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Global L^2-boundedness of a New Class of Rough Fourier Integral Operators

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Abstract. In this paper, we investigate the L^2 boundedness of Fourier integral operator $T_{\phi,a}$ with rough symbol $a \in L^\infty S^m_\rho$ and rough phase $\phi \in L^\infty \Phi^2$ which satisfies $\left|\{x : |\nabla_\xi \phi(x,\xi) - y| \le r\}\right| \le C(r^{n-1} + r^n)$ for any $\xi, y \in \mathbb{R}^n$ and $r > 0$. We obtain that $T_{\phi,a}$ is bounded on L^2 if $m < \rho(n-1)/2 - n/2$ when $0 \le \rho \le 1/2$ or $m < -(n+1)/4$ when $1/2 \le \rho \le 1$. When $\rho = 0$ or $n = 1$, the condition of m is sharp. Moreover, the maximal wave operator is a special class of $T_{\phi,a}$ which is studied in this paper. Thus, our main theorem substantially extends and improves some known results about the maximal wave operator.

1. Introduction and main results

A Fourier integral operator (FIO) is defined as

$$T_{\phi,a} f(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) \widehat{f}(\xi)\, d\xi,$$

where a is the symbol and ϕ is the phase function, and $\widehat{f}$ denotes the Fourier transform of f. As we can see, all pseudo-differential operators are of this form with $\phi(x,\xi) = x \cdot \xi$.

In the study of FIOs, one usually assume the symbol $a(x,\xi)$ belongs to Hörmander class $S^m_{\rho,\delta}$ and the phase function ϕ is in the class Φ^2 satisfying the strong non-degeneracy condition.

Definition 1.1. Let $m \in \mathbb{R}$, $0 \le \rho, \delta \le 1$. A function $a \in S^m_{\rho,\delta}$, if $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies

$$\sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{-m+\rho|\alpha|-\delta|\beta|} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| < \infty$$

for all multi-indices α, β, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

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Definition 1.2. A real-valued function $\phi(x,\xi) \in \Phi^2$, if $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, is homogeneous of order 1 in the frequency variable ξ and

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}} |\xi|^{-1+|\alpha|}|\partial_\xi^\alpha \partial_x^\beta \phi(x,\xi)| < \infty$$

for any $|\alpha| + |\beta| \geq 2$.

Definition 1.3 (Strong non-degeneracy condition)**.** A real-valued function $\phi \in C^2(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ satisfies the strong non-degeneracy condition, if there exists a constant $c > 0$ such that

$$\left|\det\left(\frac{\partial^2\phi(x,\xi)}{\partial x_j \partial \xi_k}\right)\right| \geq c \quad \text{for all } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.$$

Obviously, when x has compact support, if $\phi \in \Phi^2$ and the mixed Hessian matrix $\det\left(\frac{\partial^2\phi}{\partial x_j \partial \xi_k}\right) \neq 0$, then ϕ satisfies the strong non-degeneracy condition.

The local L^2 boundedness of FIOs with $\phi \in \Phi^2$ and satisfying the determinant of the mixed Hessian matrix is non-zero on the support of the symbol was firstly investigated by Eskin [9] for $a \in S^0_{1,0}$ and by Hörmander [13] for $a \in S^0_{\rho,1-\rho}$, $1/2 < \rho \leq 1$. Later on, Beals [2] and Greenleaf–Uhlmann [11] extended Hörmander's result to the case of $a \in S^0_{1/2,1/2}$. Meanwhile, there were many studies on the global L^2 boundedness of FIOs, such as Fujiwara [10] and Asada–Fujiwara [1]. Recently, Dos Santos Ferreira and Staubach [8] established the global L^2 boundedness with $a \in S^m_{\rho,\delta}$, $0 \leq \rho \leq 1$, $0 \leq \delta < 1$ and $m \leq \min\{0, n(\rho-\delta)/2\}$.

For the L^p boundedness of FIOs, Seeger–Sogge–Stein [18] proved the local $H^1 - L^1$ boundedness when $a \in S^{(1-n)/2}_{1,0}$ by using the well-known "dyadic-parabolic" decomposition. Moreover, they got the local L^p-boundedness when $a \in S^m_{1,0}$, $m = (1-n)|1/p - 1/2|$ and the condition of m is sharp. Later on, Ruzhansky and Sugimoto [17] proved the global L^p boundedness of FIOs with $a \in S^m_{1,0}$, $m = (1-n)|1/p-1/2|$. In [3], Castro, Israelsson and Staubach established the global L^p boundedness of FIOs with $a \in S^m_{\rho,\delta}$, $0 \leq \rho \leq 1$, $0 \leq \delta < 1$, $m = -(n-\rho)|1/p-1/2| - n\max\{0, (\delta-\rho)/2\}$ or $a \in S^m_{\rho,1}$, $0 \leq \rho \leq 1$, $m < -n(1-\rho)\max(1/p, 1/2) - (n-1)|1/p-1/2|$. Besides, there are many results about local and global L^p bondedness of FIOs, such as [4–6, 8, 15].

In [14], Kenig and Staubach introduced a class of pseudo-differential operators with the symbol belongs to rough Hörmander class was denoted by $L^\infty S^m_\rho$, and proved the sharp L^2-boundedness of this class of pseudo-differential operators. The specific definition of $L^\infty S^m_\rho$ and the result are as follows.

