Maximal Functions Along Convex Curves with Lacunary Directions

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Abstract. In this paper, we obtain the $L^p(\mathbb{R}^2)$-boundedness of the maximal functions

$$M_{I,\gamma}f(x_1, x_2) := \sup_{j \in \mathbb{Z}} \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - t, x_2 - 2^j \gamma(t))| \, dt$$

and

$$M_{II,\gamma}f(x_1, x_2) := \sup_{i,j \in \mathbb{Z}} \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - 2^i t, x_2 - 2^j \gamma(t))| \, dt,$$

where $p \in (1, \infty]$ and γ is a convex curve satisfying some suitable curvature conditions.

1. Introduction

The *maximal function M_γ along the curve $(t, \gamma(t))$* is defined by

$$M_\gamma f(x_1, x_2) := \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - t, x_2 - \gamma(t))| \, dt.$$

The question of whether this operator M_γ is bounded on $L^p(\mathbb{R}^2)$ has received much attention in the last few decades. In particular, for $\gamma(t) := t^2$, Nagel, Riviere and Wainger [22] established the $L^p(\mathbb{R}^2)$-boundedness of M_γ for all $p \in (1, \infty]$. For the cases of homogeneous curves and smooth curves, we refer to Stein [27] and Stein and Wainger [28], respectively. Later, Stein and Wainger [29] obtained the $L^p(\mathbb{R}^2)$-boundedness, $p \in (1, \infty]$, of M_γ for "well-curved" curves, they also pointed out that this boundedness is false for arbitrary C^∞ curves. Therefore, for more general curves obeying some curvature conditions, obtaining the $L^p(\mathbb{R}^2)$-boundedness of the associated maximal function M_γ has become a classical problem in harmonic analysis.

For the case of $\gamma \in C^2(\mathbb{R})$ is either odd or even, $\gamma(0) = \gamma'(0) = 0$, and convex on $(0, \infty)$, let $h(t) := t\gamma'(t) - \gamma(t)$, if h satisfies the *doubling condition*, i.e.,

(D)   There exists $1 < \varepsilon_0 < \infty$ so that $h(\varepsilon_0 t) \geq 2h(t)$ for all $0 < t < \infty$,

Nagel, Vance, Wainger and Weinberg [25] showed the $L^2(\mathbb{R}^2)$-boundedness of M_γ. Later, Carlsson, Christ, Córdoba et al. [7] extended this boundedness to all $p \in (1, \infty]$ if γ' satisfies the doubling condition (D). Furthermore, Carbery, Christ, Vance et al. [4] also obtained the $L^p(\mathbb{R}^2)$-boundedness of M_γ for all $p \in (1, \infty]$ by replacing this doubling condition (D) about γ' with the following *infinitesimally doubling condition* about h, i.e.,

(ID)   There exists $0 < \varepsilon_1 < \infty$ so that $h'(t) \geq \varepsilon_1 \dfrac{h(t)}{t}$ for all $0 < t < \infty$.

Note that both of γ' satisfies the doubling condition (D) and h satisfies the infinitesimally doubling condition (ID) imply that h satisfies the doubling condition (D). For more discussion in the case of γ is convex, we refer to [8, 10, 33], but a necessary and sufficient condition for the $L^p(\mathbb{R}^2)$-boundedness of M_γ is still not known. There are many other cases of the curves γ in obtaining the $L^p(\mathbb{R}^2)$-boundedness of M_γ; see, for example, non-convex curves [1,31], flat curves [5,6], highly monotone curves [26], oscillating curves [32]. The immediate consequence of the $L^p(\mathbb{R}^2)$-boundedness of M_γ is obtaining the *Lebesgue differentiation theorem along the curve* $(t, \gamma(t))$, i.e., for any $f \in L^p(\mathbb{R}^2)$,

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(x_1 - t, x_2 - \gamma(t)) \, \mathrm{d}t = f(x_1, x_2) \quad \text{a.e. } (x_1, x_2) \in \mathbb{R}^2.$$

In this paper, we consider two different kind of maximal functions analogue of M_γ, which are modelled on the maximal function M_γ in lacunary direction introduced by Nagel, Stein and Wainger in [23]. The first one is the *maximal function $M_{I,\gamma}$ along the curve $(t, \gamma(t))$ with the dyadic lacunary direction* $(1, 2^j)$,

$$M_{I,\gamma} f(x_1, x_2) := \sup_{j \in \mathbb{Z}} \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - t, x_2 - 2^j \gamma(t))| \, \mathrm{d}t.$$

The second one is the *maximal function $M_{II,\gamma}$ along the curve $(t, \gamma(t))$ with the dyadic lacunary direction* $(2^i, 2^j)$, which allow dyadic scaling along this dyadic lacunary direction $(1, 2^j)$ in $M_{I,\gamma}$,

$$M_{II,\gamma} f(x_1, x_2) := \sup_{i,j \in \mathbb{Z}} \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - 2^i t, x_2 - 2^j \gamma(t))| \, \mathrm{d}t.$$

Remark 1.1. Indeed, both of $M_{I,\gamma}$ and $M_{II,\gamma}$ are a special case of the following *maximal function $\mathcal{M}_{II,\gamma}$ along the curve $(t, \gamma(t))$ with the direction* (a, b),

$$\mathcal{M}_{II,\gamma} f(x_1, x_2) := \sup_{a,b \in \mathbb{R}} \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - at, x_2 - b\gamma(t))| \, \mathrm{d}t.$$

Note that $\mathcal{M}_{II,\gamma}$ arise in connection with Stein's and Bourgain's circular maximal function. In [21], Marletta and Ricci obtained that $\mathcal{M}_{II,\gamma}$ is bounded on $L^p(\mathbb{R}^2)$ if and only if

$p \in (2, \infty]$, where $\gamma(t) := t^r$ with $r \in \mathbb{R}$ and $r \neq 0, 1$. If $\gamma(t) := t$, a counter-example based on a construction of the Besicovitch–Kakeya set shows that we can not expect any $L^p(\mathbb{R}^2)$-boundednesss of $\mathcal{M}_{II,\gamma}$ for all $p \in (1, \infty)$. More related works can be found in Bourgain [2], Iosevich [16], Marletta [20], Guo, Hickman, Lie and Roos [12] and Li [19]. Here, we also want to introduce another operator, which is closely related to $\mathcal{M}_{II,\gamma}$, i.e., the corresponding *maximal Hilbert transform $\mathcal{H}_{II,\gamma}$ along the curve $(t, \gamma(t))$ with the direction (a, b),*

$$\mathcal{H}_{II,\gamma}f(x_1, x_2) := \sup_{a,b \in \mathbb{R}} \left| \mathrm{P.\,V.} \int_{-\infty}^{\infty} f(x_1 - at, x_2 - b\gamma(t)) \frac{\mathrm{d}t}{t} \right|.$$

