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Japanese Dedekind Domains Are Excellent

Chia-Fu Yu

Abstract. The well-known fundamental identity in number theory expresses the degree of an extension of global fields in terms of local information. In this article we show a generalized fundamental identity for arbitrary Dedekind domains. As an application, we show that any Japanese Dedekind domain is excellent.

1. Introduction

Let A be an S-ring of integers of a global field K, where S is a nonempty finite set of places of K containing all Archimedean ones. The well-known fundamental identity in number theory states that for any finite field extension L/K and any nonzero prime ideal $\mathfrak{p}$ of A, one has

$$\sum_{i=1}^{r} e_i f_i = [L : K], \tag{1.1}$$

where e_i and f_i are the ramification index and residue class degree of the prime ideals $\mathfrak{P}_i$ lying over $\mathfrak{p}$ (for $1 \le i \le r$), respectively. If A is an arbitrary Dedekind domain, then the fundamental identity is no longer true but instead one has the fundamental inequality $\sum_i e_i f_i \le [L : K]$ in general. It is well known that when the integral closure B of A in L is a finitely generated A-module, the equality holds; on the other hand the strict inequality can occur; see [19, Remark, p. 15].

One can rephrase the statement in terms of valuation theory, cf. [15, Chap. II, Proposition 8.5, p. 165]. In this reformation, Cohen and Zariski [6] proved a fundamental inequality for extensions of an arbitrary valuation v of a field K to a finite field extension L/K. Earlier Schmidt [18] constructed a valuation ring A in K which is non-Japanese, that is, the integral closure B of A in some finite extension L/K is not finite over A. Furthermore, Schmidt's example realizes the strict inequality in the fundamental inequality, cf. [6].

In this paper we show a modified fundamental identity for arbitrary Dedekind domains as follows.

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Theorem 1.1. *Let A be a Dedekind domain with quotient field K, L a finite extension field of K with integral closure B of A. For any nonzero prime ideal $\mathfrak{p}$ of A one has the equality*

$$\sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}} f_{\mathfrak{P}} = [(L \otimes_K K_{\mathfrak{p}}^*)_{\mathrm{ss}} : K_{\mathfrak{p}}^*], \tag{1.2}$$

where $K_{\mathfrak{p}}^*$ denotes the completion of K at $\mathfrak{p}$ and $(L\otimes_K K_{\mathfrak{p}}^*)_{\mathrm{ss}}$ denotes the semi-simplification of $L \otimes_K K_{\mathfrak{p}}^*$.

We apply Theorem 1.1 and prove the following new result.

Theorem 1.2. *Let A be a Dedekind domain with quotient field K. Then the following statements are equivalent.*

(1) *A is excellent.*
(2) *A is quasi-excellent.*
(3) *A is a Nagata ring.*
(4) *A is universally Japanese.*
(5) *A is a Japanese ring.*

We shall recall (quasi-)excellent rings, Nagata rings and (universally) Japanese rings, and some properties in Section 2 and refer to [12] for more details. In particular, one has the following well-known relations:

$$\begin{gathered}\text{(excellent rings)} \Longrightarrow \text{(quasi-excellent rings)} \Longrightarrow \text{(Nagata rings)}\\ \Longleftrightarrow \text{(Noetherian universally Japanese rings)} \Longrightarrow \text{(Japanese rings)}.\end{gathered}$$

By Theorem 1.2, when A is a Dedekind domain, the above classes of rings are actually the same. In particular, excellent Dedekind domains are the same as Japanese Dedekind domains, while the definition of the latter is much simpler. We remark that excellent Dedekind domains play a subtle role in the construction of moduli spaces: particularly in Artin's approximation theorem [2] and the existence of Néron models [3, 10.2, Theorem 2, p. 297]. Thus, for some arithmetic applications Theorem 1.2 provides a simpler way to access excellent Dedekind domains.

For a property $\mathbb{P}$ of commutative rings, we say A is locally $\mathbb{P}$ if the localization $A_{\mathfrak{p}}$ at $\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$ satisfies the property $\mathbb{P}$. Applying Theorem 1.2 to the case where A is a discrete valuation ring, we obtain the following

Corollary 1.3. *Let A be a Dedekind domain with quotient field K. Then the following statements are equivalent.*

(1) *A is locally excellent.*
(2) *A is locally quasi-excellent.*
(3) *A is a locally Nagata ring.*
(4) *A is locally universally Japanese.*
(5) *A is a locally Japanese ring.*

R. Heitmann shows that there is a locally Nagata principal ideal domain that is not Nagata. Therefore, the equivalent classes of rings in Theorem 1.2 are strictly stronger than the equivalence classes of rings in Corollary 1.3. Note that when the equality in (1.1) holds, Theorem 1.1 gives a direct connection between G-rings and locally Japanese rings; see Proposition 4.3. This is how Theorem 1.2 is proved.

We explain where (1.2) comes from. Let us fix a nonzero prime ideal $\mathfrak{p}$ of A. On B we have two linear topologies induced by two systems of the subgroups $\{\mathfrak{p}^n\Lambda\}_n$ and $\{\mathfrak{p}^n B\}_n$, respectively, where Λ is a free finite A-submodule of full rank in B, and they agree if the localization $B_\mathfrak{p}$ at $\mathfrak{p}$ is finite over $A_\mathfrak{p}$ (and as we will see from Theorem 1.1 that this is actually "if and only if"). The completion of B for the first topology gives $B\otimes_A A_\mathfrak{p}^*$, which has rank $[L\otimes_K K_\mathfrak{p}^* : K_\mathfrak{p}^*] = [L:K]$ over $A_\mathfrak{p}^*$. The completion for the second topology gives $B_\mathfrak{p}^* = \prod_{\mathfrak{P}|\mathfrak{p}} B_\mathfrak{P}^*$, which has rank $\sum_{\mathfrak{P}|\mathfrak{p}} e_\mathfrak{P} f_\mathfrak{P}$, cf. the proof of Lemma 4.1. It is clear that there is a surjective map from $B\otimes_A A_\mathfrak{p}^*$ to $B_\mathfrak{p}^*$, and using the valuation theory (see [5], also see Section 3) we show that $B_\mathfrak{p}^*$ is exactly the reduced ring of $B\otimes_A A_\mathfrak{p}^*$. The rank of the latter one is equal to $[(L\otimes_K K_\mathfrak{p}^*)_{\mathrm{ss}} : K_\mathfrak{p}^*]$.

