathsf{p}}(D_i) > 4n - 2.$$

We put an effective divisor $d_i := \varphi(D_i)$ on Σ. Then

$$\left(\Sigma, \frac{1}{4n-2}d_i + \frac{1}{2}B\right) \text{ is not log canonical at a point } \varphi(\mathsf{p}) \text{ on } \Sigma$$

by Lemma 3.1.

We consider the case $\varphi(\mathsf{p}) \notin B_1 \cup B_2 \cup B_3$. Then $\left(\Sigma, \frac{1}{4n-2}d_i\right)$ is not log canonical at $\varphi(\mathsf{p})$ which implies

$$\mathrm{glct}(\Sigma, d_i) < \frac{1}{4n-2}.$$

However, it contradicts because $d_i \sim_{\mathbb{Q}} -nK_\Sigma$ and $\mathrm{glct}(\Sigma, \Delta) \geq 1/2$ for any effective $\mathbb{Q}$-Cartier divisor $\Delta \sim_{\mathbb{Q}} -K_\Sigma$ since Σ is a nonsingular del Pezzo surface of degree 6 (see [3, Theorem 1.7]). Thus $\varphi(\mathsf{p}) \in B_1 \cup B_2 \cup B_3$.

By Proposition 1.2, we have an effective $\mathbb{Q}$-Cartier divisor $D_i - (R_j + R_k)$ for $\{i, j, k\} = \{1, 2, 3\}$. We may deal with $i = 1$.

The case $\mathsf{p} \in E_1 \cap E_2'$. We have

$$D_1 = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3' E_3' + \Omega,$$

where rational numbers $\alpha_1 \geq 0$ and $\alpha_2, \alpha_3' \geq 1$, and $E_1, E_2, E_3' \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω (denote by $\mathrm{Supp}(\Omega)$ the support of Ω). Since $\mathsf{p} \notin E_2 \cup E_3'$, the log pair $\left(S, \frac{1}{4n-2}(D_1 - \alpha_2 E_2 - \alpha_3' E_3')\right)$ is not log canonical at the point p.

Suppose $\alpha_1 = 0$, and then $2n = D_1 \cdot E_1 \geq \mathrm{mult}_{\mathsf{p}}(D_1)\,\mathrm{mult}_{\mathsf{p}}(E_1) > 4n - 2$ which is a contradiction. So $\alpha_1 \neq 0$.

Since $D_1 - (R_2 + R_3)$ is effective,

$$\Omega \cdot M_1^1 \geq (M_1^3 + M_2^3) \cdot M_1^1 = 2.$$

Thus $4n = D_1 \cdot M_1^1 = \alpha_1 + \Omega \cdot M_1^1$ implies $4n - 2 \geq \alpha_1$, and so

$$\frac{\alpha_1}{4n-2} \leq 1.$$

We have a pair $\left(S, E_1 + \frac{1}{4n-2}\Omega\right)$ is not log canonical at p. By the inversion of adjunction formula,

$$\text{the pair } \left(E_1, \frac{1}{4n-2}\Omega\Big|_{E_1}\right) \text{ is not log canonical at } \mathsf{p}.$$

This implies that

$$2n + \alpha_1 - \alpha_3' = (D_1 - \alpha_1 E_1 - \alpha_2 E_2 - \alpha_3' E_3') \cdot E_1 > 4n - 2.$$

On the other hand, since $D_1 - (R_2 + R_3)$ is effective,

$$2n = D_1 \cdot E_3' = \alpha_1 + \alpha_2 - \alpha_3' + \Omega \cdot E_3' \geq \alpha_1 + \alpha_2 - \alpha_3' + (M_1^3 + M_2^3) \cdot E_3' = \alpha_1 + \alpha_2 - \alpha_3' + 2.$$

Hence

$$\alpha_2 < 0$$

which is a contradiction.

