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Vlasov–Poisson Equation in Besov Space

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Abstract. We study the local-in-time well-posedness of Vlasov–Poisson equation in Besov space for the large initial data. To accomplish it, we establish commutator estimates in Besov space which are quite useful in dealing with the electronic term $\nabla_x \phi$. Also, the L^p-L^q type estimates for the electronic term $\nabla_x \phi$ are established, which are not only useful in the estimate for Poisson equation, but also play a fundamentally important role in commutator estimates involving the electronic term $\nabla_x \phi$.

1. Introduction

Understanding the evolution of a distribution of particles over time is a major research area of statistical physics. The Vlasov–Poisson equation is one of the key equations governing this evolution. Specifically, it models particle behaviors with long range interactions in a non-relativistic zero-magnetic field setting. Two principal types of long range interactions are Coulomb's forces, the electrostatic repulsion of similarly charged particles in a plasma, and Newtonian's forces, the gravitational attraction of stars in a galaxy. The general Cauchy's problem for the Vlasov–Poisson equation (VP) in n-dimensional space is as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \\ -\Delta_x \phi = \int_{\mathbb{R}^n} f \, dv, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.1}$$

where $f(t,x,v)$ denotes the distribution function of particles, $x \in \mathbb{R}^n$ is the position, $v \in \mathbb{R}^n$ is the velocity, $t > 0$ is the time, and the dimension $n \geq 3$.

The Cauchy problem for VP equation has been studied for several decades. In 1977 Batt [7] established the global existence for the spherically symmetric data. Shortly, in 1981 Horst [17] extended the global classical solvability to the cylindrically symmetric data. Next, in 1985, Bardos and Degond [6] obtained the global existence for "small" data. Finally, in 1989 Pfaffelmoser [25] proved the global existence of a smooth solution

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with large data. Later, more results were published by Schaeffer [27], Horst [18], Lions and Pertharne [23] and Pallard [24]. However, these literature mentioned above mainly considered solutions in L^1 and L^∞ framework. Also, there are many papers studying Vlasov–Poisson–Boltzmann (Landau) equation in L^2 setting, for example, see [2, 12, 13, 15, 16, 19, 21] and references therein. A natural question is: whether there is a solution for VP equation in L^p-setting, for example the fractional Sobolev space such as Besov space, which would not be classified into L^1 or L^∞ framework. This becomes our theme in this paper.

Let us review some known results about other partial differential equations in the fractional L^p-setting. There is a large amount of literature investigating the solutions in the fractional differential spaces, such as Besov space [5], Triebel–Lizorkin space [9] and Modulation space [31]. For instance, in [5], Bahouri et al. extensively studied solutions of Navier–Stokes equations, Schrödinger equation and wave equation in Besov space; in addition, heat equation and Navier–Stokes equation, Euler equation and wave equation in Triebel–Lizorkin space were studied in [3, 9, 10]. Furthermore, Wang et al. [31] studied solutions for Navier–Stokes equations, Schrödinger equation and wave equation systematically in Modulation spaces. However, there are rare papers investigating VP equation (as a special type of the kinetic equation) in fractional Sobolev spaces, and this becomes our motivation to do some research in this direction.

Our main objective in this paper is to prove the well-posedness of VP equation in Besov space. The specificity of our approach is to provide a commutator estimate which is proved by various techniques including Littlewood–Paley decomposition, Bony decomposition (the paradifferential calculus), and the off-diagonal L^p-L^q estimate for the electronic field term $\nabla_x\phi$.

To achieve our goal, the most difficult part lies in estimates for the electronic field term $\nabla_x\phi$. This is embodied in two aspects. One is the commutator estimate, typically, $[\Delta_j^x, \nabla_x\phi]\nabla_v f$; in our paper, we do not perform $[\Delta_j^{x,v}, \nabla_x\phi]\nabla_v f$ since $\nabla_x\phi$ does not depend on the variable v, this trick captures the anisotropy in some sense. More concretely, we will establish the commutator estimate of $\left|2^{js}\left\|[\Delta_j^x, \nabla_x\phi]\nabla_v f\right\|_{L^p_{x,v}(w)}\right|_{l_j^r}$ which is proved in the VP setting for the first time as far as we know. The other one is L^p-L^q estimate for electronic term $\nabla_x\phi$ in Besov setting. The off-diagonal L^p-L^q estimate is not only favorable to the commutator estimate but also represents the character of Poisson equation. Attentive readers could find the delicate relation between the off-diagonal estimate and the commutator estimate proved in Sections 3 and 4 separately. More precisely, we used various techniques to establish the following original and fundamentally important inequality

$$\left|2^{js}\left\|[\Delta_j^x, \nabla_x\phi^k]\nabla_v f^{k+1}\right\|_{L^p_{x,v}(w)}\right|_{l_j^r} \lesssim \|\nabla_v f^{k+1}\|_{B^{s-1}_{p,r,x}(w)} \cdot \|\nabla_x\phi^k\|_{B^s_{q,r}}.$$

This inequality embodies all the essential ingredients including an off-diagonal estimate, commutator estimates, the structure of Besov space and VP.

Before we state our main theorem, we first need to introduce some common notations and definitions.

2. Preliminaries and main theorem

2.1. Notations and definitions

First of all, we list our notations and definitions.

- Given a locally integrable function f, the maximal function Mf is defined by

$$(Mf)(x) = \sup_{\delta>0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|\, dy,$$

  where $|B(x,\delta)|$ is the volume of the ball of $B(x,\delta)$ with center x and radius δ.

- Given $f \in \mathscr{S}$, the Schwartz class, its Fourier transform $\mathscr{F} f = \widehat{f}$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx,$$

  and its inverse Fourier transform is defined by $\mathscr{F}^{-1} f(x) = \widehat{f}(-x)$.

- For $0 < \alpha < n$, we let I_α be the Riesz potential operator defined for locally integrable functions by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

- $w = \langle v\rangle^\gamma$, $\gamma \cdot p'/p > n$, $1/p' + 1/p = 1$, $\langle v\rangle = (1+|v|^2)^{1/2}$.

- $|\{a_j\}|_{l^r} = \big(\sum_{j=0}^\infty |a_j|^r\big)^{1/r}$, $\|f\|_{L^p} = \big(\int_{\mathbb{R}^n} |f(x)|^p\, dx\big)^{1/p}$, $\|f_j\|_{l^r_j(L^p)} = \big(\sum_{j=0}^\infty \|f_j\|_{L^p}^r\big)^{1/r}$ and $\|f\|_{L^p_{x,v}(w)} = \big(\int_{\mathbb{R}^{2n}} |f(x,v)|^p \cdot w\, dxdv\big)^{1/p}$.

- $A \lesssim B$ means there exists a positive constant c independent of the main parameters such that $A \le cB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2.2. Definition of Besov space

We recall the definition of Besov space [29,30]. Let $\varphi \in \mathscr{S}$ satisfying

$$\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\} \quad \text{and} \quad \varphi(\xi) > 0 \ \text{ if } 1/2 \le \xi \le 2.$$

Setting $\varphi_j(\xi)=\varphi(2^{-j}\xi)$ with $j=\{1,2,\dots,\}$, we can adjust the normalization constant in front of φ and choose $\varphi_0\in\mathscr{S}$ satisfying $\operatorname{supp}\varphi_0\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2\}$, such that

$$\sum_{j=0}^{\infty}\varphi_j(\xi)=1,\quad \forall\,\xi\in\mathbb{R}^n.$$

Given $j\in\mathbb{N}$, we define the function $S_j\in\mathscr{S}$ by

$$S_j(\xi)=1-\sum_{j'\ge j+1}\varphi_{j'}(\xi).$$

In particular, we set $\varphi_j(\xi)\equiv 0$ for $j<0$, then we have $S_j(\xi)\equiv 0$, $j<0$. We also observe

$$\operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_{j'}=\emptyset\quad \text{if } |j-j'|\ge 2. \tag{2.1}$$

Given $f\in\mathscr{S}'$, we denote $\Delta_j f=\mathscr{F}^{-1}\varphi_j\mathscr{F}f$. For $(s,p,r)\in\mathbb{R}\times[1,\infty]\times[1,\infty]$ and then we define the inhomogeneous Besov space by

$$B^s_{p,r}(\mathbb{R}^n)=\Big\{f\in\mathscr{S}':\|f\|_{B^s_{p,r}}=\big|2^{js}\|\Delta_j f\|_p\big|_{l^r_j}\le\infty\Big\}$$

with the usual interpretation for $p=\infty$ or $r=\infty$. We will use the abbreviation $B^s_{p,r}=B^s_{p,r}(\mathbb{R}^n)$ for simplicity whenever there is no confusion arising.

With notations in the definition of Besov space in mind, we also denote

$$\Delta^x_j f=\mathscr{F}_x^{-1}\varphi_j(\eta)\mathscr{F}_x f,\quad \Delta^v_j f=\mathscr{F}_v^{-1}\varphi_j(\xi)\mathscr{F}_v f,$$
$$\Delta_j f=\Delta^{x,v}_j f=\mathscr{F}^{-1}_{x,v}\varphi_j(\xi,\eta)\mathscr{F}_{x,v}f,\quad \|\nabla_x f\|_{B^s_{p,r}}=\sum_{i=1}^n\|\partial_{x_i}f\|_{B^s_{p,r}},$$

and

$$\|f\|_{B^s_{p,r,x}(w)}=\big|2^{js}\|\Delta^x_j f\|_{L^p_{x,v}(w)}\big|_{l^r_j},\quad \|f\|_{B^s_{p,r,v}(w)}=\big|2^{js}\|\Delta^v_j f\|_{L^p_{x,v}(w)}\big|_{l^r_j}.$$

In our context, we assume that

$$C_0^{-1}\big(\|f\|_{B^s_{p,r,x}(w)}+\|f\|_{B^s_{p,r,v}(w)}\big)\le\|f\|_{B^s_{p,r}(w)}\le C_0\big(\|f\|_{B^s_{p,r,x}(w)}+\|f\|_{B^s_{p,r,v}(w)}\big)$$

with $C_0>1$ is a generic constant based on the fact (see [14]) that

$$\|f\|_{B^s_{p,r}(w)}\sim\|f\|_{B^s_{p,r,x}(w)}+\|f\|_{B^s_{p,r,v}(w)}.$$

2.3. Known result on pseudo-differential operator and main theorem

The following pseudo-differential operator estimate will be used in the commutator estimate involving the electronic term $\nabla_x\phi$ and the process of proving the main theorem.