This paper is organized as follows. In Section 2 we recall several rings mentioned above and their relations. The proofs of Theorems 1.1 and 1.2 are given in Section 4.

2. Japanese, Nagata, (quasi-)excellent and G-rings

In this section we recall the definition of several fundamental rings in Introduction and their relations. Our references are Matsumura [12], and EGA IV [8,9], also cf. [20, Section 2]. All rings and algebras in this section are commutative with identity.

2.1. Japanese, universally Japanese and Nagata rings

Definition 2.1. Let A be an integral domain with quotient field K.

(1) We say that A *is N-1* if the integral closure A' of A in its quotient field K is a finite A-module.

(2) We say that A *is N-2* if for any finite field extension L over K, the integral closure A_L of A in L is a finite A-module.

The first non-N-1 one-dimensional integral domain was constructed by Akizuki [1], cf. [16]. K. Schmidt [18] and respectively Nagata (Appendix: Examples of bad Noetherian rings of [14]) constructed different Dedekind domains which are not N-2.

Definition 2.2. A ring A is said to be *Nagata* if

(1) A is Noetherian, and (2) $A/\mathfrak{p}$ is N-2 for any prime ideal $\mathfrak{p}$ of A.

Nagata rings are the same as what are called Noetherian universally Japanese rings in EGA IV [8, 23.1.1, p. 213].

Definition 2.3. (1) An integral domain A is said to be *Japanese* if it is N-2.

(2) A ring A is said to be *Japanese* if for any minimal prime ideal $\mathfrak{p}$ of A, the quotient domain $A/\mathfrak{p}$ is N-2.

(3) A ring A is said to be *universally Japanese* if any finitely generated integral domain over A is Japanese.

We clarify two notions of local finiteness of modules as follows.

Definition 2.4. Let M be a module over a commutative ring A.

(1) We say that M is *Zariski-locally finitely generated* if for any prime $\mathfrak{p} \in \operatorname{Spec} A$, there is an element $f \in A$ such that $f \notin \mathfrak{p}$ and M_f is a finite A_f-module.

(2) We say that M is *locally finitely generated* if for any prime $\mathfrak{p} \in \operatorname{Spec} A$, the localization $M_{\mathfrak{p}}$ at $\mathfrak{p}$ is a finite $A_{\mathfrak{p}}$-module.

One easily shows that if M is Zariski-locally finitely generated, then M is finitely generated. Indeed, for each maximal ideal $\mathfrak{m}$ of A there is an element $f \in A$ such that $\mathfrak{m} \notin f$ and M_f is a finite A_f-module. Since $\operatorname{Spec} A$ is quasi-compact, there exist elements $f_1, \dots, f_n \in A$ such that $(f_1, \dots, f_n)A = A$ and each M_{f_i} is a finite A_{f_i}-module. Let $S_i \subset M$ is a finite set of generators of M_{f_i} over A_{f_i}. Then the union of all S_i is a finite set of generators of M over A.

Lemma 2.5. *If for any maximal ideal $\mathfrak{m} \in \operatorname{Max}(A)$, there exists an element $f \in A$ such that $f \notin \mathfrak{m}$ and A_f is Japanese, then A is Japanese.*

Proof. Replacing A by $A/\mathfrak{p}$ for each minimal prime $\mathfrak{p}$, we may assume that A is an integral domain. Let L/K be a finite field extension with integral closure B of A in L. Since the construction of normalization commutes with localization, our assumption implies that for any $\mathfrak{m} \in \operatorname{Max}(A)$, there exists an element $f \in A$ such that $f \notin \mathfrak{m}$ and that B_f is a finite A_f-module. That is, B is Zariski-locally finitely generated and hence B is finitely generated. □

Remark 2.6. The proof of Lemma 2.5 does not show that any locally Japanese ring (that is, its localization at every prime is Japanese) is Japanese. The following is an example, due to Nagata, of a locally finitely generated module which is not finitely generated. Let k be a field of characteristic $p > 0$ such that $[k : k^p] = \infty$. Let $R := k^p[[x, y]][k]$, where x and y are indeterminates. Then the completion R^* of R at the maximal ideal (x, y) is equal to $k[[x,y]]$ and $R \subsetneq R^*$. Let $b_1,\dots,b_n,\dots$ be a sequence of p-independent elements in k, and $p_1,\dots,p_m,\dots$ be mutually non-associative prime elements (i.e., $(p_i) \neq (p_j)$ for $i \neq j$). Put $q_n := p_1 \cdots p_n$. Let $c := \sum_{i=1}^{\infty} b_i q_i \in R^*$ and let $T := R[1/x][c]$ whose normalization is denoted by T'. Then for every prime ideal $\mathfrak{p}$ of T, the localization $T'_{\mathfrak{p}}$ is a finite $T_{\mathfrak{p}}$-module while T' is not finite over T; see [14, A1.Example 8, p. 211] for more details.

2.2. G-rings, closedness of singular loci and excellent rings

Definition 2.7. [12, §33, p. 249]

(1) Let A be a Noetherian ring containing a field k. We say that A is *geometrically regular over* k if for any finite field extension k' over k, the ring $A \otimes_k k'$ is regular [12, p. 78]. This is equivalent to say that the local ring $A_{\mathfrak{m}}$ has the same property for all maximal ideals $\mathfrak{m}$ of A.

(2) Let $\phi\colon A \to B$ be a homomorphism (not necessarily of finite type) of Noetherian rings. We say that ϕ is *regular* if it is flat and for each $\mathfrak{p} \in \operatorname{Spec} A$, the fiber ring $B \otimes_A k(\mathfrak{p})$ is geometrically regular over the residue field $k(\mathfrak{p})$.