The case $\mathsf{p} \in E_1 \setminus (E_2' \cup E_3')$. We have

$$D_1 = \alpha_1 E_1 + \alpha_2' E_2' + \alpha_3' E_3' + \Omega,$$

where rational numbers $\alpha_1 \geq 0$ and $\alpha_2', \alpha_3' \geq 1$, and $E_1, E_2', E_3' \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω. Then

$$2n = D_1 \cdot E_3' = \alpha_1 - \alpha_3' + \Omega \cdot E_3' \geq \alpha_1 - \alpha_3' + (E_2 + M_1^3 + M_2^3) \cdot E_3' = \alpha_1 - \alpha_3' + 3.$$

And since

$$4n = D_1 \cdot M_1^1 = \alpha_1 + \Omega \cdot M_1^1 \geq \alpha_1 + (M_1^3 + M_2^3) \cdot M_1^1 = \alpha_1 + 2,$$

we obtain

$$2n + \alpha_1 - \alpha_2' - \alpha_3' = (D_1 - \alpha_1 E_1 - \alpha_2' E_2' - \alpha_3' E_3') \cdot E_1 > 4n - 2$$

by the inversion of adjunction formula. Hence

$$\alpha_2' < -1$$

which is a contradiction.

The case $\mathsf{p} \in M_1^1 \setminus (E_1 \cup E_1')$. The log pair

$$\left(S, \frac{1}{4n-2}(M_1^1 + M_1^3 + M_2^3 + D)\right)$$

is not log canonical at the point p, where $D_1 \sim R_2 + R_3 + D$ for some $D \in |\varphi^*(-nK_\Sigma - L_1)|$ by Proposition 1.2(i). We have

$$D = \alpha M_1^3 + \Delta$$

where a rational number $\alpha \geq 0$ and $M_1^3 \not\subset \mathrm{Supp}(\Delta)$ with an effective $\mathbb{Q}$-Cartier divisor Δ. By using a general member $\overline{M}$ of the linear system $|2M_1^2|$ such that $\overline{M} \not\subset \mathrm{Supp}(D)$,

$$8n - 12 = D \cdot \overline{M} \geq \alpha M_1^3 \cdot \overline{M} = 2\alpha.$$

Thus we can use the inversion of adjunction formula. So the log pair

$$\left(M_1^3, \frac{1}{4n-2}(M_1^1 + M_2^3 + \Delta)\Big|_{M_1^3}\right)$$

is not log canonical at p. Then

$$1 + (4n - 4) = (M_1^1 + M_2^3 + \Delta) \cdot M_1^3 \geq \mathrm{mult}_{\mathsf{p}}\left((M_1^1 + M_2^3 + \Delta)\big|_{M_1^3}\right) > 4n - 2$$

which is a contradiction.

We can induce a contradiction by using a similar argument like the above cases for each point of R. Therefore for all cases $i = 1, 2, 3$,

$$\operatorname{lct}(S, D_i) \geq \frac{1}{4n-2} \quad \text{for any } D_i \in |2nK_S|_i.$$

And the inequality is optimal because $\operatorname{lct}_{\mathsf{p}}(S, \overline{D}_i) = 1/(4n-2)$ for

$$\overline{D}_i := R_{i+1} + R_{i+2} + 2\big((2n-1)E_i' + (n-2)E_{i+1}' + (2n-3)E_{i+1} + nE_{i+2}\big) \in |2nK_S|_i$$

and

$$\mathsf{p} \in E_i' \setminus (E_{i+1} \cup E_{i+2} \cup M_1^i \cup M_2^i),$$

where the index $i \in \{1, 2, 3\}$ is considered as modulo 3.

3.2. Odd pluricanonical linear system

For a positive integer n, the complete linear system $|(2n+1)K_S|$ contains the invariant part $|(2n+1)K_S|_0$ and the anti-invariant parts $|(2n+1)K_S|_i$ with $i = 1, 2, 3$, that is,

$$|(2n+1)K_S| \supset \bigcup_{i=0}^{3} |(2n+1)K_S|_i.$$

3.2.1. Invariant part

We prove that for any $D_0' \in |(2n+1)K_S|_0$, the log pair $\big(S, \frac{1}{4n-3}D_0'\big)$ is log canonical. To obtain a contradiction, we assume that there is a member D_0' of $|(2n+1)K_S|_0$ such that the log pair $\big(S, \frac{1}{4n-3}D_0'\big)$ is not log canonical at some point p. Note that