Lemma 2.1. [11] *Define* $Af(x) =: \mathscr{F}^{-1}[a(x,\cdot)\mathscr{F}f](x)$ *with* $a \in S^m_{1,0}$. *Assume* $1 < p, q < \infty$, $s \in \mathbb{R}$, *then*

$$A\colon B^{s+m}_{p,q}(w) \to B^s_{p,q}(w)$$

is bounded, where the symbol $a(x,\xi) \in S^m_{1,0}$ *means*

$$\left|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

for any multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$ *and* $\beta = (\beta_1,\dots,\beta_n)$. *Additionally the weight* $w(x)$ *satisfies*

$$w(x) \ge 0, \quad w(x+y) \lesssim w(x)\cdot w(y).$$

Now we are ready to state our main theorem.

Theorem 2.2. *Suppose* $f_0 \in B^s_{p,r}(w)$, *then there exists* $T_0 > 0$ *such that the Cauchy problem of the Vlasov–Poisson system* (1.1) *admits a unique solution in* $L^\infty([0,T_0), B^s_{p,r}(w))$ *satisfying the initial condition* $f(x,v,0) = f_0(x,v)$, *where*

$$s > \frac{n}{p} + 1, \quad 1 < p < \frac{n}{2}, \quad n \ge 3, \quad 1 < r < \infty$$

and

$$w = \langle v\rangle^\gamma, \quad \gamma\cdot\frac{p'}{p} > n, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Remark 2.3. Note that the space $B^{2.04}_{2.9,r}(\mathbb{R}^6_{x,v})$, $1 < r < \infty$ which could not be embedded into $L^\infty(\mathbb{R}^6_{x,v})$, due to the fact $2.04\cdot 2.9 < 6$. Thus, we extend the results in the literature to more generalized function spaces in some sense.

Remark 2.4. From the local existence aspect, with our careful comparison with classical results in L^∞ or continuous function space, e.g., [6] due to C. Bardos and P. Degond, we actually improve their results from the following aspects:

- C. Bardos and P. Degond imposed the pointwise condition like

$$0 \le u_{\alpha,0}(x,v) \le \frac{\epsilon}{(1+|x|)^4\cdot(1+|v|)^4}.$$

  However, the polynomial decay in x variable is not needed at all in our proofs.

- $B^{2.04}_{2.9,r}(\mathbb{R}^6) \not\hookrightarrow C(\mathbb{R}^6)$ which implies that our regularity index can be weaker than C^2 (too strong) in [6].

- Our working space $B^s_{p,r}(\mathbb{R}^n)$ has more flexibility than $C(\mathbb{R}^n)$ or $L^\infty(\mathbb{R}^n)$ since the interplay among the parameters (s,p,r,n), and implies that we can obtain the solutions in more spaces.

Remark 2.5.
- Global existence of (1.1) is an open challenging problem. But we have a blowup criterion: if the life span T of the solution of VP system (1.1) is finite, then

$$\int_0^T \|f(\tau)\|_{B^s_{p,r,x}(w)}\, d\tau = \infty.$$

- One may notice that not all the kinetic equations are compatible with Besov space structure, for example, it is still open whether Vlasov–Poisson–Fokker–Planck equation has solutions in Besov space up to now. Additionally, as far as we know, the study especially for VP equation in fractional differential L^p-Sobolev spaces is very rare. The methods we applied in this paper are quite different from the case when the differential index is an integer.

3. L^p-L^q estimate

In this section, we are devoted to obtaining L^p-L^q estimates for the electronic term $\nabla_x \phi$ since there is no L^p estimate such that $\|\nabla_x \phi\|_{B^s_{p,r}} \lesssim \|g\|_{B^s_{p,r}}$ for the Poisson equation $-\Delta\phi = g$. The off-diagonal estimate is crucially important to obtain the commutator estimate involving the electronic term $\nabla_x\phi$, see Corollary 4.4.

Lemma 3.1 (L^p-L^q estimates)**.** *If $-\Delta\phi = g$, and suppose $1 < p < n/2$ with $1/q = 1/p - 1/n$, it holds that*

$$\|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}.$$

Proof. Note that $\nabla_x\phi = \nabla_x(I_2 * g)$ with $I_2(x) = \frac{\Gamma(n/2)}{(n-2)2\pi^{n/2}} \cdot \frac{1}{|x|^{n-2}}$, therefore there holds

$$\begin{aligned}\|\nabla_x\phi\|_{L^q(\mathbb{R}^n)} = \|\nabla_x(I_2 * g)\|_{L^q(\mathbb{R}^n)} &\lesssim \|(Mg)^{1/2}\cdot(I_2 * |g|)^{1/2}\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \|(Mg)^{1/2}\|_{L^{q_1}(\mathbb{R}^n)}\cdot\|(I_2 * |g|)^{1/2}\|_{L^{q_2}(\mathbb{R}^n)}\\ &\lesssim \|Mg\|^{1/2}_{L^{q_1/2}(\mathbb{R}^n)}\cdot\|I_2 * |g|\|^{1/2}_{L^{q_2/2}(\mathbb{R}^n)},\end{aligned}$$

where we applied Proposition 6.1 and Hölder's inequality with

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}, \quad q_i > 1.$$

On the one hand, the boundedness of Hardy–Littlewood operator yields that

$$\|Mg\|_{L^{q_1/2}(\mathbb{R}^n)} \lesssim \|g\|_{L^{q_1/2}(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)}$$

since we require that $q_1/2 = p$, i.e.,

$$\frac{2}{q_1} = \frac{1}{p}.$$

On the other hand, by Lemma 6.3, we have

$$\|I_2 * |g|\|_{L^{q_2/2}(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)},$$

where

$$\frac{2}{q_2} = \frac{1}{p} - \frac{2}{n}.$$

Consequently, $\|\nabla_x \phi\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|^{1/2}_{L^p(\mathbb{R}^n)} \cdot \|g\|^{1/2}_{L^p(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)}$. □

Remark 3.2. We summarize indices in Lemma 3.1 as follows:

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad q_i > 1,\ i = 1, 2,$$
$$\frac{2}{q_2} = \frac{1}{p} - \frac{2}{n}, \quad \frac{2}{q_1} = \frac{1}{p},$$

which implies that

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}, \quad q_1 = 2p, \quad q_2 = \frac{2np}{n-2p}.$$

The following corollaries will be used in commutator estimates in Section 4. One may observe that the L^p-L^q estimate does not result in the increase of the differential index which is an advantage. As far as we know, this property was never used before in VP equation but is quite powerful in the L^p setting.

Corollary 3.3. *With the same assumptions as in Lemma* 3.1*, we have*

$$\|\nabla_x \phi\|_{B^s_{q,r}} \lesssim \|g\|_{B^s_{p,r}}.$$

Proof. Note that

$$\Delta^x_j \nabla_x \phi = \nabla_x \Delta^x_j \phi = \nabla_x (I_2 * \Delta^x_j g).$$

Applying Lemma 3.1 with ϕ and g replaced by $\Delta^x_j \phi$ and $\Delta^x_j g$ separately, we have

$$\|\Delta^x_j \nabla_x \phi\|_{L^q_x} \lesssim \|\Delta^x_j g\|_{L^p_x}.$$

Multiplying 2^{js} and then taking the l^r norm in terms of j, the corollary is immediate. □

Corollary 3.4 (Weighted L^p-L^q estimates)**.** *If $g(x) = \int_{\mathbb{R}^n} f\,dv$ in Corollary* 3.3*, then we have*

$$\|\nabla_x \phi\|_{B^s_{q,r}} \lesssim \|f\|_{B^s_{p,r,x}(w)}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

Proof. Hölder's inequality leads to

$$|\Delta^x_j g| = \left| \int_{\mathbb{R}^n} \Delta^x_j f\,dv \right| \le \|\Delta^x_j f\|_{L^p_v(w)} \cdot \left(\int_{\mathbb{R}^n} w^{-p'/p} \right)^{1/p'}.$$

Note that $w = \langle v \rangle^\gamma$ and $\gamma \cdot p'/p > n$, which implies that

$$\left(\int_{\mathbb{R}^n} w^{-p'/p} \, dv \right)^{1/p'} \le c.$$

Thus,

$$|\Delta_j^x g| \lesssim \|\Delta_j^x f\|_{L^p_v(w)}. \tag{3.1}$$

Taking an L^p_x norm, then multiplying 2^{js}, and finally taking an l^r norm in terms of j on both sides of (3.1), we obtain

$$\left| 2^{js} \|\Delta_j^x g\|_{L^p_x} \right|_{l^r_j} \lesssim \left| 2^{js} \|\Delta_j^x f\|_{L^p_{x,v}(w)} \right|_{l^r_j}.$$

By Corollary 3.3, we get the desired result. □

An L^∞ estimate in the following is also useful in the proof of existence of the solution to (1.1).

Corollary 3.5. *If $-\Delta\phi = \int_{\mathbb{R}^n} f \, dv$, and $s > n/p - 1$, then we have*

$$\|\nabla_x \phi\|_{L^\infty} \lesssim \|f\|_{B^s_{p,r,x}(w)}.$$

Proof. Note $B^s_{q,r} \hookrightarrow L^\infty$ by Lemma 6.5 with $s > n/q$ and $1/q = 1/p - 1/n$. The result immediately follows from Corollaries 3.3 and 3.4. □

4. Commutator estimates

Commutator estimates in Bessel potential space were studied in [20, 22]. In this section, we aim to prove a general commutator estimate in Besov space which plays a key role in estimating the electronic term $\nabla_x \phi$. Our main tools are Littlewood–Paley decomposition and paradifferential calculus by J.-M. Bony splitting the commutator into several terms. In our proofs, various delicate techniques are applied for different terms.