Definition 1.4. Let $m \in \mathbb{R}$ and $0 \leq \rho \leq 1$. A function a belongs to the rough Hörmander class $L^\infty S^m_\rho$, if it satisfies

$$\sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-m+\rho|\alpha|} \|\partial_\xi^\alpha a(\,\cdot\,,\xi)\|_{L^\infty} < \infty \quad \text{for all multi index } \alpha.$$

Theorem 1.5. [14, Proposition 2.3] *When $a \in L^\infty S^m_\rho$, $0 \le \rho \le 1$, then the pseudo-differential operator T_a is bounded on L^2 if and only if $m < \frac{n}{2}(\rho - 1)$.*

Inspired by the work of Kenig and Staubach [14], Dos Santos Ferreira and Staubach [8] defined the rough phase class $L^\infty\Phi^2$ which behaves like an L^∞ function in the spatial variable x and the rough non-degeneracy condition. The specific definitions are as follows.

Definition 1.6. A real-valued function ϕ belongs to the rough phase class $L^\infty\Phi^2$, if ϕ is homogeneous of degree 1 in the frequency variable ξ and satisfies

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}} |\xi|^{k-1}\|\nabla_\xi^k\phi(\,\cdot\,,\xi)\|_{L^\infty} < \infty \quad \text{for all } k \ge 2.$$

Definition 1.7 (Rough non-degeneracy condition)**.** A real valued phase ϕ satisfies the rough non-degeneracy condition, if there exists a constant $c > 0$ such that

$$|\nabla_\xi\phi(x,\xi) - \nabla_\xi\phi(y,\xi)| \ge c|x-y|$$

for any $x, y \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

In [8], Dos Santos Ferreira and Staubach established various L^p boundedness of FIOs with $a \in L^\infty S^m_\rho$ and $\phi \in L^\infty\Phi^2$ satisfying the rough non-degeneracy condition. Here, we would like to mention the L^2 boundedness of rough FIOs.

Theorem 1.8. [8, Theorem 2.8] *When $a \in L^\infty S^m_\rho$ and $\phi \in L^\infty\Phi^2$ satisfying the rough non-degeneracy condition, $T_{\phi,a}$ is bounded on L^2 if $m < n(\rho-1)/2 - (n-1)/4$.*

On the other hand, the wave operator defined as

$$e^{it\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|)}\widehat{f}(\xi)\,d\xi$$

which is a special class of FIO with $a(x,\xi) = 1$. It is well-known that for all $f \in H^s$, if $s > 1/2$, $e^{it\sqrt{-\Delta}}f$ converges to f almost everywhere as $t \to 0$ (see [7]) and if $s \le 1/2$ the convergence fails (see [12]). The convergence is due to the following estimate of the maximal wave operator

$$\left\| \sup_{0<t<1} \left| e^{it\sqrt{-\Delta}}f \right| \right\|_{L^2} \le C\|f\|_{H^s} \tag{1.1}$$

for $s > 1/2$.

By the definition of Sobolev space, we can see that (1.1) is equivalent to $\|Tg\|_{L^2} \le C\|g\|_2$, where

$$Tg(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t(x)|\xi|)}(1+|\xi|^2)^{-s/2}\widehat{g}(\xi)\,d\xi \tag{1.2}$$

and $t(x) \in L^\infty$, $\widehat{g}(\xi) = (1+|\xi|^2)^{s/2}\widehat{f}(\xi)$. Moreover, it is easy to prove that $(1+|\cdot|^2)^{-s/2} \in L^\infty S^s_1 \subseteq L^\infty S^s_\rho$ and $x\cdot\xi + t(x)|\xi| \in L^\infty\Phi^2$ but does not satisfy the rough non-degeneracy condition. Motivated by these, we consider the L^2 boundedness of a class of FIOs which is generalized of (1.2). The following theorem is our main result in this paper.

Theorem 1.9. *Let $a \in L^\infty S^m_\rho$ and $\phi \in L^\infty\Phi^2$ satisfying*

$$\big|\{x : |\nabla_\xi\phi(x,\xi) - y| \le r\}\big| \le C(r^{n-1} + r^n) \tag{1.3}$$

for any $\xi, y \in \mathbb{R}^n$ and $r > 0$. Then $T_{\phi,a}$ is bounded on L^2 if $m < \rho(n-1)/2 - n/2$ when $0 \le \rho \le 1/2$ or $m < -(n+1)/4$ when $1/2 \le \rho \le 1$.

Remark 1.10. The reason why we replace the rough non-degeneracy condition by the condition (1.3) is that for all $t(x) \in L^\infty$, by some direct computations, we can get that $\phi(x,\xi) = x\cdot\xi + t(x)|\xi|$ does not satisfy rough non-degeneracy condition but satisfies (1.3). Moreover, we can prove that the strong non-degeneracy condition or rough non-degeneracy condition implies (1.3). So, our result extends the existing results substantially. Now, We show the proof of this conclusion below.

Proof. Since the rough non-degeneracy condition implies the strong non-degeneracy condition (see [8, Proposition 1.11]), we only need to prove the strong non-degeneracy condition implies (1.3). For this purpose, we consider the map $F_\xi\colon x \mapsto \nabla_\xi\phi(x,\xi)$. Since ϕ satisfies the strong non-degeneracy condition, setting $z = \nabla_\xi\phi(x,\xi)$ and by the inverse theorem, we have

$$\begin{aligned}
\big|\{x : |\nabla_\xi\phi(x,\xi) - y| \le r\}\big| &= \int_{\{x:|\nabla_\xi\phi(x,\xi)-y|\le r\}} dx = \int_{\{z:|z-y|\le r\}} d(F_\xi^{-1}(z)) \\
&\le \int_{\{z:|z-y|\le r\}} \left|\det\frac{\partial^2\phi(x,\xi)}{\partial x_j\partial\xi_k}\right|^{-1} dz \le C\int_{\{z:|z-y|\le r\}} dz \\
&\le Cr^n \le C(r^n + r^{n-1}).
\end{aligned}$$

□

Remark 1.11. According to [12, 16], when $\rho = 0$ or $n = 1$, the bound on m is sharp.

Throughout the paper, we use C, c to denote some positive constants that are independent of x, ξ, f and may vary from line to line. We denote by B_r the ball in $\mathbb{R}^n$ with center 0 and radius r.