(3) A Noetherian ring A is said to be a *G-ring* if for each $\mathfrak{p} \in \operatorname{Spec} A$, the natural map $\phi_{\mathfrak{p}}\colon A_{\mathfrak{p}} \to (A_{\mathfrak{p}})^*$ is regular, where $(A_{\mathfrak{p}})^*$ denotes the completion of the local ring $A_{\mathfrak{p}}$.

Note that the natural map $\phi_{\mathfrak{p}}\colon A_{\mathfrak{p}} \to (A_{\mathfrak{p}})^*$ is faithfully flat. The fibers of the natural morphism $\operatorname{Spec}(A_{\mathfrak{p}})^* \to \operatorname{Spec} A_{\mathfrak{p}}$ are called *formal fibers.* To say a Noetherian ring A is a G-ring then is equivalent to saying that all formal fibers of the canonical map $\phi_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ of A are geometrically regular. It is clear that, if A is a G-ring, then any localization $S^{-1}A$ of A and any homomorphism image A/I of A are G-rings.

Lemma 2.8. *Let K/k be any field extension. Then K is geometrically regular over k if and only if K/k is separable.*

The field extension K/k is separable if and only if for any finite field extension k'/k, the tensor product $k' \otimes_k K$ is reduced or equivalently that $k' \otimes_k K$ is regular. This proves the lemma.

For a Noetherian scheme X, let $\operatorname{Reg}(X)$ denote the subset of X that consists of regular points, which is called the regular locus of X.

Definition 2.9. Let A be a Noetherian ring.

(1) We say that A is *J-0* if $\operatorname{Reg}(\operatorname{Spec} A)$ contains a nonempty open set of $\operatorname{Spec} A$.

(2) We say that A is *J-1* if $\operatorname{Reg}(\operatorname{Spec} A)$ is open in $\operatorname{Spec} A$.

Theorem 2.10. [12, Theorem 73, p. 246] *For a Noetherian ring A, the following conditions are equivalent:*

(a) *any finitely generated A-algebra B is J-1;*

(b) *any finite A-algebra B is J-1;*

(c) *for any $\mathfrak{p} \in \operatorname{Spec} A$, and for any finite radical extension K' of $k(\mathfrak{p})$, there exists a finite A-algebra A' satisfying $A/\mathfrak{p} \subseteq A' \subseteq K'$ which is J-0 and whose quotient field is K'.*

Definition 2.11. A Noetherian ring A is said to be *J-2* if it satisfies one of the equivalent conditions in Theorem 2.10.

Lemma 2.12. *Any Noetherian Japanese ring A of dimension one is J-2.*

Proof. For each $\mathfrak{p} \in \operatorname{Spec} A$, the quotient domain $A/\mathfrak{p}$ is either a field or a Noetherian Japanese domain of dimension one. In the first case, the condition (c) holds trivially by taking $A' = K'$. In the second case, the integral closure A' of A in K' is finite over A and is a Dedekind domain, which particularly is J-0. Therefore, A is J-2. □

Theorem 2.13. (1) *Any complete Noetherian local ring is a G-ring.*

(2) *If for any maximal ideal $\mathfrak{m}$ of a Noetherian ring A, the natural map $A_{\mathfrak{m}} \to (A_{\mathfrak{m}})^*$ is regular, then A is a G-ring.*

(3) *Let A and B be Noetherian rings, and let $\phi\colon A \to B$ be a faithfully flat and regular homomorphism. If B is J-1, then so is A.*

(4) *Any semi-local G-ring is J-1.*

Proof. (1) See [12, Theorem 68, p. 225 and p. 250]. (2) See [12, Theorem 75, p. 251]. (3) and (4) See [12, Theorem 76, p. 252]. □

Theorem 2.14. (1) *Let A be a G-ring and B a finitely generated A-algebra. Then B is a G-ring.*

(2) *Let A be a G-ring which is J-2. Then A is a Nagata ring.*

Proof. (1) See [12, Theorem 77, p. 254]. (2) See [12, Theorem 78, p. 257]. □

Definition 2.15. [12, §34, p. 259] Let A be a Noetherian ring.

(1) We say that A is *quasi-excellent* if the following conditions are satisfied:

  (i) A is a G-ring;

(ii) A is J-2.

(2) We say that A is *excellent* if it satisfies (i), (ii) and the following condition

(iii) A is universally catenary [12, p. 84].

Remark 2.16. If A is catenary, then so are any localization of A and any homomorphism image of A. To show a Noetherian ring A is universally catenary, it then suffices to show that every polynomial ring $A[X_1, \ldots, X_n]$, for $n \geq 1$, is catenary. If A is Cohen–Macaulay, then A is catenary and $A[X]$ is again Cohen–Macaulay, cf. [7, Proposition 18.9 and Corollary 18.10]. Therefore, any Cohen–Macaulay ring is universally catenary. Since every regular ring is Cohen–Macaulay, every regular ring is universally catenary.

Remark 2.17. (1) Each of the conditions (i), (ii), and (iii) is stable under the localization and passage to a finitely generated algebra (see Theorems 2.10(1) and 2.14(1)).

(2) Note that (i), (ii), (iii) are conditions depending only on $A/\mathfrak{p}$, for $\mathfrak{p} \in \operatorname{Spec} A$. Thus a Noetherian ring A is (quasi-)excellent if and only if so is A_{red}.

(3) The conditions (i) and (iii) are of local nature (in the sense that if they hold for $A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$, then they hold for A), while the condition (ii) is not.

(4) Theorem 2.14(2) states that any quasi-excellent ring is a Nagata ring.

(5) It follows from Theorems 2.10 and 2.13(4) that any Noetherian local G-ring is quasi-excellent.

(6) Nagata's example of a 2-dimensional Noetherian local ring that is catenary but not universally catenary [12, (14.E), p. 87] is a G-ring, and is also a J-2 ring as any local G-ring is a J-2 ring. So it is a quasi-excellent catenary local ring that is not excellent.