From a straightforward modification of Karagulyan's counter-example in [17], we will find that the $L^p(\mathbb{R}^2)$-boundedness of $\mathcal{H}_{II,\gamma}$ fails for all $p \in (1, \infty]$ even in the case of $\gamma(t) := t^2$ or $\gamma(t) := t$. Therefore, instead of considering the $L^p(\mathbb{R}^2)$-boundedness of $\mathcal{H}_{II,\gamma}$, we always consider a new operator $\mathcal{H}_{U,\gamma}$ constructed by replacing $\sup_{a,b \in \mathbb{R}}$ with $\sup_{a,b \in U}$ in $\mathcal{H}_{II,\gamma}$, and calculate the operator norm $\|\mathcal{H}_{U,\gamma}\|_{L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)}$, where $U \subset \mathbb{R}$ is an arbitrary nonempty set. This is also a hot topic in harmonic analysis, see Demeter and Di Plinio [11], Łaba, Marinelli and Pramanik [18], Guo, Roos, Seeger and Yung [13] and the references therein. In this paper, we are not going to calculate the $L^p(\mathbb{R}^2)$-norm of this maximal singular integral operator $\mathcal{H}_{U,\gamma}$, and our results of the $L^p(\mathbb{R}^2)$-boundedness of $M_{I,\gamma}$ and $M_{II,\gamma}$ will provide an inspiration for obtaining the $L^p(\mathbb{R}^2)$-boundedness of $\mathcal{M}_{II,\gamma}$ for some $p \in (1, \infty)$ and more general class of curves γ.

We now state our results.

Theorem 1.2. *Let $\gamma \in C^2(\mathbb{R})$ be either odd or even, $\gamma(0) = \gamma'(0) = 0$, and convex on $(0, \infty)$, and satisfying*

(i) *there exists a positive constant C_1 such that $\frac{\gamma'(2t)}{\gamma'(t)} \leq C_1$ for any $t \in (0, \infty)$;*

(ii) *there exists a positive constant C_2 such that $\frac{t\gamma''(t)}{\gamma'(t)} \geq C_2$ for any $t \in (0, \infty)$.*

Then for any given $p \in (1, \infty]$, there exists a positive constant C such that

$$\|M_{I,\gamma}f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$$

for any $f \in L^p(\mathbb{R}^2)$.

Theorem 1.3. *Let $\gamma \in C^2(\mathbb{R})$ be either odd or even, $\gamma(0) = \gamma'(0) = 0$, and convex on $(0, \infty)$, and satisfying*

(i) *there exists a positive constant C_1' such that $\frac{t\gamma'(t)}{\gamma(t)} \leq C_1'$ for any $t \in (0, \infty)$;*

(ii) *there exists a positive constant C_2 such that $\frac{t\gamma''(t)}{\gamma'(t)} \geq C_2$ for any $t \in (0, \infty)$.*

Then for any given $p \in (1, \infty]$, there exists a positive constant C such that

$$\|M_{II,\gamma}f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$$

for any $f \in L^p(\mathbb{R}^2)$.

Remark 1.4. The conditions of Theorem 1.2 imply the conditions of Theorem 1.3. Indeed, from (2.1), as in [24, Lemma 2], we have

$$\frac{\gamma'(2t)}{\gamma'(t)} \geq e^{C_2/2} \quad \Longrightarrow \quad h(t) \geq \lambda t\gamma'(t) \quad \Longrightarrow \quad h\left(\frac{2}{\lambda}t\right) \geq 2h(t) \quad \text{for any } t \in (0, \infty),$$

where $h(t) = t\gamma'(t) - \gamma(t)$ and $\lambda := \frac{1-e^{-C_2/2}}{2}$. Furthermore, as in [24, Lemma 1], we have

$$h\left(\frac{2}{\lambda}t\right) \geq 2h(t) \quad \Longrightarrow \quad \frac{2}{\lambda}t\big(\gamma'(s) - \gamma'(t)\big) \geq h(t) \ \text{ with } \ s \geq \frac{2}{\lambda}t$$

$$\Longrightarrow \quad \gamma\left(\frac{4}{\lambda}t\right) \geq \frac{4}{\lambda}t\gamma'(t) \quad \text{for any } t \in (0, \infty).$$

This, combined with Theorem 1.2(i), implies that $\frac{t\gamma'(t)}{\gamma(t)} \leq (C_1)^{\log_2 \frac{4}{\lambda}}$ for any $t \in (0, \infty)$. Therefore, we obtain Theorem 1.3(i) with $C_1' := (C_1)^{\log_2 \frac{4}{\lambda}}$.

From this fact, it is easy to see that Theorem 1.2 can be covered by Theorem 1.3. However, the method of proving Theorem 1.2 is totally different from the method applied in proving Theorem 1.3. Therefore, we will retain Theorem 1.2.

Example 1.5. Let us list some examples of curves satisfying the conditions in Theorems 1.2 and 1.3. Since $\gamma(t)$ is odd or even, and $\gamma(0) = 0$, we write only the part for $t > 0$.

(1) for any $t > 0$, $\gamma(t) := t^\alpha$, where $\alpha > 1$;

(2) for any $k \in \mathbb{N}$ and $t > 0$, $\gamma(t) := \sum_{i=1}^k t^{\alpha_i}$, where $\alpha_i > 1$ for all $i = 1, 2, \ldots, k$;

(3) for any $t > 0$, $\gamma(t) := t^\alpha \ln(1 + t)$, or $\gamma(t) := t^\alpha \arctan t$, where $\alpha > 1$;

(4) for any $t > 0$, $\gamma(t) := (t \sin t)\mathbf{1}_{\{0 < t < \varepsilon_0\}}(t)$, or $(t - \sin t)\mathbf{1}_{\{0 < t < \varepsilon_0\}}(t)$, or $(t - \ln(1 + t))\mathbf{1}_{\{0 < t < \varepsilon_0\}}(t)$, where ε_0 is small enough.

