TAIWANESE JOURNAL OF MATHEMATICS
Vol. xx, No. x, pp. 1–15, xx 20xx
DOI: 10.11650/tjm/220401

On Almost Self-centered Graphs and Almost Peripheral Graphs

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Abstract. An almost self-centered graph is a connected graph of order n with exactly $n-2$ central vertices, and an almost peripheral graph is a connected graph of order n with exactly $n-1$ peripheral vertices. We determine (1) the maximum girth of an almost self-centered graph of order n; (2) the maximum independence number of an almost self-centered graph of order n and radius r; (3) the minimum order of a k-regular almost self-centered graph; (4) the maximum size of an almost peripheral graph of order n; (5) possible maximum degrees of an almost peripheral graph of order n and (6) the maximum number of vertices of maximum degree in an almost peripheral graph of order n with maximum degree $n-4$ which is the second largest possible. Whenever the extremal graphs have a neat form, we also describe them.

1. Introduction

We consider finite simple graphs. The *order* of a graph is its number of vertices, and the *size* its number of edges. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G, respectively. Denote by $d_G(u,v)$ the distance between two vertices u and v in G. The *eccentricity* of a vertex v in a graph G, denoted by $\mathrm{ecc}_G(v)$, is the distance to a vertex farthest from v; that is, $\mathrm{ecc}_G(v) = \max\{d_G(v,u) \mid u \in V(G)\}$. If the graph G is clear from the context, we omit the subscript G. If $\mathrm{ecc}(v) = d(v,x)$, then the vertex x is called an *eccentric vertex* of v. The *radius* of a graph G, denoted by $\mathrm{rad}(G)$, is the minimum eccentricity of all the vertices in $V(G)$, whereas the *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity. A vertex v is a *central vertex* of G if $\mathrm{ecc}(v) = \mathrm{rad}(G)$. The *center* of a graph G, denoted by $C(G)$, is the set of all central vertices of G. A vertex u is a *peripheral vertex* of G if $\mathrm{ecc}(u) = \mathrm{diam}(G)$. The *periphery* of G is the set of all peripheral vertices of G. A graph with a finite radius or diameter is necessarily connected.

If $\mathrm{rad}(G) = \mathrm{diam}(G)$, then the graph G is called *self-centered*. Thus, a self-centered graph is a graph in which every vertex is a central vertex. This class of graphs have been extensively studied. See [2] and the references therein. Since a nontrivial graph has at least two peripheral vertices, a connected non-self-centered graph of order n has at most $n-2$ central vertices. The following concept was introduced in [7].

Received August 12, 2021; Accepted April 10, 2022.
Communicated by Daphne Der-Fen Liu.
2020 *Mathematics Subject Classification.* 05C35, 05C07, 05C69.
Key words and phrases. almost self-centered graph, almost peripheral graph, girth, independence number.
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Definition 1.1. A connected graph of order n is called *almost self-centered* if it has exactly $n-2$ central vertices.

Since every graph has at least one central vertex, a connected non-self-centered graph of order n has at most $n-1$ peripheral vertices. The following concept was introduced in [8].

Definition 1.2. A connected graph of order n is called *almost peripheral* if it has exactly $n-1$ peripheral vertices.

In this paper we investigate several extremal problems on these two classes of graphs. In particular, we determine (1) the maximum girth of an almost self-centered graph of order n; (2) the maximum independence number of an almost self-centered graph of order n and radius r; (3) the minimum order of a k-regular almost self-centered graph; (4) the maximum size of an almost peripheral graph of order n; (5) possible maximum degrees of an almost peripheral graph of order n and (6) the maximum number of vertices of maximum degree in an almost peripheral graph of order n with maximum degree $n-4$ which is the second largest possible. Whenever the extremal graphs have a neat form, we also describe them. For related research, see [4–6, 11].

In Section 2 we treat almost self-centered graphs, and in Section 3 we treat almost peripheral graphs.

For graphs G and H, the notation $G+H$ means the disjoint union of G and H. A *dominating vertex* in a graph of order n is a vertex of degree $n-1$. Two vertices u and v on a cycle C of length n are called *antipodal vertices* if $d_C(u,v)=\lfloor n/2 \rfloor$. An (x,y)-*path* is a path with endpoints x and y. A *diametral path* in a graph G is a shortest (x,y)-path of length $\mathrm{diam}(G)$. We list some notations which will be used:

C_n : the cycle of order n,

P_n : the path of order n,

K_n : the complete graph of order n,

$\overline{G}$: the complement of the graph G,

$e(G)$: the size of the graph G,

$\deg(v)$: the degree of the vertex v,

$N(v)$: the neighborhood of the vertex v,

$\delta(G)$: the minimum degree of vertices of the graph G,

$\Delta(G)$: the maximum degree of vertices of the graph G,

$\alpha(G)$: the independence number of the graph G,

$g(G)$: the girth of the graph G,

$N[v]$: the closed neighborhood of the vertex v; i.e., $N[v] = N(v) \cup \{v\}$,

$N_i(v)$: the i-th neighborhood of the vertex v; i.e., $N_i(v) = \{x \in V(G) \mid d(v,x) = i\}$.

It is known [9, p. 288] that if G is a connected graph satisfying $\mathrm{diam}(G) \geq \mathrm{rad}(G)+2$, then every integer k with $\mathrm{rad}(G) < k < \mathrm{diam}(G)$ is the eccentricity of some vertex. Thus if G is almost self-centered or almost peripheral, then the vertices of G have only two distinct eccentricities and hence $\mathrm{diam}(G) = \mathrm{rad}(G) + 1$.