$$|(2n+1)K_S|_0 = R + |2(n-1)K_S|$$

(see Proposition 1.2 and apply $B \sim -3K_\Sigma$ and $K_S \sim_{\mathbb{Q}} \varphi^*(K_\Sigma + \frac{1}{2}B)$). Thus there is the member D' of the complete linear system $|2(n-1)K_S|$ such that $D_0' = R + D'$. Since the global log canonical threshold of the pair $(S, 2(n-1)K_S)$ is $1/(4n-4)$ (see [6, Theorem 1.3]), p is contained in R. We consider the following cases.

The case $\mathsf{p} \in E_3 \cap E_1'$. The log pair $\big(S, \frac{1}{4n-3}(E_3 + E_1' + D')\big)$ is not log canonical at the point p. For the effective divisor

$$N := (4n-3)E_3 + (4n-3)E_1' + (2n-2)E_2 + (2n-2)E_2' \sim E_3 + E_1' + D',$$

the log canonical threshold of the log pair (S, N) is $1/(4n-3)$. By Lemma 2.1, there is an effective $\mathbb{Q}$-Cartier divisor $N' \sim_{\mathbb{Q}} N$ such that at least one component of N is not contained in the support of N' and the log pair $\left(S, \frac{1}{4n-3}N'\right)$ is not log canonical at p. Thus at least one of E_2, E_3, E_1' and E_2' is not contained in $\mathrm{Supp}(N')$.

We can represent

$$N' = \alpha_3 E_3 + \alpha_1' E_1' + \Omega,$$

where rational numbers $\alpha_3, \alpha_1' \geq 0$ and $E_3, E_1' \not\subset \mathrm{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor Ω.

Suppose $E_2 \not\subset \mathrm{Supp}(N')$. Then

$$2n-1 = N' \cdot E_2 \geq \alpha_1' E_1' \cdot E_2 = \alpha_1'$$

By the inversion of adjunction formula, the log pair

$$\left(E_1', \frac{1}{4n-3}(\alpha_3 E_3 + \Omega)\Big|_{E_1'}\right)$$

is not log canonical at p. Thus

$$(2n-2) + \alpha_1' = (\alpha_3 E_3 + \Omega) \cdot E_1' \geq \mathrm{mult}_{\mathsf{p}}\left((\alpha_3 E_3 + \Omega)\big|_{E_1'}\right) > 4n-3$$

which is a contradiction.

For each case E_2', E_3 or $E_1' \not\subset \mathrm{Supp}(N')$, we also get a contradiction by using a similar argument as above. We remark that $E_3 \not\subset \mathrm{Supp}(N')$ (resp. $E_1' \not\subset \mathrm{Supp}(N')$) means $\alpha_3 = 0$ (resp. $\alpha_1' = 0$).

The case $\mathsf{p} \in E_3 \setminus (E_1' \cup E_2')$. The log pair $\left(S, \frac{1}{4n-3}(E_3 + M_1^3 + M_2^3 + D')\right)$ is not log canonical at the point p. We have

$$D' = \alpha_3 E_3 + \alpha_1' E_1' + \alpha_2' E_2' + \Delta,$$

where rational numbers $\alpha_3, \alpha_1', \alpha_2' \geq 0$ and $E_3, E_1', E_2' \not\subset \mathrm{Supp}(\Delta)$ with an effective $\mathbb{Q}$-Cartier divisor Δ. Let $\widetilde{M}$ be a general member of the linear system $|2M_1^3|$ such that $\widetilde{M} \not\subset \mathrm{Supp}(D')$. Then

$$8n-8 = D' \cdot \widetilde{M} \geq \alpha_3 E_3 \cdot \widetilde{M} = 2\alpha_3$$

implies that $4n-4 \geq a_3$. By the inversion of adjunction formula, the log pair

