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The L^p-L^q Boundedness and Compactness of Bergman Type Operators

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Abstract. We investigate Bergman type operators on the complex unit ball, which are singular integral operators induced by the modified Bergman kernel. We consider the L^p-L^q boundedness and compactness of Bergman type operators. The results of boundedness can be viewed as the Hardy–Littlewood–Sobolev (HLS) type theorem in the case unit ball. We also give some sharp norm estimates of Bergman type operators which in fact gives the upper bounds of the optimal constants in the HLS type inequality on the unit ball. Moreover, a trace formula is given.

1. Introduction

Let $\mathbb{B}^d$ be the unit ball on the d-dimensional complex Euclidian space $\mathbb{C}^d$ with the normalized Lebesgue measure dv. For $\alpha \in \mathbb{R}$, the α-order Bergman type kernel function is given by

$$k_\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^\alpha}.$$

Clearly the $(d+1)$-order Bergman type kernel function $k_{d+1}(z,w)$ is the standard Bergman kernel on $\mathbb{B}^d$. The Bergman type integral operator K_α on $L^1(\mathbb{B}^d, dv)$ is defined by

$$K_\alpha f(z) = \int_{\mathbb{B}^d} k_\alpha(z,w) f(w)\, dv(w),$$

where $\langle z,w\rangle = z_1\overline{w}_1 + \cdots + z_d\overline{w}_d$ is the standard Hermitian inner product on $\mathbb{C}^d$. Such operators K_α play an important role in complex analysis of several variables and operator theory; in particular, when $\alpha = d+1$, K_{d+1} is the standard Bergman projection on the unit ball $\mathbb{B}^d$. Indeed, for any $\alpha > 0$, if we restrict K_α to Bergman spaces, then every K_α is a special form of fractional radial differential operator, see Lemma 2.8 below. The fractional radial differential operators have many applications in the function space and operator theory; see for examples [23, 24]. On the other hand, the operators K_α play a

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significant role in the characterization of the weighted Bloch spaces and Lipschitz spaces on the unit ball; see [23–25]. We also consider the kernel integral operator K_α^+ on $L^1(\mathbb{B}^d, dv)$ for any $\alpha \in \mathbb{R}$, which is given by

$$K_\alpha^+ f(z) = \int_{\mathbb{B}^d} \frac{f(w)}{|1-\langle z, w\rangle|^\alpha}\, dv(w).$$

The operators K_α^+ can be regarded as Riesz potential operators on the bounded domain $\mathbb{B}^d$. Comparing to the classical Riesz potential operators on the real Euclidian space $\mathbb{R}^d$, whose basic result concerning mapping properties is the Hardy–Littlewood–Sobolev (HLS) theorem or inequality which essentially describes the boundedness of Riesz potential operators $R_\alpha\colon L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$; see [14, 17, 20, 21] and references therein. One of the interesting questions involving the HLS inequality is the estimate of the optimal constant [13, 14].

For abbreviation, we replace $L^p(\mathbb{B}^d, dv)$ by $L^p(\mathbb{B}^d)$ or L^p for any $1 \le p \le \infty$ without confusion arises. The present paper mainly concerns the L^p-L^q boundedness, compactness and norm estimates of Bergman type operators $K_\alpha, K_\alpha^+\colon L^p \to L^q$ for $1 \le p, q \le \infty$. Analogously, the results of L^p-L^q boundedness can be viewed as the HLS type theorem with respect to K_α^+ on $\mathbb{B}^d$. Actually, on a more general bounded domain Ω in $\mathbb{C}^d$, the L^p-L^q boundedness of Bergman type operators and in particular L^p-L^q boundedness of the standard Bergman projection have attracted much interest in the past few decades. As we all know, the standard Bergman projection is bounded for any bounded domain when $p = q = 2$. However, the results would be subtle in general. Nevertheless, the known results depend strongly on the geometric property of the domain Ω; we refer the reader to [1, 2, 9, 16, 18, 24] for more recent progress on this issue.

Now return to our unit ball setting. In [8], X. Fang and Z. Wang established a relation between the boundedness of standard Bergman projection and Berezin transform on the weighted Bergman spaces on the unit disc $\mathbb{D} = \mathbb{B}^1$. The compactness of standard Bergman projection $K_2\colon L^\infty(\mathbb{D}) \to L^q(\mathbb{D})$ for $1 \le q < \infty$ was observed by K. Zhu in [25, Section 3.6]. Recently, X. Fang and G. Cheng et al. [3] completely solved the L^p-L^q boundedness problem of K_α on the unit disc $\mathbb{D}$, and they also considered the L^p-L^q boundedness of the Bergman type operator on the upper half plane. G. Cheng and X. Hou et al. [4] solved the L^p-L^q boundedness problem of K_α in the special case of the $\alpha = 1$ on the unit ball $\mathbb{B}^d$ for any $d \ge 1$. Soon afterwards H. Kaptanoğlu and A. Üreyen [11] solved the L^p-L^q boundedness problem of Bergman type operators in the general case. The results of boundedness of K_α not only give a positive answer to the conjecture proposed in [3] but also extend some classical results [6, 15, 18, 24, 25], the results of boundedness of K_α^+ are essentially the HLS type theorem as mentioned before. In the present paper we will describe the L^p-L^q compactness of K_α on the unit ball $\mathbb{B}^d$ $(d \ge 1)$; especially the relationship between the boundedness and compactness. Meanwhile, we estimate norms of Bergman type operators which in fact gives the upper bounds of the optimal constants in the HLS type inequalities on the unit ball. First, it is trivial that $K_\alpha, K_\alpha^+\colon L^p \to L^q$ are compact for any $1 \le p, q \le \infty$ if $\alpha \le 0$. Thus we mainly concern the case of $\alpha > 0$. The following theorems are main results of L^p-L^q boundedness and compactness of Bergman type operators; however, the norm estimates of Bergman type operators shall be given in the following Section 5.

Theorem 1.1. *If $d+1 < \alpha < d+2$, then the following conditions are equivalent:*

(1) $K_\alpha\colon L^p \to L^q$ *is bounded;*

(2) $K_\alpha^+\colon L^p \to L^q$ *is bounded;*

(3) $K_\alpha\colon L^p \to L^q$ *is compact;*

(4) *p, q satisfy one of the following inequalities:*

 (a) $1/(d+2-\alpha) < p < \infty$, $1/q > 1/p + \alpha - (d+1)$;

 (b) $p = \infty$, $q < 1/(\alpha - (d+1))$.

As a consequence of Theorem 1.1, the following HLS type inequality is established on the bounded domain $\mathbb{B}^d$.

HLS 1.2. *For any $1 < p, s < \infty$, $1/s + 1/p + \alpha < d + 2$ and $d + 1 < \alpha < d + 2$, then there exists a constant C which depends only on p, α, d, s such that*

$$\left| \int_{\mathbb{B}^d} \int_{\mathbb{B}^d} \frac{f(w)g(z)}{|1 - \langle z, w\rangle|^\alpha}\, dv(w)dv(z) \right| \le C\|f\|_{L^p}\|g\|_{L^s} \tag{1.1}$$

for all $f \in L^p(\mathbb{B}^d)$, $g \in L^s(\mathbb{B}^d)$.

Theorem 1.3. [11] *If $0 < \alpha \le d+1$, then the following conditions are equivalent:*

(1) $K_\alpha\colon L^p \to L^q$ *is bounded;*

(2) $K_\alpha^+\colon L^p \to L^q$ *is bounded;*

(3) *p, q satisfy one of the following inequalities:*

 (a) $p = 1$, $q < (d+1)/\alpha$;

 (b) $1 < p < (d+1)/(d+1-\alpha)$, $1/q \ge 1/p + \alpha/(d+1) - 1$;

 (c) $p = (d+1)/(d+1-\alpha)$, $q < \infty$;

 (d) $(d+1)/(d+1-\alpha) < p \le \infty$.

Note that $K_\alpha, K_\alpha^+ \colon L^1 \to L^{(d+1)/\alpha}$ are both unbounded under the condition of Theorem 1.3, but it turns out that K_α is weak type $(1,(d+1)/\alpha)$, i.e., $K_\alpha, K_\alpha^+ \colon L^1 \to L^{(d+1)/\alpha,\infty}$ are both bounded on $\mathbb{B}^d$, see the following Proposition 6.1, which is a generalization of the result that the standard Bergman projection is weak type $(1,1)$ on some bounded domains [6,15]. By Theorem 1.3, this implies the HLS type inequality on $\mathbb{B}^d$ as below.

HLS 1.4. *For any $1 < p, s < \infty$, $1/s+1/p+\alpha/(d+1) \leq 2$ and $\alpha \leq d+1$, then there exists a constant C that depends only on p, α, d, s such that (1.1) holds for all $f \in L^p(\mathbb{B}^d)$, $g \in L^s(\mathbb{B}^d)$.*

Comparing HLS 1.2 and HLS 1.4 with the classical HLS inequality [13,14,17,20,21] on $\mathbb{R}^d$.

Theorem 1.5. *If $0 < \alpha \leq d+1$, then the following conditions are equivalent:*

(1) *$K_\alpha \colon L^p \to L^q$ is compact;*

(2) *p, q satisfy one of the following inequalities:*

  (a) *$p = 1$, $q < (d+1)/\alpha$;*
  (b) *$1 < p < (d+1)/(d+1-\alpha)$, $1/q > 1/p + \alpha/(d+1) - 1$;*
  (c) *$p = (d+1)/(d+1-\alpha)$, $q < \infty$;*
  (d) *$(d+1)/(d+1-\alpha) < p \leq \infty$.*

Theorem 1.6. *For $\alpha \in \mathbb{R}$, then the following conditions are equivalent:*

(1) *$\alpha < d+2$;*

(2) *there exist $1 \leq p, q \leq \infty$ such that $K_\alpha \colon L^p \to L^q$ is bounded;*

(3) *there exist $1 \leq p, q \leq \infty$ such that $K_\alpha^+ \colon L^p \to L^q$ is bounded;*

(4) *there exist $1 \leq p, q \leq \infty$ such that $K_\alpha \colon L^p \to L^q$ is compact.*

Theorem 1.7. *If $0 < \alpha < (d+2)/2$, then the following holds.*

(1) *$K_\alpha, K_\alpha^+ \colon L^2 \to L^2$ are Hilbert–Schmidt.*

(2) *Moreover, if $d = 1$ and $0 < \alpha < 3/2$, then we have the trace formula,*

$$\operatorname{Tr}(K_\alpha^* K_\alpha) = \|K_{2\alpha}^+\|_{L^\infty \to L^1} = \frac{1}{(\alpha-1)^2}\left(\frac{\Gamma(3-2\alpha)}{\Gamma^2(2-\alpha)} - 1\right),$$

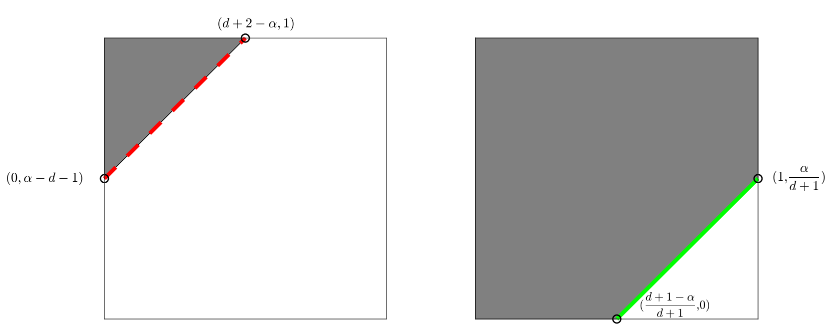
where Γ is the usual Gamma function. When $\alpha = 1$, the quantity on the right side should be interpreted as $\pi^2/6$.