(7) Rotthaus [17] constructed a regular local ring R of dimension three which contains a field and which is Nagata, but not quasi-excellent.

3. Valuations, completions and extensions

In this section, a group will mean an abelian group unless stated otherwise.

3.1. Valuations and valuation rings

For a totally ordered abelian group Γ written additively, we denote by $\Gamma_\infty = \Gamma \cup \{\infty\}$ the totally ordered commutative monoid with $\gamma \leq \infty$ for all $\gamma \in \Gamma$ and $\gamma + \infty = \infty + \gamma = \infty$ for all $\gamma \in \Gamma_\infty$.

Definition 3.1. (1) An integral domain A with quotient field K is called a *valuation ring* or a *valuation ring of* K if for any $x \in K^\times$ either $x \in A$ or $x^{-1} \in A$.

(2) A *valuation* of a field K is a group homomorphism $v\colon K^\times \to \Gamma$, where Γ is a totally ordered abelian group, such that

$$v(x+y) \ge \min\{v(x), v(y)\}, \quad \forall\, x, y \in K^\times. \tag{3.1}$$

We extend v to a function $v\colon K \to \Gamma_\infty$ by putting $v(0) = \infty$. Clearly, the condition (3.1) holds for all $x, y \in K$. The homomorphism image $v(K^\times)$ is called the *value group* of the valuation v. Clearly, $A := \{x \in K : v(x) \ge 0\}$ is a valuation ring and $\mathfrak{m} := \{x \in A : v(x) > 0\}$ is its maximal ideal. We call A and $\kappa := A/\mathfrak{m}$ the *valuation ring* and *residue field* of v, respectively.

(3) A valuation v of K is said to be *discrete* if its value group is isomorphic to $\mathbb{Z}$ compatible with the orders.

(4) Two valuations v_1 and v_2 of K with value groups Γ_1 and Γ_2 are said to be *equivalent* if there is an isomorphism $\alpha\colon \Gamma_1 \xrightarrow{\sim} \Gamma_2$ of ordered groups such that $v_2 = \alpha \circ v_1$.

The construction $v \mapsto A$ gives rise to a map from the set of equivalence classes of valuations of K to the set of valuation rings of K. The reverse construction is as follows: For a given valuation ring $(A, \mathfrak{m})$, define $\Gamma := K^\times/A^\times$ and $P := \mathfrak{m}/A^\times$ the set of positive elements, then Γ is a totally ordered group and the natural projection $v\colon K^\times \to \Gamma$ is a valuation of K so that the valuation ring of v is equal to A. It is easy to see the above map is bijection, cf. [5, VI, §3.2, Proposition 3].

Proposition 3.2. *Let A be a valuation ring of a field K.*

(1) *The set of primes ideals of A is totally ordered by the order of inclusion.*

(2) *If $B \supset A$ is a subring of K, then B is a valuation ring and the maximal ideal $\mathfrak{m}(B)$ of B is a prime ideal of A. Moreover, the map $B \mapsto \mathfrak{m}(B)$ is a order-reversing bijection between the totally ordered set of subrings of K containing A and the totally ordered set of prime ideals of A. The inverse map is given by $\mathfrak{p} \mapsto A_\mathfrak{p}$, the localization of A at $\mathfrak{p}$.*

Proof. (1) See [5, VI, §1.2, Theorem 1(e)]. (2) See [5, VI, §4.1, Proposition 1 and Corollary]. □

Definition 3.3. A subgroup H of an ordered group G is said to be *isolated* if the relation $0 \le y \le x$ with $x \in H$ implies $y \in H$.

Proposition 3.4. *Let G be an ordered group and P the set of its positive elements.*

(1) *The kernel of an increasing homomorphism of G to an ordered group is an isolated subgroup of G.*

(2) *Conversely, let H be an isolated subgroup of G and $g\colon G \to G/H$ the canonical homomorphism. Then $g(P)$ is the set of positive elements of an ordered group structure on G/H. Moreover, if G is totally ordered, so is G/H.*

If G is totally ordered, then the set of isolated subgroups of G are totally ordered by the order of inclusion. For otherwise, there is a positive element x in one isolated subgroup H but not in H', and a positive element $x' \in H' \setminus H$. Suppose for example $x \leq x'$, then $x \in H'$, a contradiction.

Definition 3.5. (1) Let G be a totally ordered group. If the number of isolated subgroups of G distinct from G is finite and is equal to n, G is said to be *of height* n. If this number is infinite, G is said to be *of infinite height*. Denote by $h(G)$ the height of G.

(2) The *height* of a valuation v of K is defined as the height of its value group.

The height of the groups $\mathbb{Z}$ and $\mathbb{R}$ are of height 1. If G is a totally ordered group and H is an isolated subgroup, then $h(G) = h(H) + h(G/H)$. In particular, if G is the lexicographic product of two totally ordered groups H and H', then $h(G) = h(H) + h(H')$. Thus, the lexicographic product $\mathbb{Z} \times \mathbb{Z}$ is of height 2.

Fix a valuation ring of A of K with the canonical valuation $v_A\colon K^\times \to \Gamma_A := K^\times/A^\times$. For subring B of K containing A, B is a valuation ring and $A^\times \subset B^\times$. Let $\lambda\colon \Gamma_A \to \Gamma_B$ be the natural projection. As $A \subset B$, λ maps the positive elements of Γ_A to positive elements of Γ_B. Thus, λ is a morphism of ordered groups and the kernel H_B of λ is an isolated subgroup of Γ_A. The mapping $B \mapsto H_B$ is an increasing bijection of the set of subrings of of K containing A onto the set of isolated subgroups of Γ_A, cf. [5, VI, §4.3, Proposition 4]. Combining with Proposition 3.2, these two sets are in deceasing bijection with the set of primes ideals of A. In particular, $h(\Gamma_A) = \mathrm{ht}(\mathfrak{m}(A)) = \dim A$, cf. [5, VI, §4.4, Proposition 5].