2. Almost self-centered graphs

A *binocle* is a graph that consists of two cycles C, D and a (u, v)-path P such that $V(P) \cap V(C) = \{u\}$, $V(P) \cap V(D) = \{v\}$ and $V(C) \cap V(D) \subseteq V(P)$. Here we allow the possibility that P has length 0; i.e., P is a vertex. Note also that if P is nontrivial, then C and D are vertex-disjoint. A *theta* (or *theta graph*) is a graph that consists of three internally vertex-disjoint paths sharing the same two endpoints. $\theta_{a,b,c}$ will denote the theta consisting of three paths with lengths a, b and c, respectively. A binocle and a theta are depicted in Figure 2.1.

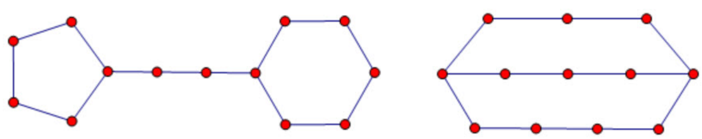

Figure 2.1: A binocle and $\theta_{4,4,5}$.

We make the convention that the girth of an acyclic graph is undefined. Thus whenever we talk about the girth of a graph, the graph is not acyclic. A connected graph is said to be *unicyclic* if it contains exactly one cycle. Recall that a connected graph of order n is unicyclic if and only if it has size n [10, p. 77].

Lemma 2.1. *Let G be a unicyclic graph of order $n \geq 6$. Then G is almost self-centered if and only if n is odd and G is the graph obtained from C_{n-1} by attaching one edge.*

Proof. Suppose G is almost self-centered. We have $e(G) = n$ and $G \neq C_n$. It is known [3] that the center of any connected graph lies within one block. Let B be the block of G in which $C(G)$ lies. Then B is unicyclic and $\delta(B) = 2$. Thus B is a cycle of order $n-2$ or $n-1$. If $B = C_{n-2}$, let $V(G) \setminus V(B) = \{x, y\}$. Then x and y are leaves. Since x and y are the only two peripheral vertices, their neighbors are a pair of antipodal vertices of the cycle B. But then B contains a vertex whose eccentricity in G is $\mathrm{diam}(G) - 2$, contradicting the assumption that G is almost self-centered. Hence $B = C_{n-1}$ and G is the graph obtained from C_{n-1} by attaching one edge. Since G is almost self-centered, n is odd.

Conversely, it is easy to verify that this graph is almost self-centered. □

Lemma 2.2. *If G is a connected graph of order n and size $n+1$ with $\delta(G)=2$, then G is either a binocle or a theta.*

Proof. Since $\sum_{x\in V(G)} \deg(x) = 2n+2$ and $\delta(G)=2$, the degree sequence of G is $(2,2,\dots,2,4)$ or $(2,2,\dots,2,3,3)$. In the former case, G is a graph consisting of two cycles sharing a common vertex, which is a binocle, while in the latter case, G is either a binocle or a theta. □

Lemma 2.2 can also be proved easily using induction on the order.

Lemma 2.3. *Let a, b, c be positive integers with $a \le b \le c$. Then* $\mathrm{rad}(\theta_{a,b,c}) = \lfloor (a+c)/2 \rfloor$ *and* $\mathrm{diam}(\theta_{a,b,c}) = \lfloor (b+c)/2 \rfloor$. *Consequently, $\theta_{a,b,c}$ is self-centered if and only if $b=a$ if $a+c$ is odd and $b \le a+1$ if $a+c$ is even. Also* $\mathrm{diam}(\theta_{a,b,c}) = \mathrm{rad}(\theta_{a,b,c})+1$ *if and only if $a+1 \le b \le a+2$ if $a+c$ is odd and $a+2 \le b \le a+3$ if $a+c$ is even.*

Proof. Easy verification. □

Lemma 2.4. *Let G be a connected graph of order n and size $n+1$ with $\delta(G)=2$. Then G is almost self-centered if and only if n is even and $G=\theta_{1,2,n-2}$.*

Proof. By Lemma 2.2, G is either a binocle or a theta. Suppose that G is almost self-centered. It is easy to see that an almost self-centered graph with minimum degree 2 is 2-connected. Since a binocle has connectivity 1, we deduce that G is a theta. Let $G=\theta_{a,b,c}$ with $a \le b \le c$. Since G is almost self-centered, $\mathrm{diam}(G)=\mathrm{rad}(G)+1$. By Lemma 2.3, $a+1 \le b \le a+2$ if $a+c$ is odd, and $a+2 \le b \le a+3$ if $a+c$ is even. First suppose that $a+c$ is odd. We assert that $a=1$. To the contrary, assume $a \ge 2$. Then $b \ge a+1 \ge 3$. Let $G=\theta_{a,b,c}$ consist of the three (x,y)-paths P_1, P_2, P_3 of lengths a, b, c, respectively. Denote $r=\mathrm{rad}(G)$ and $d=r+1=\mathrm{diam}(G)$. Note that x and y are central vertices of G; i.e., $\mathrm{ecc}(x)=\mathrm{ecc}(y)=r$. Let w_1 be the neighbor of x on P_2 and let w_2 be the neighbor of y on P_2. Let x_1 and x_2 be the two antipodal vertices of x on the odd cycle $C=P_1 \cup P_3$ where $d_C(x_2,y)=d_C(x_1,y)+1$. Then $d_G(w_1,x_2) \ge r+1=d$. Thus both w_1 and x_2 are peripheral vertices. Similarly, w_2 is a peripheral vertex. But then G contains at least three peripheral vertices, a contradiction.