Proposition 4.1. *Suppose $s > n/p + 1$, $1/p = 1/q + 1/n$, $1 < p, q < \infty$, $1 < r < \infty$ and $f \in B^s_{q,r}(\mathbb{R}^n)$, $g \in B^{s-1}_{p,r}(\mathbb{R}^n)$, then we have*

$$\left| 2^{js} \|[\Delta_j, f] g\|_{L^p(\mathbb{R}^n)} \right|_{l^r_j} \lesssim \|g\|_{B^{s-1}_{p,r}(\mathbb{R}^n)} \|f\|_{B^s_{q,r}(\mathbb{R}^n)}.$$

Proof. We decompose $f(x) = S_{j-2} f(x) + (f(x) - S_{j-2} f(x))$ and then use Bony's formula [4, 8] for the paraproduct of two functions:

$$fg = T_f g + T_g f + R(f, g),$$

where we set

$$T_f g = \sum_{j\in\mathbb{N}} S_{j-2} f \Delta_j g, \quad T_g f = \sum_{j\in\mathbb{N}} S_{j-2} g \Delta_j f, \quad R(f,g) = \sum_{|i-j|\le 1} \Delta_i f \Delta_j g,$$

then we have

$$\begin{aligned}[\Delta_j, f]g &= \Delta_j(fg) - f\Delta_j g \\ &= \Delta_j(T_g f) - [T_f, \Delta_j]g - T_{\Delta_j g}(f - S_{j-2}f) \\ &\quad - R(f - S_{j-2}f, \Delta_j g) + \Delta_j R(f,g) - R(S_{j-2}f, \Delta_j g) \\ &= I - II - III - IV + V - VI.\end{aligned} \tag{4.1}$$

We estimate term by term of (4.1) individually as below.

Estimate of I. We start with estimating the first term I in (4.1). From the definition of the paraproduct and the support condition (2.1), we obtain

$$\begin{aligned}2^{js}\|I\|_{L^p} &= 2^{js}\|\Delta_j(T_g f)\|_{L^p} \le 2^{js} \sum_{|j'-j|\le 3} \|\Delta_j(S_{j'-2} g \cdot \Delta_{j'} f)\|_{L^p} \\ &\lesssim 2^{js} \sum_{|j'-j|\le 3} \|S_{j'-2} g \cdot \Delta_{j'} f\|_{L^p} \lesssim 2^{js} \sum_{|j'-j|\le 3} \|S_{j'-2} g\|_{L^{p_1}} \|\Delta_{j'} f\|_{L^q},\end{aligned} \tag{4.2}$$

where $1/n + 1/q = 1/p$.

Taking an l^r norm on both sides of (4.2) yields

$$\begin{aligned}\left|2^{js}\|I\|_{L^p}\right|_{l^r_j} &= \left|2^{js}\|\Delta_j(T_g f)\|_{L^p}\right|_{l^r_j} \lesssim \|g\|_{L^n} \left| \sum_{|j'-j|\le 3} 2^{j's}\|\Delta_{j'} f\|_{L^q} \right|_{l^r_j} \\ &\lesssim \|g\|_{L^n} \left|2^{j's}\|\Delta_{j'} f\|_{L^q}\right|_{l^r_{j'}} \lesssim \|g\|_{L^n}\|f\|_{B^s_{q,r}} \lesssim \|g\|_{B^{s-1}_{p,r}}\|f\|_{B^s_{q,r}},\end{aligned}$$

where we applied Lemma 6.4 with

$$s - 1 > n\left(\frac{1}{p} - \frac{1}{n}\right),$$

i.e., $s > n/p$.

Remark 4.2. We give a comment on dealing with indices. Note that the upper bound is $\|g\|_{B^{s-1}_{p,r}}\|f\|_{B^s_{q,r}}$ instead of $\|g\|_{L^\infty}\|f\|_{B^s_{p,r}}$. This trick is very useful in our situation as shown in Corollary 4.4. Indeed, we are going to take f as $\nabla_x \phi^k$, g as $\nabla_v f^{k+1}$ in Corollary 4.4.

Estimate of II. More careful attention is needed to obtain the estimate of the second term II in (4.1). Roughly speaking, our technique is to rewrite

$$g = (\widetilde{I} - \Delta_y)(\widetilde{I} - \Delta_y)^{-1} g,$$

and then transfer the derivative $(\widetilde{I} - \Delta_y)$ to f. In what follows, we will show in detail about dealing with the transfer.

Note that

$$II = [T_f, \Delta_j]g = \sum_{j'\in\mathbb{N}}[S_{j'-2}f, \Delta_j]\Delta_{j'}g \quad \text{and} \quad [S_{j'-2}f, \Delta_j]\Delta_{j'}g = 0 \quad \text{if } |j-j'| \geq 4.$$

Thus, we have

$$\begin{aligned}
&[T_f, \Delta_j]g \\
= &\sum_{|j'-j|\leq 3}[S_{j'-2}f, \Delta_j]\Delta_{j'}g \\
= &\sum_{|j'-j|\leq 3} 2^{jn}\int_{\mathbb{R}^n} h(2^j(x-y))\cdot(S_{j'-2}f(x)-S_{j'-2}f(y))(\widetilde{I}-\Delta_y)(\widetilde{I}-\Delta_y)^{-1}g\Delta_{j'}g\,dy \\
= &\sum_{|j'-j|\leq 3} 2^{jn}\int_{\mathbb{R}^n}(\widetilde{I}-\Delta_y)\big[h(2^j(x-y))\cdot(S_{j'-2}f(x)-S_{j'-2}f(y))\big]\Delta_{j'}\big[(\widetilde{I}-\Delta_y)^{-1}g\big]\,dy,
\end{aligned}$$

where $\widetilde{I}$ is an identity operator and $h = \mathscr{F}^{-1}\varphi$.

We also note that

$$\begin{aligned}
&(\widetilde{I}-\Delta_y)\big[h(2^j(x-y))\cdot(S_{j'-2}f(x)-S_{j'-2}f(y))\big] \\
&= h(2^j(x-y))\cdot(S_{j'-2}f(x)-S_{j'-2}f(y)) \\
&\quad -\big[(\Delta h)(2^j(x-y))\cdot 2^{2j}(S_{j'-2}f(x)-S_{j'-2}f(y))\big] \\
&\quad - 2\cdot 2^j(\nabla h)(2^j(x-y))\cdot\nabla(S_{j'-2}f(x)-S_{j'-2}f(y)) \\
&\quad - h(2^j(x-y))\cdot\Delta(S_{j'-2}f(x)-S_{j'-2}f(y)) \\
&= II_1+II_2+II_3+II_4.
\end{aligned} \tag{4.3}$$

In order to get the estimate of II, we need estimate those four terms in (4.3). In the sequel, we are using the notation $\widetilde{g} = (\widetilde{I}-\Delta_y)^{-1}g$ for simplicity.

For II_1, using the integral representation of the convolution, we have

$$\begin{aligned}
&\Big\|2^{js}\sum_{|j'-j|\leq 3}2^{jn}\int_{\mathbb{R}^n} II_1\cdot\Delta_{j'}\widetilde{g}\,dy\Big\|_{l^r_j(L^p)} \\
&\lesssim \Big|\sum_{|j'-j|\leq 3}2^{js}2\|S_{j'-2}f\|_{L^\infty}\|2^{jn}h(2^jy)\|_{L^1}\|\Delta_{j'}\widetilde{g}\|_{L^p}\Big|_{l^r_j} \\
&\lesssim 2^{3s}\|f\|_{L^\infty}\Big|\sum_{|j'-j|\leq 3}2^{j's}\|\Delta_{j'}\widetilde{g}\|_{L^p}\Big|_{l^r_j} \lesssim \|f\|_{L^\infty}\|h\|_{L^1}\Big|\sum_{|j'-j|\leq 3}2^{j's}\|\Delta_{j'}\widetilde{g}\|_{L^p}\Big|_{l^r_j} \\
&\lesssim \|f\|_{L^\infty}\|\widetilde{g}\|_{B^s_{p,r}} \lesssim \|f\|_{L^\infty}\|g\|_{B^{s-2}_{p,r}}.
\end{aligned}$$

For II_2, note that

$$S_{j'-2}f(x)-S_{j'-2}f(y) = \int_0^1[(\nabla S_{j'-2}f)(x+\tau(y-x))\cdot(x-y)]\,d\tau.$$

Thus,

$$\begin{aligned}
&\left\| 2^{js} \sum_{|j'-j|\le 3} 2^{jn} \int_{\mathbb{R}^n} II_2 \cdot \Delta_{j'}\widetilde{g}\, dy \right\|_{l_j^r(L^p)} \\
&\lesssim 2^{3s} \|\nabla S_{j'-2} f\|_{L^\infty} \left\| \sum_{|j'-j|\le 3} 2^{j's} 2^{jn} 2^{2j} \int_{\mathbb{R}^n} |(\Delta h)(2^j(x-y))| \cdot |x-y| \cdot |\Delta_{j'}\widetilde{g}|\, dy \right\|_{l_j^r(L^p)} \\
&\lesssim \|\nabla f\|_{L^\infty} \left\| \sum_{|j'-j|\le 3} 2^{j's} 2^{-j} 2^{2j} \int_{\mathbb{R}^n} |(\Delta h(z))| \cdot |z| \cdot |(\Delta_{j'}\widetilde{g})(x - 2^{-j}z)|\, dz \right\|_{l_j^r(L^p)} \\
&\lesssim \|\nabla f\|_{L^\infty} \left| \sum_{|j'-j|\le 3} 2^{j'(s+1)} \|(\Delta_{j'}\widetilde{g})(x)\|_{L^p} \right|_{l_j^r} \lesssim \|\nabla f\|_{L^\infty} \|\widetilde{g}\|_{B^{s+1}_{p,r}} \lesssim \|\nabla f\|_{L^\infty} \|g\|_{B^{s-1}_{p,r}},
\end{aligned}$$

where we applied changing variable $z = 2^j(x-y)$ in the third line, and used the lifting property [26, 29] in the last inequality above.