It is easy to see that, for the case of $\gamma(t) := t^d$ under $d \in \mathbb{N}$, the $L^p(\mathbb{R}^2)$-boundedness of $M_{I,\gamma}$ and $M_{II,\gamma}$ are equivalent. But for more general curves, this equivalence relation does not hold. Therefore, we consider $M_{I,\gamma}$ and $M_{II,\gamma}$, respectively. The history of these operators goes back to the case of $\gamma(t) := t$. Córdoba and Fefferman [9] and Strömberg [30] used a suitable geometric argument to obtain the $L^p(\mathbb{R}^2)$-boundedness of $M_{II,\gamma}$ for all $p \in [2, \infty]$. Later, Nagel, Stein and Wainger [23] extended this result to all $p \in (1, \infty]$

by using a Littlewood–Paley decomposition. Furthermore, Carbery [3] extended these two-dimensional results [9, 23, 30] to arbitrary dimension. For the case of $\gamma(t) := t^2$, Guo, Roos, Seeger and Yung [13, Lemma 5.1] obtained the $L^p(\mathbb{R}^2)$-boundedness of $M_{I,\gamma}$ for all $p \in (1, \infty]$, and the same boundedness for the case of $\gamma(t) := [t]^\alpha$ can also been obtained with some standard modifications, where $[t]^\alpha$ stands for either $|t|^\alpha$ or $\operatorname{sgn}(t)|t|^\alpha$, $\alpha > 0$ and $\alpha \neq 1$. There are also some other types of $M_{II,\gamma}$; see, for example,

$$M_{II}^Q f(x_1, x_2) := \sup_{i,j \in \mathbb{Z}} \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x_1 - 2^i Q(t), x_2 - 2^j Q(t))| \, \mathrm{d}t,$$

where Q is a real polynomial with $Q(0) = 0$. Hare and Ricci [14] obtained the $L^p(\mathbb{R}^2)$-boundedness of M_{II}^Q for all $p \in (1, \infty]$. More recently, Hong, Kim and Yang [15] extended M_{II}^Q to more general polynomial curves and arbitrary dimension.

Our proofs of Theorems 1.2 and 1.3 rely on the non-stationary phase method in Carlsson, Christ, Córdoba et al. [7], the Littlewood–Paley theory, and a bootstrapping argument similar that of Nagel, Stein and Wainger [23]. In Section 2, we prove Theorem 1.2 by comparing it with a smoother operator. Theorem 1.3 is proved in Section 3, we split it as the low-frequency part and the high-frequency part. The low-frequency part can be viewed as a one-dimensional convolution type operator, and the high-frequency part is proved by interpolating between a decay estimate under $p \in [2, \infty)$ and a non-decay estimate under $p \in (1, 2)$.

Throughout this paper, we use "C" to denote a *positive constant* that is independent of the main parameters involved but whose value may vary from line to line. The *positive constants with subscripts*, such as C_1 and C_2, are the same in different occurrences. For two real functions f and g, we use $f \lesssim g$ or $g \gtrsim f$ to denote $f \leq Cg$ and, if $f \lesssim g \lesssim f$, we then write $f \approx g$. We use $\mathbb{Z}$ to denote the *set of integers*, and $\mathbb{N} := \{1, 2, 3, \ldots\}$. We also use $a \Rightarrow b$ to denote that a implies b. For any set E, we use $\mathbf{1}_E$ to denote the *characteristic function* of E.

2. Proof of Theorem 1.2

We start by introducing some simple properties of the curve γ which we need in the course of proof. Indeed, let $G(t) := \ln \gamma'(t)$ for any $t \in (0, \infty)$, by Theorem 1.2(ii), we then have $G'(t) \geq \frac{C_2}{t}$ for any $t \in (0, \infty)$. By the Lagrange mean value theorem, there exists a constant $\theta \in [1, 2]$ such that $G(2t) - G(t) = G'(\theta t)t \geq \frac{C_2}{2}$, which further leads to

$$(2.1) \qquad \frac{\gamma'(2t)}{\gamma'(t)} \geq e^{C_2/2} \quad \text{for any } t \in (0, \infty).$$

On the other hand, since $\gamma(0) = \gamma'(0) = 0$, by the Cauchy mean value theorem, there exists $T \in (0, t)$ such that $\frac{t\gamma'(t)}{\gamma(t)} = \frac{t\gamma'(t) - 0\gamma'(0)}{\gamma(t) - \gamma(0)} = \frac{\gamma'(T) + T\gamma''(T)}{\gamma'(T)}$ for any $t \in (0, \infty)$. Thus, we

apply Theorem 1.2(ii) to conclude that

$$(2.2) \qquad \frac{t\gamma'(t)}{\gamma(t)} \geq C_2 + 1 \quad \text{for any } t \in (0, \infty).$$

We will make use of some ideas from Carlsson, Christ, Córdoba et al. [7] to prove our theorem. Indeed, it is naturally to consider the $L^2(\mathbb{R}^2)$-boundedness fristly, and also should control $\sup_{j,k \in \mathbb{Z}} M_{I,\gamma}^{j,k} f(x_1, x_2)$ by $\sum_{j,k \in \mathbb{Z}} M_{I,\gamma}^{j,k} f(x_1, x_2)$. From Plancherel's theorem, it is sufficient to bound the corresponding multiplier $\sum_{j,k \in \mathbb{Z}} m_{I,\gamma}^{j,k}(\xi_1, \xi_2)$. Furthermore, by applying van der Corput's lemma, we may bound the multiplier $m_{I,\gamma}^{j,k}(\xi_1, \xi_2)$ by $|(\varphi_{I,\gamma}^{j,k})'(t)|^{-1}$ or $|(\varphi_{I,\gamma}^{j,k})''(t)|^{-1/2}$, where $\varphi_{I,\gamma}^{j,k}(t)$ is the corresponding phase function of the multiplier $m_{I,\gamma}^{j,k}(\xi_1, \xi_2)$, but it is not enough to bound $\sum_{j,k \in \mathbb{Z}} m_{I,\gamma}^{j,k}(\xi_1, \xi_2)$. Therefore, to bound this sum, we want to bound the multiplier $m_{I,\gamma}^{j,k}(\xi_1, \xi_2)$ by $|\varphi_{I,\gamma}^{j,k}(t)|$. Note that $\left| e^{i\varphi_{I,\gamma}^{j,k}(t)} - 1 \right| \leq \left| \varphi_{I,\gamma}^{j,k}(t) \right|$, this implies that we should define a new multiplier $\overline{m}_{I,\gamma}^{j,k}(\xi_1, \xi_2)$ to compare our multiplier $m_{I,\gamma}^{j,k}(\xi_1, \xi_2)$, it is also the reason of defining the operator σ_k in [7]. For the general $L^p(\mathbb{R}^2)$-boundedness, as in Nagel, Stein and Wainger [23], the bootstapping argument is a useful tool, and our proof also relies on this argument.