A totally ordered group G is of height ≤ 1 if and only if it is isomorphic to a subgroup of $\mathbb{R}$, cf. [5, VI, §4.5, Proposition 8].

3.2. Topological fields and completions

We first discuss the notion of completeness of a Hausdorff commutative topological group and its completion. Our references are [4] and [11, Chap. 10].

Definition 3.6. Let X be a nonempty set.

(1) A *filter* of X is a nonempty subset $\mathfrak{F} \subset P(X)$ of the power set of X satisfying

(F1) If $A, B \in \mathfrak{F}$, then $A \cap B \in \mathfrak{F}$;

(F2) If $A \in \mathfrak{F}$ and $A \subset A' \subset X$, then $A' \in \mathfrak{F}$;

(F3) $\emptyset \notin \mathfrak{F}$.

(2) A *filter base* of X is a nonempty subset $\mathfrak{B} \subset P(X)$ satisfying

(FB1) If $A, B \in \mathfrak{F}$, then there exists $C \in \mathfrak{B}$ such that $C \subset A \cap B$;

(FB2) $\emptyset \notin \mathfrak{B}$.

Condition (F3) implies that any finite intersection of members in $\mathfrak{F}$ is nonempty. For a filter base $\mathfrak{B}$, the set $\mathfrak{F} := \{F \subset X : \exists\, B \in \mathfrak{B}, B \subset F\}$ is a filter of X, called the filter generated by $\mathfrak{B}$, and $\mathfrak{B}$ is called a base of $\mathfrak{F}$. A filter $\mathfrak{F}'$ is called a *refinement* of $\mathfrak{F}$ if $\mathfrak{F} \subset \mathfrak{F}'$.

Example 3.7. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in X. Let $\mathfrak{F}$ be the set consisting of all subsets E such that there exists $N \geq 1$ such that $x_k \in E$ for all $k \geq N$. Then $\mathfrak{F}$ is a filter. Let $\mathfrak{B}$ be the set consisting of all subsets $\{x_k, x_{k+1}, \dots\}$ for some k. Then $\mathfrak{B}$ is a filter base which generates $\mathfrak{F}$.

Example 3.8. Let X be a topological space. For every point $x \in X$, let $\mathfrak{N}_x$ denote the collection of all neighborhoods E of x (there exists an open subset $U \ni x$ contained in E). Then $\mathfrak{N}_x$ is a filter. Any fundamental system of neighborhoods of x is a filter base of $\mathfrak{N}_x$.

Definition 3.9. Let X be a topological space. We say a filter base $\mathfrak{F}$ *converges to a point* $x \in X$, denoted by $\mathfrak{F} \to x$, if every $E \in \mathfrak{N}_x$ contains a member $F \in \mathfrak{F}$. In this case, by (FB1), x is in the closure of every member $F \in \mathfrak{F}$. If $\mathfrak{F}$ is a filter, this is equivalent to say that $\mathfrak{F}$ is a refinement of $\mathfrak{N}_x$.

A topological space X is Hausdorff if and only if every filter converges to at most one point. Let $f\colon X \to Y$ be a map of topological spaces and $\mathfrak{F}$ a filter of X. Set

$$f\mathfrak{F} := \{F \subset Y : \exists\, E \in \mathfrak{F}, f(E) \subset F\},$$

which is a filter as $f(A \cap B) \subset f(A) \cap f(B)$. Then the map f is continuous if and only if for every $x \in X$ one has $(\mathfrak{F} \to x) \Longrightarrow (f\mathfrak{F} \to f(x))$.

Definition 3.10. Let $(A, +)$ be a Hausdorff commutative topological group.

(1) A filter $\mathfrak{F}$ of X is called a *Cauchy filter* if for any neighborhood $U \in \mathfrak{N}_0$ there exists $E \in \mathfrak{F}$ such that

$$E - E := \{x - y : x, y \in E\} \subset U.$$

(2) A is said to be *complete* if every Cauchy filter converges.

(3) A *completion* of A is pair (A^*, ι), where A^* is a complete topological abelian group and $\iota\colon A \to A^*$ is a morphism of topological groups, satisfying the following conditions.

   (a) $\iota\colon A \to \iota(A)$ is a homeomorphism.

   (b) $\iota(A) \subset A^*$ is dense.

The condition (a) says that ι is injective and topology of A is the same as the topology induced by A^*. It is proved [4, III, §3.5, Theorem 2 and §3.4, Proposition 8] that a completion (A^*, ι) exists and satisfies the functorial property: for any pairing (B, f) where B is a complete topological abelian group and $f\colon A \to B$ is a morphism of topological groups, then there exists a unique morphism $g\colon A^* \to B$ such that $g \circ \iota = f$. In particular, a completion (A^*, ι) is unique up to a unique isomorphism; such a pair (A^*, ι) is called *the completion of* A. If A is a Hausdorff topological ring, then the completion A^* of $(A, +)$ is a complete topological ring (complete for the underlying topological group $(A^*, +)$), cf. [4, III, §6].

Let v be a valuation of a field K with value group G. For all $\alpha \in G$, let

$$V_\alpha := \{x \in K : v(x) > \alpha\} \quad \text{and} \quad V_{\geq\alpha} := \{x \in K : v(x) \geq \alpha\}$$

which are clearly additive subgroups of K. There exists a unique linear topology $\mathfrak{T}_v$ on K for which the sets V_α form a fundamental system of neighborhoods of 0. If v is trivial, then $\mathfrak{T}_v$ is the discrete topology. Equipped with this topology K is a Hausdorff topological field.

Let K^* be the completion of K, which is a complete topological ring. Note that if v is of height 1, or more generally, there exists a *countable* fundamental system of neighborhoods of 0. Then the notion of completeness and the construction of completion can be made by the usual Cauchy sequences, which is similar to the classical construction of $\mathbb{R}$ from $\mathbb{Q}$.

Proposition 3.11. *Let v be a valuation of a field K with value group G and equip G with the discrete topology.*

(1) *The complete ring K^* of K is a topological field.*

(2) *The continuous map $v\colon K \to G_\infty$ can be extended uniquely to a continuous map $v^*\colon K^* \to G_\infty$ which is a valuation of K^*.*

(3) *The topology on* K^* *is the topology defined by the valuation* v^*.