The above theorems show that $K_\alpha\colon L^p \to L^q$ is bounded if and only if $K_\alpha^+\colon L^p \to L^q$ is bounded. We know from Theorem 1.1 that, when $d+1 < \alpha < d+2$, $K_\alpha\colon L^p \to L^q$ is compact if and only if $K_\alpha\colon L^p \to L^q$ is bounded. However, it is no longer true if $0 < \alpha \leq d+1$ by Theorems 1.3 and 1.5. In particular, the standard Bergman projection $K_{d+1}\colon L^p \to L^q$ is compact if and only if $1 \leq q < p \leq \infty$ on $\mathbb{B}^d$.

We now explain briefly the main ideas of the proofs. Let us consider the above boundedness problem in the following viewpoint. Denote $G(K_\alpha)$ by the set of $(1/p, 1/q) \in E$ such that $K_\alpha\colon L^p \to L^q$ is bounded, where E is given by

$$E = \{(x,y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\},$$

i.e., E is a unit square in the real plane $\mathbb{R}^2$. Following by T. Tao [21], $G(K_\alpha)$ is called the type diagram of the operator K_α, see Figure 1.1. The similar idea was adopted in [4,11]. As we shall see that every $G(K_\alpha)$ is convex and axisymmetric on the inside of E. We see that the proof of the above boundedness theorems is equivalent to solving the corresponding type diagrams. The convexities and axisymmetries of the type diagrams will make the process simpler. Then combining with several embedding theorems of holomorphic function spaces and some estimates of Bergman kernel on the unit ball, we completely characterize the L^p-L^q boundedness and compactness of K_α. Similarly, we can define the type diagrams $G(K_\alpha^+)$ of operators K_α^+, which are also convex and is axisymmetric; see Figure 1.1. The above main theorems show in fact that $G(K_\alpha^+) = G(K_\alpha)$ for any $\alpha \in \mathbb{R}$. After characterizing the boundedness and compactness of K_α, by using the hypergeometric function theory and the interpolation theory, we give some sharp norm estimates of K_α, K_α^+. It is in fact that we estimate the upper bounds of the optimal constants in the inequalities of HLS 1.2 and HLS 1.4. The results of this paper can be generalized to cover some weighted Lesbegue integrable spaces and more general kernel operators on the unit ball.



(1) $d+1 < \alpha < d+2$ (2) $0 < \alpha \leq d+1$

Figure 1.1: Type diagrams $G(K_\alpha)$ and $G(K_\alpha^+)$.

The paper is organized as follows. In Section 2, we give some basic properties of the operators K_α. Section 3 provides the proof of Theorem 1.1. Section 4 is devoted to the proofs of Theorems 1.5 and 1.6. In Section 5, we give some sharp norm estimates of the operators K_α, K_α^+. In Section 6, we prove the weak type boundedness result.

2. Basic properties of K_α

In this section, we prove some results which will be frequently used in the sequel. We first take a rough look at the property of type diagram $G(K_\alpha)$ of the operator K_α. We prove that every $G(K_\alpha)$ is convex and is axisymmetric on the inside of E as mentioned before. Let l_E be the diagonal line of the square E which connects points $(0,1)$ and $(1,0)$. Clearly $G(K_\alpha) \subset E$ for any $\alpha \in \mathbb{R}$.

Proposition 2.1. (1) *If $G(K_\alpha) \neq \emptyset$, then $(0,1) \in G(K_\alpha)$; if $(1,0) \in G(K_\alpha)$, then $G(K_\alpha) = E$.*

(2) *For any $\alpha \in \mathbb{R}$, the type diagram $G(K_\alpha)$ is convex and is axisymmetric about l_E on the inside of E.*

Proof. (1) This comes from the the continuity of embeddings of L-integrable spaces, i.e., the embedding $L^p \subset L^q$ is continuous whenever $p \geq q \geq 1$.

(2) To show that $G(K_\alpha)$ is convex, it suffices to show that if $(1/p_1, 1/q_1), (1/p_2, 1/q_2) \in G(K_\alpha)$, then $\theta(1/p_1, 1/q_1) + (1-\theta)(1/p_2, 1/q_2) \in G(K_\alpha)$ for any $0 \leq \theta \leq 1$. Indeed, it is a direct corollary of the following Lemma 2.2 which is a classical complex interpolation result. We now turn to the symmetry. By Fubini's theorem, this implies that K_α is adjoint. Then, for $1 < p, q < \infty$, the boundedness of $K_\alpha\colon L^p \to L^q$ is equivalent to the boundedness of $K_\alpha\colon L^{q'} \to L^{p'}$, where p', q' are the conjugate numbers of p, q, respectively. This means that $(1/p, 1/q) \in G(K_\alpha)$ if and only if $(1/q', 1/p') \in G(K_\alpha)$. It is easy to check that $(1/p, 1/q)$ and $(1/q', 1/p')$ are symmetric about l_E by the conjugate relationship. $\square$

Lemma 2.2. [24] *Suppose $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. Let T be a linear operator such that $T\colon L^{p_1} \to L^{q_1}$ is bounded with norm M_1 and $T\colon L^{p_2} \to L^{q_2}$ is bounded with norm M_2. Then $T\colon L^p \to L^q$ is bounded with norm no more than $M_1^\theta M_2^{1-\theta}$, where $\theta \in (0,1)$ satisfying*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Remark 2.3. Proposition 2.1 shows that the type diagram $G(K_\alpha)$ is a bounded convex set in the plane $\mathbb{R}^2$, then it suffices to find out all extreme points or the boundary points of $G(K_\alpha)$. The symmetry of $G(K_\alpha)$ shows that is only need to find out a half. On the other hand, Proposition 2.1 holds for more general domains and adjoint operators.

Corollary 2.4. (1) *If $G(K_\alpha^+) \neq \emptyset$, then $(0,1) \in G(K_\alpha^+)$; if $(1,0) \in G(K_\alpha^+)$, then $G(K_\alpha^+) = E$.*

(2) *For any $\alpha \in \mathbb{R}$, the type diagram $G(K_\alpha^+)$ is convex and is axisymmetric about l_E on the inside of E.*

Corollary 2.5. *If $\alpha \le 0$, then $G(K_\alpha) = G(K_\alpha^+) = E$.*

Corollary 2.5 indicates that $K_\alpha, K_\alpha^+ \colon L^p \to L^q$ are bounded for any $1 \le p, q \le \infty$ if $\alpha \le 0$. For any $\beta > -1$, denote $dv_\beta(z) = c_\beta(1-|z|^2)^\beta \, dv(z)$, where $c_\beta = \frac{\Gamma(d+\beta+1)}{\Gamma(d+1)\Gamma(\beta+1)}$. For $1 \le p \le \infty$, let $A_\beta^p = H(\mathbb{B}^d) \cap L^p(dv_\beta)$ be the weighted Bergman space on $\mathbb{B}^d$, where $H(\mathbb{B}^d)$ is the holomorphic function space on $\mathbb{B}^d$; in particular, $A_\beta^\infty = H^\infty$ is just the bounded holomorphic function space. Recall that K_{d+1} is the Bergman projection from L^p onto A_0^p, a well known result is that $K_{d+1}(L^p) = A_0^p$ for $1 < p < \infty$. We now establish a general result for $\alpha \ge d+1$.

Proposition 2.6. *Suppose that $\alpha \ge d+1$ and $1 < p < \infty$, then*

$$K_\alpha(L^p) = K_\alpha(A_0^p) = A^p_{p(\alpha-d-1)}.$$

To prove Proposition 2.6, we need some lemmas. The following Lemma 2.7 was proved [3] in the case $d = 1$, by the same method, it can be proved in the general case, see [3, Lemma 11] for more details.

Lemma 2.7. *If $\alpha > 0$ and $1 < p < \infty$, then*

$$K_\alpha K_{d+1} = K_\alpha \quad \text{on } L^p.$$

Lemma 2.7 shows that for $1 < p, q < \infty$, $K_\alpha \colon L^p \to L^q$ is bounded if and only if $K_\alpha \colon A_0^p \to A_0^q$ is bounded. We now turn to the behavior of K_α on holomorphic function spaces. Recall first the definition of fractional radial differential operator $R^{s,t}$ on $H(\mathbb{B}^d)$. For any two real parameters s and t with the property that neither $d+s$ nor $d+s+t$ is a negative integer, the invertible operator $R^{s,t}$ is given by

$$R^{s,t} f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(d+1+s)\Gamma(d+1+n+s+t)}{\Gamma(d+1+s+t)\Gamma(d+1+n+s)} f_n(z)$$

for any $f = \sum_{n=0}^\infty f_n \in H(\mathbb{B}^d)$ with homogeneous expansion. In fact, it can be checked by the direct calculation that the invertible operator of $R^{s,t}$ is just $R^{s+t,-t}$. Be careful that the invertible operator here merely means it is linear. We refer the reader to [24] for more details.