2. Proof of Theorem 1.9

Before proving the main theorem, we need the following two lemmas for the low frequency of $T_{\phi,a}$.

Lemma 2.1. [8, Lemma 1.17] *Suppose that* $u \in C_c^\infty(B_1)$ *and satisfies that*

$$|\nabla^k u(x)| \le C_k |x|^{1-k} \quad \text{for all } k \in \mathbb{N}^+,$$

then for any $0 \le \mu < 1$, *we have*

$$\left| \int_{\mathbb{R}^n} e^{ix\cdot y} u(x)\, dx \right| \le C \langle y \rangle^{-n-\mu}.$$

Lemma 2.2. *Suppose* a *and* ϕ *satisfy the assumptions of Theorem* 1.9, *then for any* $\eta \in C_c^\infty(B_1)$, *the following operator*

$$S_{0,\phi,a} f(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)\eta(\xi) f(\xi)\, d\xi$$

is bounded on L^2.

Proof. By standard dual argument, we have $\|S_{0,\phi,a}\|_{L^2\to L^2}^2 = \|S_{0,\phi,a}S_{0,\phi,a}^*\|_{L^2\to L^2}$, where $S_{0,\phi,a}S_{0,\phi,a}^* f(x) = \int_{\mathbb{R}^n} k_0(x,y) f(y)\, dy$ and

$$k_0(x,y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(y,\xi)\eta^2(\xi)\, d\xi.$$

By Schur's theorem, to prove the L^2 boundedness of $S_{0,\phi,a}S_{0,\phi,a}^*$, it suffices to show that

$$\sup_y \int_{\mathbb{R}^n} |k_0(x,y)|\, dx < \infty \quad \text{and} \quad \sup_x \int_{\mathbb{R}^n} |k_0(x,y)|\, dy < \infty.$$

By choosing some $\xi_0 \in S^{n-1}$ and setting $h_x(\xi) = \phi(x,\xi) - \nabla_\xi \phi(x,\xi_0)\cdot\xi$, $h_y(\xi) = \phi(y,\xi) - \nabla_\xi\phi(y,\xi_0)\cdot\xi$, we have

$$k_0(x,y) = \int_{\mathbb{R}^n} e^{i\langle \nabla_\xi\phi(x,\xi_0) - \nabla_\xi\phi(y,\xi_0),\xi\rangle} e^{i(h_x(\xi)-h_y(\xi))} a(x,\xi)\overline{a}(y,\xi)\eta^2(\xi)\, d\xi.$$

We claim that h_x satisfies the following estimate

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{-1+k} |\nabla_\xi^k h_x(\xi)| < \infty \quad \text{for all } k \ge 1. \tag{2.1}$$

Indeed, since $\phi \in L^\infty\Phi^2$, using the mean value theorem, we have

$$|\nabla_\xi h_x(\xi)| = |\nabla_\xi\phi(x,\xi) - \nabla_\xi\phi(x,\xi_0)| = |\nabla_\xi\phi(x,\xi/|\xi|) - \nabla_\xi\phi(x,\xi_0)| < \infty.$$

When $k \ge 2$, we have

$$|\nabla_\xi^k h_x(\xi)| = |\nabla_\xi^k \phi(x,\xi)| \le C|\xi|^{1-k}$$

as desired. Similarly, $h_y(\xi)$ has the same estimate.

Applying (2.1) and the fact $a \in L^\infty S^m_\rho$, we can get

$$
\begin{aligned}
&|\nabla_\xi^k (e^{i(h_x(\xi)-h_y(\xi))} a(x,\xi)\overline{a}(y,\xi)\eta^2(\xi))| \\
&\le C_k \sum_{k_1+k_2+k_3=k} |\nabla_\xi^{k_1} e^{i(h_x(\xi)-h_y(\xi))}||\nabla_\xi^{k_2}(a(x,\xi)\overline{a}(y,\xi))||\nabla_\xi^{k_3}\eta^2(\xi)| \\
&\le C_k \sum_{k_1\le k} |\nabla_\xi^{k_1} e^{i(h_x(\xi)-h_y(\xi))}| \\
&\le C_k \sum_{k_1\le k} \sum_{s=1}^{k_1} \sum_{\substack{t_1+\cdots+t_s=k_1 \\ t_1,\dots,t_s>0}} |\nabla_\xi^{t_1}(h_x(\xi)-h_y(\xi))\cdots\nabla_\xi^{t_s}(h_x(\xi)-h_y(\xi))| \\
&\le C_k \sum_{k_1\le k} \sum_{s=1}^{k_1} \sum_{\substack{t_1+\cdots+t_s=k_1 \\ t_1,\dots,t_s>0}} |\xi|^{1-t_1}\cdots|\xi|^{1-t_s} \\
&\le C_k \sum_{k_1\le k} \sum_{s=1}^{k_1} \sum_{\substack{t_1+\cdots+t_s=k_1 \\ t_1,\dots,t_s>0}} |\xi|^{s-t_1-\cdots-t_s} \\
&\le C_k|\xi|^{1-k}.
\end{aligned}
$$

Then by Lemma 2.1, for any $y \in \mathbb{R}^n$ and $0 \le \mu < 1$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} |k_0(x,y)|\,dx &\le C \int \left(1+|\nabla_\xi\phi(x,\xi_0)-\nabla_\xi\phi(y,\xi_0)|^2\right)^{-(n+\mu)/2} \\
&\le C\left(\int_{\{x:|\nabla_\xi\phi(x,\xi_0)-\nabla_\xi\phi(y,\xi_0)|<1\}} + \int_{\{x:|\nabla_\xi\phi(x,\xi_0)-\nabla_\xi\phi(y,\xi_0)|\ge1\}}\right) \\
&\qquad \left(1+|\nabla_\xi\phi(x,\xi_0)-\nabla_\xi\phi(y,\xi_0)|^2\right)^{-(n+\mu)/2} \\
&= I + II.
\end{aligned}
$$