(4) *For all* $\alpha \in G$, *the closures* $\overline{V}_\alpha$ *and* $\overline{V}_{\geq\alpha}$ *of* V_α *and* $V_{\geq\alpha}$ *are the subsets of* K^* *defined by* $v^*(x) > \alpha$ *and* $v^*(x) \geq \alpha$, *respectively.*

(5) *The valuation ring of* v^* *is the completion* A^* *of* A*; its maximal ideal is the completion* $\mathfrak{m}^*$ *of the maximal ideal* $\mathfrak{m}$ *of* A.

(6) $A^* = A + \mathfrak{m}^*$*; the residue field of* v^* *is canonically identified with that of* v.

Proof. See [5, VI, §5.3, Proposition 5]. □

3.3. Extensions of valuations and the fundamental inequality

Let v be a valuation of a field K, A the valuation ring of v, $\mathfrak{m}$ its maximal ideal, and Γ_v the value group. Let L/K be a finite field extension and w be a valuation of L which extends v. Denote by Γ_w the value group, A' the valuation ring and $\mathfrak{m}'$ the maximal ideal of w, respectively. Write $\kappa(v)$ and $\kappa(w)$ for the residue fields of v and w, respectively. The completion of K at v (resp. of L at w) is denoted by K^* (resp. L^*_w). The valuation of K^* extending v is denoted by v^*.

Definition 3.12. (1) The *ramification index of* w *over* v is defined as $e(w/v) := [\Gamma_w : \Gamma_v]$.

(2) The *residue class degree* of w over v is defined as $f(w/v) := [\kappa(w) : \kappa(v)]$.

Lemma 3.13. *Let* K, v, L *and* w *be as above.*

(1) *The inequality* $e(w/v)f(w/v) \leq [L : K]$ *holds. In particular,* $e(w/v)$ *and* $f(w/v)$ *are finite.*

(2) *The height of* w *is equal to that of* v.

(3) *The valuation* w *is trivial (resp. discrete) if and only if so is* v.

Proof. See [5, VI, §8.1]. □

Definition 3.14. [5, VI, §7.2] Two valuations v and v' of K are said to be *independent* if the subring generated by their valuation rings is equal to K; and *dependent* otherwise.

The trivial valuation is independent of any valuation of K. Two valuations are dependent if and only if there is a relation $A \subset A' \subsetneq K$ among their valuation rings A and A'. If two non-trivial valuations v and v' are of same finite height, then they are dependent if and only if they are equivalent. Indeed, if v and v' are equivalent, then $A = A' \subsetneq K$ and they are dependent. Conversely if they are dependent then up to switching the order one has $A \subset A' \subsetneq K$. Therefore, $A = A'$ for otherwise $\mathrm{ht}(v') < \mathrm{ht}(v)$, contradiction.

Let Σ_v be a complete set of representatives of equivalence classes of extensions of a valuation v of K on L. If v is trivial then Σ_v consists of the trivial valuation of L. Also write $w|v$ if $w \in \Sigma_v$. If $w_1, w_2 \in \Sigma_v$ with $w_1 \neq w_2$ (this implies that v must be non-trivial), then there is no inclusion relation for their valuation rings, cf. [6, (A), p. 2]. Thus, every two distinct valuations in Σ_v are independent.

Proposition 3.15. *Let v be a valuation of K and L/K a finite field extension.*

(1) *For every $w \in \Sigma_v$, one has $e(w^*/v^*) = e(w/v)$, $f(w^*/v^*) = f(w/v)$, $[L^*_w : K^*] \leq [L : K]$ and $e(w/v)f(w/v) \leq [L^*_w : K^*]$.*

(2) *Every set of pairwise independent valuations of L extending a non-trivial valuation v is finite. Let $\{w_1, \dots, w_r\}$ be a maximal set of pairwise independent valuations of L extending a non-trivial valuation v. Then the canonical mapping $\phi\colon K^* \otimes_K L \to \prod_{i=1}^r L^*_{w_i}$ (extending by continuity the diagonal map $L \to \prod_{i=1}^r L^*_{w_i}$) is surjective, its kernel is the Jacobson radical of $K^* \otimes_K L$ and*

$$\sum_{i=1}^{r} [L^*_{w_i} : K^*] \leq [L : K].$$

Proof. See [5, VI, §8.2, Proposition 2]. □

Corollary 3.16 (The fundamental inequality)**.** *Let v be a valuation of K and L/K a finite extension. We have*

$$\sum_{w|v} e(w/v)f(w/v) \leq [L : K]. \tag{3.2}$$

Proof. If v is trivial, then w is trivial and one has $\Gamma_v = \Gamma_w = \{0\}$, $\kappa(v) = K$ and $\kappa(w) = L$. Therefore, $\sum_{w|v} e(w/v)f(w/v) = [L : K]$. Suppose that v is non-trivial. By the remark above Proposition 3.15, Σ_v is a maximal set $\{w_1, \dots, w_r\}$ of pairwise independent valuations of L extending v. Then (3.2) follows from Proposition 3.15. □

Definition 3.17. [5, VI, §3.5, §8.4]

(1) Let G be an ordered set. A subset M of G is called *major* if the relations $x \in M$ and $y \geq x$ imply $y \in M$.

(2) Let G be a totally ordered commutative group and H a subgroup of finite index. Denote by $G_{>0} \subset G$ the subset of strictly positive elements. The *initial index of H in G*, denoted by $\varepsilon(G, H)$, is the number of major subsets M of G such that $H_{>0} \subset M \subset G_{>0}$.

If $G = \mathbb{Z}$ and $H = m\mathbb{Z}$ with $m > 0$, letting $M(x) := \{y \in G : y \geq x\}$, then $M(1), M(2), \dots, M(m)$ are all major subsets of G satisfying the property in Definition 3.17 and $\varepsilon(G, H) = m$.