We now begin the proof of Theorem 1.2. Since $M_{I,\gamma}$ is a positive operator we may assume that f is non-negative. Furthermore, we can bound $M_{I,\gamma} f$ by

$$\sup_{j \in \mathbb{Z}} \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \sum_{k:\, 2^k \leq \varepsilon} 2^k \frac{1}{2^k} \int_{2^k < |t| \leq 2^{k+1}} f(x_1 - t, x_2 - 2^j \gamma(t)) \, \mathrm{d}t$$

$$\lesssim \sup_{j,k \in \mathbb{Z}} \frac{1}{2^k} \int_{2^k < |t| \leq 2^{k+1}} f(x_1 - t, x_2 - 2^j \gamma(t)) \, \mathrm{d}t =: \sup_{j,k \in \mathbb{Z}} M_{I,\gamma}^{j,k} f(x_1, x_2).$$

Therefore, it suffices to prove the $L^p(\mathbb{R}^2)$-boundedness of the operator $\sup_{j,k \in \mathbb{Z}} M_{I,\gamma}^{j,k}$. As in [7], we define a smoother operator $\overline{M}_{I,\gamma}^{j,k}$ by

$$\overline{M}_{I,\gamma}^{j,k} f(x_1, x_2) := \frac{1}{2^{2k}} \int_{2^k < |t| \leq 2^{k+1}} \int_{2^k < |\tau| \leq 2^{k+1}} f(x_1 - t, x_2 - 2^j \gamma(\tau)) \, \mathrm{d}t \mathrm{d}\tau,$$

which keeps many the same characteristics from $M_{I,\gamma}^{j,k}$. More importantly, from (2.2) and the fact that $(\gamma^{-1})'(t)\gamma'(\gamma^{-1}(t)) = 1$ where γ^{-1} is the inverse function of γ, by letting $s := 2^j \gamma(\tau)$, we can bound $\overline{M}_{I,\gamma}^{j,k} f(x_1, x_2)$ by

$$\frac{1}{2^{2k}} \int_{2^k < |t| \leq 2^{k+1}} \int_{2^j \gamma(2^k) < |s| \leq 2^j \gamma(2^{k+1})} f(x_1 - t, x_2 - s) \left| \frac{\gamma(\gamma^{-1}(2^{-j}s))}{\gamma'(\gamma^{-1}(2^{-j}s))\gamma^{-1}(2^{-j}s)} \right|$$

$$\times \left| \frac{\gamma^{-1}(2^{-j}s)2^{-j}}{\gamma(\gamma^{-1}(2^{-j}s))} \right| \mathrm{d}t \mathrm{d}s.$$

Furthermore, from (2.2) and Theorem 1.2(i), by simple calculation, one has

$$\frac{1}{2^{2k}} 2^{k+1} 2^j \gamma(2^{k+1}) \left| \frac{\gamma(\gamma^{-1}(2^{-j}s))}{\gamma'(\gamma^{-1}(2^{-j}s))\gamma^{-1}(2^{-j}s)} \right| \cdot \left| \frac{\gamma^{-1}(2^{-j}s)2^{-j}}{\gamma(\gamma^{-1}(2^{-j}s))} \right| \lesssim 1,$$

which further implies that $\overline{M}_{I,\gamma}^{j,k}f$ can be bounded by $M^{(1)}M^{(2)}f$. Here and hereafter, we use $M^{(1)}$ and $M^{(2)}$ denote the *Hardy–Littlewood maximal operators applied in the first variable and the second variable*, respectively. This implies the following pointwise estimate

$$\sup_{j,k\in\mathbb{Z}} \overline{M}_{I,\gamma}^{j,k}f(x_1,x_2) \lesssim M^{(1)}M^{(2)}f(x_1,x_2).$$

From the $L^p(\mathbb{R}^2)$-boundedness of $M^{(1)}$ and $M^{(2)}$, we have

$$(2.3) \qquad \left\| \sup_{j,k\in\mathbb{Z}} \overline{M}_{I,\gamma}^{j,k}f \right\|_{L^p(\mathbb{R}^2)} \lesssim \left\| M^{(1)}M^{(2)}f \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1,\infty]$. Therefore, it suffices to prove

$$(2.4) \qquad \left\| \sup_{j,k\in\mathbb{Z}} \left| \left(M_{I,\gamma}^{j,k} - \overline{M}_{I,\gamma}^{j,k} \right)f \right| \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1,\infty]$.

2.1. $L^2(\mathbb{R}^2)$-boundedness

For $p = 2$ in (2.4), it is naturally to consider the multipliers of $M_{I,\gamma}^{j,k}$ and $\overline{M}_{I,\gamma}^{j,k}$ by Plancherel's theorem. Indeed, a calculation gives that the multiplier of $M_{I,\gamma}^{j,k}$ can be written as

$$m_{I,\gamma}^{j,k}(\xi_1,\xi_2) := \int_{1\le|t|<2} e^{-i2^k t\xi_1 - i2^j\gamma(2^k t)\xi_2}\, \mathrm{d}t$$

and the multiplier of $\overline{M}_{I,\gamma}^{j,k}$ can be written as

$$\overline{m}_{I,\gamma}^{j,k}(\xi_1,\xi_2) := \int_{1\le|t|<2} \int_{1\le|\tau|<2} e^{-i2^k t\xi_1 - i2^j\gamma(2^k \tau)\xi_2}\, \mathrm{d}t\mathrm{d}\tau.$$

We now bound the left-hand side of (2.4) by

$$\left\| \left[\sum_{j,k\in\mathbb{Z}} \left| \left(M_{I,\gamma}^{j,k} - \overline{M}_{I,\gamma}^{j,k} \right)f \right|^2 \right]^{1/2} \right\|_{L^2(\mathbb{R}^2)}.$$

Because the $l^2(\mathbb{Z}^2)$ and $L^2(\mathbb{R}^2)$ norms commute, from Plancherel's theorem, it is enough to show that

$$(2.5) \qquad \sum_{j,k\in\mathbb{Z}} \left| m_{I,\gamma}^{j,k}(\xi_1,\xi_2) - \overline{m}_{I,\gamma}^{j,k}(\xi_1,\xi_2) \right|^2 \lesssim 1$$

for all $(\xi_1, \xi_2) \in \mathbb{R}^2$. Furthermore, let us set $\xi_1 := 2^M \widetilde{\xi}_1$ and $\xi_2 := 2^N \widetilde{\xi}_2$, where $M, N \in \mathbb{Z}$ and $|\widetilde{\xi}_1|, |\widetilde{\xi}_2| \in [1, 2)$, then the left-hand side of (2.5) is equal to

$$
\sum_{j,k\in\mathbb{Z}} \left| \int_{1\le|t|<2} e^{-i2^{k+M}t\widetilde{\xi}_1 - i2^{j+N}\gamma(2^k t)\widetilde{\xi}_2}\, dt - \int_{1\le|t|<2}\int_{1\le|\tau|<2} e^{-i2^{k+M}t\widetilde{\xi}_1 - i2^{j+N}\gamma(2^k \tau)\widetilde{\xi}_2}\, dtd\tau \right|^2
$$

$$
= \sum_{j,k\in\mathbb{Z}} \left| \int_{1\le|t|<2} e^{-i2^k t\widetilde{\xi}_1 - iC_M 2^j\Gamma_M(2^k t)\widetilde{\xi}_2}\, dt - \int_{1\le|t|<2}\int_{1\le|\tau|<2} e^{-i2^k t\widetilde{\xi}_1 - iC_M 2^j\Gamma_M(2^k \tau)\widetilde{\xi}_2}\, dtd\tau \right|^2
$$

$$
=: \sum_{j,k\in\mathbb{Z}} \left| \mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) \right|^2,
$$

where $\Gamma_M(t) := \frac{\gamma(2^{-M}t)}{\gamma(2^{-M})}$, $C_M \in [1, 2)$ is a constant depending only on M.