For I, by (1.3), we have

$$
I \le \left|\{x : |\nabla_\xi\phi(x,\xi_0)-\nabla_\xi\phi(y,\xi_0)| < 1\}\right| < \infty.
$$

For II, we have

$$
\begin{aligned}
II &= \sum_{s=1}^\infty \int_{\{x:2^{s-1}\le|\nabla_\xi\phi(x,\xi_0)-\nabla_\xi\phi(y,\xi_0)|<2^s\}} \left(1+|\nabla_\xi\phi(x,\xi_0)-\nabla_\xi\phi(y,\xi_0)|^2\right)^{-(n+\mu)/2} \\
&\le C\sum_{s=1}^\infty 2^{-(s-1)(n+\mu)}\left|\{x : |\nabla_\xi\phi(x,\xi_0)-\nabla_\xi\phi(y,\xi_0)| < 2^s\}\right| \\
&\le C\sum_{s=1}^\infty 2^{-(s-1)(n+\mu)}(2^{s(n-1)}+2^{sn}) < \infty.
\end{aligned}
$$

By the same method, we can also prove $\sup_y \int_{\mathbb{R}^n} |k_0(x,y)|\,dy < \infty$. Then it follows that $S_{0,\phi,a}$ is bounded on L^2. □

Now we turn to prove Theorem 1.9.

Proof of Theorem 1.9. First, we write $T_{\phi,a}$ as $T_{\phi,a} = S_{\phi,a}\mathcal{F}$, where

$$S_{\phi,a}f(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) f(\xi)\, d\xi$$

and $\mathcal{F}(f) = \widehat{f}$. By Plancherel's theorem, it is enough to prove the L^2 boundedness of $S_{\phi,a}$.

Decomposing $S_{\phi,a}$ as

$$\begin{aligned} S_{\phi,a}f(x) &= \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)\chi_0(\xi) f(\xi)\, d\xi + \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)(1-\chi_0(\xi)) f(\xi)\, d\xi \\ &= S_{0,\phi,a}f(x) + S_{1,\phi,a}f(x), \end{aligned}$$

where $\chi_0 \in C_c^\infty(B_2)$ and $\chi_0 = 1$ in B_1.

We can get the L^2 boundedness of $S_{0,\phi,a}f$ directly from Lemma 2.2. So, it remains to prove the L^2 boundedness of $S_{1,\phi,a}f$. By standard dual argument, we only need to prove L^2 boundedness of $S_{1,\phi,a}S_{1,\phi,a}^*$, where $S_{1,\phi,a}S_{1,\phi,a}^* f(x) = \int_{\mathbb{R}^n} k_1(x,y) f(y)\, dy$ and

$$k_1(x,y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(x,\xi)(1-\chi_0(\xi))^2\, d\xi.$$

By the well-known Littlewood–Paley decomposition, we can obtain that $(1-\chi_0(\xi))^2 = \sum_{j=1}^\infty \chi_j(\xi)$, where

$$\chi_j \in C_c^\infty(B_{2^{j+1}} \setminus B_{2^{j-1}}), \quad |\nabla_\xi^k \chi_j(\xi)| \le C_k 2^{-jk} \quad \text{for all } k \in \mathbb{N}.$$

Then $k_1(x,y)$ can be decomposed as

$$\begin{aligned} k_1(x,y) &= \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(x,\xi)(1-\chi_0(\xi))^2\, d\xi \\ &= \sum_{j=1}^\infty \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(x,\xi)\chi_j(\xi)\, d\xi \\ &= \sum_{j=1}^\infty k_{1,j}(x,y). \end{aligned}$$

Next, we will show that

$$\sup_y \int_{\mathbb{R}^n} |k_1(x,y)|\, dx < \infty, \quad \sup_x \int_{\mathbb{R}^n} |k_1(x,y)|\, dy < \infty. \tag{2.2}$$

Then by (2.2) and Schur's theorem, we have

$$\|T_{1,\phi,a}\|_{L^2\to L^2} = \|S_{1,\phi,a}\|_{L^2\to L^2} = \|S_{1,\phi,a}S_{1,\phi,a}^*\|_{L^2\to L^2}^{1/2}.$$

Case 1*:* $0 \le \rho \le 1/2$. For any $j \in \mathbb{N}$, B_j^ν denote a ball $B(\xi_j^\nu, 2^{j(1-\rho)})$ with $2^{j-1} \le |\xi_j^\nu| < 2^{j+1}$. We can observe that there are no more than $J = C2^{j\rho n}$ points $\xi_j^\nu \in B_{2^{j+1}} \setminus B_{2^{j-1}}$ and cut-off functions $\psi_j^\nu \in C_c^\infty(B_j^\nu)$ such that

$$\sum_{\nu=1}^{J} \psi_j^\nu(\xi) = 1, \quad |\nabla_\xi^k \psi_j^\nu(\xi)| \le C_k 2^{-jk(1-\rho)} \quad \text{for all } k \in \mathbb{N}. \tag{2.3}$$

Then $k_{1,j}$ can be decomposed as

$$\begin{aligned} k_{1,j}(x,y) &= \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(x,\xi)\chi_j(\xi)\, d\xi \\ &= \sum_{\nu=1}^{J} \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(x,\xi)\chi_j(\xi)\psi_j^\nu(\xi)\, d\xi \\ &= \sum_{\nu=1}^{J} k_{1,j,\nu}(x,y). \end{aligned}$$