Proposition 3.18. *Let G be a totally ordered commutative group and H a subgroup of finite index.*

(1) *If the set $G_{>0}$ has no least element, then $\varepsilon(G, H) = 1$;*

(2) *If the set $G_{>0}$ has the least element x_0, then $\varepsilon(G, H) = [G_0 : (G_0 \cap H)]$, where G_0 is the cyclic subgroup generated by x_0.*

In particular, $\varepsilon(G, H)$ divides $[G : H]$.

Proof. See [5, VI, §8.4, Proposition 3]. □

Definition 3.19. Let v be a valuation of K and $w|v$ a valuation of a finite extension L/K with value groups Γ_v and Γ_w, respectively. The *initial ramification index of w with respect to v (or w over v)* is defined as $\varepsilon(w/v) := \varepsilon(\Gamma_w, \Gamma_v)$.

Theorem 3.20. *Let L/K be a and let v a valuation of K with valuation ring A and maximal ideal $\mathfrak{m}$. Let L/K be a finite field extension with integral closure B of A in L. The following conditions are equivalent:*

(a) *B is a finite A-module;*

(b) *B is a free A-module;*

(c) $[B/\mathfrak{m}B : \kappa(\mathfrak{m})] = [L : K]$;

(d) $\sum_{w|v} \varepsilon(w/v) f(w/v) = [L : K]$.

Proof. See [5, VI, §8.5, Theorem 2]. □

Remark 3.21. (1) The fundamental inequality (Corollary 3.16) was first proved by Cohen and Zariski for an arbitrary valuation v [6].

(2) The condition (d) is equivalent to (d') $\sum_{w|v} e(w/v) f(w/v) = [L : K]$ and $\varepsilon(w/v) = e(w/v)$ for all $w|v$. When v is discrete, one has $\varepsilon(w/v) = e(w/v)$ and the condition (d) is equivalent to $\sum_{w|v} e(w/v) f(w/v) = [L : K]$. In this special case, Theorem 3.20 was first proved by Cohen and Zariski [6].

4. Proofs of Theorems 1.1 and 1.2

4.1. Proof of Theorem 1.1

Theorem 1.1 follows from Lemma 4.1 and Proposition 4.2.

Lemma 4.1. *Let A be a Dedekind domain with quotient field K, and L a finite field extension of K of degree n with integral closure B of A in L. For each nonzero prime ideal $\mathfrak{p}$ of A, one has*

$$\sum_{i=1}^{r} e_i f_i = \dim_{k(\mathfrak{p})} B/\mathfrak{p}B \le n,$$

where $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ are the prime ideals of B over $\mathfrak{p}$, e_i and f_i are the ramification index and the residue class degree of $\mathfrak{P}_i$ over $\mathfrak{p}$. Moreover, if B is a finite A-module, then the equality holds.

Proof. This is well-known (cf. [19]); we include a proof for the reader's convenience. We localize the Dedekind domains A and B at $\mathfrak{p}$ and get a discrete valuation ring $A_\mathfrak{p}$ and a semi-local Dedekind domain $B_\mathfrak{p}$ with same number of maximal ideals $\mathfrak{P}_i B_\mathfrak{p}$. Then $B_\mathfrak{p}$ is the integral closure of $A_\mathfrak{p}$ in L and the numerical invariants r, e_i, f_i remain the same. Therefore, after replacing A by $A_\mathfrak{p}$, we can assume that A is a discrete valuation ring with uniformizer π. By the Chinese Remainder Theorem, $B/\mathfrak{p}B \simeq \prod_{i=1}^{r} B/\mathfrak{P}_i^{e_i}$. We filter each $k(\mathfrak{p})$-vector space $B/\mathfrak{P}_i^{e_i}$ by the decreasing subspaces $\mathfrak{P}_i^j/\mathfrak{P}_i^{e_i}$ and obtain

$$\dim_{k(\mathfrak{p})} B/\mathfrak{p}B = \sum_{i=1}^{r} \sum_{j=0}^{e_i-1} \dim_{k(\mathfrak{p})} \mathfrak{P}_i^j/\mathfrak{P}_i^{j+1}. \tag{4.1}$$

The ideal $\mathfrak{P}_i^j/\mathfrak{P}_i^{j+1}$ generated by one element a_j as the quotient ring B/J for any nonzero ideal J is a principal ideal ring. The map $1 \mapsto a_j$ induces an isomorphism $A/\mathfrak{P}_i \simeq \mathfrak{P}_i^j/\mathfrak{P}_i^{j+1}$ and hence $\dim_{k(\mathfrak{p})} \mathfrak{P}_i^j/\mathfrak{P}_i^{j+1} = f_i$. It follows from (4.1) that $\dim_{k(\mathfrak{p})} B/\mathfrak{p}B = \sum_{i=1}^{r} e_i f_i$.

We prove the inequality in (4.1) by showing that if $\overline{x}_1, \ldots, \overline{x}_s$ are $k(\mathfrak{p})$-linearly independent, then their liftings $x_1, \ldots, x_s$ are K-linearly independent. Suppose not, then

$$a_1 x_1 + a_2 x_2 + \cdots + a_s x_s = 0$$

for some nonzero element a_i in K. Multiplying a suitable power of π, we can assume that $a_i \in A$ for all i but $a_i \notin \mathfrak{p}$ for some i. Modulo $\mathfrak{p}$ we get a non-trivial linear relation $\overline{a}_1\overline{x}_1 + \overline{a}_2\overline{x}_2 + \cdots + \overline{a}_s\overline{x}_s = 0$, a contradiction.