The case when $a+c$ is even can be treated similarly. Hence $a=1$. Lemma 2.3 implies that $b \ge 2$ if $a+c$ is odd, and $b \ge 3$ if $a+c$ is even. If $b \ge 3$, using the above argument we obtain contradictions. Thus $a+c=1+c$ is odd and $b=2$. It follows that $n=a+b+c-1=1+(1+c)$ is even and $G=\theta_{1,2,n-2}$.

Conversely, it is easy to verify that if n is even then the theta $\theta_{1,2,n-2}$ is almost self-centered. □

Now we are ready to state and prove the first main result.

Theorem 2.5. *Let $g(n)$ denote the maximum girth of an almost self-centered graph of order n with $n \geq 5$. Then*

$$g(n) = \begin{cases} n-1 & \text{if } n \text{ is odd}, \\ 4\lfloor n/6 \rfloor & \text{if } n \text{ is even and } n \neq 10, \\ 5 & \text{if } n = 10. \end{cases}$$

Furthermore, if $n \geq 12$ and 6 divides n, then $g(n)$ is attained uniquely by the graph obtained from $\theta_{n/3,n/3,n/3}$ by attaching an edge to a vertex of degree three.

Proof. Let G be an almost self-centered graph of order $n \geq 5$. Clearly $G \neq C_n$. Hence $g(n) \leq n-1$. On the other hand, if n is odd, then the graph obtained from C_{n-1} by attaching an edge is almost self-centered and has girth $n-1$. Hence $g(n) = n-1$ if n is odd.

Now suppose that n is even. Note that adding edges to a graph does not increase its girth. The cases $n \leq 16$ can be verified by a computer search. Indeed, using Lemma 2.1 and the fact [1, p. 195] that a graph of order n and size $n+3$ has girth at most $\lfloor 4(n+3)/9 \rfloor$, we need only check the sizes $n+1$ and $n+2$ for a graph of order $n \leq 16$.

Next suppose that n is even and $n \geq 18$. We first show that $g(G) \leq 4\lfloor n/6 \rfloor$. It is known [1, p. 195] that a graph of order n and size $n+2$ has girth at most $\lfloor n/2 \rfloor + 1$. The inequality $\lfloor n/2 \rfloor + 1 \leq 4\lfloor n/6 \rfloor$ for $n \geq 18$ implies that if $e(G) \geq n+2$, then $g(G) \leq 4\lfloor n/6 \rfloor$. Also, Lemma 2.1 excludes the possibility that $e(G) = n$. It remains to consider the case when $e(G) = n+1$, and from now on we make this assumption. It is known [3] that the center of any connected graph lies within one block. Let B be the block of G in which $C(G)$ lies. Since $|C(G)| = n-2$ and $e(G) = n+1$, the size of B equals its order plus one. Since $n \geq 18$, B is 2-connected and $\delta(B) = 2$. By Lemma 2.2, B is a theta. Let $B = \theta_{a,b,c}$, which consists of three (x,y)-paths P_1, P_2, P_3 whose lengths are a, b, c, respectively with $a \leq b \leq c$.

Since the eccentricities of two adjacent vertices differ by at most one, every leaf of G is a peripheral vertex. Hence G has at most two leaves. We first exclude the possibility of two leaves. To the contrary, assume that G has two distinct leaves u and v whose neighbors are s and t, respectively.

Denote $d = \operatorname{diam}(B)$ and $f = \operatorname{diam}(G)$. Then $d+1 \leq f \leq d+2$. Clearly it is impossible that $f < d$. It is also impossible that $f = d$, since otherwise G would have at least four peripheral vertices, a contradiction. Hence $f \geq d+1$. The inequality $f \leq d+2$

follows from the fact that adding two leaves to B can increase its diameter by at most 2. We distinguish two cases.

Case 1*:* $f = d + 2$. In this case $\mathrm{rad}(G) = d + 1$. Since adding leaves to B can increase the eccentricity of any vertex of B by at most 1, we deduce that B is self-centered. Clearly $d_B(s,t) = d \geq 2$. Let w be an internal vertex on a shortest (s,t)-path in B. Then $\mathrm{ecc}_G(w) = d$, contradicting $\mathrm{rad}(G) = d + 1$.

Case 2*:* $f = d + 1$. Since $\mathrm{rad}(G) = d$, we have $\mathrm{rad}(B) \geq d - 1$. We further consider two subcases.

Subcase 2.1*:* $\mathrm{rad}(B) = d$*; i.e., B is self-centered.* Let s' be an eccentric vertex of s in B. Then $d_B(s', s) = d$, implying that $d_G(s', u) = d+1 = f$. But then G has at least three peripheral vertices u, v, s', a contradiction.

Subcase 2.2*:* $\mathrm{rad}(B) = d - 1$. By Lemma 2.3, $a + 1 \leq b \leq a + 2$ if $a + c$ is odd, and $a + 2 \leq b \leq a + 3$ if $a + c$ is even. It suffices to consider the two cases: $b \geq a + 2$; $b = a + 1$ and $a + c$ is odd.

First suppose $b \geq a + 2$. Since $\mathrm{ecc}_B(x) = \mathrm{rad}(B) = d - 1$ and $\mathrm{rad}(G) = d$, one of the two leaves, say u, must be an eccentric vertex of x in G. Let p be the neighbor of x on P_2. Using the structure of the theta B and the condition $b \geq a + 2$, we deduce that $d_G(p, u) \geq d + 1$. Consequently, G has at least three peripheral vertices u, v, p, which is a contradiction.

Next suppose that $b = a+1$ and $a+c$ is odd. If in G, x and y have a common eccentric vertex (which must be one of the two leaves), say u, then s is a common eccentric vertex of x and y in B. Note that now s lies in P_3. Such a situation occurs only if $a = 1$, and hence $b = 2$. Let q be the internal vertex of P_2. Then $d_G(q, u) = d + 1$. Thus G has at least three peripheral vertices u, v, q, which is a contradiction.