By the fact that

$$
\left| \mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) \right| \le \sum_{j,k\in\mathbb{Z}} \left| \mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) \right|,
$$

it suffices to prove

$$
(2.6) \qquad\qquad \sum_{j,k\in\mathbb{Z}} \left| \mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) \right| \lesssim 1
$$

for all $|\widetilde{\xi}_1|, |\widetilde{\xi}_2| \in [1, 2)$, where the implicit constant is independent of M.

We now define the phase function of $\mathfrak{m}_{I,\gamma}^{j,k}$ by

$$
\varphi_{I,\gamma}^{j,k}(t) := 2^k t\widetilde{\xi}_1 + C_M 2^j\Gamma_M(2^k t)\widetilde{\xi}_2.
$$

Then

$$
(2.7) \qquad\qquad \left(\varphi_{I,\gamma}^{j,k} \right)'(t) = 2^k\widetilde{\xi}_1 + C_M 2^j\Gamma_M'(2^k t)2^k\widetilde{\xi}_2
$$

and

$$
\left(\varphi_{I,\gamma}^{j,k} \right)''(t) = C_M 2^j\Gamma_M''(2^k t)2^{2k}\widetilde{\xi}_2.
$$

To establish (2.6), we consider the following seven cases according to the values of k and j. More precisely, for any given $k \ge 0$, we split $\mathbb{R}$ as a union of the following four intervals: $(-\infty, 1/8]$, $(1/8, \Gamma_M'(2^{k+1})]$, $\left(\Gamma_M'(2^{k+1}), \frac{1}{8}(2C_1)^{2k}\Gamma_M'(1) \right]$ and $\left(\frac{1}{8}(2C_1)^{2k}\Gamma_M'(1), \infty \right)$, and divide (2.6) into four cases according to whether or not $\frac{1}{8}2^{-j}$ belong to one of these intervals. For $k < 0$, similarly, we will divide (2.6) into three cases.

Case 1: $k \ge 0$ and $\frac{1}{8}2^{-j} \le \frac{1}{8}$ (i.e., $j \ge 0$). From Theorem 1.2(ii), this implies that

$$
\left| \left(\varphi_{I,\gamma}^{j,k} \right)''(t) \right| = \left| C_M 2^j \frac{\Gamma_M''(2^k t)2^k t}{\Gamma_M'(2^k t)} \frac{\Gamma_M'(2^k t)}{2^k t}2^{2k}\widetilde{\xi}_2 \right| \gtrsim 2^j 2^k\Gamma_M'(2^k).
$$

By van der Corput's lemma, we obtain

$$
\left| \mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) \right| \lesssim \left(2^j 2^k\Gamma_M'(2^k) \right)^{-1/2}.
$$

The same estimate also holds for $\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)$, by (2.2), we then have

$$\sum_{\text{Case 1}} \left|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)\right| \lesssim \sum_{\text{Case 1}} \left(2^j 2^k \Gamma_M'(2^k)\right)^{-1/2} \lesssim \sum_{\text{Case 1}} \left(2^j 2^k\right)^{-1/2} \lesssim 1$$

for all $|\widetilde{\xi}_1|, |\widetilde{\xi}_2| \in [1, 2)$.

Case 2: $k \geq 0$ and $\frac{1}{8} < \frac{1}{8}2^{-j} \leq \Gamma_M'(2^{k+1})$. By Remark 1.4 and Theorem 1.2(i), we have $-\log_2(8C_1 C_1') - k\log_2 C_1 \leq j < 0$ and $2^j \Gamma_M'(2^k) \geq \frac{1}{8C_1}$. As in Case 1, it follows that

$$\sum_{\text{Case 2}} \left|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)\right| \lesssim \sum_{\text{Case 2}} \left(2^j 2^k \Gamma_M'(2^k)\right)^{-1/2} \lesssim \sum_{\text{Case 2}} 2^{-k/2}$$

$$\lesssim \sum_{k \geq 0} \left(\log_2(8C_1 C_1') + k\log_2 C_1\right) 2^{-k/2} \lesssim 1$$

for all $|\widetilde{\xi}_1|, |\widetilde{\xi}_2| \in [1, 2)$.

Case 3: $k \geq 0$ and $\Gamma_M'(2^{k+1}) < \frac{1}{8}2^{-j} \leq \frac{1}{8}(2C_1)^{2k}\Gamma_M'(1)$. From $\Gamma_M'(2^{k+1}) < \frac{1}{8}2^{-j}$ and (2.7), this implies that

$$\left|(\varphi_{I,\gamma}^{j,k})'(t)\right| \geq 2^k\left(|\widetilde{\xi}_1| - C_M 2^j |\Gamma_M'(2^k t)\widetilde{\xi}_2|\right) \geq 2^k.$$

This, by van der Corput's lemma and the fact that $(\varphi_{I,\gamma}^{j,k})'(t)$ is monotonic on $[1, 2)$ or $(-2, -1]$, leads to

$$\left|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)\right| \lesssim 2^{-k}.$$

For $\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)$, by van der Corput's lemma, it is easy to see that $\left|\int_{1 \leq |t| < 2} e^{-i2^k t \widetilde{\xi}_1}\, dt\right| \lesssim 2^{-k}$. This, combined with $\left|\int_{1 \leq |\tau| < 2} e^{-iC_M 2^j \Gamma_M(2^k \tau)\widetilde{\xi}_2}\, d\tau\right| \lesssim 1$, further implies that

$$\left|\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)\right| \lesssim 2^{-k}.$$

Note that $|j| \lesssim k$, we have

$$\sum_{\text{Case 3}} \left|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)\right| \lesssim \sum_{\text{Case 3}} 2^{-k} \lesssim \sum_{k \geq 0} k 2^{-k} \lesssim 1$$

for all $|\widetilde{\xi}_1|, |\widetilde{\xi}_2| \in [1, 2)$.