Lemma 2.8. *For $\alpha > 0$ and $1 < p < \infty$, the following holds on A_0^p,*

$$K_\alpha = R^{0,\alpha-d-1}.$$

Proof. Suppose $f = \sum_{n=0}^{\infty} f_n \in A_0^p$ with the homogeneous expansion. By direct calculation, this implies that

$$K_\alpha f = \sum_{n=0}^{\infty} \frac{\Gamma(d+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(d+1+n)} f_n. \tag{2.1}$$

This leads to the desired result. □

Proof of Proposition 2.6. Lemma 2.7 implies that $K_\alpha(L^p) = K_\alpha(A_0^p)$. We now prove $K_\alpha(A_0^p) = A^p_{p(\alpha-d-1)}$. By [23, Theorem 14], which is a characterization of Bergman space, we infer that $f \in A_0^p$ if and only if $R^{0,\alpha-d-1} f \in L^p(dv_{p(\alpha-d-1)})$, namely $f \in A_0^p$ if and only if $R^{0,\alpha-d-1} f \in A^p_{p(\alpha-d-1)}$. Note that $K_\alpha = R^{0,\alpha-d-1}$ by Lemma 2.8, it follows that $f \in A_0^p$ if and only if $K_\alpha f \in A^p_{p(\alpha-d-1)}$. This shows that $K_\alpha(A_0^p) \subset A^p_{p(\alpha-d-1)}$. To prove another direction, suppose that $g \in A^p_{p(\alpha-d-1)}$. Since $K_\alpha = R^{0,\alpha-d-1}$ is invertible on $H(\mathbb{B}^d)$, i.e., there exists $f \in H(\mathbb{B}^d)$ such that $K_\alpha f = R^{0,\alpha-d-1} f = g$. From [24, Theorem 2.19], there exists a positive constant c which only depends on α, d, p such that

$$\|f\|_{L^p} \le c\|g\|_{A^p_{p(\alpha-d-1)}}.$$

This means that $f \in A_0^p$. Thus $A^p_{p(\alpha-d-1)} \subset K_\alpha(A_0^p)$. This completes the proof. □

Corollary 2.9. *Suppose that $\alpha \ge d+1$ and $1 < p < \infty$, then for any $\gamma > -1$, the following holds,*

$$K_\alpha(L^p(dv_\gamma)) = K_\alpha(A_\gamma^p) = A^p_{\gamma+p(\alpha-d-1)}.$$

The following Proposition 2.10 gives the image of K_α in case of $p = \infty$. Let $\mathcal{B}_\beta$ denote the weighted Bloch space on $\mathbb{B}^d$ for $\beta > 0$, see the definition in [24, Section 7.1].

Proposition 2.10. *For $\alpha \ge d+1$, then $K_\alpha(H^\infty) \subsetneq K_\alpha(L^\infty) = \mathcal{B}_{\alpha-d}$.*

Proof. Observe that $K_\alpha(L^\infty) = \mathcal{B}_{\alpha-d}$ by [24, Theorem 7.1]. If $\alpha = d+1$, then $K_{d+1}(H^\infty) = H^\infty$, thus $K_{d+1}(H^\infty) \subsetneq \mathcal{B}_{\alpha-d}$. Now, we turn to the case $\alpha > d+1$. Note that $K_\alpha(H^\infty) \subset K_\alpha(A_0^p)$ for any $1 < p < \infty$, this implies by Proposition 2.6 that

$$K_\alpha(H^\infty) \subset \bigcap_{1<p<\infty} A^p_{p(\alpha-d-1)}. \tag{2.2}$$

On the other hand, from [24, Theorem 2.1], which is a pointwise estimates of functions in the weighted Bergman spaces, we know that

$$A_\gamma^p \subset \mathcal{B}_{(d+1+\gamma)/p}. \tag{2.3}$$

Combining (2.2) with (2.3), this implies that

$$K_\alpha(H^\infty) \subset \bigcap_{1<p<\infty} \mathcal{B}_{(\alpha-d)+(d+1)/p-1}.$$

Together with the fact that the weighted Bloch space is strictly increased, namely $\mathcal{B}_\beta \subsetneq \mathcal{B}_{\beta'}$ whenever $0 < \beta < \beta'$, this implies that $K_\alpha(H^\infty) \subsetneq \mathcal{B}_{\alpha-d}$. □

Remark 2.11. The monotonicity of the weighted Bloch space can be obtained as follows. It is easy to see that the weighted Bloch space is increased, so it suffices to show that is strict. For any $0 < \beta < \beta'$, there exist $p > 1$ and $\varepsilon > 0$ such that

$$\beta < \beta - 1 + \frac{d+\varepsilon}{p} < \beta'.$$

Combining (2.3) and the following Lemma 3.2, this implies that

$$\mathcal{B}_\beta \subsetneq A^p_{p(\beta-1)-1+\varepsilon} \subset \mathcal{B}_{\beta'}.$$

The reader can also consult [10].

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Although L^p-L^q bounded results have been proved in [11], for the self-contained purpose, we still give the different proofs here whose methods shall be further applied to the compactness and norm estimates in the following sections. We need several embedding theorems of holomorphic function spaces on the unit ball $\mathbb{B}^d$. For convenience, we state them without proof as follows.

Lemma 3.1. [23] *Let $0 < q < p < \infty$. Then $A^p_\beta \subset A^q_\gamma$ if and only if $(\beta+1)/p < (\gamma+1)/q$. And in this case the inclusions are strict.*

Lemma 3.2. [22, 23] *Suppose that $\beta > 0$, $\gamma > -1$, $p \geq 1$, then $\mathcal{B}_\beta \subset A^p_\gamma$ if and only if $\beta - 1 < (1+\gamma)/p$. And in this case the inclusions are strict.*

We also need the following lemmas.

Lemma 3.3. *If $d+1 < \alpha < d+2$, then $K_\alpha\colon L^\infty \to L^q$ is bounded if and only if $q < 1/(\alpha-(d+1))$.*

Proof. We first show that $K_\alpha\colon L^\infty \to L^q$ is bounded if $q < 1/(\alpha-(d+1))$. Then, for $f \in L^\infty$, by [19, Proposition 1.4.10] and Hölder's inequality, this implies that

$$|K_\alpha f(z)| \leq \|f\|_\infty \int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\, dv(w) \leq C_{d,\alpha}\|f\|_\infty (1-|z|^2)^{d+1-\alpha}, \quad |z| \to 1^-, \tag{3.1}$$

where $C_{d,\alpha}$ is a constant. The condition $q < 1/(\alpha-(d+1))$ means that $q((d+1)-\alpha) > -1$. Then (3.1) follows that $K_\alpha f(z) \in L^q$ and $K_\alpha\colon L^\infty \to L^q$ is bounded. We now turn to prove that $K_\alpha\colon L^\infty \to L^q$ is unbounded if $q \geq 1/(\alpha-(d+1))$. By Hölder's inequality, it is enough to prove that $K_\alpha\colon L^\infty \to L^{1/(\alpha-(d+1))}$ is unbounded. It suffices to show that $K_\alpha(L^\infty) \not\subset L^{1/(\alpha-(d+1))}$. Since $K_\alpha(L^\infty) = \mathcal{B}_{\alpha-d}$, it suffices to show that $\mathcal{B}_{\alpha-d} \not\subset A_0^{1/(\alpha-(d+1))}$. Indeed, it is a fact from Lemma 3.2. □

Corollary 3.4. *If $d+1<\alpha<d+2$, then $K_\alpha\colon L^p\to L^1$ is bounded if and only if $p>1/((d+2)-\alpha)$.*

Proof. First, suppose $p>1/((d+2)-\alpha)$. From Lemma 3.3 and K_α is an adjoint operator, we know that $K_\alpha\colon L^p\to(L^\infty)^*$ is bounded if $p>1/((d+2)-\alpha)$. Proposition 2.6 implies that $K_\alpha(L^p)=A^p_{p(\alpha-d-1)}$. Since $(p(\alpha-d-1)+1)/p<(\alpha-d-1)+(d+2)-\alpha=1$, it follows by Lemma 3.1 that $A^p_{p(\alpha-d-1)}\subset A^1_0$. Thus $K_\alpha(L^p)\subset L^1$. Note that $L^1\subset(L^\infty)^*$, this implies that $K_\alpha\colon L^p\to L^1$ is bounded.

Conversely, suppose that $K_\alpha\colon L^p\to L^1$, $p\neq\infty$ is bounded. Then $K_\alpha\colon L^\infty\to L^{p'}$ is bounded, where $p'=p/(p-1)$. From Lemma 3.3, this implies that $p/(p-1)=p'<1/(\alpha-(d+1))$, this means that $p>1/((d+2)-\alpha)$. Clearly the case of $p=\infty$ is trivial by Lemma 3.3. □

Corollary 3.5. *If $d+1<\alpha<d+2$, then*

(1) $K_\alpha^+\colon L^\infty\to L^q$ *is bounded if and only if* $q<1/(\alpha-(d+1))$*;*

(2) $K_\alpha^+\colon L^p\to L^1$ *is bounded if and only if* $p>1/((d+2)-\alpha)$*.*

Proof. (1) For $f\in L^\infty$, by [19, Proposition 1.4.10] and Hölder's inequality, this implies that

$$|K_\alpha^+f(z)|\le\|f\|_\infty\int_{\mathbb{B}^d}\frac{1}{|1-\langle z,w\rangle|^\alpha}\,dv(w)\le C_{d,\alpha}\|f\|_\infty(1-|z|^2)^{d+1-\alpha},\quad |z|\to1^-, \tag{3.2}$$

where $C_{d,\alpha}$ is a constant. So, if $q<1/(\alpha-(d+1))$, i.e., $q((d+1)-\alpha)>-1$, then (3.2) implies that $K_\alpha f(z)\in L^q$ and $K_\alpha^+\colon L^\infty\to L^q$ is bounded. This means that

$$\big\{(0,1/q):1/q>\alpha-(d+1)\big\}\subset G(K_\alpha^+).$$

On the other hand, Lemma 3.3 implies that the point $(0,1/q)\in G(K_\alpha)$ if and only if $1/q>\alpha-(d+1)$. Note that $|K_\alpha(f)|\le K_\alpha^+(|f|)$, this implies immediately that $G(K_\alpha^+)\subset G(K_\alpha)$. Hence, it follows that $(0,1/q)\in G(K_\alpha)$ if and only if $1/q>\alpha-(d+1)$. This leads the desired result.

(2) The proof is similar to (1). □

Lemma 3.6. *Suppose that $d+1<\alpha<d+2$ and $1/q\le1/p+\alpha-(d+1)$, then $K_\alpha\colon L^p\to L^q$ is unbounded.*

Proof. By the continuity of the embeddings of L-integrable spaces, it suffices to show that $K_\alpha\colon L^p\to L^q$ is unbounded if $d+1<\alpha<d+2$, $1/q=1/p+\alpha-(d+1)$. The cases of $p=\infty$ or $q=1$ have been proved in Lemma 3.3 and Corollary 3.4. For the case of $1<p,q<\infty$, it suffices to show that $K_\alpha(L^p)\not\subset L^q$. On the other hand, Proposition 2.6

shows that $K_\alpha(L^p) = A^p_{p(\alpha-d-1)}$, which is a holomorphic function space. Thus, it suffices to show that

$$K_\alpha(L^p) = A^p_{p(\alpha-d-1)} \not\subset A^q_0. \tag{3.3}$$

Since $(p(\alpha-d-1)+1)/p = 1/q$, it follows that (3.3) holds by Lemma 3.1. This completes the proof. $\square$

Proof of Theorem 1.1. *Step* 1. Proof of the equivalence that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4).