If in G, x and y do not have a common eccentric vertex, then one of u and v is an eccentric vertex of x and the other is an eccentric vertex of y. The conditions $a + b + c = n - 1$, $b = a + 1$, and $n \geq 18$ imply that $a + c \geq 8$. Hence $d - 1 = \lfloor (a + c)/2 \rfloor \geq 4$. Since $d_G(u, v) = d + 1$, we have $d_B(s, t) = d - 1 \geq 4$. Note that s and t lie in P_3. Choose two adjacent vertices v_1 and v_2 on P_3 between s and t. Since the cycle $P_1 \cup P_3$ is odd, v_1 and v_2 have a common antipodal vertex z on P_1. It is easy to verify that $\mathrm{ecc}_G(z) = d - 1$, contradicting the fact that $\mathrm{rad}(G) = d$.

If G has no leaf, by Lemma 2.4, $G = \theta_{1,2,n-2}$. Thus $g(G) = 3 < 4\lfloor n/6 \rfloor$.

Finally, we consider the case when G has exactly one leaf. Let u be the leaf and let s be its neighbor. Note that $s \in V(B)$, since otherwise the vertices of G would have at least three distinct eccentricities, contradicting the assumption that G is almost self-centered. We continue using the notations $d = \mathrm{diam}(B)$ and $f = \mathrm{diam}(G)$. If $f = d$, then G would have at least three peripheral vertices, a contradiction. It is also impossible that $f \geq d+2$, since adding a leaf increases the eccentricity of any vertex by at most 1. Hence $f = d+1$. Clearly $\mathrm{rad}(B) \geq d-1$.

We assert that B is self-centered; i.e., $\mathrm{rad}(B) = d$. To the contrary, suppose $\mathrm{rad}(B) = d-1$. Then $\mathrm{ecc}_B(x) = \mathrm{ecc}_B(y) = d-1$. Since $\mathrm{rad}(G) = d$, we deduce that u must be the common eccentric vertex of x and y, implying that s is a common antipodal vertex of x and y on the cycle $P_1 \cup P_3$. As argued above, $a+c$ is odd and $a = 1$. By Lemma 2.3, $b = 2$ or $b = 3$. If $b = 2$, let z be a neighbor of s on P_3. Then $\mathrm{ecc}_G(z) = d-1$, contradicting the fact that $\mathrm{rad}(G) = d$. If $b = 3$, let w_1 be the neighbor of x on P_2 and let w_2 be the neighbor of y on P_2. Then it is easy to check that G has at least three peripheral vertices u, w_1, w_2, which is a contradiction again. Thus B is self-centered.

By Lemma 2.3, $b = a$ if $a+c$ is odd and $b \leq a+1$ if $a+c$ is even. We have $a+b+c = n$, and clearly $g(G) = a+b$. There are two possibilities: (1) $b = a$; (2) $b = a+1$ and $a+c$ is even. Denote $k = \lfloor n/6 \rfloor$.

(1) Suppose $b = a$. We have $3a \leq n$, implying that $a \leq n/3$. If $n = 6k$ or $n = 6k+2$, we obtain $g(G) = 2a \leq 4k$. If $n = 6k+4$, we have $a \leq 2k+1$. The case $a = 2k+1$ will be excluded. Assume $a = 2k+1$. Then $b = 2k+1$ and $c = 2k+2$. Thus $a+c = b+c$ is odd. But then G has at least three peripheral vertices, a contradiction. Hence $a \leq 2k$ and $g(G) = 2a \leq 4k$.

(2) Suppose $b = a+1$ and $a+c$ is even. We have $a \leq (n-1)/3 \leq 2k+1$, where in the second inequality we have used $n \leq 6k+4$. But it is impossible that $a = 2k+1$, since otherwise $c = 2k+1 < 2k+2 = b$, contradicting our assumption that $b \leq c$. The conditions $a+b+c = n$, $b = a+1$ and that both n and $a+c$ are even imply that a is odd. Thus $a = 2k$ is also impossible. It follows that $a \leq 2k-1$ and consequently $g(G) = a+b = 2a+1 \leq 4k-1$.

Finally, we prove that the upper bound $4\lfloor n/6 \rfloor$ can be attained and when 6 divides n, the extremal graph is unique. Denote $k = \lfloor n/6 \rfloor$. Let G be the graph obtained from $\theta_{2k,2k,n-4k}$ by attaching an edge to one of the two vertices of degree three. Then G is an almost self-centered graph of order n with girth $4\lfloor n/6 \rfloor$.

Suppose G is an almost self-centered graph of order $n = 6k \geq 18$ with girth $4k$. Then the above analysis shows that G is a graph obtained from $\theta_{a,b,c}$ by attaching an edge where $a = b$. Since $g(G) = a+b = 2a = 4k$, we have $a = b = 2k$. The condition $a+b+c = n$ further implies $c = 2k$. Thus the theta is $\theta_{2k,2k,2k}$. There is only one way to attach an edge to this theta so that the resulting graph is almost self-centered; i.e., attach the edge to a vertex of degree three. This shows that the extremal graph is unique. The proof is complete. □

One conclusion in Theorem 2.5 states that if $n \geq 12$ and 6 divides n, then the extremal graph for $g(n)$ is unique. We remark that if n is even with $n \geq 14$ and 6 does not divide n, then there are at least three extremal graphs for $g(n)$. This can be seen as follows. Using the notations in the proof of Theorem 2.5, we may attach an edge to any vertex on P_1 of the theta $\theta_{2k,2k,n-2k}$ to obtain an extremal graph.