For II_3, again using the integral representation of the convolution, we obtain

$$\begin{aligned}
&\left\| 2^{js} \sum_{|j'-j|\le 3} 2^{jn} \int_{\mathbb{R}^n} II_3 \cdot \Delta_{j'}\widetilde{g}\, dy \right\|_{l_j^r(L^p)} \\
&\lesssim 2^{3s} \left\| \sum_{|j'-j|\le 3} 2^{j's} 2^{jn} 2^{j} \int_{\mathbb{R}^n} |(\nabla h)(2^j(x-y))| \|\nabla S_{j'-2} f\|_{L^\infty} |\Delta_{j'}\widetilde{g}|\, dy \right\|_{l_j^r(L^p)} \\
&\lesssim \|\nabla f\|_{L^\infty} \|\nabla h\|_{L^1} \left| \sum_{|j'-j|\le 3} 2^{j'(s+1)} \|\Delta_{j'}\widetilde{g}\|_{L^p}\, dy \right|_{l_j^r} \\
&\lesssim \|\nabla f\|_{L^\infty} \|\nabla h\|_{L^1} \|\widetilde{g}\|_{B^{s+1}_{p,r}} \lesssim \|\nabla f\|_{L^\infty} \|g\|_{B^{s-1}_{p,r}}.
\end{aligned}$$

For II_4, using the integral representation of the convolution one more time, we obtain

$$\begin{aligned}
&\left\| 2^{js} \sum_{|j'-j|\le 3} 2^{jn} \int_{\mathbb{R}^n} II_4 \cdot \Delta_{j'}\widetilde{g}\, dy \right\|_{l_j^r(L^p)} \\
&\lesssim 2^{3s} \left\| \sum_{|j'-j|\le 3} 2^{j's} 2^{jn} \int_{\mathbb{R}^n} |h(2^j(x-y))| \|\Delta S_{j'-2} f\|_{L^\infty} |\Delta_{j'}\widetilde{g}|\, dy \right\|_{l_j^r(L^p)} \\
&\lesssim \|\Delta f\|_{L^\infty} \|h\|_{L^1} \left| \sum_{|j'-j|\le 3} 2^{j's} \|\Delta_{j'}\widetilde{g}\|_{L^p} \right|_{l_j^r} \lesssim \|\Delta f\|_{L^\infty} \|\widetilde{g}\|_{B^{s}_{p,r}} \lesssim \|\Delta f\|_{L^\infty} \|g\|_{B^{s-2}_{p,r}}.
\end{aligned}$$

In summary, combining all the estimates from II_1 to II_4 with applications of embedding results in Lemmas 6.4 and 6.5, we have for II,

$$\left| 2^{js} \|II\|_{L^p} \right|_{l_j^r} \lesssim \|f\|_{B^s_{q,r}} \|g\|_{B^{s-1}_{p,r}},$$

where $s - 2 > n/q$, i.e., $s > n/p + 1$.

Estimate of III. We are in the position to estimate the third term III in (4.1). The strategy is to balance the differential index s of f and g. We observe that, from the definition of the paraproduct and the support condition (2.1),

$$\begin{aligned} T_{\Delta_j g}(f - S_{j-2}f) &= \sum_{j'\in\mathbb{N}} S_{j'-1}\Delta_j g \cdot \Delta_{j'}(f - S_{j-2}f) \\ &= \sum_{j'\in\mathbb{N}} S_{j'-1}\Delta_j g \cdot \sum_{|j'-m|\le 4} \Delta_{j'}\Delta_m f. \end{aligned}$$

Again successively applying Hölder's inequality and Young's inequality, we have

$$\begin{aligned} 2^{js}\|III\|_{L^p} &= 2^{js}\|T_{\Delta_j g}(f - S_{j-2}f)\|_{L^p} \\ &\le 2^{js_1} \sum_{j'\ge j-4} \|S_{j'-1}\Delta_j g\|_{L^{p_1}} 2^{js_2}\|\Delta_{j'}(f - S_{j-2}f)\|_{L^{p_2}} \\ &\lesssim 2^{js_1}\|\Delta_j g\|_{L^{p_1}} \sum_{j'\ge j-4}\sum_{|j'-m|\le 4} 2^{j's_2}\|\Delta_{j'}\Delta_m f\|_{L^{p_2}} \\ &\lesssim 2^{js_1}\|\Delta_j g\|_{L^{p_1}} \sum_{j'\ge 0} 2^{j's_2}\|\Delta_{j'} f\|_{L^{p_2}} \\ &\lesssim 2^{js_1}\|\Delta_j g\|_{L^{p_1}} \|f\|_{B^{s_2}_{p_2,1}}, \end{aligned} \tag{4.4}$$

where $s_1 + s_2 = s$, $s_i > 0$, $i = 1, 2$ and $1/p_1 + 1/p_2 = 1/p$. We also require that

$$s - 1 - \frac{n}{p} > s_1 - \frac{n}{p_1}, \quad s - \frac{n}{q} > s_2 - \frac{n}{p_2}, \quad q < p_2.$$

Taking the l^r norm on both sides of (4.4) yields

$$\left|2^{js}\|III\|_{L^p}\right|_{l^r_j} \lesssim \|g\|_{B^{s_1}_{p_1,r}}\|f\|_{B^{s_2}_{p_2,1}} \lesssim \|g\|_{B^{s-1}_{p,r}}\|f\|_{B^{s}_{q,r}}.$$

Estimate of IV. We move on to the estimate of the fourth term IV in (4.1). Again, from the definition of the paraproduct and the support condition (2.1), we obtain

$$\begin{aligned} 2^{js}\|IV\|_{L^p} &= 2^{js}\|R(f - S_{j-2}f, \Delta_j g)\|_{L^p} \\ &= 2^{js}\Big\| \sum_{|j'-j''|\le 1} \Delta_{j'}(f - S_{j-2}f)\cdot \Delta_{j''}\Delta_j g)\Big\|_{L^p} \\ &= 2^{js}\Big\| \sum_{|j'-j|\le 3}\sum_{|j''-j|\le 1}\sum_{|m-j|\le 5} (\Delta_m\Delta_{j'} f \cdot \Delta_{j''}\Delta_j g)\Big\|_{L^p} \\ &\le 2^{js} \sum_{|j'-j|\le 3}\sum_{|j''-j|\le 1}\sum_{|m-j|\le 5} \|(\Delta_m\Delta_{j'} f \cdot \Delta_{j''}\Delta_j g)\|_{L^p} \\ &\le \sum_{|j'-j|\le 3}\sum_{|j''-j|\le 1}\sum_{|m-j|\le 5} 2^{js_1}\|\Delta_{j''}\Delta_j g\|_{L^{p_1}} 2^{js_2}\|\Delta_m\Delta_{j'} f\|_{L^{p_2}} \\ &\lesssim 2^{js_1}\|\Delta_j g\|_{L^{p_1}} \sum_{j'\in\mathbb{N}} 2^{j's_2}\|\Delta_m\Delta_{j'} f\|_{L^{p_2}}, \end{aligned} \tag{4.5}$$

where $s_1+s_2=s$, $s_i>0$, $i=1,2$ and $1/p_1+1/p_2=1/p$, Hölder's inequality and Young's inequality were applied as well. Taking the l^r norm on both sides of (4.5) yields

$$\big|2^{js}\|IV\|_{L^p}\big|_{l^r} \lesssim \|g\|_{B^{s_1}_{p_1,r}}\|f\|_{B^{s_2}_{p_2,1}} \lesssim \|g\|_{B^{s-1}_{p,r}}\|f\|_{B^{s}_{q,r}}.$$

Estimate of V. We now estimate the fifth term V in (4.1). Note that

$$\Delta_j R(f,g) = \Delta_j \sum_{j'\in\mathbb{N}}\sum_{|j''|\le 1} \Delta_{j'-j''}f\cdot\Delta_{j'}g = \sum_{j'\ge j-5}\sum_{|j''|\le 1}\Delta_j(\Delta_{j'-j''}f\cdot\Delta_{j'}g).$$

From this and with Hölder's inequality, we deduce that

$$\begin{aligned}
2^{js}\|V\|_{L^p} &= 2^{js}\|\Delta_j R(f,g)\|_{L^p}\\
&\lesssim 2^{j(s_1+s_2)}\Big\|\sum_{j'\ge j-5}\sum_{|j''|\le 1}\Delta_j(\Delta_{j'-j''}f\cdot\Delta_{j'}g)\Big\|_{L^p}\\
&\lesssim \sum_{j'\ge j-5}\sum_{|j''|\le 1} 2^{j(s_1+s_2)}\|\Delta_{j'-j''}f\cdot\Delta_{j'}g\|_{L^p}\\
&\lesssim \sum_{j'\ge j-5}\sum_{|j''|\le 1} 2^{(j-j')(s_1+s_2)}2^{j's_1}\|\Delta_{j'}g\|_{L^{p_1}}2^{(j'-j'')s_2}\|\Delta_{j'-j''}f\|_{L^{p_2}},
\end{aligned}\tag{4.6}$$

where $s_1+s_2=s$, $s_i>0$, $i=1,2$ and $1/p_1+1/p_2=1/p$. Taking the l^r norm on both sides of (4.6) and using Young's inequality and embedding theorems [26] yields

$$\big|2^{js}\|V\|_{L^p}\big|_{l^r} \lesssim \|g\|_{B^{s_1}_{p_1,r_1}}\|f\|_{B^{s_2}_{p_2,r_2}} \lesssim \|g\|_{B^{s-1}_{p,r}}\|f\|_{B^{s}_{q,r}},$$

where $1/r_1+1/r_2=1/r$, $s-1-n/p>s_1-n/p_1$, $s-n/q>s_2-n/p_2$ and $p<q<p_2$.