Case 4: $k \geq 0$ and $\frac{1}{8}(2C_1)^{2k}\Gamma_M'(1) < \frac{1}{8}2^{-j}$. The method of establishing (2.6) in this case is different from these cases above. Here, $\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)$ plays a key role in this method, and we should consider $\left|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)\right|$, instead of considering $\left|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)\right|$ and $\left|\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)\right|$ separately. Indeed, we can rewrite $\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2) - \overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1, \widetilde{\xi}_2)$ as

$$\int_{1 \leq |t| < 2} e^{-i2^k t \widetilde{\xi}_1}\left(e^{-iC_M 2^j \Gamma_M(2^k t)\widetilde{\xi}_2} - 1\right) dt$$

$$- \int_{1 \leq |t| < 2} e^{-i2^k t \widetilde{\xi}_1} \int_{1 \leq |\tau| < 2} \left(e^{-iC_M 2^j \Gamma_M(2^k \tau)\widetilde{\xi}_2} - 1\right) d\tau dt.$$

By (2.2), Remark 1.4 and Theorem 1.2(i), we have

$$\big|e^{-iC_M2^j\Gamma_M(2^kt)\widetilde{\xi}_2}-1\big|\leq\big|C_M2^j\Gamma_M(2^kt)\widetilde{\xi}_2\big|\leq\frac{8C_1}{C_2+1}C_1'2^j(2C_1)^k,$$

$$\big|e^{-iC_M2^j\Gamma_M(2^k\tau)\widetilde{\xi}_2}-1\big|\leq\big|C_M2^j\Gamma_M(2^k\tau)\widetilde{\xi}_2\big|\leq\frac{8C_1}{C_2+1}C_1'2^j(2C_1)^k.$$

Furthermore, an application of $\sum_{j:2^j\lesssim(2C_1)^{-2k}}2^j\lesssim(2C_1)^{-2k}$ then shows that

$$\sum_{\text{Case 4}}\big|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)-\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)\big|\lesssim\sum_{\text{Case 4}}2^j(2C_1)^k\lesssim\sum_{k\geq0}(2C_1)^{-k}\lesssim1$$

for all $|\widetilde{\xi}_1|,|\widetilde{\xi}_2|\in[1,2)$.

Case 5: $k<0$ *and* $(2e^{C_2/2})^{k/2}\leq2^{-j}$. As in Case 4, we may then apply (2.1), (2.2) and Remark 1.4 to conclude that

$$\big|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)-\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)\big|\lesssim2^j\big(2e^{C_2/2}\big)^k.$$

Therefore, by $\sum_{j:2^j\lesssim(2e^{C_2/2})^{-k/2}}2^j\lesssim(2e^{C_2/2})^{-k/2}$, it follows that

$$\sum_{\text{Case 5}}\big|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)-\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)\big|\lesssim\sum_{\text{Case 5}}2^j\big(2e^{C_2/2}\big)^k\lesssim\sum_{k<0}\big(2e^{C_2/2}\big)^{k/2}\lesssim1$$

for all $|\widetilde{\xi}_1|,|\widetilde{\xi}_2|\in[1,2)$.

Case 6: $k<0$ *and* $(2C_1)^{2k}<2^{-j}<(2e^{C_2/2})^{k/2}$. In this case, we should rewrite $\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)-\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)$ as

$$\int_{1\leq|t|<2}\big(e^{-i2^kt\widetilde{\xi}_1}-1\big)e^{-iC_M2^j\Gamma_M(2^kt)\widetilde{\xi}_2}\,\mathrm{d}t$$

$$-\int_{1\leq|\tau|<2}e^{-iC_M2^j\Gamma_M(2^k\tau)\widetilde{\xi}_2}\int_{1\leq|t|<2}\big(e^{-i2^kt\widetilde{\xi}_1}-1\big)\,\mathrm{d}t\mathrm{d}\tau.$$

From $\big|e^{-i2^kt\widetilde{\xi}_1}-1\big|\leq\big|2^kt\widetilde{\xi}_1\big|\lesssim2^k$, this implies $\big|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)-\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)\big|\lesssim2^k$. Thus, noting the fact that $|j|\lesssim|k|$, we have

$$\sum_{\text{Case 6}}\big|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)-\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)\big|\lesssim\sum_{\text{Case 6}}2^k\lesssim\sum_{k<0}|k|2^k\lesssim1$$

for all $|\widetilde{\xi}_1|,|\widetilde{\xi}_2|\in[1,2)$.

Case 7: $k<0$ *and* $2^{-j}\leq(2C_1)^{2k}$. As in Case 1, by $\sum_{j:2^{-j}\leq(2C_1)^{2k}}2^{-j/2}\lesssim(2C_1)^k$ and (2.2), we have

$$\sum_{\text{Case 7}}\big|\mathfrak{m}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)-\overline{\mathfrak{m}}_{I,\gamma}^{j,k}(\widetilde{\xi}_1,\widetilde{\xi}_2)\big|\lesssim\sum_{\text{Case 7}}\big(2^j2^k\Gamma_M'(2^k)\big)^{-1/2}\lesssim\sum_{\text{Case 7}}\big(2^j(2C_1)^k\big)^{-1/2}$$

$$\lesssim\sum_{k<0}(2C_1)^{k/2}\lesssim1$$

for all $|\widetilde{\xi}_1|,|\widetilde{\xi}_2|\in[1,2)$.

Altogether, we obtain (2.4) with $p=2$.

2.2. $L^p(\mathbb{R}^2)$-boundedness

In Subsection 2.1, we have obtained the $L^2(\mathbb{R}^2)$-boundedness of $\sup_{j,k\in\mathbb{Z}}\left|(M_{I,\gamma}^{j,k}-\overline{M}_{I,\gamma}^{j,k})f\right|$. Therefore, the $L^p(\mathbb{R}^2)$-boundedness, $p\in(2,\infty)$, of $\sup_{j,k\in\mathbb{Z}}\left|(M_{I,\gamma}^{j,k}-\overline{M}_{I,\gamma}^{j,k})f\right|$ in (2.4) can be followed by interpolating with the trivial $L^\infty(\mathbb{R}^2)$-boundedness. The purpose of this subsection is to prove (2.4) for all $p\in(1,2)$ in terms of the method of bootstrapping an iterated interpolation argument in the spirit of Nagel, Stein and Wainger in [23].

As in Carlsson, Christ, Córdoba et al. [7], we should first describe a Littlewood–Paley decomposition. Let us set

$$\alpha_k := \gamma'(2^k) \quad \text{and} \quad \widehat{\mathbb{P}_k f}(\xi_1,\xi_2) := \Phi_k(\xi_1,\xi_2)\widehat{f}(\xi_1,\xi_2),$$

where

$$\operatorname{supp}\Phi_k \subseteq \left\{(\xi_1,\xi_2)\in\mathbb{R}^2 : \alpha_{k-1}\le|\xi_1/\xi_2|\le\alpha_{k+1}\right\}.$$

From [7], we also have

$$(2.8) \qquad\qquad \sum_{k\in\mathbb{Z}}\mathbb{P}_k f = f$$

and

$$(2.9) \qquad\qquad \left\|\left(\sum_{k\in\mathbb{Z}}|\mathbb{P}_k f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p\in(1,\infty)$.