Next we consider the independence number. There is only one almost self-centered graph of order n and radius 1; i.e., the graph obtained from K_n by deleting an edge.

Theorem 2.6. *The maximum independence number of an almost self-centered graph of order n and radius r with $r \geq 2$ is $n - r$.*

Proof. Let G be an almost self-centered graph of order n and radius r. First recall that $\text{diam}(G) = \text{rad}(G) + 1 = r + 1$. Let P be a diametral path of G. If $r = 2$, P has order 4. Any independent set can contain at most 2 of the four vertices on P. Thus $\alpha(G) \leq n - 2$.

Suppose $r \geq 3$. Let x be a central vertex of the path P. Now P has order at least 5 and no vertex on P is an eccentric vertex of x. Let y be an eccentric vertex of x. It is known [3] that the center of any connected graph lies within one block. Let B be the block of G in which $C(G)$ lies. Then $x, y \in V(B)$. By Menger's theorem [10, p. 167], there are two internally disjoint (x, y)-paths Q_1 and Q_2. Denote by k the length of the cycle $D = Q_1 \cup Q_2$. Then $k \geq 2r$. Any independent set can contain at most $\lfloor k/2 \rfloor$ vertices on D. Thus $\alpha(G) \leq \lfloor k/2 \rfloor + (n - k) = n - \lceil k/2 \rceil \leq n - (k/2) \leq n - r$.

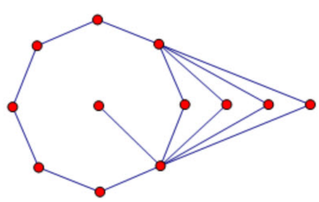

Figure 2.2: The graph $Z(12, 4)$.

Conversely, we construct a graph to show that the upper bound $n - r$ can be attained. Attaching an edge to the cycle $v_1, v_2, \ldots, v_{2r}$ at the vertex v_1 we obtain a graph H. Adding $n - 2r - 1$ new vertices to H such that each of them has v_1 and v_3 as neighbors, we obtain the graph $Z(n, r)$. It is easy to see that $Z(n, r)$ is an almost self-centered graph of order n and radius r with independence number $n - r$. The graph $Z(12, 4)$ is depicted in Figure 2.2. □

Corollary 2.7. *The maximum independence number of an almost self-centered graph of order n with $n \geq 5$ is $n - 2$, and there are exactly two extremal graphs.*

Proof. By Theorem 2.6 and the fact that the almost self-centered graph of order n and radius 1 has independence number 2, we deduce that the maximum independence number is $n - 2$.

Suppose G is an almost self-centered graph of order n whose independence number is $n-2$. By Theorem 2.6, $\mathrm{rad}(G)=2$ and consequently $\mathrm{diam}(G)=3$. Let $P: x_1, x_2, x_3, x_4$ be a diametral path of G. Then x_1 and x_4 are the two peripheral vertices of G. Denote $S = V(G) \setminus \{x_1, x_2, x_3, x_4\}$. G has only one maximum independent set; i.e., $S \cup T$ where T consists of two vertices from P. There are three possible choices for T: $\{x_1, x_3\}$, $\{x_2, x_4\}$ and $\{x_1, x_4\}$, the first two of which will yield isomorphic graphs. Since every leaf of an almost self-centered graph is a peripheral vertex, every vertex in S has degree at least 2. If $T = \{x_1, x_3\}$, then every vertex in S has x_2 and x_4 as neighbors; if $T = \{x_1, x_4\}$, then every vertex in S has x_2 and x_3 as neighbors. Conversely, it is easy to see that these two graphs satisfy all the requirements. □

Now we consider regular almost self-centered graphs.

Theorem 2.8. *Let $r(k)$ denote the minimum order of a k-regular almost self-centered graph. Then*

$$r(k) = \begin{cases} 12 & \text{if } k = 3, \\ 2k+2 & \text{if } k \geq 4. \end{cases}$$

Proof. Let G be a k-regular almost self-centered graph of order n, and let x and y be the two peripheral vertices of G. There is only one almost self-centered graph of order n and diameter at most 2; i.e., the graph obtained from K_n by deleting an edge. Thus $\mathrm{diam}(G) \geq 3$, implying that $N[x] \cap N[y] = \phi$. It follows that

$$n \geq |N[x]| + |N[y]| = (k+1) + (k+1) = 2k+2.$$

We first show $r(3) = 12$. Suppose $k = 3$. Then n is even and $n \geq 2 \times 3 + 2 = 8$. We will exclude the two orders 8 and 10. If $n = 8$, then $\mathrm{diam}(G) = 3$ and $G - \{x, y\}$ is a 2-regular graph of order 6, which must be C_6 or $2C_3$. In each case, G has at least four peripheral vertices, a contradiction.

If $n = 10$, we deduce that $\mathrm{diam}(G) = 3$, since otherwise either G has a vertex of degree at least 4 or G has three peripheral vertices. Recall that $N_i(x) = \{v \in V(G) \mid d(x, v) = i\}$. We have $|N_1(x)| = 3$ and $|N_3(x)| = 1$, implying $|N_2(x)| = 5$. Here we have used the fact that G has exactly two peripheral vertices. Note that each vertex in $N_2(x)$ has at least one neighbor in $N_1(x)$. Analyzing possible adjacency relations in $G - \{x, y\}$, we deduce that G has at least four peripheral vertices, a contradiction.

Thus we have proved that $n \geq 12$. On the other hand, the graph depicted in Figure 2.3 is a 3-regular almost self-centered graph of order 12. This shows $r(3) = 12$.