By setting $h_x(\xi) = \phi(x,\xi) - \nabla_\xi \phi(x,\xi_j^\nu)\cdot\xi$, $h_y(\xi) = \phi(y,\xi) - \nabla_\xi\phi(y,\xi_j^\nu)\cdot\xi$ and $b_j^\nu(x,y,\xi) = e^{i(h_x(\xi)-h_y(\xi))} a(x,\xi)\overline{a}(x,\xi)\chi_j(\xi)\psi_j^\nu(\xi)$, we can rewrite $k_{1,j,\nu}(x,y)$ as

$$\begin{aligned} k_{1,j,\nu}(x,y) &= \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(x,\xi)\chi_j(\xi)\psi_j^\nu(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} e^{i\langle \nabla_\xi\phi(x,\xi_j^\nu) - \nabla_\xi\phi(y,\xi_j^\nu), \xi\rangle} b_j^\nu(x,y,\xi)\, d\xi. \end{aligned}$$

Since $\phi \in L^\infty\Phi^2$, $\xi \in B_j^\nu$, using the mean value theorem, we have

$$|\nabla_\xi h_x(\xi)| \le C|\xi - \xi_j^\nu| \sup_{\zeta\in B_j^\nu} |\nabla_\xi^2 \phi(x,\zeta)| \le C2^{j(1-\rho)}2^{-j} = C2^{-j\rho}. \tag{2.4}$$

For $k \ge 2$, since $0 \le \rho \le 1/2$, we get

$$|\nabla_\xi^k h_x(\xi)| = |\nabla_\xi^k \phi(x,\xi)| \le C2^{j(1-k)} \le C2^{-j\rho k}. \tag{2.5}$$

Obviously, h_y has the same estimates as (2.4) and (2.5).

In addition, since $a \in L^\infty S_\rho^m$ and ψ_j^ν satisfies (2.3), we obtain that

$$\begin{aligned} &|\nabla_\xi^k (a(x,\xi)\overline{a}(x,\xi)\chi_j(\xi)\psi_j^\nu(\xi))| \\ &\le \sum_{k_1+\cdots+k_4=k} |\nabla_\xi^{k_1} a(x,\xi)||\nabla_\xi^{k_2}\overline{a}(x,\xi)||\nabla_\xi^{k_3}\chi_j(\xi)||\nabla_\xi^{k_4}\psi_j^\nu(\xi)| \\ &\le C \sum_{k_1+\cdots+k_4=k} 2^{j(2m-\rho(k_1+k_2))}2^{-jk_3}2^{-j(1-\rho)k_4} \\ &\le C2^{j(2m-\rho k)}. \end{aligned} \tag{2.6}$$

In the last inequality, we use the fact $1-\rho \ge \rho$ when $0 \le \rho \le 1/2$.

Next, we define a self adjoint operator L as

$$L = I - 2^{2j\rho}\nabla_\xi^2.$$

For any $N \in \mathbb{N}$, by (2.4), (2.5) and (2.6), we have

$$\begin{aligned}
&|L^N b_j^\nu(x,y,\xi)| \\
&\le C \sum_{N_1+N_2\le 2N} 2^{j\rho(N_1+N_2)} \big|\nabla_\xi^{N_1}[a(x,\xi)\overline{a}(x,\xi)\psi_j^\nu(\xi)\chi_j(\xi)]\big|\big|\nabla_\xi^{N_2} e^{i(h_x(\xi)-h_y(\xi))}\big| \\
&\le C \sum_{N_1+N_2\le 2N} 2^{j\rho(N_1+N_2)} 2^{j(2m-\rho N_1)} \\
&\quad \times \sum_{t=1}^{N_2} \sum_{\substack{k_1+\cdots+k_t=N_2 \\ k_1,\dots,k_t>0}} |\nabla_\xi^{k_1}(h_x(\xi)-h_y(\xi))\cdots\nabla_\xi^{k_t}(h_x(\xi)-h_y(\xi))| \\
&\le C \sum_{N_1+N_2\le 2N} 2^{j(2m+\rho N_2)} \sum_{t=1}^{N_2} \sum_{\substack{k_1+\cdots+k_t=N_2 \\ k_1,\dots,k_t>0}} 2^{-jk_1\rho}\cdots 2^{-jk_t\rho} \\
&\le C2^{2jm}.
\end{aligned}$$

Since L is self adjoint and ξ-support of b_j^ν is contained in B_j^ν, for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
|k_{1,j,\nu}(x,y)| &= \left|\int_{\mathbb{R}^n} e^{i\langle \nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu),\xi\rangle} b_j^\nu(x,y,\xi)\,d\xi\right| \\
&= \big(1+2^{2j\rho}|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|^2\big)^{-N} \\
&\quad \times \left|\int_{\mathbb{R}^n} e^{i\langle \nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu),\xi\rangle} L^N b_j^\nu(x,y,\xi)\,d\xi\right| \\
&\le C\big(1+2^{2j\rho}|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|^2\big)^{-N} 2^{j(2m+n(1-\rho))}.
\end{aligned}$$

Now, we estimate $\int_{\mathbb{R}^n} |k_{1,j,\nu}(x,y)|\,dx$. For any $y \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big(1+2^{2j\rho}|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|^2\big)^{-N}\,dx \\
&= \left(\int_{\{x:|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|<2^{-j\rho}\}} + \int_{\{x:|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|\ge 2^{-j\rho}\}}\right) \\
&\quad \big(1+2^{2j\rho}|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|^2\big)^{-N}\,dx \\
&= I + II.
\end{aligned}$$

For I, by (1.3), we have

$$I \le \big|\{x : |\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)| < 2^{-j\rho}\}\big| \le C(2^{-j\rho(n-1)}+2^{-j\rho n}) \le C2^{-j\rho(n-1)}.$$