On the other hand, for any given $k\in\mathbb{Z}$, let us define a function $\psi_k(j)$ satisfying

$$(2.10) \qquad\qquad 2^j\gamma'(2^k) = \gamma'(2^{k+\psi_k(j)}).$$

It is easy to see that ψ_k is monotonic and $\psi_k(0)=0$, which leads to the fact that there exists the inverse function $(\psi_k)^{-1}$ of ψ_k. From (2.8), we write

$$\sup_{j,k\in\mathbb{Z}}\left|(M_{I,\gamma}^{j,k}-\overline{M}_{I,\gamma}^{j,k})f\right| = \sup_{j,k\in\mathbb{Z}}\left|(M_{I,\gamma}^{j,k}-\overline{M}_{I,\gamma}^{j,k})\left(\sum_{l\in\mathbb{Z}}\mathbb{P}_{l+k+\psi_k(j)}f\right)\right|.$$

Therefore, for (2.4), by the triangle inequality, it is enough to prove that there exists a positive constant α such that

$$\left\|\sup_{j,k\in\mathbb{Z}}\left|(M_{I,\gamma}^{j,k}-\overline{M}_{I,\gamma}^{j,k})(\mathbb{P}_{l+k+\psi_k(j)}f)\right|\right\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\alpha|l|}\|f\|_{L^p(\mathbb{R}^2)}$$

for all $p\in(1,2)$, which is equivalent to

$$\left\|\sup_{j,k\in\mathbb{Z}}\left|(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}-\overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k})(\mathbb{P}_j f)\right|\right\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\alpha|l|}\|f\|_{L^p(\mathbb{R}^2)}.$$

Furthermore, by (2.9), it suffices to obtain

$$\left\|\left[\sum_{j\in\mathbb{Z}}\sup_{k\in\mathbb{Z}}\left|\left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}-\overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)(\mathbb{P}_jf)\right|^2\right]^{1/2}\right\|_{L^p(\mathbb{R}^2)}$$

$$\lesssim 2^{-\alpha|l|}\left\|\left(\sum_{j\in\mathbb{Z}}|\mathbb{P}_jf|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^2)}$$

for all $p\in(1,2)$. By interpolation, we need to establish only the following estimates

$$(2.11)\quad \left\|\left[\sum_{j\in\mathbb{Z}}\sup_{k\in\mathbb{Z}}\left|\left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}-\overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)(\mathbb{P}_jf)\right|^2\right]^{1/2}\right\|_{L^2(\mathbb{R}^2)}$$
$$\lesssim 2^{-\alpha|l|}\left\|\left(\sum_{j\in\mathbb{Z}}|\mathbb{P}_jf|^2\right)^{1/2}\right\|_{L^2(\mathbb{R}^2)}$$

and

$$(2.12)\quad \left\|\left[\sum_{j\in\mathbb{Z}}\sup_{k\in\mathbb{Z}}\left|\left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}-\overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)(\mathbb{P}_jf)\right|^2\right]^{1/2}\right\|_{L^p(\mathbb{R}^2)}$$
$$\lesssim \left\|\left(\sum_{j\in\mathbb{Z}}|\mathbb{P}_jf|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^2)}$$

for all $p\in(1,2)$.

Proposition 2.1. *We have*

(1) (2.11) *followed from*

(2.13)
$$\left\|\sup_{k\in\mathbb{Z}}\left|\left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}-\overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)(\mathbb{P}_jf)\right|\right\|_{L^2(\mathbb{R}^2)}\lesssim 2^{-\alpha|l|}\|\mathbb{P}_jf\|_{L^2(\mathbb{R}^2)}.$$

(2) (2.12) *followed from*

$$(2.14)\quad \left\|\sup_{k\in\mathbb{Z}}\left|\left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}-\overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)(\mathbb{P}_jf)\right|\right\|_{L^p(\mathbb{R}^2)}\lesssim\|\mathbb{P}_jf\|_{L^p(\mathbb{R}^2)}$$

for all $p\in(1,\infty)$.

Proof. For $(2.13)\Rightarrow(2.11)$, which comes from the commutative property between $l^2(\mathbb{Z})$ and $L^2(\mathbb{R}^2)$ norms. For $(2.14)\Rightarrow(2.12)$, we first consider the more general estimate

$$(2.15)\quad \left\|\left[\sum_{j\in\mathbb{Z}}\sup_{k\in\mathbb{Z}}\left|\left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}-\overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)(\mathbb{P}_jf)\right|^{q_1}\right]^{1/q_1}\right\|_{L^{q_2}(\mathbb{R}^2)}$$
$$\lesssim \left\|\left(\sum_{j\in\mathbb{Z}}|\mathbb{P}_jf|^{q_1}\right)^{1/q_1}\right\|_{L^{q_2}(\mathbb{R}^2)}$$

for certain $q_1 \in (1, \infty]$ and $q_2 \in (1, \infty)$.

For the case of $q_1 = \infty$ and $q_2 = 2$, by (2.9), it follows that

$$\left\| \sup_{j \in \mathbb{Z}} |\mathbb{P}_j f| \right\|_{L^p(\mathbb{R}^2)} \leq \left\| \left(\sum_{l \in \mathbb{Z}} |\mathbb{P}_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$. Recall that in Subsection 2.1, we have obtained

$$\left\| \sup_{j,k \in \mathbb{Z}} M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}$$

for all $f \in L^2(\mathbb{R}^2)$. Therefore,

(2.16)
$$\left\| \sup_{j,k \in \mathbb{Z}} M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} (\mathbb{P}_j f) \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| \sup_{j,k \in \mathbb{Z}} M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} \left(\sup_{j \in \mathbb{Z}} |\mathbb{P}_j f| \right) \right\|_{L^2(\mathbb{R}^2)}$$
$$\lesssim \left\| \sup_{j \in \mathbb{Z}} |\mathbb{P}_j f| \right\|_{L^2(\mathbb{R}^2)}.$$

From (2.3), we also have

(2.17)
$$\left\| \sup_{j,k \in \mathbb{Z}} \overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} (\mathbb{P}_j f) \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| M^{(1)} M^{(2)} \left(\sup_{j \in \mathbb{Z}} |\mathbb{P}_j f| \right) \right\|_{L^2(\mathbb{R}^2)}$$
$$\lesssim \left\| \sup_{j \in \mathbb{Z}} |\mathbb{P}_j f| \right\|_{L^2(\mathbb{R}^2)}.$$

Thus, from (2.16) and (2.17), we obtain (2.15) in the case of $q_1 = \infty$ and $q_2 = 2$.