Next suppose $k \geq 4$. We have proved above that any k-regular almost self-centered graph has order at least $2k+2$. To show $r(k) = 2k+2$, it suffices to construct such a graph R of order $2k+2$. Let $V(R) = \{x_0, y_0\} \cup A \cup B$, where $A = \{x_1, x_2, \ldots, x_k\}$ and

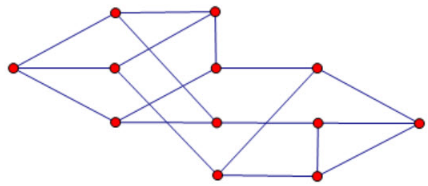

Figure 2.3: A cubic ASC graph of order 12.

$B = \{y_1, y_2, \ldots, y_k\}$. We use the notation $u \leftrightarrow v$ to mean that the two vertices u and v are adjacent. If k is even, the adjacency of R is defined as follows:

$$N(x_0) = A, \quad N(y_0) = B, \quad N(x_i) = \{x_{i+k/2}, y_i, y_{i+1}, \ldots, y_{i+k-3}\} \text{ if } 1 \le i \le k/2,$$
$$N(x_j) = \{x_{j-k/2}, y_j, y_{j+1}, \ldots, y_{j+k-3}\} \text{ if } k/2+1 \le j \le k, \quad y_i \leftrightarrow y_{i+k/2} \text{ if } 1 \le i \le k/2.$$

If k is odd, the adjacency of R is defined as follows:

$$N(x_0) = A, \quad N(y_0) = B, \quad N(x_1) = \{x_2, x_3, \ldots, x_{(k+1)/2}\} \cup \{y_1, y_2, \ldots, y_{(k-1)/2}\},$$
$$N(x_i) = \{x_1\} \cup (B \setminus \{y_{i-1}, y_k\}) \quad \text{if } 2 \le i \le (k+1)/2,$$
$$N(x_j) = B \setminus \{y_{j-(k+1)/2}\} \quad \text{if } (k+3)/2 \le j \le k,$$
$$N(y_k) = \{x_{(k+3)/2}, \ldots, x_k\} \cup \{y_1, y_2, \ldots, y_{(k-1)/2}\}.$$

Here the subscripts of the vertices are taken modulo k. It is easy to verify that R is a k-regular almost self-centered graph of order $2k+2$ with periphery $\{x_0, y_0\}$ and center $A \cup B$. □

3. Almost peripheral graphs

Theorem 3.1. *The maximum size of an almost peripheral graph of order n is $\lfloor (n-1)^2/2 \rfloor$. If n is odd, this maximum size is attained uniquely by the graph $\overline{K_1 + ((n-1)/2)K_2}$; if n is even, this maximum size is attained uniquely by the graph $\overline{K_1 + ((n-4)/2)K_2 + P_3}$.*

Proof. Use the fact that an almost peripheral graph can have at most one dominating vertex and the degree sum formula. □

In the following result we determine which numbers are possible for the maximum degree of an almost peripheral graph with a given order.

Theorem 3.2. *There exists an almost peripheral graph of order $n \ge 7$ with maximum degree Δ if and only if $\Delta \in \{3, 4, \ldots, n-4, n-1\}$.*

Proof. Suppose that G is an almost peripheral graph of order $n \ge 7$ with maximum degree Δ. Clearly $3 \le \Delta \le n-1$. We first exclude the two values $n-2$ and $n-3$ for Δ. To the contrary suppose $\Delta = n-2$ or $n-3$. Note that $\text{rad}(G) \geq 2$ and $\text{diam}(G) = \text{rad}(G)+1 \geq 3$. Let $x \in V(G)$ with $\deg(x) = \Delta$. There exists a vertex y with $y \notin N[x]$ such that y and x have a common neighbor w.

If $\Delta = n-2$, then both x and w have eccentricity at most 2, implying that they are central vertices, a contradiction.

Suppose $\Delta = n-3$. Let z be the vertex outside $N[x] \cup \{y\}$. We always have $\text{rad}(G) = 2$ and hence $\text{diam}(G) = \text{rad}(G) + 1 = 3$.

If z and y are adjacent, then w is the central vertex. Since $\text{ecc}(x) = 3$ and z is the only possible eccentric vertex of x, we deduce that z is nonadjacent to any vertex in $N[x]$. It follows that z is a leaf. Since $\text{ecc}(z) = 3$, y is another central vertex, a contradiction.

If z and y are nonadjacent, then $N(x) \cap N(z) \neq \phi$. In this case, x is the central vertex. Since $\text{diam}(G) = 3$, there exists a (y, z)-path P of length 2 or 3. Then any internal vertex of P is a central vertex different from x, a contradiction.

Conversely, we will show that every number in $\{3, 4, \ldots, n-4, n-1\}$ can be attained. The star of order n is an almost peripheral graph with maximum degree $n-1$. Next, for each Δ with $3 \leq \Delta \leq n-4$ we construct an almost peripheral graph $G(n, \Delta)$ of order n with maximum degree Δ. We will first construct all $G(n, 3)$ for $n = 7, 8, \ldots$, and then inductively construct the remaining $G(n, \Delta)$ with $\Delta \geq 4$.

$G(7, 3)$, $G(8, 3)$, $G(9, 3)$, and $G(10, 3)$ are depicted in Figure 3.1.

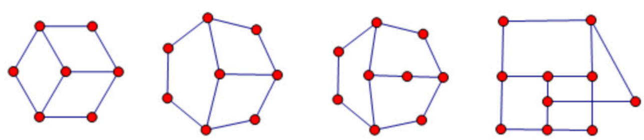

Figure 3.1: $G(n, 3)$ with $n = 7, 8, 9, 10$.

We will need the four preliminary graphs in Figure 3.2.