Estimate of VI. Finally, we arrive at estimating the last term VI in (4.1). Since

$$\begin{aligned}
&R(S_{j-2}f,\Delta_j g)\\
&= \sum_{|j'-j''|\le 1}\Delta_{j'}S_{j-2}f\cdot\Delta_{j''}\Delta_j g\\
&= \sum_{|j'-j''|\le 1}\Big(\Delta_{j'}\sum_{m\le j-2}\Delta_m f\Big)\cdot\Delta_{j''}\Delta_j g \sum_{|j'-j|\le 2}\sum_{|j''-j|\le 1}\sum_{|m-j|\le 3}\Delta_{j'}(\Delta_m f)\cdot\Delta_{j''}\Delta_j g,
\end{aligned}$$

we have

$$\begin{aligned}
2^{js}\|VI\|_{L^p} &= 2^{js}\|R(S_{j-2}f,\Delta_j g)\|_{L^p}\\
&= 2^{js}\Big\|\sum_{|j'-j|\le 2}\sum_{|j''-j|\le 1}\sum_{|m-j|\le 3}\Delta_{j'}(\Delta_m f)\cdot\Delta_{j''}\Delta_j g\Big\|_{L^p}\\
&\le 2^{js}\sum_{|j'-j|\le 2}\sum_{|j''-j|\le 1}\sum_{|m-j|\le 3}\|\Delta_m\Delta_{j'}f\|_{L^{p_1}}\|\Delta_{j''}\Delta_j g\|_{L^{p_2}}\\
&\lesssim 2^{js}\|\Delta_j g\|_{L^{p_1}}\sum_{|m-j|\le 3}\|\Delta_m f\|_{L^{p_2}},
\end{aligned}\tag{4.7}$$

where Hölder's inequality was applied with $1/p_1 + 1/p_2 = 1/p$. Taking the l^r norm on both sides of (4.7), then using Hölder's inequality and Young's inequality, we have

$$\big|2^{js}\|VI\|_{L^p}\big|_{l^r} \lesssim \|g\|_{B^{s_1}_{p_1,r_1}}\|f\|_{B^{s_2}_{p_2,r_2}} \lesssim \|g\|_{B^{s-1}_{p,r}}\|f\|_{B^s_{q,r}},$$

where $1/r_1 + 1/r_2 = 1/r$.

Collecting all the estimates from I to VI, the conclusion is immediate. □

Remark 4.3. To be convenient for the reader, we summarize indices in our proofs as follows:

$$\begin{aligned}
&s = s_1 + s_2, \quad s_i > 0, \quad i = 1, 2,\\
&s > s_1 + \frac{n}{p_2} + 1,\\
&s > s_2 + \frac{n}{p_1} - 1,\\
&s > \frac{n}{p} + 1,\\
&\frac{1}{p} = \frac{1}{q} + \frac{1}{n}, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \quad 1 < q < p_2, \quad p < p_1 < n,\\
&n \geq 3.
\end{aligned}$$

Indeed, for any given s, p and n in our working space $B^s_{p,r}(\mathbb{R}^n)$ such that $s > n/p + 1$, $1 < p < n$ and $n \geq 3$, we can designate with $\delta > 0$ arbitrarily small such that we can take,

$$s_1 = 1, \quad s_2 = s - 1, \quad p_1 = n - \delta, \quad p_2 = \frac{(n-\delta)p}{n-\delta-p}, \quad q = \frac{np}{n-p}, \quad p < n.$$

Taking f as $\nabla_x\phi^k$, g as $\nabla_v f^{k+1}$ with the L^p norm replaced by the $L^p_{x,v}(w)$ norm in Proposition 4.1, we have the following corollary. In this corollary, the L^p-L^q estimate plays a fundamentally important role to control the electronic term $\nabla_x\phi$, i.e.,

$$\|\nabla_x\phi^k\|_{B^s_{q,r}} \lesssim \|f^k\|_{B^s_{p,r,x}(w)}.$$

This is the main reason that we use g to balance the integral index.

Corollary 4.4. *Let s, p, r be as in Proposition 4.1, it holds that*

$$\Big|2^{js}\big\|[\Delta^x_j, \nabla_x\phi^k]\nabla_v f^{k+1}\big\|_{L^p_{x,v}(w)}\Big|_{l^r_j} \lesssim \|f^{k+1}\|_{B^s_{p,r}(w)}\|f^k\|_{B^s_{p,r,x}(w)}.$$

Proof. Note

$$\begin{aligned}
\Big|2^{js}\big\|[\Delta^x_j, \nabla_x\phi^k]\nabla_v f^{k+1}\big\|_{L^p_{x,v}(w)}\Big|_{l^r_j} &\lesssim \|\nabla_v f^{k+1}\|_{B^{s-1}_{p,r,x}(w)} \cdot \|\nabla_x\phi^k\|_{B^s_{q,r}}\\
&\lesssim \|f^{k+1}\|_{B^s_{p,r}(w)} \cdot \|f^k\|_{B^s_{p,r,x}(w)},
\end{aligned}$$

where in the last line we applied Lemma 2.1 and Corollary 3.4. □

Now we move on to proving the commutator estimate about Δ_j^v. The trick bases on a good observation $[\Delta_j^v, v]\nabla_x f^{k+1} = \sum_{|j-j'|\le 3}[\Delta_j^v, v]\Delta_{j'}^v\nabla_x f^{k+1}$. It brings us an extra decay factor 2^{-j} which is fundamentally important in estimating J_5 shortly.

Proposition 4.5.

$$\big\|[\Delta_j^v, v]\nabla_x f^{k+1}\big\|_{L^p_{x,v}(w)} \lesssim \sum_{|j-j'|\le 3} 2^{-j}\big\|\Delta_{j'}^v\nabla_x f^{k+1}\big\|_{L^p_{x,v}(w)}.$$

Proof. Since

$$\begin{aligned}[\Delta_j^v, v]\Delta_{j'}^v\nabla_x f^{k+1} &= 2^{jn}\int_{\mathbb{R}^n} h(2^j(v-y))\cdot(y-v)\cdot(\Delta_{j'}^v\nabla_x f^{k+1})(y)\,dy\\ &= -\int_{\mathbb{R}^n} h(z)\cdot(2^{-j}z)\cdot(\Delta_{j'}^v\nabla_x f^{k+1})(v-2^{-j}z)\,dz \quad (z=2^j(v-y)),\end{aligned}$$

and $w(v)=\langle v\rangle^\gamma$, we have

$$\begin{aligned}&\big\|[\Delta_j^v, v]\Delta_{j'}^v\nabla_x f^{k+1}\big\|_{L^p_{x,v}(w)}\\ &\lesssim \int_{\mathbb{R}^n}|h(z)|\cdot(2^{-j}|z|)\left(\int_{\mathbb{R}^n}|(\Delta_{j'}^v\nabla_x f^{k+1})(v-2^{-j}z)|^p\langle v\rangle^\gamma\,dv\right)^{1/p} dz\\ &\lesssim \int_{\mathbb{R}^n}|h(z)|\cdot(2^{-j}|z|)\left(\int_{\mathbb{R}^n}|(\Delta_{j'}^v\nabla_x f^{k+1})(\widetilde{v})|^p\langle \widetilde{v}+2^{-j}z\rangle^\gamma\,d\widetilde{v}\right)^{1/p} dz\\ &\lesssim \int_{\mathbb{R}^n}|h(z)|\cdot(2^{-j}|z|)\left(\int_{\mathbb{R}^n}|(\Delta_{j'}^v\nabla_x f^{k+1})(\widetilde{v})|^p\langle \widetilde{v}\rangle^\gamma\langle 2^{-j}z\rangle^\gamma\,d\widetilde{v}\right)^{1/p} dz\\ &\lesssim 2^{-j}\left(\int_{\mathbb{R}^n}|h(z)||z|\langle z\rangle^{\gamma/p}\,dz\right)\cdot\big\|\Delta_{j'}^v\nabla_x f^{k+1}\big\|_{L^p_{x,v}(w)}.\end{aligned}$$

Consequently,

$$\big\|[\Delta_j^v, v]\Delta_{j'}^v\nabla_x f^{k+1}\big\|_{L^p_{x,v}(w)} \lesssim \sum_{|j-j'|\le 3} 2^{-j}\big\|\Delta_{j'}^v\nabla_x f^{k+1}\big\|_{L^p_{x,v}(w)},$$

which implies the proposition. □

5. Proof of the main theorem

5.1. Existence

With the above powerful commutator estimates, we are ready to prove the main theorem. In this part, we adopt the L^p-version energy method in Besov space and the iteration method to prove the existence of the solution to (1.1). Note our working space is $\|f\|_{B^s_{p,r}(w)} \sim \|f\|_{B^s_{p,r,x}(w)} + \|f\|_{B^s_{p,r,v}(w)}$. This observation enables us to estimate $\|f\|_{B^s_{p,r,x}(w)}$ and $\|f\|_{B^s_{p,r,v}(w)}$ individually. In fact, to the best of our knowledge, it is unknown whether there is an effective way to deal with $[\Delta_j^{x,v}, \nabla_x\phi^k]\nabla_x f^{k+1}$ since $\nabla_x\phi$ does not depend on v, while $\Delta_j^{x,v}$ depends on both x and v.