For II, choosing $N > n/2$, we can get

$$\begin{aligned}
II &= \int_{\{x:|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|\geq 2^{-j\rho}\}} \left(1+2^{2j\rho}|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|^2\right)^{-N} dx \\
&= \sum_{s=1}^{\infty} \int_{\{x:2^{s-1}2^{-j\rho}\leq|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|<2^s 2^{-j\rho}\}} \left(1+2^{2j\rho}|\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)|^2\right)^{-N} dx \\
&\leq \sum_{s=1}^{\infty} 2^{-2N(s-1)} \left|\{x : |\nabla_\xi\phi(x,\xi_j^\nu)-\nabla_\xi\phi(y,\xi_j^\nu)| < 2^s 2^{-j\rho}\}\right| \\
&\leq C\sum_{s=1}^{\infty} 2^{-2N(s-1)} [(2^s 2^{-j\rho})^{n-1} + (2^s 2^{-j\rho})^n] \\
&\leq C2^{-j\rho(n-1)}.
\end{aligned}$$

So $\int_{\mathbb{R}^n} |k_{1,j,\nu}(x,y)|\,dx \leq C2^{j(2m+n(1-\rho))}2^{-j\rho(n-1)}$. Therefore, for any $y \in \mathbb{R}^n$, since $J = C2^{j\rho n}$, when $m < \rho(n-1)/2 - n/2$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |k_1(x,y)|\,dx \leq \sum_{j=1}^{\infty}\sum_{\nu=1}^{J} \int_{\mathbb{R}^n} |k_{1,j,\nu}(x,y)|\,dx &\leq C\sum_{j=1}^{\infty}\sum_{\nu=1}^{J} 2^{j(2m+n(1-\rho))}2^{-j\rho(n-1)} \\
&\leq C\sum_{j=1}^{\infty} 2^{j(2m+n-\rho(n-1))} < \infty.
\end{aligned}$$

By the same method, we also can get $\sup_x \int_{\mathbb{R}^n} |k_1(x,y)|\,dy < \infty$ when $m < \rho(n-1)/2 - n/2$. So if $m < \rho(n-1)/2 - n/2$ when $0 \leq \rho \leq 1/2$, $T_{\phi,a}$ is bounded on L^2.

Case 2*:* $1/2 \leq \rho \leq 1$. First, let us recall the well-known "dyadic-parabolic" decomposition [18]. For $j \in \mathbb{N}$, fix a collection $\{\xi_j^\nu\}_\nu$ of unit vectors, that satisfy

(i) $|\xi_j^{v_1} - \xi_j^{v_2}| \geq 2^{-j/2}$, $v_1 \neq v_2$;

(ii) if $\xi \in \mathbb{S}^{n-1}$, then there exists an ξ_j^ν so that $|\xi - \xi_j^\nu| < 2^{-j/2}$.

For each $j \in \mathbb{N}$, set $\Gamma_j^\nu = \{\xi : |\xi/|\xi| - \xi_j^\nu| \leq 2\cdot 2^{-j/2}\}$. Then we can construct an associated partition of unity given by ψ_j^ν, such that each ψ_j^ν is homogeneous of degree 0, supported in Γ_j^ν and satisfies that

$$\sum_{\nu=1}^{J} \psi_j^\nu(\xi) \equiv 1 \ \text{ for all } \ \xi \neq 0 \quad \text{and} \quad |\nabla_\xi^k \psi_j^\nu(\xi)| \leq C_k|\xi|^{-k}2^{jk/2}, \ \ k \in \mathbb{N}. \tag{2.7}$$

Then we can decompose $k_1(x,y)$ as

$$\begin{aligned}
k_1(x,y) &= \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(x,\xi)(1-\chi_0(\xi))^2\,d\xi \\
&= \sum_{j=1}^{\infty}\sum_{\nu=1}^{J} \int_{\mathbb{R}^n} e^{i(\phi(x,\xi)-\phi(y,\xi))} a(x,\xi)\overline{a}(x,\xi)\chi_j(\xi)\psi_j^\nu(\xi)\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \sum_{\nu=1}^{J} k_{1,j,\nu}(x, y) \\
&= \sum_{j=1}^{\infty} \sum_{\nu=1}^{J} \int_{\mathbb{R}^n} e^{i\langle \nabla_\xi \phi(x,\xi_j^\nu) - \nabla_\xi \phi(y,\xi_j^\nu), \xi\rangle} b_j^\nu(x, y, \xi)\, d\xi,
\end{aligned}
$$

where $b_j^\nu(x, y, \xi) = e^{i(h_x(\xi)-h_y(\xi))} a(x, \xi)\overline{a}(x, \xi)\chi_j(\xi)\psi_j^\nu(\xi)$ and $h_x(\xi) = \phi(x, \xi) - \nabla_\xi \phi(x, \xi_j^\nu) \cdot \xi, h_y(\xi) = \phi(y, \xi) - \nabla_\xi \phi(y, \xi_j^\nu) \cdot \xi$.

Without loss of generality, by rotating coordinate axes, we can assume that $\xi_j^\nu = (1, 0, \ldots, 0)$. If we denote $\xi' = (0, \xi_2, \ldots, \xi_n)$, then we have

Lemma 2.3. [19, pp. 406–407] *For $\phi \in L^\infty \Phi^2$ and $\xi \in \Gamma_j^\nu \cap \{\xi : 2^{j-1} < |\xi| \le 2^{j+1}\}$, we have*

$$
\begin{aligned}
|\partial_{\xi_1}^N \nabla_{\xi'}^M (h_x(\xi) - h_y(\xi))| &\le C 2^{-j(N+M/2)} \qquad \text{if } N, M \ge 0 \text{ with } N + M \ge 1, \\
|\partial_{\xi_1}^N \nabla_{\xi'}^M \psi_j^\nu(\xi)| &\le C 2^{-j(N+M/2)} \qquad \text{if } N, M \ge 0.
\end{aligned}
$$