First, we prove that (1) is equivalent to (4). As mentioned before, it is equivalent to proving that $G(K_\alpha)$ is exactly the triangle region $D_1 \subset E$ which determined by the equations in Theorem 1.1(4), namely $G(K_\alpha) = D_1$. Lemma 3.3, Corollary 3.4 and the convexity of $G(K_\alpha)$ imply that $D_1 \subset G(K_\alpha)$. On the other hand, Lemma 3.6 and the convexity of $G(K_\alpha)$ imply that $E - D_1 \subset E - G(K_\alpha)$, it follows that $G(K_\alpha) \subset D_1$. Thus $G(K_\alpha) = D_1$. We now turn to prove that (2) is equivalent to (4), it is equivalent to proving that $G(K_\alpha^+) = D_1$. Corollary 3.5 and the convexity of $G(K_\alpha^+)$ implies that $D_1 \subset G(K_\alpha^+)$. Combining the fact that $G(K_\alpha^+) \subset G(K_\alpha) = D_1$, then $G(K_\alpha^+) = D_1$. This completes the proof.

Step 2. We prove that (1) $\Leftrightarrow$ (3).

Since compact operators must be bounded, it suffices to prove that

$$K_\alpha \colon L^p \to L^q \quad \text{is compact, if } (1/p, 1/q) \in G(K_\alpha).$$

We first prove the following claim.

Claim. $K_\alpha \colon L^\infty \to L^q$ *is compact if and only if* $q < 1/(\alpha - (d+1))$.

If $K_\alpha \colon L^\infty \to L^q$ is compact, is immediate from Corollary 3.4 that $q < 1/(\alpha-(d+1))$. We now prove the reverse, that is, to prove that $K_\alpha \colon L^\infty \to L^q$ is compact if $q < 1/(\alpha - (d+1))$. We need to show that for any bounded sequence in L^∞, there is a subsequence such that whose image under K_α converges in L^q. Suppose that $\{f_n\} \in L^\infty$ is an arbitrary bounded sequence and K is an arbitrary compact subset of $\mathbb{B}^d$. Moreover, we assume that $\|f_n\|_\infty \le C$ for any $n \ge 1$, where C is a positive constant. Then we obtain

$$\sup_{z\in K} |K_\alpha f_n(z)| \le \|f_n\|_\infty \sup_{z\in K} \int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\, dv(w) \le \|f_n\|_\infty \sup_{z\in K} \frac{1}{(1-|z|)^\alpha} < \infty.$$

Combining with that the image of K_α is holomorphic, this implies that $\{K_\alpha f_n\}$ is a normal family. Hence $\{f_n\}$ has a subsequence $\{f_{n_j}\}$ such that $K_\alpha f_{n_j}$ converges uniformly on compact subsets of $\mathbb{B}^d$ to a holomorphic function g. By Fatou's Lemma and boundedness of K_α, it follows that

$$\int_{\mathbb{B}^d} |g|^q\, dv \le \varliminf_{j\to\infty} \int_{\mathbb{B}^d} |K_\alpha f_{n_j}|^q\, dv \le \|K_\alpha\|^q_{L^\infty \to L^q} \varliminf_{j\to\infty} \|f_{n_j}\|^q_\infty < \infty. \tag{3.4}$$

This means that $g \in L^q$. We now prove that there exists positive function $g_1 \in L^q$ such that $|K_\alpha f_{n_j}| \le g_1$. We first observe that [19, Proposition 1.4.10] and the condition $q < 1/(\alpha - (d+1))$ imply that

$$\left(\int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\,dv(w)\right)^q \in L^1.$$

Then by easy estimate, this implies that

$$|K_\alpha f_{n_j}(z)| \le \|f_{n_j}\|_\infty \int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\,dv(w) \le C\int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\,dv(w). \tag{3.5}$$

Thus (3.5) shows that it is enough to take $g_1 = C\int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\,dv(w)$. Combining (3.4) with (3.5), this implies that

$$|K_\alpha f_{n_j} - g|^q \le (g_1 + |g|)^q \in L^1, \quad \forall\, j \ge 1.$$

By dominated convergence theorem, it gives that

$$\begin{aligned}\lim_{j\to\infty} \|K_\alpha f_{n_j} - g\|_q &= \lim_{j\to\infty}\left(\int_{\mathbb{B}^d} |K_\alpha f_{n_j} - g|^q\,dv\right)^{1/q}\\ &= \left(\int_{\mathbb{B}^d} \lim_{j\to\infty} |K_\alpha f_{n_j} - g|^q\,dv\right)^{1/q} = 0,\end{aligned}$$

and the claim follows.

Combining the last claim with the basic fact that an operator is compact if and only if its adjoint operator is still compact, thus we get that $K_\alpha\colon L^p \to L^1$ is compact if and only if $p < 1/(d+2-\alpha)$. Then by the following Lemma 3.7, which is an interpolation result of the compact operators, this implies that $K_\alpha\colon L^p \to L^q$ is compact if $(1/p, 1/q) \in G(K_\alpha)$. $\square$

Lemma 3.7. [5,12] *Suppose that $1 \le p_1, p_2, q_1, q_2 \le \infty$ and $q_1 \ne \infty$. If a linear operator T such that $T\colon L^{p_1} \to L^{q_1}$ is bounded and $T\colon L^{p_2} \to L^{q_2}$ is compact, then $T\colon L^p \to L^q$ is compact, where $\theta \in (0,1)$ satisfying*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Remark 3.8. The compactness of $K_\alpha\colon L^p \to L^q$ for $1 < p, q < \infty$ can be also proved by the Carleson type measure theory on Bergman spaces, see definition in [23,24]. This strategy will be adopted as we shall see in Section 4.

4. Proofs of Theorems 1.5 and 1.6

Theorem 1.3 characterizes the L^p-L^q boundedness of K_α, K_α^+ when $0 < \alpha \le d+1$. We will completely characterize the L^p-L^q compactness of K_α when $0 < \alpha \le d+1$. It is equivalent to solving the set $F(K_\alpha)$, where $F(K_\alpha)$ is defined by

$$F(K_\alpha) = \left\{(1/p, 1/q) \in E : K_\alpha\colon L^p \to L^q \text{ is compact}\right\}.$$

It is easy to see that $F(K_\alpha)$ is a subset of $G(K_\alpha)$. Theorem 1.5 in fact shows that $F(K_\alpha)$ and $G(K_\alpha)$ differ only by a segment on the boundary of $G(K_\alpha)$. Thus we always show first that K_α is compact on the other part of the boundary of $G(K_\alpha)$. In the end of this section, we give the proof of Theorem 1.6.

Proposition 4.1. *$K_{d+1}\colon L^p \to L^q$ is compact if and only if $1 \le q < p \le \infty$.*

Proof. From Theorem 1.3, we know that $K_{d+1}\colon L^p \to L^q$ is bounded if and only if $q \le p$. Since K_{d+1} is the standard Bergman projection, it is easy to see $K_{d+1}\colon L^p \to L^p$ is not compact for any $1 < p < \infty$. Thus it suffices to show that $K_\alpha\colon L^p \to L^q$ is compact if $q < p$. Indeed, this can be proved by the similar method we used in Step 2 of proof of Theorem 1.1, thus we omit it. □

Now, we recall some results on the hypergeometric function theory which we shall use later on. For complex numbers α, β, γ and complex variable z, we use the classical notation ${}_2F_1(\alpha, \beta; \gamma; z)$ to denote

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j (\beta)_j}{j!(\gamma)_j} z^j \tag{4.1} $$

with $\gamma \ne 0, -1, -2, \ldots$, where $(\alpha)_j = \Pi_{k=0}^{j-1}(\alpha + k)$ is the Pochhammer symbol for any complex number α. The following lemma is in fact a restatement of [19, Proposition 1.4.10].

Lemma 4.2. [19] *Suppose $\beta \in \mathbb{R}$ and $\gamma > -1$, then*

$$ \int_{\mathbb{B}^d} \frac{(1 - |w|^2)^\gamma}{|1 - \langle z, w\rangle|^{2\beta}} \, dv(w) = \frac{\Gamma(1 + d)\Gamma(1 + \gamma)}{\Gamma(1 + d + \gamma)} \, {}_2F_1(\beta, \beta; 1 + d + \gamma; |z|^2). $$

We also need the following lemma.

Lemma 4.3. [7, Chapter 2] *The following three identities hold.*

(1) ${}_2F_1(\alpha, \beta; \gamma; z) = (1 - z)^{\gamma - \alpha - \beta} \, {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma; z)$*;*

(2) ${}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$ *if* $\mathrm{Re}(\gamma - \alpha - \beta) > 0$*;*

(3) $\frac{d}{dz} \, {}_2F_1(\alpha, \beta; \gamma; z) = \frac{\alpha\beta}{\gamma} \, {}_2F_1(\alpha + 1, \beta + 1; \gamma + 1; z)$.

Now, we introduce the following auxiliary function I_α for any $\alpha < d + 1$. The function $I_\alpha(r, z)$ on $[0, 1) \times \overline{\mathbb{B}^d}$ is denoted by

$$ I_\alpha(r, z) = \int_{r \le |w| < 1} \frac{1}{|1 - \langle z, w\rangle|^\alpha} \, dv(w). $$

Since $\alpha < d+1$, it follows by (4.1) and Lemma 4.2 that

$$\begin{aligned} I_\alpha(r,z) &\leq \int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\,dv(w) \\ &= \sum_{j=0}^\infty \left(\frac{\Gamma(j+\alpha/2)}{\Gamma(\alpha/2)}\right)^2 \frac{\Gamma(d+1)}{\Gamma(j+1)\Gamma(j+d+1)} \\ &= \frac{\Gamma(d+1)\Gamma(d+1-\alpha)}{\Gamma^2(d+1-\alpha/2)} \end{aligned} \tag{4.2}$$

for any $(r,z) \in [0,1) \times \overline{\mathbb{B}^d}$, which means that I_α is finite on $[0,1) \times \overline{\mathbb{B}^d}$. Moreover, I_α is increasing on $[0,1) \times \overline{\mathbb{B}^d}$ in the following sense.

Lemma 4.4. *Suppose $r \in [0,1)$, then*

$$I_\alpha(r,z_1) \leq I_\alpha(r,z_2),$$

whenever $z_1, z_2 \in \overline{\mathbb{B}^d}$ and $|z_1| \leq |z_2|$.