$$\left(E_3, \frac{1}{4n-3}(M_1^3 + M_2^3 + \Delta)\Big|_{E_3}\right)$$

is not log canonical at p. Thus

$$(2n-1) + \alpha_3 - \alpha_1' - \alpha_2' \geq \left((M_1^3 + M_2^3 + \Delta) \cdot E_3\right)_{\mathsf{p}} \geq \mathrm{mult}_{\mathsf{p}}\left((M_1^3 + M_2^3 + \Delta)\big|_{E_3}\right) > 4n-3$$

which implies $\alpha_3 > (2n-2) + \alpha_1' + \alpha_2'$. Meanwhile, the inequality

$$(2n-2) - \alpha_3 + \alpha_1' = \Delta \cdot E_1' \geq 0$$

implies that $(2n-2)+\alpha_1' \geq \alpha_3$ which is a contradiction.

The case $\mathsf{p} \in M_1^1 \setminus (E_1 \cup E_1')$. Set $M := M_1^2 + M_2^2 + M_1^3 + M_2^3$. Then the log pair

$$\left(S, \frac{1}{4n-3}(M_1^1 + M + D')\right)$$

is not log canonical at the point p. We have

$$D' = \alpha M_1^1 + \Delta,$$

where a rational number $\alpha \geq 0$ and $M_1^1 \not\subset \operatorname{Supp}(\Delta)$ with an effective $\mathbb{Q}$-Cartier divisor Δ. By using a general member $\widehat{M}$ of the linear system $|2M_1^2|$ such that $\widehat{M} \not\subset \operatorname{Supp}(D')$,

$$8n-8 = D' \cdot \widehat{M} \geq \alpha M_1^1 \cdot \widehat{M} = 2\alpha$$

which implies $4n-4 \geq \alpha$. By the inversion of adjunction formula, the log pair

$$\left(M_1^1, \frac{1}{4n-3}(M+\Delta)\Big|_{M_1^1}\right)$$

is not log canonical at p. Then

$$1 + (4n-4) \geq \big((M+\Delta)\cdot M_1^1\big)_{\mathsf{p}} \geq \operatorname{mult}_{\mathsf{p}}((M+\Delta)|_{M_1^1}) > 4n-3$$

which is a contradiction.

We can induce a contradiction by using a similar argument like the above cases for each point of R. Hence

$$\operatorname{lct}(S, D_0') \geq \frac{1}{4n-3} \quad \text{for any } D_0' \in |(2n+1)K_S|_0.$$

And the inequality is optimal because $\operatorname{lct}_{\mathsf{p}}(S, \overline{D}_0') = 1/(4n-3)$ for

$$\overline{D}_0' := R + 2(n-1)(2E_2' + E_3' + 2E_1 + E_3) \in |(2n+1)K_S|_0$$

and

$$\mathsf{p} \in E_2' \setminus (E_1 \cup E_3 \cup M_1^2 \cup M_2^2).$$

3.2.2. Anti-invariant part

For a positive integer n and $i = 1, 2, 3$, $|(2n+1)K_S|_i$ is represented by

$$R_i + |\varphi^*((1-n)K_\Sigma + L_i)|$$

(see Proposition 1.2 and apply $B \sim -3K_\Sigma$).

We may consider for $i = 1$. The divisor

$$\overline{D}_1' := E_1 + E_1' + M_1^2 + M_2^2 + 2\big((2n+1)E_2' + (n-1)E_1' + nE_1 + 2nE_3\big)$$

is in $|(2n+1)K_S|_1$. The log canonical threshold of the log pair $(S, \overline{D}_1')$ is $1/(4n+2)$. Note that the global log canonical threshold of the log pair (S, K_S) is $1/2$ (see [6, Theorem 1.3]). This means that the infimum of the set

$$\{\mathrm{lct}(S, D_1') \mid D_1' \in |(2n+1)K_S|_1\}$$

is $1/(4n+2)$. Thus

$$\inf\{\mathrm{lct}(S, D_i') \mid D_i' \in |(2n+1)K_S|_i\} = \frac{1}{4n+2}$$

for each $i = 1, 2, 3$.

Acknowledgments

The authors are very grateful to the referees for valuable suggestions and comments. The first author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (No. 2020R1A2C4002510). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2020R1I1A1A01074847) and by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1A4A3033098).

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