By Lemma 2.3, for any $k, l \ge 0$, we have

$$
\begin{aligned}
&|\partial_{\xi_1}^k \nabla_{\xi'}^l e^{i(h_x(\xi)-h_y(\xi))}| \\
&\le C \sum_{t=1}^{k+l} \sum_{\substack{k_1+\cdots+k_t=k \\ l_1+\cdots+l_t=l \\ k_1+l_1,\ldots,k_t+l_t>0}} |\partial_{\xi_1}^{k_1} \nabla_{\xi'}^{l_1} (h_x(\xi) - h_y(\xi)) \cdots \partial_{\xi_1}^{k_t} \nabla_{\xi'}^{l_t} (h_x(\xi) - h_y(\xi))| \\
&\le C \sum_{t=1}^{k+l} \sum_{\substack{k_1+\cdots+k_t=k \\ l_1+\cdots+l_t=l \\ k_1+l_1,\ldots,k_t+l_t>0}} 2^{-j(k_1+l_1/2)} \cdots 2^{-j(k_t+l_t/2)} \qquad (2.8) \\
&\le C \sum_{t=1}^{k+l} 2^{-j(k+l/2)} \le C 2^{-j(k+l/2)}.
\end{aligned}
$$

Now we define an operator as

$$
L = I - 2^{2j\rho} \partial_{\xi_1}^2 - 2^j \nabla_{\xi'}^2.
$$

Since $a \in L^\infty S_\rho^m$, $1/2 \le \rho \le 1$, applying (2.7) and (2.8), we have

$$
\begin{aligned}
&|L^N b_j^\nu(x, y, \xi)| \\
&\le C \sum_{N_1+N_2 \le N} 2^{2j\rho N_1} 2^{jN_2} |\partial_{\xi_1}^{2N_1} \nabla_{\xi'}^{2N_2} (e^{i(h_x(\xi)-h_y(\xi))} a(x, \xi)\overline{a}(y, \xi)\chi_j(\xi)\psi_j^\nu(\xi))| \\
&\le C \sum_{N_1+N_2 \le N} 2^{2j\rho N_1} 2^{jN_2} \\
&\quad \times \sum_{\substack{k_1+k_2+k_3=2N_1 \\ l_1+l_2+l_3=2N_2}} |\partial_{\xi_1}^{k_1} \nabla_{\xi'}^{l_1} e^{i(h_x(\xi)-h_y(\xi))}||\partial_{\xi_1}^{k_2} \nabla_{\xi'}^{l_2} (a(x, \xi)\overline{a}(y, \xi)\chi_j(\xi))||\partial_{\xi_1}^{k_3} \nabla_{\xi'}^{l_3} \psi_j^\nu(\xi)|
\end{aligned}
$$

$$\le C \sum_{N_1+N_2\le N} 2^{2j\rho N_1} 2^{jN_2} \sum_{\substack{k_1+k_2+k_3=2N_1\\ l_1+l_2+l_3=2N_2}} 2^{-j(k_1+l_1/2)} 2^{j(2m-\rho(k_2+l_2))} 2^{-j(k_3+l_3/2)}$$
$$\le C \sum_{N_1+N_2\le N} 2^{2j\rho N_1} 2^{jN_2} \sum_{\substack{k_1+k_2+k_3=2N_1\\ l_1+l_2+l_3=2N_2}} 2^{j(2m-\rho(k_1+k_2+k_3)-(l_1+l_2+l_3)/2)} \le C2^{jm}.$$

In the second to last line of above estimate, we use the condition $1/2 \le \rho \le 1$ to estimate the power term. It is easy to see that $|\{\xi \mid (x,y,\xi) \in \operatorname{supp} b_j^\nu \text{ for some } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n\}| \le C2^{j(n+1)/2}$. Then for any $N \in \mathbb{N}$, we have

$$\tag{2.9}\begin{aligned}
&|k_{1,j,\nu}(x,y)|\\
&= \big(1 + 2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)|^2 + 2^j|\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(x,\xi_j^\nu)|^2\big)^{-N}\\
&\quad \times \left| \int_{\mathbb{R}^n} e^{i\langle \nabla_\xi\phi(x,\xi_j^\nu) - \nabla_\xi\phi(y,\xi_j^\nu),\xi\rangle} L^N b_j^\nu(x,y,\xi)\, d\xi \right|\\
&\le C\big(1 + 2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)|^2 + 2^j|\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)|^2\big)^{-N}\\
&\quad \times \int_{\mathbb{R}^n} |L^N b_j^\nu(x,y,\xi)|\, d\xi\\
&\le C\big(1 + 2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)|^2 + 2^j|\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)|^2\big)^{-N}\\
&\quad \times 2^{j(2m+(n+1)/2)}.
\end{aligned}$$

Now we begin to estimate $\int_{\mathbb{R}^n} |k_{1,j,\nu}(x,y)|\, dx$. First, we denote

$$\begin{aligned}
E_1 &= \{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| \le 2^{-j\rho},\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| \le 2^{-j/2}\},\\
E_2 &= \{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| \ge 2^{-j\rho},\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| \le 2^{-j/2}\},\\
E_3 &= \{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| \le 2^{-j\rho},\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| \ge 2^{-j/2}\},\\
E_4 &= \{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| \ge 2^{-j\rho},\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| \ge 2^{-j/2}\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big(1 + 2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)|^2 + 2^j|\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)|^2\big)^{-N}\, dx\\
&= \left(\int_{E_1} + \int_{E_2} + \int_{E_3} + \int_{E_4}\right)\\
&\qquad \big(1 + 2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)|^2 + 2^j|\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)|^2\big)^{-N}\, dx\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Fix $y \in \mathbb{R}^n$, $r > 0$, we observe that the rectangle $\{z : |z_1 - \partial_{\xi_1}\phi(y,\xi_j^\nu)| \le 2^{-j\rho}r, |z' - \nabla_{\xi'}\phi(y,\xi_j^\nu)| \le 2^{-j/2}r\}$ can be covered by no more than $C2^{j(n-1)(\rho-1/2)}$ balls with radius $2^{-j\rho}r$. Then, since ϕ satisfies (1.3), we obtain that

$$\tag{2.10}\begin{aligned}
&\big|\{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| \le 2^{-j\rho}r,\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu) \le 2^{-j/2}r\}\big|\\
&\le C2^{j(n-1)(\rho-1/2)}[(2^{-j\rho}r)^{n-1} + (2^{-j\rho}r)^n].
\end{aligned}$$