Proof. From (4.2), we know that I_α is finite on $[0,1) \times \overline{\mathbb{B}^d}$. We now calculate its exact value. It follows from [24, Lemma 1.8, 1.11] and the unitary invariance of the Lebesgue measure that

$$\begin{aligned} I_\alpha(r,z) &= \int_{r\leq|w|<1} \frac{1}{\left|1-|z|w_1\right|^\alpha}\,dv(w) \\ &= \int_{r\leq|w|<1} \sum_{j=0}^\infty \left(\frac{\Gamma(j+\alpha/2)}{\Gamma(\alpha/2)\Gamma(j+1)}\right)^2 |z|^{2j}|w_1^j|^2\,dv(w) \\ &= \sum_{j=0}^\infty \left(\frac{\Gamma(j+\alpha/2)}{\Gamma(\alpha/2)\Gamma(j+1)}\right)^2 |z|^{2j} \cdot 2d\int_r^1 t^{2d+2j-1}\,dt \int_{\mathbb{S}^d} |\xi_1^j|^2\,d\sigma(\xi) \\ &= \sum_{j=0}^\infty \left(\frac{\Gamma(j+\alpha/2)}{\Gamma(\alpha/2)}\right)^2 \frac{\Gamma(d+1)(1-r^{2(j+d)})}{\Gamma(j+1)\Gamma(j+d+1)}|z|^{2j} \end{aligned} \tag{4.3}$$

for any $z \in \overline{\mathbb{B}^d}$. This leads to the desired result since all coefficients of the power series expansion about $|z|$ in (4.3) are positive. $\square$

Lemma 4.5. *If $0 < \alpha < d+1$, then $K_\alpha\colon L^\infty \to L^q$ is compact for any $1 \leq q \leq \infty$.*

Proof. Since the continuity of the embeddings of L-integrable spaces, it suffices to prove that $K_\alpha\colon L^\infty \to L^\infty$ is compact. We first prove that, for any $f \in L^\infty$, then $K_\alpha f \in A(\mathbb{B}^d)$, where $A(\mathbb{B}^d) = H(\mathbb{B}^d) \cap C(\overline{\mathbb{B}^d})$ is the ball algebra. For $f \in L^\infty$, it is clear that $K_\alpha f$ is holomorphic on the ball, i.e., $K_\alpha f \in H(\mathbb{B}^d)$. From Lemma 4.2 and Lemma 4.3(2), this implies that $K_\alpha f(\eta)$ exists for any $\eta \in \partial\mathbb{B}^d$ and

$$|K_\alpha f(\eta)| \leq \|f\|_\infty \frac{\Gamma(d+1)\Gamma(d+1-\alpha)}{\Gamma^2(d+1-\alpha/2)}.$$

We now prove that $K_\alpha f$ is actually continuous on the closed ball $\overline{\mathbb{B}^d}$. It suffices to prove that $K_\alpha f$ is continuous on $\partial\mathbb{B}^d$, namely we need to prove that, for any $\eta \in \partial\mathbb{B}^d$ and for any point sequence $\{z_n\}$ in $\overline{\mathbb{B}^d}$ satisfying $z_n \to \eta$, we have $K_\alpha f(z_n) \to K_\alpha f(\eta)$ as $n \to \infty$. By Lemma 4.2 and Lemma 4.3(2) again, we have

$$\begin{aligned} |K_\alpha f(z)| &\le \|f\|_\infty \int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\, dv(w) \le \|f\|_\infty \int_{\mathbb{B}^d} \frac{1}{|1-\langle \eta,w\rangle|^\alpha}\, dv(w) \\ &= \|f\|_\infty \frac{\Gamma(d+1)\Gamma(d+1-\alpha)}{\Gamma^2(d+1-\alpha/2)} \end{aligned} \tag{4.4}$$

for any $z \in \mathbb{B}^d$. Due to the absolute continuity of the integral, this implies that, for any $\varepsilon > 0$, there exists $0 < \delta < 1$ satisfying

$$\int_F \frac{dv(w)}{|1-\langle \eta,w\rangle|^\alpha} \le \frac{\varepsilon}{4} \tag{4.5}$$

whenever $v(F) < \delta$. Denote $F_\delta = \left\{z \in \mathbb{B}^d : \sqrt[d]{1-\delta/2} < |z| < 1\right\}$. Note that $v(F_\delta) = \delta/2 < \delta$ and

$$\frac{1}{(1-\langle z_n,w\rangle)^\alpha} \to \frac{1}{(1-\langle \eta,w\rangle)^\alpha} \quad \text{uniformly on } \mathbb{B}^d \setminus F_\delta \quad \text{as } n \to \infty.$$

Then there exists $N > 0$ such that, for any $n > N$,

$$\int_{\mathbb{B}^d\setminus F_\delta} \left| \frac{1}{(1-\langle z_n,w\rangle)^\alpha} - \frac{1}{(1-\langle \eta,w\rangle)^\alpha} \right| dv(w) \le \frac{\varepsilon}{2}.$$

Combining this with (4.2), (4.4), (4.5) and Lemma 4.4, this implies that, for any $n > N$,

$$\begin{aligned} |K_\alpha f(z_n) - K_\alpha f(\eta)| &\le \|f\|_\infty \int_{\mathbb{B}^d\setminus F_\delta} \left| \frac{1}{(1-\langle z_n,w\rangle)^\alpha} - \frac{1}{(1-\langle \eta,w\rangle)^\alpha} \right| dv(w) \\ &\quad + \|f\|_\infty \int_{F_\delta} \left| \frac{1}{(1-\langle z_n,w\rangle)^\alpha} - \frac{1}{(1-\langle \eta,w\rangle)^\alpha} \right| dv(w) \\ &\le \|f\|_\infty \int_{\mathbb{B}^d\setminus F_\delta} \left| \frac{1}{(1-\langle z_n,w\rangle)^\alpha} - \frac{1}{(1-\langle \eta,w\rangle)^\alpha} \right| dv(w) \\ &\quad + 2\|f\|_\infty \int_{F_\delta} \frac{1}{|1-\langle \eta,w\rangle|^\alpha}\, dv(w) \\ &\le \|f\|_\infty \frac{\varepsilon}{2} + 2\|f\|_\infty \frac{\varepsilon}{4} \\ &= \varepsilon \|f\|_\infty. \end{aligned} \tag{4.6}$$

This completes the proof of what $K_\alpha f$ is continuous on the closed ball $\overline{\mathbb{B}^d}$. We turn to prove that, for any bounded sequence in L^∞, there exists a subsequence satisfying its image under K_α is convergent in L^∞. Suppose that $\{f_n\}$ is a bounded sequence in L^∞, then we have $\{K_\alpha f_n\}$ is in $C(\overline{\mathbb{B}^d})$ and $\{K_\alpha f_n\}$ is uniformly bounded by (4.4). We are in a position to prove that $\{K_\alpha f_n\}$ is also equicontinuous. From (4.6), we know that

$$\lim_{\mathbb{B}^d \ni z \to \eta} \int_{\mathbb{B}^d} \left| \frac{1}{(1-\langle z,w\rangle)^\alpha} - \frac{1}{(1-\langle \eta,w\rangle)^\alpha} \right| dv(w) = 0 \tag{4.7}$$

for arbitrary fixed $\eta \in \partial\mathbb{B}^d$. Combining (4.7) with the unitary invariance of Lebsgue measure and the symmetry of the unit ball, this implies that, for any $\epsilon > 0$, there exists $0 < \delta' < 1$ satisfying

$$\int_{\mathbb{B}^d} \left| \frac{1}{(1-\langle z,w\rangle)^\alpha} - \frac{1}{(1-\langle \eta,w\rangle)^\alpha} \right| dv(w) \le \frac{\epsilon}{2} \tag{4.8}$$

whenever $z \in \mathbb{B}^d$, $\eta \in \partial\mathbb{B}^d$ and $|z-\eta| < \delta'$. Denote $B_{1-\delta'/2} = \{z \in \mathbb{C}^d : |z| \le 1-\delta'/2\}$ and $C_{\delta'/2} = \{z \in \mathbb{C}^d : 1-\delta'/2 < |z| \le 1\}$. Then the closed ball $\overline{\mathbb{B}^d}$ has the following decomposition

$$\overline{\mathbb{B}^d} = B_{1-\delta'/2} \cup C_{\delta'/2} \quad \text{and} \quad B_{1-\delta'/2} \cap C_{\delta'/2} = \emptyset.$$

Since the function $\frac{1}{(1-\langle z,w\rangle)^\alpha}$ is uniformly continuous on compact set $B_{1-\delta'/2} \times \overline{\mathbb{B}^d}$, then there exists $0 < \delta'' < 1$ such that

$$\left| \frac{1}{(1-\langle z_1,w\rangle)^\alpha} - \frac{1}{(1-\langle z_2,w\rangle)^\alpha} \right| \le \epsilon \tag{4.9}$$

whenever $(z_1,w),(z_2,w) \in B_{1-\delta'/2} \times \overline{\mathbb{B}^d}$ and $|z_1-z_2| < \delta''$. Set $\delta''' = \min\{\delta'/2, \delta''\}$. We now prove that, for any $z_1, z_2 \in \overline{\mathbb{B}^d}$ such that $|z_1-z_2| < \delta'''$, then we have

$$\int_{\mathbb{B}^d} \left| \frac{1}{(1-\langle z_1,w\rangle)^\alpha} - \frac{1}{(1-\langle z_2,w\rangle)^\alpha} \right| dv(w) \le \epsilon. \tag{4.10}$$

In fact, there are two cases need to be considered. The first case is $z_1 \in C_{\delta'/2}$ or $z_2 \in C_{\delta'/2}$. Without loss of generality, we can assume that $z_1 \in C_{\delta'/2}$, then there exists an $\eta \in \partial\mathbb{B}^d$ satisfying $|z_1-\eta| < \delta''' \le \delta'/2$. By triangle inequality, this implies that $|z_2-\eta| \le |z_2-z_1| + |z_1-\eta| < \delta'$. Together with (4.8), this implies that

$$\begin{aligned}
&\int_{\mathbb{B}^d} \left| \frac{1}{(1-\langle z_1,w\rangle)^\alpha} - \frac{1}{(1-\langle z_2,w\rangle)^\alpha} \right| dv(w) \\
\le &\int_{\mathbb{B}^d} \left| \frac{1}{(1-\langle z_1,w\rangle)^\alpha} - \frac{1}{(1-\langle \eta,w\rangle)^\alpha} \right| dv(w) \\
&+ \int_{\mathbb{B}^d} \left| \frac{1}{(1-\langle \eta,w\rangle)^\alpha} - \frac{1}{(1-\langle z_2,w\rangle)^\alpha} \right| dv(w) \\
\le &\ \epsilon.
\end{aligned}$$

The second case is $z_1, z_2 \in B_{1-\delta'/2}$. By (4.9), this implies that

$$\int_{\mathbb{B}^d} \left| \frac{1}{(1-\langle z_1,w\rangle)^\alpha} - \frac{1}{(1-\langle z_2,w\rangle)^\alpha} \right| dv(w) \le \epsilon \int_{\mathbb{B}^d} dv = \epsilon.$$

This proves (4.10). Combining with

$$|K_\alpha f_n(z_1) - K_\alpha f_n(z_2)| \le \|f_n\|_\infty \int_{\mathbb{B}^d} \left| \frac{1}{(1-\langle z_1, w\rangle)^\alpha} - \frac{1}{(1-\langle z_2, w\rangle)^\alpha} \right| dv(w),$$

this implies that $\{K_\alpha f_n\}$ is equicontinuous. Thus by Arzelà–Ascoli theorem, this implies that $\{K_\alpha f_n\}$ has a convergency subsequence in the supremum norm. □

Corollary 4.6. *If $0 < \alpha < d+1$, then the following holds:*

(1) $K_\alpha \colon L^p \to L^1$ *is compact for any* $1 \le p \le \infty$.