Suppose B is a finite A-module. Since B is torsion free and A is a principal ideal domain, B is a free A-module of rank s. Then one has $\dim_K L = \operatorname{rank}_A B = \dim_{k(\mathfrak{p})} B/\mathfrak{p}B$. This proves the desired equality. □

Proposition 4.2. *Let the notation be as in Lemma* 4.1*. Then*

$$\dim_{k(\mathfrak{p})} B/\mathfrak{p}B = \sum_{i=1}^{r} \operatorname{rank}_{A_\mathfrak{p}^*} B_{\mathfrak{P}_i}^* = \sum_{i=1}^{r} \dim_{K_\mathfrak{p}^*} L_{\mathfrak{P}_i}^* = [(L \otimes_K K_\mathfrak{p}^*)_{\mathrm{ss}} : K_\mathfrak{p}^*],$$

where $K_\mathfrak{p}^*$ and $A_\mathfrak{p}^*$ (resp. $L_{\mathfrak{P}_i}^*$ and $B_{\mathfrak{P}_i}^*$) denote the completion of K and A (resp. L and B) at $\mathfrak{p}$ (resp. $\mathfrak{P}_i$), respectively.

Proof. The ramification index and residue class degree remain the same after the completion, that is, $e(\mathfrak{P}_i/\mathfrak{p}) = e(\mathfrak{P}_i^*/\mathfrak{p}^*)$ and $f(\mathfrak{P}_i/\mathfrak{p}) = f(\mathfrak{P}_i^*/\mathfrak{p}^*)$, where $\mathfrak{p}^* = \mathfrak{p}A_\mathfrak{p}^*$ and $\mathfrak{P}_i^* = \mathfrak{P}_i B_{\mathfrak{P}_i}^*$. Since $K_\mathfrak{p}^*$ is complete and $L_{\mathfrak{P}_i}^*$ is a finite field extension of $K_\mathfrak{p}^*$, we have $e_i f_i = e(\mathfrak{P}_i^*/\mathfrak{p}^*) f(\mathfrak{P}_i^*/\mathfrak{p}^*) = [L_{\mathfrak{P}_i}^* : K_\mathfrak{p}^*]$ [19, Chap. II, §2, Corollary 1]. Note that $B_{\mathfrak{P}_i}^*$ is the integral closure of $A_\mathfrak{p}^*$ in $L_{\mathfrak{P}_i}^*$. Since $B_{\mathfrak{P}_i}^*$ is a finite free $A_\mathfrak{p}^*$-module say of rank m, we have $L_{\mathfrak{P}_i}^* = B_{\mathfrak{P}_i}^* \otimes_{A_\mathfrak{p}^*} K_\mathfrak{p}^* \simeq (K_\mathfrak{p}^*)^m$ and $\operatorname{rank}_{A_\mathfrak{p}^*} B_{\mathfrak{P}_i}^* = [L_{\mathfrak{P}_i}^* : K_\mathfrak{p}^*] = e_i f_i$. This and Lemma 4.1 prove the first two equalities. The last equality follows from the isomorphism $\prod_{i=1}^r L_{\mathfrak{P}_i}^* \simeq (L \otimes_K K_\mathfrak{p}^*)_{\mathrm{ss}}$; see Proposition 3.15. □

4.2. Proof of Theorem 1.2

Theorem 1.2 will follow immediately from Theorem 4.4.

Proposition 4.3. *Let A be a Dedekind domain with quotient field K. Then the following statements are equivalent:*

(1) *For any finite field extension L/K and any nonzero prime ideal $\mathfrak{p}$ of A, one has $\sum_{\mathfrak{P}|\mathfrak{p}} e_\mathfrak{P} f_\mathfrak{P} = [L : K]$.*

(2) *For any finite field extension L/K and any nonzero prime ideal $\mathfrak{p}$ of A, the tensor product $L \otimes_K K_\mathfrak{p}^*$ is a semi-simple $K_\mathfrak{p}^*$-algebra.*

(3) *A is a G-ring.*

(4) *For every nonzero prime ideal $\mathfrak{p}$ of A, the localization $A_\mathfrak{p}$ is Japanese.*

Proof. The equivalence of (1) and (2) follows from Theorem 1.1.

We prove the equivalence of (2) and (3). To show A is a G-ring, one needs to show that every formal fiber of $\phi_\mathfrak{p}\colon A_\mathfrak{p} \to A_\mathfrak{p}^*$ is geometrically regular. If $\mathfrak{p} = 0$, the formal fiber $K \to K^* = K$ is clearly geometrically regular. Suppose $\mathfrak{p}$ is a nonzero prime ideal. The special formal fiber $k(\mathfrak{p}) \to k(\mathfrak{p}^*) = k(\mathfrak{p})$ is clearly geometrically regular and the generic formal fiber is given by $K \to K_\mathfrak{p}^*$. By Lemma 2.8, $K_\mathfrak{p}^*/K$ is geometrically regular if and only if it is separable. Thus, A is a G-ring if and only if for any nonzero prime ideal $\mathfrak{p}$ of A, the field extension $K_\mathfrak{p}^*/K$ is separable. This is equivalent to that for any finite extension L/K and for any nonzero prime ideal $\mathfrak{p}$ of A, the tensor product $L \otimes_K K_\mathfrak{p}^*$ is semi-simple.

We prove the equivalence of (1) and (4). The direction (4) ⇒ (1) follows from Lemma 4.1 and we show the other direction. For each nonzero prime ideal $\mathfrak{p}$, since $\sum_{i=1}^r e_i f_i = [L : K]$, by Theorem 3.20 the integral closure $B_\mathfrak{p}$ of $A_\mathfrak{p}$ in L is a finite $A_\mathfrak{p}$-module. Thus, for any finite field extension L/K the integral closure of $A_\mathfrak{p}$ in L is a finite A-module, and hence $A_\mathfrak{p}$ is Japanese. This completes the proof of the proposition. □

Theorem 4.4. *Any Japanese Dedekind domain A is excellent.*

Proof. It follows immediately from Proposition 4.3 and Lemma 2.12 that A is quasi-excellent. It is well-known that any Dedekind domain is universally catenary (cf. Remark 2.16). Therefore, A is excellent. □

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Chia-Fu Yu
Institute of Mathematics, Academia Sinica, Astronomy Mathematics Building, No. 1, Roosevelt Rd. Sec. 4, Taipei 10617, Taiwan
E-mail address: `chiafu@math.sinica.edu.tw`