By (2.10), for I_1, we have

$$\begin{aligned} I_1 &\leq \left|\{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| \leq 2^{-j\rho}, ; |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| \leq 2^{-j/2}\}\right| \\ &\leq C2^{j(n-1)(\rho-1/2)}(2^{-j\rho(n-1)} + 2^{-j\rho n}) \leq C2^{-j(n-1)/2}. \end{aligned}$$

For II_2, since $E_2 \subseteq \bigcup_{s=1}^\infty E_s$, where

$$\begin{aligned} E_s = \{x : 2^{s-1}2^{-j\rho} &\leq |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| < 2^s 2^{-j\rho}, \\ &|\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| \leq 2^{-j/2}\}, \end{aligned}$$

then when $N > n/2$, we have

$$\begin{aligned} I_2 &\leq \sum_{s=1}^\infty \int_{E_s} \left(1 + 2^{2j\rho}|\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)|^2\right)^{-N} \\ &\leq \sum_{s=1}^\infty 2^{-2N(s-1)} \\ &\qquad \times \left|\{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| < 2^s 2^{-j\rho},\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| \leq 2^s 2^{-j/2}\}\right| \\ &\leq C\sum_{s=1}^\infty 2^{-2N(s-1)} 2^{j(n-1)(\rho-1/2)}[(2^s 2^{-j\rho})^{n-1} + (2^s 2^{-j\rho})^n] \\ &\leq C\sum_{s=1}^\infty 2^{-s(2N-n)} 2^{-j(n-1)/2} \leq C2^{-j(n-1)/2}. \end{aligned}$$

The estimate for I_3 is similar to I_2, we omit the details here.

For I_4, since $E_4 \subseteq \bigcup_{t=1}^\infty \bigcup_{s=1}^\infty E_{t,s}$, where

$$\begin{aligned} E_{t,s} = \{x : 2^{s-1}2^{-j\rho} &\leq |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| < 2^s 2^{-j\rho}, \\ 2^{t-1}2^{-j/2} &\leq |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| < 2^t 2^{-j/2}\}. \end{aligned}$$

Then when $N > n$, we get that

$$\begin{aligned} I_4 &\leq \sum_{s=1}^\infty \sum_{t=1}^\infty 2^{-N(s-1)} 2^{-N(t-1)} \\ &\qquad \times \left|\{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| < 2^s 2^{-j\rho},\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| < 2^t 2^{-j/2}\}\right| \\ &\leq \sum_{s=1}^\infty \sum_{t\leq s} 2^{-N(s-1)} 2^{-N(t-1)} \\ &\qquad \times \left|\{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| < 2^s 2^{-j\rho},\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| < 2^s 2^{-j/2}\}\right| \\ &\quad + \sum_{s=1}^\infty \sum_{t\geq s} 2^{-N(s-1)} 2^{-N(t-1)} \\ &\qquad \times \left|\{x : |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)| < 2^t 2^{-j\rho},\ |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)| < 2^t 2^{-j/2}\}\right| \end{aligned}$$

$$
\begin{aligned}
&\le C \sum_{s=1}^{\infty} \sum_{t\le s}^{\infty} 2^{-N(s-1)} 2^{-N(t-1)} 2^{j(n-1)(\rho-1/2)} [(2^s 2^{-j\rho})^{n-1} + (2^s 2^{-j\rho})^{n-1}] \\
&\quad + \sum_{s=1}^{\infty} \sum_{t\ge s}^{\infty} 2^{-N(s-1)} 2^{-N(t-1)} 2^{j(n-1)(\rho-1/2)} [(2^t 2^{-j\rho})^{n-1} + (2^t 2^{-j\rho})^{n-1}] \\
&\le C 2^{-j(n-1)/2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \left(1 + 2^{2j\rho} |\partial_{\xi_1}\phi(x,\xi_j^\nu) - \partial_{\xi_1}\phi(y,\xi_j^\nu)|^2 + 2^j |\nabla_{\xi'}\phi(x,\xi_j^\nu) - \nabla_{\xi'}\phi(y,\xi_j^\nu)|^2\right)^{-N} dx \\
&\le C 2^{-j(n-1)/2}, \qquad (2.11) \\
&\int_{\mathbb{R}^n} |k_{1,j,\nu}(x,y)|\, dx \le C 2^{j(2m+1)}.
\end{aligned}
$$

For any $y \in \mathbb{R}^n$, by (2.9), (2.11) and with the fact $J = C2^{(n-1)/2}$, then when $m < -(n+1)/4$, we can get

$$
\int_{\mathbb{R}^n} |k_1(x,y)|\, dx \le \sum_{j=1}^{\infty} \sum_{\nu=1}^{J} \int |k_{1,j,\nu}(x,y)|\, dx \le C \sum_{j=1}^{\infty} 2^{j(2m+(n+1)/2)} < \infty.
$$

By symmetry, it is easy to get that for any $x \in \mathbb{R}^n$, when $m < -(n+1)/4$, $\int_{\mathbb{R}^n} |k_1(x,y)|\, dy < \infty$. So when $1/2 \le \rho \le 1$, if $m < -(n+1)/4$, then $T_{\phi,a}$ is bounded on L^2. □

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