(2) $K_\alpha \colon L^1 \to L^q$ *is compact if and only if* $q < (d+1)/\alpha$.

(3) $K_\alpha \colon L^p \to L^\infty$ *is compact if and only if* $p > (d+1)/(d+1-\alpha)$.

Proof. This comes from Lemmas 3.7, 4.5 and the fact that K_α is adjoint. □

It remains to deal with the case $1 < p, q < \infty$, we need the following result about Carleson type measures for Bergman spaces on the unit ball.

Lemma 4.7. [23] *Suppose $1 \le p \le q < \infty$ and μ is a positive Borel measure on $\mathbb{B}^d$. Then the following conditions are equivalent:*

(1) *If $\{f_n\}$ is a bounded sequence in A_0^p and $f_n(z) \to 0$ for every $z \in \mathbb{B}^d$, then*

$$\lim_{n\to\infty} \int_{\mathbb{B}^d} |f_n|^q \, d\mu = 0.$$

(2) *For every (or some) $s > 0$, we have*

$$\lim_{|z|\to 1^-} \int_{\mathbb{B}^d} \frac{(1-|z|^2)^s}{|1-\langle z, w\rangle|^{s+q(d+1)/p}} \, d\mu(w) = 0.$$

The Borel measure in Lemma 4.7 is in fact the so-called vanishing Carleson measure. If denote $A^q(d\mu)$ by the weighted Bergman space $A^q(d\mu) = H(\mathbb{B}^d) \cap L^q(\mathbb{B}^d, d\mu)$, then Lemma 4.7(1) guarantees (or is equivalent to) that the embedding $\mathrm{Id} \colon A_0^p \to A^q(d\mu)$ is compact.

Proposition 4.8. *If $0 < \alpha < d+1$ and $1 < p \le q < \infty$, then the following are equivalent.*

(1) $K_\alpha \colon L^p \to L^q$ *is compact.*

(2) $K_\alpha \colon A_0^p \to A_0^q$ *is compact.*

(3) *The embedding* $\mathrm{Id} \colon A_0^p \to A^q_{q(d+1-\alpha)}$ *is compact.*

(4) $1/q > 1/p + \alpha/(d+1) - 1$.

Proof. We first prove that (1) is equivalent to (2). Clearly (1) implies (2). To prove the reverse, note that $0 < \alpha < d+1$, then Theorem 1.3 gives us that $K_{d+1}\colon L^p \to A_0^p$ is bounded. Suppose that $\{f_n\}$ is an arbitrary bounded sequence in L^p, thus we have $\{K_{d+1}f_n\}$ is a bounded sequence in A_0^p. Then the compactness of operator $K_\alpha\colon A_0^p \to A_0^q$ implies that, there exists a subsequence $\{f_{n_j}\}$ such that $\{K_\alpha(K_{d+1}f_{n_j})\}$ is convergent in A_0^q. Combining this with Lemma 2.7, we infer that $\{K_\alpha f_{n_j}\}$ is convergent in A_0^q. This proves that (2) implies (1).

We now prove that (2) is equivalent to (3). Similar to the proof of Proposition 2.6, by [23, Theorem 14] and [24, Theorem 2.19], it can be proved that

$$R^{\alpha-d-1,d+1-\alpha}\colon A_0^q \to A^q_{q(d+1-\alpha)}$$

and its inverse operator are bounded. Note that $K_\alpha = R^{0,\alpha-d-1}$ on A_0^p and

$$R^{\alpha-d-1,d+1-\alpha}R^{0,\alpha-d-1}f = f, \quad \forall\, f \in A_0^p.$$

Then we have the following decomposition for the embedding Id,

$$\mathrm{Id}\colon A_0^p \xrightarrow{K_\alpha = R^{0,\alpha-d-1}} A_0^q \xrightarrow{R^{\alpha-d-1,d+1-\alpha}} A^q_{q(d+1-\alpha)}, \quad \mathrm{Id} = R^{\alpha-d-1,d+1-\alpha}K_\alpha. \tag{4.11}$$

Combining (4.11) with the fact that $R^{\alpha-d-1,d+1-\alpha}$ has bounded inverse, this implies that $K_\alpha\colon A_0^p \to A_0^q$ is compact if and only if the embedding $\mathrm{Id}\colon A_0^p \to A^q_{q(d+1-\alpha)}$ is compact.

In the next, we prove that (3) is equivalent to (4). Suppose that $\{f_n\}$ is an arbitrary bounded sequence in A_0^p, then by [23, Theorem 20], the locally estimate for functions in A_0^p, this implies that $\{f_n\}$ is a normal family. Hence, by Fatou's lemma, similar to the proof of Theorem 1.1, there exists a subsequence $\{f_{n_j}\}$ and $g \in A_0^p$ such that f_{n_j} converges uniformly to g on any compact subset of $\mathbb{B}^d$. Then $\{f_{n_j} - g\}$ is in A_0^p and $f_{n_j} - g \to 0$ pointwise as $j \to \infty$. Together with Lemma 4.7, it follows that the embedding $\mathrm{Id}\colon A_0^p \to A^q_{q(d+1-\alpha)}$ is compact if and only if

$$\lim_{|z|\to 1^-} \int_{\mathbb{B}^d} \frac{(1-|z|^2)^s}{|1-\langle z, w\rangle|^{s+q(d+1)/p}}\, dv_{q(d+1-\alpha)}(w) = 0 \tag{4.12}$$

for any $s > 0$. On the other hand, by [19, Proposition 1.4.10], this implies that (4.12) is equivalent to $1/q > 1/p + \alpha/(d+1) - 1$. This completes the proof. □

Proof of Theorem 1.5. When $\alpha = d+1$. Theorem 1.5 degenerates into Proposition 4.1.

Now, we turn to the case $0 < \alpha < d+1$. We first prove that (2) implies (1). In fact, it is an immediate corollary from Lemmas 3.7, 4.5 and Corollary 4.6. To see the reverse, note that Theorem 1.3 and Proposition 4.8 give that (1) is not held if (2) is not held, this implies that (1) implies (2), completing the proof. □

Proof of Theorem 1.6. By Theorems 1.1, 1.3 and 1.5, it is easy to see that (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Leftrightarrow$ (3). Thus we only need to show that (2) $\Rightarrow$ (1). It is equivalent to proving that (2) is not true if neither is (1). It suffices to show that $K_\alpha\colon L^\infty \to L^1$ is not bounded if $\alpha \geq d+2$. Suppose $\alpha \geq d+2$. In view to Proposition 2.10 or [24, Theorem 7.1], this implies that $K_\alpha(L^\infty) = \mathcal{B}_{\alpha-d}$. From Lemma 3.2, we know that $\mathcal{B}_{\alpha-d} \not\subset L^1$, this means that $K_\alpha\colon L^\infty \to L^1$ is not bounded. This completes the proof. $\square$

5. Norm estimates

In the previous sections, we have completely characterized the L^p-L^q boundedness of K_α, K_α^+ and compactness of K_α. In the present section, we will state and prove some sharp norm estimates of K_α, K_α^+, which give essentially the upper bounds of the optimal constants in the HLS type inequalities.

Proposition 5.1. *If $d+1<\alpha<d+2$ and $K_\alpha\colon L^p \to L^q$ is bounded, then*

$$\|K_\alpha\|_{L^p\to L^q} \leq \frac{\Gamma(d+1)^{1+1/q-1/p}\Gamma(\alpha-(d+1))\Gamma\left(\frac{1}{q^{-1}-p^{-1}}(d+1-\alpha)+1\right)^{1/q-1/p}}{\Gamma(\alpha/2)^2\Gamma\left(\frac{1}{q^{-1}-p^{-1}}(d+1-\alpha)+d+1\right)^{1/q-1/p}}.$$

To prove Proposition 5.1 we first establish the following lemma.

Lemma 5.2. *Suppose that $d+1<\alpha<d+2$ and $(0,1/q)\in G(K_\alpha)=G(K_\alpha^+)$, then the following holds.*

(1) $\|K_\alpha\|_{L^\infty\to L^q} \leq \|K_\alpha^+\|_{L^\infty\to L^q} = \left\| \int_{\mathbb{B}^d} k_\alpha^+(\cdot\,,w)\,dv(w)\right\|_{L^q}$.

(2) *In particular, when $d=1$,*

$$\|K_\alpha\|_{L^\infty\to L^1} \leq \|K_\alpha^+\|_{L^\infty\to L^1} = \frac{4}{(\alpha-2)^2}\left(\frac{\Gamma(3-\alpha)}{\Gamma^2(2-\alpha/2)}-1\right). \tag{5.1}$$

(3) *For any general $(0,1/q)\in G(K_\alpha)=G(K_\alpha^+)$,*

$$\begin{aligned}\|K_\alpha\|_{L^\infty\to L^q} &\leq \|K_\alpha^+\|_{L^\infty\to L^q}\\ &\leq \frac{\Gamma(d+1)^{1+1/q}\Gamma(\alpha-(d+1))\Gamma(q(d+1-\alpha)+1)^{1/q}}{\Gamma(\alpha/2)^2\Gamma(q(d+1-\alpha)+d+1)^{1/q}}.\end{aligned} \tag{5.2}$$

Proof. (1) Since $|K_\alpha(f)| \leq K_\alpha^+(|f|)$, this implies that $\|K_\alpha\|_{L^\infty\to L^q} \leq \|K_\alpha^+\|_{L^\infty\to L^q}$ if K_α and K_α^+ are bounded. Note that $|K_\alpha^+ f|(z) \leq \|f\|_\infty \int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^\alpha}\,dv(w)$ for any $f\in L^\infty$, hence

$$\|K_\alpha\|_{L^\infty\to L^q} \leq \|K_\alpha^+\|_{L^\infty\to L^q} \leq \left\|\int_{\mathbb{B}^d} k_\alpha^+(\cdot\,,w)\,dv(w)\right\|_{L^q}.$$

To see the reverse, we note that

$$\|K_\alpha^+\|_{L^\infty\to L^q} \geq \|K_\alpha^+1\|_{L^q} = \left\| \int_{\mathbb{B}^d} k_\alpha^+(\,\cdot\,,w)\,dv(w) \right\|_{L^q}.$$

This leads to the desired result.