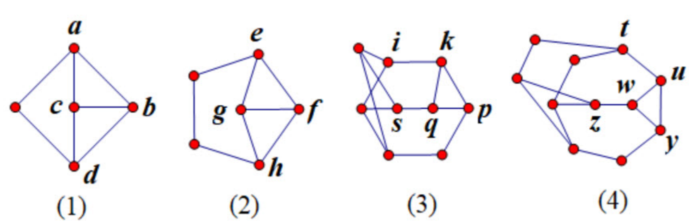

Figure 3.2: Preliminary graphs for $G(n, 3)$.

Now let $n \geq 11$ and denote $k = \lfloor (n+5)/4 \rfloor$. If $n \equiv 3 \pmod 4$, $G(n, 3)$ is obtained from the graph in Figure 3.2(1) by replacing the edges ab and bd by a path of length $k-1$, and replacing the edges ac and cd by a path of length $k-2$; if $n \equiv 0 \pmod 4$, $G(n, 3)$

is obtained from the graph in Figure 3.2(2) by replacing the edges ef and fh by a path of length $k-1$, and replacing the edges eg and gh by a path of length $k-2$; if $n \equiv 1 \pmod 4$, $G(n,3)$ is obtained from the graph in Figure 3.2(3) by replacing the edges ik, kp, sq and qp by a path of length $k-2$, $k-1$, $k-3$ and $k-2$, respectively; if $n \equiv 2 \pmod 4$, $G(n,3)$ is obtained from the graph in Figure 3.2(4) by replacing the edges tu, uy, zw and wy by a path of length $k-3$, $k-1$, $k-3$ and $k-2$, respectively.

For a vertex v in a graph, the operation *duplicating* v means that adding a new vertex x and adding edges incident to x such that $N(x) = N(v)$.

Note that for $n = 7$, $n-4 = 3$ and that every $G(n,3)$ constructed above contains a vertex of degree 3 that has a non-central neighbor of degree 2. Now suppose that we have constructed $G(n,3), G(n,4), \ldots, G(n,n-4)$, where $G(n,\Delta)$ contains a vertex of degree Δ that has a non-central neighbor x_Δ of degree 2, $\Delta = 3, \ldots, n-4$, Then in $G(n,\Delta)$, duplicate the vertex x_Δ to obtain a new graph which we denote by $G(n+1, \Delta+1)$. Thus we can construct $G(n+1,3), G(n+1,4), \ldots, G(n+1,n-3)$ which satisfy all the requirements and the additional condition of containing a vertex of maximum degree that has a non-central neighbor of degree 2. Thus the inductive steps can continue. □

Finally, we consider the maximum number of vertices of maximum degree in an almost peripheral graph.

Definition 3.3. *Blowing up a vertex v in a graph into the complete graph K_t* is the operation of replacing v by K_t and adding edges joining each vertex in $N(v)$ to each vertex in K_t.

Definition 3.4. A vertex v in a graph G is called a *top vertex* if $\deg(v) = \Delta(G)$.

Theorem 3.5. *The maximum number of top vertices in an almost peripheral graph of order $n \geq 8$ with maximum degree $n-4$ is $n-5$ and this maximum number is uniquely attained by the graph obtained from the graph of order 7 in Figure 3.1 by blowing up a non-central vertex of degree 3 into K_{n-6}.*

Proof. First, it is easy to verify that the extremal graph given in Theorem 3.5 is an almost peripheral graph of order n with maximum degree $n-4$ that has $n-5$ top vertices. Let G be an almost peripheral graph of order $n \geq 8$ with maximum degree $n-4$. We may suppose that G has at least three top vertices, since otherwise the number of top vertices in G is less than $n-5$. Recall that $\mathrm{diam}(G) = \mathrm{rad}(G) + 1 \geq 2$.

Let x be a peripheral vertex of degree $n-4$. Then there are only three vertices outside $N[x]$. We will use the fact that every vertex in $N_i(x)$ has at least one neighbor in $N_{i-1}(x)$ for $1 \leq i \leq \mathrm{diam}(G)$. The proof consists of a series of claims.

Claim 1*:* $\mathrm{diam}(G) = 3$.

Clearly $\mathrm{diam}(G) = \mathrm{ecc}(x) \leq 4$. If $\mathrm{ecc}(x) = 4$, let x, r, s, p, q be a diametral path. Then $\mathrm{ecc}(r) \leq 3$ and $\mathrm{ecc}(s) \leq 3$, implying that both r and s are central vertices, a contradiction. Thus $\mathrm{diam}(G) \leq 3$. On the other hand, it is impossible that $\mathrm{diam}(G) = 2$, since otherwise $\mathrm{rad}(G) = 1$, implying that $\Delta(G) = n - 1$, a contradiction. Hence $\mathrm{diam}(G) = 3$.

Claim 2: The vertex x has only one eccentric vertex, which is not a leaf.

If $|N_3(x)| = 2$, then $|N_2(x)| = 1$. Now the vertex in $N_2(x)$ and its neighbors in $N(x)$ are central vertices, a contradiction. Thus x has only one eccentric vertex, which we denote by w. Let $N_2(x) = \{u, v\}$. If w is a leaf, without loss of generality, suppose u is the neighbor of w. Since $\mathrm{ecc}(w) \leq 3$, we deduce that $\mathrm{ecc}(u) \leq 2$; i.e., u is a central vertex. If u and v are adjacent, then every neighbor of u in $N(x)$ is also a central vertex, a contradiction; if u and v are nonadjacent, then $d(u, v) = 2$, implying that u and v have a common neighbor y in $N(x)$. But then y is also a central vertex, a contradiction again.

Claim 2 shows that $N(w) = \{u, v\}$.