Proof of existence. We consider the following iterating sequence for solving the VP system (1.1),

$$\begin{aligned} &\partial_t f^{k+1} + v\cdot\nabla_x f^{k+1} + \nabla_x\phi^k\cdot\nabla_v f^{k+1} = 0 \\ &-\Delta\phi^k = \int_{\mathbb{R}^n} f^k\,dv, \quad f^{k+1}(0,x,v) = f_0(x,v). \end{aligned} \tag{5.1}$$

Step 1: Applying Δ_j^x to (5.1) we have

$$\partial_t\Delta_j^x f^{k+1} + v\cdot\nabla_x\Delta_j^x f^{k+1} + [\Delta_j^x, \nabla_x\phi^k]\nabla_v f^{k+1} + \nabla_x\phi^k\cdot\nabla_v\Delta_j^x f^{k+1} = 0. \tag{5.2}$$

Multiplying $|\Delta_j^x f^{k+1}|^{p-2}(\Delta_j^x f^{k+1})w$ on both sides of (5.2), and then integrating over $\mathbb{R}_x^n\times\mathbb{R}_v^n$ yields

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\Delta_j^x f^{k+1}\|_{L^p_{x,v}(w)}^p + \underbrace{\big\langle v\cdot\nabla_x\Delta_j^x f^{k+1}, |\Delta_j^x f^{k+1}|^{p-2}(\Delta_j^x f^{k+1})w\big\rangle}_{J_1} \\ &+ \underbrace{\big\langle [\Delta_j^x, \nabla_x\phi^k]\nabla_v f^{k+1}, |\Delta_j^x f^{k+1}|^{p-2}(\Delta_j^x f^{k+1})w\big\rangle}_{J_2} \\ &+ \underbrace{\big\langle \nabla_x\phi^k\cdot\nabla_v\Delta_j^x f^{k+1}, |\Delta_j^x f^{k+1}|^{p-2}(\Delta_j^x f^{k+1})w\big\rangle}_{J_3} = 0. \end{aligned} \tag{5.3}$$

We now estimate (5.3) term by term. For J_1, we have

$$J_1 = \int_{\mathbb{R}^{2n}} v\cdot\nabla_x|\Delta_j^x f^{k+1}|^p w\,dxdv = 0. \tag{5.4}$$

For J_2, we get

$$2^{jsp}J_2 \lesssim 2^{js}\big\|[\Delta_j^x, \nabla_x\phi^k]\nabla_v f^{k+1}\big\|_{L^p_{x,v}(w)}\cdot 2^{js(p-1)}\|\Delta_j^x f^{k+1}\|_{L^p_{x,v}(w)}^{p-1}. \tag{5.5}$$

For J_3, note that $|\nabla_v w|\lesssim w$, integration by parts yields

$$J_3 \lesssim \int_{\mathbb{R}^{2n}} |\nabla_x\phi^k|\cdot|\Delta_j^x f^{k+1}|^p w\,dxdv \lesssim \|\nabla_x\phi^k\|_{L^\infty}\|\Delta_j^x f^{k+1}\|_{L^p_{x,v}(w)}^p. \tag{5.6}$$

Multiplying 2^{jsp} on both sides of (5.3), plugging (5.4), (5.5) and (5.6), and then integrating with respect to t, we have

(5.7)

$$\begin{aligned} 2^{js}\|\Delta_j^x f^{k+1}\|_{L^p_{x,v}(w)} \le 2^{js}\|\Delta_j^x f_0\|_{L^p_{x,v}(w)} + C\bigg(&\int_0^t \|\nabla_x\phi^k\|_{L^\infty}\cdot 2^{js}\|\Delta_j^x f^{k+1}\|_{L^p_{x,v}(w)}\,d\tau \\ &+ \int_0^t 2^{js}\big\|[\Delta_j^x, \nabla_x\phi^k]\nabla_x f^{k+1}\big\|_{L^p_{x,v}(w)}\,d\tau\bigg), \end{aligned}$$

where we used an elementary fact that

$$\frac{d}{dt}\|\Delta_j^x f^{k+1}\|_{L^p_{x,v}(w)}^p = p\|\Delta_j^x f^{k+1}\|_{L^p_{x,v}(w)}^{p-1}\frac{d}{dt}\|\Delta_j^x f^{k+1}\|_{L^p_{x,v}(w)}.$$

Taking an l_j^r norm on both sides of (5.7), we have

$$\begin{aligned}\|f^{k+1}\|_{B^s_{p,r,x}(w)} \le \|f_0\|_{B^s_{p,r,x}(w)} + C\bigg(\int_0^t \|f^k\|_{B^s_{p,r,x}(w)}\cdot\|f^{k+1}\|_{B^s_{p,r,v}(w)}\,d\tau \\ + \int_0^t \|f^k\|_{B^s_{p,r,x}(w)}\cdot\|f^{k+1}\|_{B^s_{p,r}(w)}\,d\tau\bigg).\end{aligned} \tag{5.8}$$

Step 2: Applying Δ_j^v to (5.1) we have

$$\partial_t\Delta_j^v f^{k+1} + v\cdot\nabla_x\Delta_j^v f^{k+1} + [\Delta_j^v, v]\nabla_x f^{k+1} + \nabla_x\phi^k\cdot\nabla_v\Delta_j^v f^{k+1} = 0. \tag{5.9}$$

Multiplying $|\Delta_j^v f^{k+1}|^{p-2}(\Delta_j^v f^{k+1})w$ on both sides of (5.9), and then integrating over $\mathbb{R}^n_x\times\mathbb{R}^n_v$ yields

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\Delta_j^v f^{k+1}\|_{L^p_{x,v}(w)}^p + \underbrace{\big\langle v\cdot\nabla_x\Delta_j^v f^{k+1}, |\Delta_j^v f^{k+1}|^{p-2}(\Delta_j^v f^{k+1})w\big\rangle}_{J_4} \\ &+ \underbrace{\big\langle[\Delta_j^v, v]\nabla_x f^{k+1}, |\Delta_j^v f^{k+1}|^{p-2}(\Delta_j^v f^{k+1})w\big\rangle}_{J_5} \\ &+ \underbrace{\big\langle\nabla_x\phi^k\cdot\nabla_v\Delta_j^v f^{k+1}, |\Delta_j^v f^{k+1}|^{p-2}(\Delta_j^v f^{k+1})w\big\rangle}_{J_6} = 0.\end{aligned} \tag{5.10}$$

We now estimate (5.10) term by term. For J_4, it is easy to see

$$J_4 = \int_{\mathbb{R}^{2n}} v\cdot\nabla_x|\Delta_j^v f^{k+1}|^p w\,dxdv = 0. \tag{5.11}$$

For J_5, we obtain

$$J_5 \lesssim \big\|[\Delta_j^v, v]\nabla_x f^{k+1}\big\|_{L^p_{x,v}(w)}\big\|\Delta_j^v f^{k+1}\big\|_{L^p_{x,v}(w)}^{p/p'}, \quad \frac{1}{p}+\frac{1}{p'} = 1.$$

Now the commutator estimate comes into play, by Proposition 4.5, we have

$$\begin{aligned}2^{jsp}J_5 &\lesssim 2^{jsp}\cdot 2^{-j}\sum_{|j-j'|\le 1}\big\|\Delta_{j'}^v(\nabla_x f^{k+1})\big\|_{L^p_{x,v}(w)}\cdot\big\|\Delta_j^v f^{k+1}\big\|_{L^p_{x,v}(w)}^{p-1} \\ &\lesssim \sum_{|j-j'|\le 1} 2^{j'(s-1)}\big\|\Delta_{j'}^v(\nabla_x f^{k+1})\big\|_{L^p_{x,v}(w)}\cdot 2^{js(p-1)}\|\Delta_j^v f^{k+1}\|_{L^p_{x,v}(w)}^{p-1}.\end{aligned} \tag{5.12}$$

For J_6, note that $|\nabla_v w| \lesssim w$, integration by parts yields

$$J_6 \lesssim \int_{\mathbb{R}^{2n}} |\nabla_x\phi^k|\cdot|\Delta_j^v f^{k+1}|^p w\,dxdv \lesssim \|\nabla_x\phi^k\|_{L^\infty}\|\Delta_j^v f^{k+1}\|_{L^p_{x,v}(w)}^p. \tag{5.13}$$

Multiplying 2^{jsp} on both sides of (5.10), plugging (5.11), (5.12) and (5.13) into (5.10), then integrating with respect to t, we have

(5.14)
$$2^{js}\|\Delta_j^v f^{k+1}\|_{L^p_{x,v}(w)} \leq 2^{js}\|\Delta_j^v f_0\|_{L^p_{x,v}(w)} + C\bigg(\int_0^t \|\nabla_x \phi^k\|_{L^\infty} \cdot 2^{js}\|\Delta_j^v f^{k+1}\|_{L^p_{x,v}(w)}\,d\tau + \int_0^t \sum_{|j-j'|\leq 1} 2^{j'(s-1)}\big\|\Delta_{j'}^v(\nabla_x f^{k+1})\big\|_{L^p_{x,v}(w)}\,d\tau\bigg),$$

where we applied the following fact

$$\frac{d}{dt}\|\Delta_j^v f^{k+1}\|^p_{L^p_{x,v}(w)} = p\|\Delta_j^v f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}\frac{d}{dt}\|\Delta_j^v f^{k+1}\|_{L^p_{x,v}(w)}.$$

Taking an l_j^r norm on both sides of (5.14), we have

(5.15)
$$\|f^{k+1}\|_{B^s_{p,r,v}(w)} \leq \|f_0\|_{B^s_{p,r,v}(w)} + C\left(\int_0^t \|f^k\|_{B^s_{p,r,x}(w)} \cdot \|f^{k+1}\|_{B^s_{p,r,v}(w)}\,d\tau + \int_0^t \|f^{k+1}\|_{B^s_{p,r}(w)}\,d\tau\right).$$

Combining (5.8) and (5.15), we deduce that

$$\|f^{k+1}(t)\|_{B^s_{p,r}(w)} \leq C_0^2\|f_0\|_{B^s_{p,r}(w)} + C\int_0^t (1+\|f^k\|_{B^s_{p,r}(w)}) \cdot \|f^{k+1}\|_{B^s_{p,r}(w)}\,d\tau.$$

Let us take

$$T_0 = \frac{C_1}{1+2C_0^2\|f_0\|_{B^s_{p,r}(w)}} \quad \text{with } C_1 C < \frac{1}{4}.$$

Inductively, we assume

$$\sup_{0\leq t\leq T_0}\|f^k(t)\|_{B^s_{p,r}(w)} \leq 2C_0^2\|f_0\|_{B^s_{p,r}(w)},$$

then it follows that

$$\sup_{0\leq t\leq T_0}\|f^{k+1}(t)\|_{B^s_{p,r}(w)} \leq C_0^2\|f_0\|_{B^s_{p,r}(w)} + CT_0(1+2C_0^2\|f_0\|_{B^s_{p,r}(w)}) \cdot \sup_{0\leq t\leq T_0}\|f^{k+1}(t)\|_{B^s_{p,r}(w)},$$

i.e.,

$$\big(1 - CT_0(1+2C_0^2\|f_0\|_{B^s_{p,r}(w)})\big)\sup_{0\leq t\leq T_0}\|f^{k+1}(t)\|_{B^s_{p,r}(w)} \leq C_0^2\|f_0\|_{B^s_{p,r}(w)}.$$

Note

$$1 - CT_0\big(1+2C_0^2\|f_0\|_{B^s_{p,r}(w)}\big) \geq \frac{3}{4},$$

we have

$$\sup_{0\le t\le T_0}\|f^{k+1}(t)\|_{B^s_{p,r}(w)}\le 2C_0^2\|f_0\|_{B^s_{p,r}(w)}.$$

This implies that

$$\sup_k\sup_{0\le t\le T_0}\|f^{k}(t)\|_{B^s_{p,r}(w)}\le 2C_0^2\|f_0\|_{B^s_{p,r}(w)},$$

i.e., we get a uniform-in-k estimate. As a routine, let $k\to\infty$, we obtain the solution and completes the proof of existence. □

5.2. Uniqueness

We are in the position to prove the uniqueness of the solution to (1.1). The proof of the uniqueness is quite analogous to the existence. But this time, we are working in $B^{s-1}_{p,r}(w)$ other than in $B^s_{p,r}(w)$ space. Indeed, if we worked in $B^s_{p,r}(w)$ space, we could not control $\big|2^{js}\|\Delta_j^x\nabla_v g\|_{L^{p'}_{x,v}(w)}\big|_{l^r_j}$.