(2) We now turn to calculate the norm in the case of $d=1$. It follows, by Lemma 4.3(2) and what we have proven, that

$$\begin{aligned}\|K_\alpha\|_{L^\infty\to L^1} \leq \|K_\alpha^+\|_{L^\infty\to L^1} &= \int_{\mathbb{B}^d} {}_2F_1(\alpha/2,\alpha/2;d+1;|z|^2)\,dv(z)\\ &= d\int_0^1 {}_2F_1(\alpha/2,\alpha/2;d+1;r)r^{d-1}\,dr,\end{aligned}$$

in the last equality we apply the integration in polar coordinates, see [24, Lemma 1.8], and the unitary invariance of hypergeometric function ${}_2F_1(\alpha/2,\alpha/2;d+1;|z|^2)$. We now use the differential properties listed in Lemma 4.3 to calculate the integral in the case of $d=1$. We observe Lemma 4.3(3), it gives that

$$\frac{d}{dr}\big({}_2F_1(\alpha/2-1,\alpha/2-1;1;r)\big) = \Big(\frac{\alpha}{2}-1\Big)^2\,{}_2F_1(\alpha/2,\alpha/2;2;r).$$

Integrating the two sides of the above equality, we get

$$\int_0^1 {}_2F_1(\alpha/2,\alpha/2;2;r)\,dr = \frac{4}{(\alpha-2)^2}\big({}_2F_1(\alpha/2-1,\alpha/2-1;1;1)-1\big).$$

Together with Lemma 4.3(2) yields the desired result.

(3) Combining (1) with Lemma 4.2 and Lemma 4.3(1)(2), it follows that

$$\begin{aligned}&\|K_\alpha^+\|_{L^\infty\to L^q}\\ &= \left(\int_{\mathbb{B}^d}\left(\int_{\mathbb{B}^d}\frac{1}{|1-\langle z,w\rangle|^\alpha}\,dv(w)\right)^q dv(z)\right)^{1/q}\\ &= \left(\int_{\mathbb{B}^d} {}_2F_1(\alpha/2,\alpha/2;d+1;|z|^2)^q\,dv(z)\right)^{1/q}\\ &= \left(\int_{\mathbb{B}^d}(1-|z|^2)^{q(d+1-\alpha)}\,{}_2F_1(d+1-\alpha/2,d+1-\alpha/2;d+1;|z|^2)^q\,dv(z)\right)^{1/q}\\ &\leq {}_2F_1(d+1-\alpha/2,d+1-\alpha/2;d+1;1)\left(\int_{\mathbb{B}^d}(1-|z|^2)^{q(d+1-\alpha)}\,dv(z)\right)^{1/q}\\ &= \frac{\Gamma(d+1)^{1+1/q}\Gamma(\alpha-(d+1))\Gamma(q(d+1-\alpha)+1)^{1/q}}{\Gamma(\alpha/2)^2\Gamma(q(d+1-\alpha)+d+1)^{1/q}}.\end{aligned}$$

This leads to (5.2). □

Proof of Proposition 5.1. Suppose $K_\alpha^+\colon L^p \to L^q$ is bounded, it is equivalent to $(1/p, 1/q) \in G(K_\alpha^+)$. Then Theorem 1.1(3) guarantees $1/q - 1/p > \alpha - (d+1)$. Using Theorem 1.1(3) again, we see that

$$(0, 1/q - 1/p), (1 - (1/q - 1/p), 1) \in G(K_\alpha^+) \tag{5.3}$$

and there exists $0 \le \theta \le 1$ satisfying

$$(1/p, 1/q) = \theta \cdot (0, 1/q - 1/p) + (1-\theta) \cdot (1 - (1/q - 1/p), 1). \tag{5.4}$$

Combining (5.3), (5.4) with Lemma 2.2, it follows that

$$\|K_\alpha^+\|_{L^p \to L^q} \le \|K_\alpha^+\|^{\theta}_{L^\infty \to L^{\frac{1}{q^{-1}-p^{-1}}}} \|K_\alpha^+\|^{1-\theta}_{L^{\frac{1}{1-(q^{-1}-p^{-1})}} \to L^1}. \tag{5.5}$$

We observe that the adjoint operator of $K_\alpha^+\colon L^\infty \to L^{\frac{1}{q^{-1}-p^{-1}}}$ is exactly the operator $K_\alpha^+\colon L^{\frac{1}{1-(q^{-1}-p^{-1})}} \to L^1$, which means that

$$\|K_\alpha^+\|_{L^\infty \to L^{\frac{1}{q^{-1}-p^{-1}}}} = \|K_\alpha^+\|_{L^{\frac{1}{1-(q^{-1}-p^{-1})}} \to L^1}.$$

Applying this to (5.5), yields

$$\|K_\alpha^+\|_{L^p \to L^q} \le \|K_\alpha^+\|_{L^\infty \to L^{\frac{1}{q^{-1}-p^{-1}}}}. \tag{5.6}$$

Combining (5.6) with (5.3) and applying Lemma 5.2, this leads to the desired conclusion. $\square$

Corollary 5.3. *Suppose C_1 is the optimal constant in HLS* 1.2*, then*

$$C_1 \le \frac{\Gamma(d+1)^{2-1/s-1/p}\Gamma(\alpha-(d+1))\Gamma\big(\frac{1}{1-s^{-1}-p^{-1}}(d+1-\alpha)+1\big)^{1-1/s-1/p}}{\Gamma\big(\frac{\alpha}{2}\big)^2\Gamma\big(\frac{1}{1-s^{-1}-p^{-1}}(d+1-\alpha)+d+1\big)^{1-1/s-1/p}}.$$

We now turn to handle the case of $0 < \alpha < d+1$. Let $k_\alpha^+(z,w) = \frac{1}{|1-\langle z,w\rangle|^\alpha}$, $z, w \in \mathbb{B}^d$. Obviously k_α^+ is the integral kernel function of the integral operator K_α^+.

Proposition 5.4. *If $0 < \alpha < d+1$ and $1/p - (1-\alpha/(d+1)) < 1/q \le 1/p$, then*

$$\|K_\alpha\|_{L^p \to L^q} \le \|K_\alpha^+\|_{L^p \to L^q} \le \left(\frac{\Gamma(d+1)\Gamma\big(d+1-\frac{\alpha}{1-(p^{-1}-q^{-1})}\big)}{\Gamma^2\big(d+1-\frac{\alpha}{2(1-(p^{-1}-q^{-1}))}\big)}\right)^{1-(1/p-1/q)}. \tag{5.7}$$

In particular, when $q = \infty$, the inequality (5.7) *is an equality.*

Proof. We first prove that (5.7) is in fact equality in the case of $q = \infty$. From [21, Proposition 5.4], we know that

$$\|K_\alpha\|_{L^p\to L^\infty} = \|K_\alpha^+\|_{L^p\to L^\infty} = \sup_{z\in\mathbb{B}^d}\left(\int \frac{dv(w)}{|1-\langle z,w\rangle|^{\frac{p\alpha}{p-1}}}\right)^{\frac{p-1}{p}}. \tag{5.8}$$

On the other hand, Lemma 4.2 and Lemma 4.3(2) yield

$$\begin{aligned}\int_{\mathbb{B}^d} \frac{dv(w)}{|1-\langle z,w\rangle|^{\frac{p\alpha}{p-1}}} &= {}_2F_1\left(\frac{p\alpha}{2(p-1)}, \frac{p\alpha}{2(p-1)}; d+1; |z|^2\right)\\ &\leq {}_2F_1\left(\frac{p\alpha}{2(p-1)}, \frac{p\alpha}{2(p-1)}; d+1; 1\right)\\ &= \frac{\Gamma(d+1)\Gamma\big(d+1-\frac{p\alpha}{p-1}\big)}{\Gamma^2\big(d+1-\frac{p\alpha}{2(p-1)}\big)}.\end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9), this implies that

$$\|K_\alpha\|_{L^p\to L^\infty} = \|K_\alpha^+\|_{L^p\to L^\infty} = \left(\frac{\Gamma(d+1)\Gamma\big(d+1-\frac{p\alpha}{p-1}\big)}{\Gamma^2\big(d+1-\frac{p\alpha}{2(p-1)}\big)}\right)^{\frac{p-1}{p}}. \tag{5.10}$$

We now turn to prove (5.7) in the general case. Note first that $|K_\alpha(f)| \leq K_\alpha^+(|f|)$, this implies that $\|K_\alpha\|_{L^p\to L^q} \leq \|K_\alpha^+\|_{L^p\to L^q}$ if K_α and K_α^+ are bounded. Since $1/p - (1 - \alpha/(d+1)) < 1/q \leq 1/p$, Theorem 1.3 implies that

$$(1/p, 1/q), (1/p - 1/q, 0), (1, 1-(1/p-1/q)) \in G(K_\alpha^+) \tag{5.11}$$

and there exists $0 \leq \theta \leq 1$ satisfying

$$(1/p, 1/q) = \theta\cdot(1/p-1/q, 0) + (1-\theta)\cdot(1, 1-(1/p-1/q)). \tag{5.12}$$

Combining (5.11), (5.12) with Lemma 2.2, it follows that

$$\|K_\alpha^+\|_{L^p\to L^q} \leq \|K_\alpha^+\|^{\theta}_{L^{\frac{1}{p^{-1}-q^{-1}}}\to L^\infty}\|K_\alpha^+\|^{1-\theta}_{L^1\to L^{\frac{1}{1-(p^{-1}-q^{-1})}}}. \tag{5.13}$$

Observe that the adjoint operator of $K_\alpha^+ \colon L^{\frac{1}{p^{-1}-q^{-1}}} \to L^\infty$ is exactly the operator $K_\alpha^+ \colon L^1 \to L^{\frac{1}{1-(p^{-1}-q^{-1})}}$, hence

$$\|K_\alpha^+\|_{L^{\frac{1}{p^{-1}-q^{-1}}}\to L^\infty} = \|K_\alpha^+\|_{L^1\to L^{\frac{1}{1-(p^{-1}-q^{-1})}}}. \tag{5.14}$$

Thus by (5.13) and (5.14), it follows that

$$\|K_\alpha^+\|_{L^p\to L^q} \leq \|K_\alpha^+\|_{L^{\frac{1}{p^{-1}-q^{-1}}}\to L^\infty}.$$

Together with (5.10), this completes the proof. $\square$

Corollary 5.5. *Suppose that* C_2 *is the optimal constant in HLS* 1.4, *then the following holds.*

(1) *If* $1/p < 1 - 1/s$, *then*

$$C_2 \leq \frac{\Gamma(d+1)\Gamma(d+1-\alpha)}{\Gamma^2(d+1-\alpha/2)}.$$

(2) *If* $1/p - (1-\alpha/(d+1)) < 1 - 1/s \leq 1/p$, *then*

$$C_2 \leq \left(\frac{\Gamma(d+1)\Gamma\big(d+1-\frac{\alpha}{2-p^{-1}-s^{-1}}\big)}{\Gamma^2\big(d+1-\frac{\alpha}{2(2-p^{-1}-s^{-1})}\big)}\right)^{2-(1/p-1/s)}.$$

Proof of Theorem 1.7. When $\alpha < (d+2)/2$, by [19, Proposition 1.4.10], this implies that the kernel function $k_\alpha^+ \in L^2(\mathbb{B}^d\times\mathbb{B}^d, dv\times dv)$, thus $K_\alpha, K_\alpha^+ \colon L^2 \to L^2$ are Hilbert–Schmidt. Note that

$$\operatorname{Tr}(K_\alpha^* K_\alpha) = \int_{\mathbb{B}^d}\int_{\mathbb{B}^d} \frac{1}{|1-\langle z,w\rangle|^{2\alpha}}\, dv(w)dv(z). \tag{5.15}$$

When $\alpha \neq 1$, similar to (5.1), yields the trace formula. We now deal with the spacial case $\alpha = 1$. Combining Lemma 4.2 with (5.15), this implies that

$$\operatorname{Tr}(K_1^* K_1) = \int_0^1 {}_2F_1(1,1;2;r)\, dr = \sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{6}. \qquad \square$$

Remark 5.6. By Proposition 4.3(3) and inductive method, we can get explicit trace formulas for any dimension $d \geq 1$.