Claim 3: u and v have at most one common neighbor in $N(x)$.

This holds since every common neighbor of u and v in $N(x)$ is a central vertex.

Claim 4: u and v are nonadjacent.

To the contrary, assume that u and v are adjacent. Then any neighbor of either u or v in $N(x)$ is a central vertex. It follows that u and v have a common neighbor y in $N(x)$ and y is their only neighbor in $N(x)$. Now any vertex $z \in N(x) \setminus \{y\}$ must be adjacent to y, since $d(z, w) \leq 3$. Consequently, $\deg(y) = n - 2 > n - 4 = \Delta(G)$, a contradiction.

Claim 5: Neither u nor v is a top vertex.

To the contrary, assume $\deg(u) = n - 4$. By Claim 4, $|N(u) \cap N(x)| = n - 5$. Let $z \in N(x)$ be the nonneighbor of u. If $d(u, z) = 2$, then u is a central vertex and u, v have no common neighbor in $N(x)$. Hence v is adjacent to z. Let y be a common neighbor of u and z. Then y is also a central vertex, a contradiction. Hence $d(u, z) = 3$. The condition $d(z, w) \leq 3$ implies that z is adjacent to v. If u and v have no common neighbor in $N(x)$, then G is self-centered, a contradiction. If u and v have a common neighbor in $N(x)$, then x and u are the only two vertices with degree $n - 4$, contradicting our assumption that G has at least three top vertices.

Similarly, we can prove that $\deg(v) < n - 4$.

Claim 6: Each of u and v has at least two neighbors in $N(x)$.

To the contrary, assume that $N(u) \cap N(x) = \{y\}$. Then for any vertex $z \in N(x) \setminus \{y\}$, z cannot be adjacent to both y and v, since otherwise y and z are central vertices. Considering $d(z, u)$ and $d(z, w)$ we deduce that z is a peripheral vertex. If y and v are nonadjacent, then G is self-centered, a contradiction. If y and v are adjacent, then y is the central vertex. Since $d(z, w) \leq 3$, we obtain $d(z, v) \leq 2$. It follows that v is also a central vertex, a contradiction.

Similarly, we can prove that v has at least two neighbors in $N(x)$.

Claim 7: G has at most $n-5$ top vertices and the extremal graph is unique.

By Claims 3 and 6, u has a neighbor y in $N(x)$ that is nonadjacent to v, and v has a neighbor z in $N(x)$ that is nonadjacent to u. Note that if f is a neighbor of u in $N(x)$ and g is a neighbor of v in $N(x)$ with $f \neq g$, then f and g are nonadjacent, since otherwise f and g are central vertices. Using Claim 6 again we deduce that neither y nor z has maximum degree. Thus G has at least the five vertices y, z, u, v, w with degrees less than $n-4$. It follows that G has at most $n-5$ top vertices.

Conversely, suppose G has $n-5$ top vertices. Then the above analysis shows that (1) each of u and v has exactly two neighbors in $N(x)$; (2) u and v have exactly one common neighbor h in $N(x)$; and (3) the closed neighborhood of every vertex in $N(x) \setminus \{y, z, h\}$ is equal to $N[x]$. Consequently, G is the graph obtained from the graph of order 7 in Figure 3.1 by blowing up a non-central vertex of degree 3 into K_{n-6}. This completes the proof. □

The extremal graph of order 10 in Theorem 3.5 is depicted in Figure 3.3.

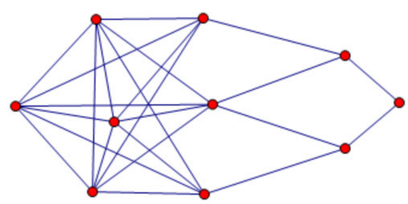

Figure 3.3: The extremal graph of order 10.

Acknowledgments

This research was supported by the NSFC grant 11671148 and Science and Technology Commission of Shanghai Municipality (STCSM) grant 18dz2271000.

References

[1] B. Bollobás and E. Szemerédi, *Girth of sparse graphs*, J. Graph Theory **39** (2002), no. 3, 194–200.

[2] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1990.

[3] F. Harary and R. Z. Norman, *The dissimilarity characteristic of Husimi trees*, Ann. of Math. (2) **58** (1953), no. 1, 134–141.

[4] Y. Hu and X. Zhan, *Possible cardinalities of the center of a graph*, Bull. Malays. Math. Sci. Soc. **44** (2021), no. 6, 3629–3636.

[5] S. Klavžar, H. Liu, P. Singh and K. Xu, *Constructing almost peripheral and almost self-centered graphs revisited*, Taiwanese J. Math. **21** (2017), no. 4, 705–717.

[6] S. Klavžar, K. P. Narayankar, S. P. Narayankar and D. Shubhalakshmi, *Simplified constructions of almost peripheral graphs and improved embeddings into them*, Filomat **32** (2018), no. 4, 1193–1198.

[7] S. Klavžar, K. P. Narayankar and H. B. Walikar, *Almost self-centered graphs*, Acta Math. Sin. (Engl. Ser.) **27** (2011), no. 12, 2343–2350.

[8] S. Klavžar, K. P. Narayankar, H. B. Walikar and S. B. Lokesh, *Almost-peripheral graphs*, Taiwanese J. Math. **18** (2014), no. 2, 463–471.

[9] L. M. Lesniak-Foster, *Eccentric sequences in graphs*, Period. Math. Hungar. **6** (1975), no. 4, 287–293.

[10] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 1996.

[11] K. Xu, H. Liu, K. C. Das and S. Klavžar, *Embeddings into almost self-centered graphs of given radius*, J. Comb. Optim. **36** (2018), no. 4, 1388–1410.

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