Proof of uniqueness. Assume another solution g exists and satisfies the condition $\|g\|_{L^\infty([0,T_0),B^s_{p,r}(w))}<\infty$, taking the difference between f and g, we have

$$\begin{aligned}&(\partial_t+v\cdot\nabla_x+\nabla_x\phi_f\cdot\nabla_v)(f-g)+(\nabla_x\phi_f-\nabla_x\phi_g)\cdot\nabla_v g=0,\\&-\Delta_x(\phi_f-\phi_g)=\int_{\mathbb{R}^n}(f-g)\,dv,\quad f(0,x,v)=g(0,x,v).\end{aligned}\tag{5.16}$$

Step 1: Applying Δ_j^x to (5.16), we have

$$\begin{aligned}&\partial_t\Delta_j^x(f-g)+v\cdot\nabla_x\Delta_j^x(f-g)+[\Delta_j^x,\nabla_x\phi_f]\nabla_v(f-g)+\nabla_x\phi_f\cdot\nabla_v\Delta_j^x(f-g)\\&\quad+[\Delta_j^x,\nabla_x(\phi_f-\phi_g)]\nabla_v g+\nabla_x(\phi_f-\phi_g)\cdot\Delta_j^x\nabla_v g=0.\end{aligned}\tag{5.17}$$

Multiplying $|\Delta_j^x(f-g)|^{p-2}(\Delta_j^x(f-g))w$ on both sides of (5.17), and then integrating over $\mathbb{R}^n_x\times\mathbb{R}^n_v$ yields

$$\begin{aligned}&\frac1p\frac{d}{dt}\|\Delta_j^x(f-g)\|^p_{L^p_{x,v}(w)}+\underbrace{\big\langle v\cdot\nabla_x\Delta_j^x(f-g),|\Delta_j^x(f-g)|^{p-2}(\Delta_j^x(f-g))w\big\rangle}_{J_7}\\&\quad+\underbrace{\big\langle[\Delta_j^x,\nabla_x\phi_f]\nabla_v(f-g),|\Delta_j^x(f-g)|^{p-2}(\Delta_j^x(f-g))w\big\rangle}_{J_8}\\&\quad+\underbrace{\big\langle\nabla_x\phi_f\cdot\nabla_v\Delta_j^x(f-g),|\Delta_j^x(f-g)|^{p-2}(\Delta_j^x(f-g))w\big\rangle}_{J_9}\\&\quad+\underbrace{\big\langle[\Delta_j^x,\nabla_x(\phi_f-\phi_g)]\nabla_v g,|\Delta_j^x(f-g)|^{p-2}(\Delta_j^x(f-g))w\big\rangle}_{J_{10}}\\&\quad+\underbrace{\big\langle\nabla_x(\phi_f-\phi_g)\cdot\Delta_j^x\nabla_v g,|\Delta_j^x(f-g)|^{p-2}(\Delta_j^x(f-g))w\big\rangle}_{J_{11}}=0.\end{aligned}\tag{5.18}$$

We could repeat estimates in the proof of existence except for some special terms. Thus we would like to write down estimates directly without the details.

Obviously, $J_7 = 0$. For J_8 and J_{10}, which are similar to the estimate of J_2, we have

$$2^{j(s-1)p} J_8 \lesssim 2^{j(s-1)} \|[\Delta_j^x, \nabla_x \phi_f] \nabla_v (f-g)\|_{L^p_{x,v}(w)} \cdot 2^{j(s-1)(p-1)} \|\Delta_j^x (f-g)\|^{p-1}_{L^p_{x,v}(w)},$$

and

$$2^{j(s-1)p} J_{10} \lesssim 2^{j(s-1)} \|[\Delta_j^x, \nabla_x (\phi_f - \phi_g)] \nabla_v g\|_{L^p_{x,v}(w)} \cdot 2^{j(s-1)(p-1)} \|\Delta_j^x (f-g)\|^{p-1}_{L^p_{x,v}(w)}.$$

For J_9, which is similar to the estimate of J_3, with Corollary 3.5 we have

$$\begin{aligned} 2^{j(s-1)p} J_9 &\lesssim 2^{j(s-1)p} \|\nabla_x \phi_f\|_{L^\infty} \|\Delta_j^x (f-g)\|^p_{L^p_{x,v}(w)} \\ &\lesssim \|f\|_{B^s_{p,r,x}(w)} 2^{j(s-1)p} \|\Delta_j^x (f-g)\|^p_{L^p_{x,v}(w)}. \end{aligned}$$

For J_{11}, by Corollary 3.5, we obtain

$$\begin{aligned} 2^{j(s-1)p} J_{11} &\lesssim \|\nabla_x (\phi_f - \phi_g)\|_{L^\infty} \cdot 2^{j(s-1)} \|\Delta_j^x \nabla_v g\|_{L^p_{x,v}(w)} 2^{j(s-1)(p-1)} \|\Delta_j^x (f-g)\|^{p-1}_{L^p_{x,v}(w)} \\ &\lesssim \|f-g\|_{B^{s-1}_{p,r,x}(w)} \cdot 2^{j(s-1)} \|\Delta_j^x \nabla_v g\|_{L^p_{x,v}(w)} 2^{j(s-1)(p-1)} \|\Delta_j^x (f-g)\|^{p-1}_{L^p_{x,v}(w)}, \end{aligned}$$

where $s > n/p$.

Note that

$$\frac{d}{dt} \|\Delta_j^x (f-g)\|^p_{L^p_{x,v}(w)} = p \|\Delta_j^x (f-g)\|^{p-1}_{L^p_{x,v}(w)} \frac{d}{dt} \|\Delta_j^x (f-g)\|_{L^p_{x,v}(w)}.$$

Multiplying $2^{j(s-1)p}$ on both sides of (5.18), plugging all the estimates from J_7 to J_{11}, and then integrating with respect to t, we have

$$\begin{aligned} 2^{j(s-1)} \|\Delta_j^x (f-g)\|_{L^p_{x,v}(w)} \lesssim &\int_0^t 2^{j(s-1)} \big\|[\Delta_j^x, \nabla_x \phi_f] \nabla_v (f-g)\big\|_{L^p_{x,v}(w)} \, d\tau \\ &+ \int_0^t 2^{j(s-1)} \big\|[\Delta_j^x, \nabla_x (\phi_f - \phi_g)] \nabla_v g\big\|_{L^p_{x,v}(w)} \, d\tau \\ &+ \int_0^t \|f\|_{B^s_{p,r,x}(w)} 2^{j(s-1)} \|\Delta_j^x (f-g)\|_{L^p_{x,v}(w)} \, d\tau \\ &+ \int_0^t \|f-g\|_{B^{s-1}_{p,r,x}(w)} 2^{j(s-1)} \|\Delta_j^x \nabla_v g\|_{L^p_{x,v}(w)} \, d\tau. \end{aligned} \tag{5.19}$$

Applying the commutator estimate Proposition 4.1, and taking the l^r_j norm on both sides of (5.19), we have

$$\begin{aligned} &\|f-g\|_{B^{s-1}_{p,r,x}(w)} \\ \lesssim &\int_0^t \|f\|_{B^{s-1}_{p,r}(w)} \|f-g\|_{B^{s-1}_{p,r}(w)} \, d\tau + \int_0^t \|(f-g)\|_{B^{s-1}_{p,r,x}(w)} \|g\|_{B^{s-1}_{p,r}(w)} \, d\tau \\ &+ \int_0^t \|f\|_{B^s_{p,r,x}(w)} \|f-g\|_{B^{s-1}_{p,r,x}(w)} \, d\tau + \int_0^t \|f-g\|_{B^{s-1}_{p,r,x}(w)} \|g\|_{B^{s-1}_{p,r,x}(w)} \, d\tau. \end{aligned} \tag{5.20}$$

Step 2: Applying Δ_j^v to (5.16), we have

$$\begin{aligned}&\partial_t\Delta_j^v(f-g)+v\cdot\nabla_x\Delta_j^v(f-g)+[\Delta_j^v,v]\nabla_x(f-g)\\&\quad+\nabla_x\phi_f\cdot\Delta_j^v\nabla_v(f-g)+\nabla_x(\phi_f-\phi_g)\cdot\nabla_v\Delta_j^v g=0.\end{aligned}\tag{5.21}$$

Multiplying $|\Delta_j^v(f-g)|^{p-2}(\Delta_j^v(f-g))w$ on both sides of (5.21), and then integrating over $\mathbb{R}_x^n\times\mathbb{R}_v^n$ yields