As a consequence of Theorem 1.7 we obtain the following generalized Euler–Jacobi identity.

Corollary 5.7. *Suppose* $0 < \alpha < 3/2$, *then*

$$\sum_{j=0}^\infty \left(\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(2+j)}\right)^2 = \frac{1}{(\alpha-1)^2}\left(\frac{\Gamma(3-2\alpha)}{\Gamma^2(2-\alpha)} - 1\right). \tag{5.16}$$

When $\alpha = 1$, the identity (5.16) is the well known Euler–Jacobi identity

$$\sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{6}.$$

When $d = 1$, $0 < \alpha < 3/2$, we know that $K_\alpha \colon L^2 \to L^2$ is compact by Theorem 1.1 or Theorem 1.7. It is trivial to see that the spectrum $\sigma(K_\alpha)$ of the operator K_α is exactly the point spectrum. Note that every K_α is adjoint, then combining (2.1) with (5.16), we have the following.

Corollary 5.8. *Suppose that* $d = 1$ *and* $0 < \alpha < 3/2$, *then* $K_\alpha\colon L^2 \to L^2$ *is compact and*

$$\sigma(K_\alpha) = \bigcup_{j=0}^{\infty} \left\{ \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(2 + j)} \right\}.$$

Moreover, in this case,

$$\|K_\alpha\|_{L^2 \to L^2} = \max_{0 \le j \le \infty} \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(2 + j)}.$$

6. The weak type boundedness result

In the last section we shall establish the weak type boundedness result for Bergman type operators as mentioned above. Recall that k_α, k_α^+ are the integral kernel functions of the integral operators K_α, K_α^+ respectively. For $p \ge 1$, the Lorentz space $L^{p,\infty}$ on $\mathbb{B}^d$ is defined by

$$L^{p,\infty} = \left\{ f : \sup_{\lambda > 0} \lambda d_f^{1/p}(\lambda) < \infty \right\},$$

where $d_f(\lambda) = v\{z \in \mathbb{B}^d : |f(z)| > \lambda\}$. Note that $L^{p,\infty} \subset L^{q,\infty}$ if $p > q$, and the inclusion is continuous.

Proposition 6.1. *If* $0 < \alpha \le d + 1$, *then* $K_\alpha, K_\alpha^+ \colon L^1 \to L^{(d+1)/\alpha,\infty}$ *are bounded.*

Before proving the proposition, we first establish the following lemma.

Lemma 6.2. *There exists a constant* C *that only depends on* α *and* d *such that*

$$\|k_\alpha(z, \cdot\,)\|_{L^{(d+1)/\alpha,\infty}} = \|k_\alpha(\,\cdot\,, z)\|_{L^{(d+1)/\alpha,\infty}} < C$$

for any $z \in \mathbb{B}^d$.

Proof. By the unitary invariance of Lebesgue measure, we need only to consider the case $z = (|z|, 0, \ldots, 0)$. Observe that

$$d_{k_\alpha(\,\cdot\,,z)}(\lambda) = v\left\{ w \in \mathbb{B}^d : \frac{1}{|1 - \langle w, z\rangle|^\alpha} > \lambda \right\} = v\left\{ w : \left| \frac{1}{|z|} - w_1 \right| < \frac{1}{|z|} \lambda^{-\frac{1}{\alpha}} \right\}. \tag{6.1}$$

Hence $\frac{1}{|1 - \langle w, z\rangle|^\alpha} < 2^\alpha$, when $|z| < 1/2$. It follows that $d_{k_\alpha(\,\cdot\,,z)}(\lambda) = 0$ if $\lambda \ge 2^\alpha$. Thus

$$\|k_\alpha(\,\cdot\,, z)\|_{L^{(d+1)/\alpha,\infty}} \le 2^\alpha$$

when $|z| < 1/2$. We now turn to the case $1/2 \le |z| < 1$. The conclusion comes immediately from the following estimate

$$\lambda d_{k_\alpha(\,\cdot\,,z)}^{\alpha/(d+1)}(\lambda) \le \begin{cases} 1, & \lambda \le 1, \\ (d \cdot 2^{3d-1})^{\alpha/(d+1)}, & 1 < \lambda < \frac{1}{(1-|z|)^\alpha}, \\ 0, & \lambda \ge \frac{1}{(1-|z|)^\alpha}. \end{cases} \tag{6.2}$$

We now prove (6.2). Let $dV(w) = \left(\frac{i}{2}\right)^d \prod_{n=1}^d dw_n \wedge d\overline{w}_n$ be the volume form. Then $dV = \frac{\pi^d}{\Gamma(d+1)}\, dv$. Observe that $\lambda d_{k_\alpha(\,\cdot\,,z)}^{\alpha/(d+1)}(\lambda) < 1$ if $\lambda \leq 1$. We denote I by the subset in the unit disk such that

$$I = \left\{ w_1 \in \mathbb{D} : \left| \frac{1}{|z|} - w_1 \right| < \frac{1}{|z|} \lambda^{-\frac{1}{\alpha}} \right\}.$$

When $1 < \lambda < \frac{1}{(1-|z|)^\alpha}$, by (6.1) and Fubini's theorem, we see that

$$\begin{aligned}
d_{k_\alpha(\,\cdot\,,z)}(\lambda) &= v\left\{ w : \left| \frac{1}{|z|} - w_1 \right| < \frac{1}{|z|} \lambda^{-\frac{1}{\alpha}} \right\} \\
&\leq \frac{\Gamma(d+1)}{\pi^d} \left(\frac{i}{2}\right)^d \int_I dw_1 \wedge d\overline{w}_1 \int_{|w_2|^2+\cdots+|w_d|^2<1-|w_1|^2} \prod_{n=2}^d dw_n \wedge d\overline{w}_n \\
&= d \int_I (1-|w_1|^2)^{d-1}\, dv(w_1) \\
&< d\left(1 - \frac{1}{|z|^2} + 2\frac{1}{|z|^2}\frac{1}{\lambda^{1/\alpha}} - \frac{1}{|z|^2\lambda^{2/\alpha}}\right)^{d-1} \int_I dv(w_1) \\
&< d \cdot 2^{3d-3} \frac{1}{\lambda^{(d-1)/\alpha}} \frac{4}{\lambda^{2/\alpha}} \\
&= \frac{d \cdot 2^{3d-1}}{\lambda^{(d+1)/\alpha}}.
\end{aligned} \tag{6.3}$$

Then (6.3) implies that $\lambda d_{k_\alpha(\,\cdot\,,z)}^{\alpha/(d+1)}(\lambda) < (d \cdot 2^{3d-1})^{\alpha/(d+1)}$ if $1 < \lambda < \frac{1}{(1-|z|)^\alpha}$. When $\lambda \geq \frac{1}{(1-|z|)^\alpha}$, it is easy to see that $d_{k_\alpha(\,\cdot\,,z)}(\lambda) = 0$. So $\lambda d_{k_\alpha(\,\cdot\,,z)}^{\alpha/(d+1)}(\lambda) = 0$ if $\lambda \geq \frac{1}{(1-|z|)^\alpha}$. □

Corollary 6.3. *There exists a constant C that only depends on α and d such that, for any $z \in \mathbb{B}^d$,*

$$\|k_\alpha^+(z,\cdot\,)\|_{L^{(d+1)/\alpha,\infty}} = \|k_\alpha^+(\,\cdot\,,z)\|_{L^{(d+1)/\alpha,\infty}} < C.$$

We now modify [21, Proposition 6.1] to suit our setting.

Lemma 6.4. [21] *Suppose that $k\colon \mathbb{B}^d \times \mathbb{B}^d \to \mathbb{C}$ is measurable such that*

$$\|k(z,\cdot\,)\|_{L^{r,\infty}} \leq C, \quad z \in \mathbb{B}^d, \ \text{a.e.}$$

and

$$\|k(\,\cdot\,,w)\|_{L^{r,\infty}} \leq C, \quad w \in \mathbb{B}^d, \ \text{a.e.}$$

for some $1 < r < \infty$ and $C > 0$. Then the operator T defined as

$$Tf(z) = \int_{\mathbb{B}^d} k(z,w) f(w)\, dv(w)$$

is bounded from L^1 to $L^{r,\infty}$. Moreover, if $1 < p < q < \infty$ such that $1/p + 1/r = 1/q + 1$, then T is bounded from L^p to L^q.

Proof of Proposition 6.1. $\alpha = d+1$, K_{d+1} is the Bergman projection, then $K_{d+1}\colon L^1 \to L^{1,\infty}$ is bounded by the proof of [15, Theorem 6]. Indeed, similar to the proof of [15, Theorem 6], by the Calderón–Zygmund decomposition, it can be proved that $K^+_{d+1}\colon L^1 \to L^{1,\infty}$ is bounded. When $0 < \alpha < d+1$, by Lemmas 6.2 and 6.4, this implies that $K_\alpha, K^+_\alpha\colon L^1 \to L^{(d+1)/\alpha,\infty}$ are bounded. This completes the proof. □

Remark 6.5. The sufficiency part of Theorem 1.3 can be also proved with the help of Lemmas 6.2 and 6.4. On the other hand, the necessity part of Theorem 1.3 can be reduced to the case of unit disk $\mathbb{D}$ by the natural isometric embedding from $A^p_{d-1}(\mathbb{D})$ into $A^p_0(\mathbb{B}^d)$. It in fact provides an alternative approach to prove Theorem 1.3.

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