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\Delta_j^v(f-g)\|^p_{L^p_{x,v}(w)}+\underbrace{\big\langle v\cdot\nabla_x\Delta_j^v(f-g),|\Delta_j^v(f-g)|^{p-2}(\Delta_j^v(f-g))w\big\rangle}_{J_{12}}\\&\quad+\underbrace{\big\langle[\Delta_j^v,v]\nabla_x(f-g),|\Delta_j^v(f-g)|^{p-2}(\Delta_j^v(f-g))w\big\rangle}_{J_{13}}\\&\quad+\underbrace{\big\langle\nabla_x\phi_f\cdot\Delta_j^v\nabla_v(f-g),|\Delta_j^v(f-g)|^{p-2}(\Delta_j^v(f-g))w\big\rangle}_{J_{14}}\\&\quad+\underbrace{\big\langle\nabla_x(\phi_f-\phi_g)\cdot\nabla_v\Delta_j^v g,|\Delta_j^v(f-g)|^{p-2}(\Delta_j^v(f-g))w\big\rangle}_{J_{15}}=0.\end{aligned}\tag{5.22}$$

We estimate (5.22) term by term. Obviously, $J_{12}=0$. For J_{13}, which is similar to the estimate of J_5, we have

$$2^{j(s-1)p}J_{13}\lesssim\sum_{|j-j'|\le1}2^{j'(s-2)}\|\Delta_{j'}^v\nabla_x(f-g)\|_{L^p_{x,v}(w)}\cdot2^{j(s-1)(p-1)}\|\Delta_j^v f^{k+1}\|^{p-1}_{L^p_{x,v}(w)}.$$

For J_{14}, which is similar to the estimate of J_6, we have

$$2^{j(s-1)p}J_{14}\lesssim\|f\|_{B^{s-1}_{p,r,x}(w)}2^{j(s-1)p}\cdot\|\Delta_j^v(f-g)\|^p_{L^p_{x,v}(w)},$$

where $s>n/p$.

For J_{15}, we have

$$\begin{aligned}&2^{j(s-1)p}J_{15}\\&\lesssim\|f-g\|_{B^{s-1}_{p,r,x}(w)}\cdot2^{j(s-1)}\|\Delta_j^v\nabla_v g\|_{L^p_{x,v}(w)}\cdot2^{j(s-1)(p-1)}\|\Delta_j^v(f-g)\|^{p-1}_{L^p_{x,v}(w)},\quad s>\frac{n}{p}.\end{aligned}$$

Note that

$$\frac{d}{dt}\|\Delta_j^v(f-g)\|^p_{L^p_{x,v}(w)}=p\|\Delta_j^v(f-g)\|^{p-1}_{L^p_{x,v}(w)}\frac{d}{dt}\|\Delta_j^v(f-g)\|_{L^p_{x,v}(w)}.$$

Multiplying $2^{j(s-1)p}$ on both sides of (5.22), plugging all the estimates from J_{12} to J_{15} into (5.22), and integrating with respect to t, we have

$$\begin{aligned}2^{j(s-1)}\|\Delta_j^v(f-g)\|_{L^p_{x,v}(w)}&\lesssim\int_0^t\sum_{|j-j'|\le1}2^{j'(s-2)}\|\Delta_{j'}^v\nabla_x(f-g)\|_{L^p_{x,v}(w)}\,d\tau\\&\quad+\int_0^t\|f\|_{B^{s-1}_{p,r,x}(w)}2^{j(s-1)}\|\Delta_j^v(f-g)\|_{L^p_{x,v}(w)}\,d\tau\\&\quad+\int_0^t\|f-g\|_{B^{s-1}_{p,r,x}(w)}2^{j(s-1)}\|\Delta_j^v\nabla_v g\|_{L^p_{x,v}(w)}\,d\tau.\end{aligned}\tag{5.23}$$

Taking the l^r_j norm on both sides of (5.23), we have

$$\begin{aligned}\|f-g\|_{B^{s-1}_{p,r,v}(w)} &\lesssim \int_0^t \|\nabla_x(f-g)\|_{B^{s-2}_{p,r,v}(w)}\,d\tau + \int_0^t \|f\|_{B^{s-1}_{p,r,x}(w)}\|f-g\|_{B^{s-1}_{p,r,v}(w)}\,d\tau \\ &\quad + \int_0^t \|f-g\|_{B^{s-1}_{p,r,x}(w)}\|g\|_{B^{s}_{p,r}(w)}\,d\tau. \end{aligned} \tag{5.24}$$

Note $\|f\|_{B^{s-1}_{p,r}(w)} \sim \|f\|_{B^{s-1}_{p,r,x}(w)} + \|f\|_{B^{s-1}_{p,r,v}(w)}$, and $f(0,x,v) = g(0,x,v)$. Combining (5.20) and (5.24), we have

$$\|f-g\|_{B^{s-1}_{p,r}(w)} \lesssim C(f_0,g)\int_0^t \|(f-g)(\tau)\|_{B^{s-1}_{p,r}(w)}\,d\tau.$$

By Gronwall's inequality, we have $\|f-g\|_{B^{s-1}_{p,r}(w)} \equiv 0$ implying $f \equiv g$, which completes the proof of uniqueness part. Thus we end the proof of Theorem 2.2. □

6. Appendix

In this appendix, we cite some classical results for Riesz potential [1,28] and some embedding results as well. They are very important in deriving L^p-L^q estimates and commutator estimates, see Sections 3 and 4 separately.

6.1. Pointwise estimate of Riesz potential

We first give the pointwise estimate of Riesz potential, for more details, see [1].

Proposition 6.1. [1] *For any multi-index ξ with $|\xi| < \alpha < n$, there is a constant A such that for any $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and almost every x, we have*

$$|D^\xi(I_\alpha * f(x))| \le AMf(x)^{|\xi|/\alpha} \cdot (I_\alpha * |f|(x))^{1-|\xi|/\alpha},$$

where $I_\alpha = \frac{\gamma_\alpha}{|x|^{n-\alpha}}$, $\gamma_\alpha = \frac{\Gamma((n-\alpha)/2)}{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}$.

Remark 6.2. In our paper, we consider $-\Delta\phi = \int_{\mathbb{R}^n} f\,dv =: g$, $n \ge 3$. Thus, in our context, I_α can be taken

$$I_2(x) = \frac{1}{(n-2)\omega_{n-1}} \cdot \frac{1}{|x|^{n-2}}, \quad \text{i.e., } \alpha = 2,$$

where $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the $(n-1)$-dimensional area of the unit sphere in $\mathbb{R}^n$, then we have

$$|D^\xi(I_2 * g(x))| \le cMg(x)^{|\xi|/2} \cdot (I_2 * |g|(x))^{1-|\xi|/2}.$$

Next we present the boundedness of Riesz potential operator.

Lemma 6.3. [28] *If* $-\Delta\phi = g \in L^p(\mathbb{R}^n)$*, then* $\phi = I_2 * g$ *and*

$$\|I_2 * g\|_{L^{\widetilde{q}}(\mathbb{R}^n)} \leq c\|g\|_{L^p(\mathbb{R}^n)},$$

where $1 < p < n/2$*,* $c = c(p, \widetilde{q})$ *and*

$$\frac{1}{\widetilde{q}} = \frac{1}{p} - \frac{2}{n}.$$

6.2. Embedding theorem

The following embedding results were applied in commutator estimates, see Section 4. For completeness, we give their proofs.

Lemma 6.4. *Let* $s \in \mathbb{R}$*,* $1 \leq p_1, p, r \leq \infty$ *and* $p < p_1$*,* $n(1/p - 1/p_1) < s$*. We have the following embedding result*

$$B^s_{p,r} \hookrightarrow L^{p_1}.$$

Proof.

$$\begin{aligned}\|f\|_{L^{p_1}} &= \Big\|\sum_{j\in\mathbb{N}} \Delta_j f\Big\|_{L^{p_1}} = \Big\|\sum_{j\in\mathbb{N}} \Delta_j \sum_{|j-j'|\leq 1} \Delta_{j'} f\Big\|_{L^{p_1}} \leq \sum_{j\in\mathbb{N}} \sum_{|j-j'|\leq 1} \|\Delta_{j'}\Delta_j f\|_{L^{p_1}} \\ &\leq \sum_{j\in\mathbb{N}} \sum_{|j-j'|\leq 1} \|\mathscr{F}^{-1}\varphi_{j'}\|_{L^{\widetilde{p}}}\|\Delta_j f\|_{L^p} = \sum_{j\in\mathbb{N}} \sum_{|j-j'|\leq 1} 2^{j'n(1/p-1/p_1)} 2^{-js} 2^{js}\|\Delta_j f\|_{L^p} \\ &\lesssim |\{a_j\}|_{l^{r'}} \left(\sum_{j=0}^{\infty} 2^{jsr}\|\Delta_j f\|^r_{L^p}\right)^{1/r} \lesssim \|f\|_{B^s_{p,r}},\end{aligned}$$

where we applied Young's inequality with $1/\widetilde{p} + 1/p = 1/p_1 + 1$, and Hölder's inequality with $1/r' + 1/r = 1$. In addition,

$$a_j = \sum_{|j-j'|\leq 1} 2^{j'n(1/p-1/p_1)} 2^{-js}.$$ □

Lemma 6.5. [31] *Let* $1 \leq q < \infty$*,* $1 \leq r \leq \infty$ *and* $s > n/q$*. We have the embedding result*

$$B^s_{q,r} \hookrightarrow L^\infty.$$

Proof.

$$\begin{aligned}\|f\|_{L^\infty} &= \Big\|\sum_{j\in\mathbb{N}} \Delta_j f\Big\|_{L^\infty} \leq \sum_{j\in\mathbb{N}} \|\Delta_j f\|_{L^\infty} \leq \sum_{j\in\mathbb{N}} 2^{jn/q}\|\Delta_j f\|_{L^q} \\ &= \sum_{j\in\mathbb{N}} 2^{j(n/q-s)} 2^{js}\|\Delta_j f\|_{L^q} \leq \sum_{j\in\mathbb{N}} 2^{j(n/q-s)}\|f\|_{B^s_{q,\infty}} \leq \|f\|_{B^s_{q,r}},\end{aligned}$$

where we applied the assumption $n/q - s < 0$, and the fact that $B^s_{q,r} \hookrightarrow B^s_{q,\infty}$ in the last line. □

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