For the case of $q_1 = q_2$ and $q_1, q_2 \in (1, \infty)$, because the $l^p(\mathbb{Z})$ and $L^p(\mathbb{R}^2)$ norms commute, we have that (2.15) is equivalent to (2.14). Then, by the interpolation argument, we may obtain that (2.12) holds for all $p \in (4/3, 2)$. Repeating the interpolation argument and using $q_1 = \infty$ and $q_2 \in (4/3, 2)$, we may also obtain that (2.12) holds for all $p \in (8/7, 2)$. Reiterating this process sufficiently many times we can show that (2.12) holds for all $p \in (1, 2)$. Hence, we obtain (2.14) $\Rightarrow$ (2.12). $\square$

Therefore, it is enough to prove (2.13) and (2.14). Indeed, since the $L^p(\mathbb{R}^2)$-boundedness of $\sup_{j,k \in \mathbb{Z}} \left| (M_{I,\gamma}^{j,k} - \overline{M}_{I,\gamma}^{j,k}) f \right|$ has been obtained for all $p \in [2, \infty]$, we can restrict $p \in (1, 2)$ in (2.14). We first bound the left-hand side of (2.14) by

$$\left\| \left[\sum_{k \in \mathbb{Z}} \left| \left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} - \overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} \right) (\mathbb{P}_j f) \right|^2 \right]^{1/2} \right\|_{L^p(\mathbb{R}^2)},$$

and define a finer frequency decomposition operator $\mathbb{P}_{j,-k+m}$ by restricting the frequency region to

$$\left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : 2^{-k+m-1} \leq |\xi_1|, \gamma'(2^j)|\xi_2| \leq 2^{-k+m+1} \right\}.$$

Furthermore, we can write

$$\mathbb{P}_j f = \sum_{m \in \mathbb{Z}} \mathbb{P}_{j,-k+m} f.$$

Thus, for (2.14), by the triangle inequality, it suffices to prove that there exists a positive constant β such that

$$
(2.18) \quad
\left\| \left[\sum_{k \in \mathbb{Z}} \left| \left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} - \overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} \right) (\mathbb{P}_{j,-k+m} f) \right|^2 \right]^{1/2} \right\|_{L^p(\mathbb{R}^2)}
$$
$$
\lesssim 2^{-\beta|m|} \|\mathbb{P}_j f\|_{L^p(\mathbb{R}^2)}
$$

for all $p \in (1,2)$. As a result, we obtain $(2.18) \Rightarrow (2.14)$.

Proposition 2.2. *We have that* (2.13) *and* (2.18) *can be followed from*

$$
(2.19) \quad
\left\| \left[\sum_{k \in \mathbb{Z}} \left| \left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} - \overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} \right) (\mathbb{P}_{j,-k+m} f) \right|^2 \right]^{1/2} \right\|_{L^2(\mathbb{R}^2)}
$$
$$
\lesssim 2^{-\alpha|l|} 2^{-\beta|m|} \|\mathbb{P}_j f\|_{L^2(\mathbb{R}^2)}
$$

for some positive constants α and β.

Proof. For $(2.19) \Rightarrow (2.13)$, which can be proved as $(2.18) \Rightarrow (2.14)$. For $(2.19) \Rightarrow (2.18)$, by interpolation and the Littlewood–Paley theory, it suffices to obtain

$$
(2.20) \quad
\left\| \left[\sum_{k \in \mathbb{Z}} \left| \left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} - \overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} \right) (\mathbb{P}_{j,-k+m} f) \right|^2 \right]^{1/2} \right\|_{L^p(\mathbb{R}^2)}
$$
$$
\lesssim \left\| \left(\sum_{k \in \mathbb{Z}} |\mathbb{P}_{j,-k+m} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)}
$$

for all $p \in (1,2)$. Similarly, we consider the more general estimate

$$
(2.21) \quad
\left\| \left[\sum_{k \in \mathbb{Z}} \left| \left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} - \overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} \right) (\mathbb{P}_{j,-k+m} f) \right|^{q_1} \right]^{1/q_1} \right\|_{L^{q_2}(\mathbb{R}^2)}
$$
$$
\lesssim \left\| \left(\sum_{k \in \mathbb{Z}} |\mathbb{P}_{j,-k+m} f|^{q_1} \right)^{1/q_1} \right\|_{L^{q_2}(\mathbb{R}^2)}
$$

for certain $q_1 \in (1,\infty]$ and $q_2 \in (1,\infty)$.

As in (2.16) and (2.17), we have that (2.21) can be established for the case of $q_1 = \infty$ and $q_2 = 2$. On the other hand, for the case of $q_1 = q_2$ and $q_1, q_2 \in (1,\infty)$, it is easy to see that (2.21) is equivalent to

$$
\left\| \left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} - \overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} \right) (\mathbb{P}_{j,-k+m} f) \right\|_{L^{q_2}(\mathbb{R}^2)} \lesssim \|\mathbb{P}_{j,-k+m} f\|_{L^{q_2}(\mathbb{R}^2)}
$$

for all $q_2 \in (1,\infty)$, which is a trivial consequence of Minkowski's inequality. Therefore, as in (2.12), we obtain (2.20), which leads to $(2.19) \Rightarrow (2.18)$. $\qquad \square$

In conclusion, we will only prove (2.19). For this aim, it is sufficient to show the following estimate

$$\left\|\left(M_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k} - \overline{M}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)(\mathbb{P}_{j,-k+m}f)\right\|_{L^2(\mathbb{R}^2)} \lesssim 2^{-\alpha|l|}2^{-\beta|m|}\|f\|_{L^2(\mathbb{R}^2)}.$$

This, by Plancherel's theorem, can be followed from

$$(2.22) \qquad \left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2) - \overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 2^{-\alpha|l|}2^{-\beta|m|}$$

for all $(\xi_1,\xi_2) \in \mathbb{R}^2$ satisfying

$$\alpha_{j-1} \le |\xi_1/\xi_2| \le \alpha_{j+1} \quad \text{and} \quad 2^{-k+m-1} \le |\xi_1|, \ \gamma'(2^j)|\xi_2| \le 2^{-k+m+1}.$$

On the other hand, we may assume that $|l|$ and $|m|$ large enough, since we have the following trivial estimates

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 1 \quad \text{and} \quad \left|\overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 1$$

for all $(\xi_1,\xi_2) \in \mathbb{R}^2$.

To prove (2.22), we consider the following five cases according to the values of m and l.

Case I: $m \ge 0$ *and* $l \le \frac{m}{2\log_2 C_1}$. From Theorem 1.2(ii) and (2.10), this implies that

$$\left|\left(\varphi_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)''(t)\right| = \left|2^{(\psi_k)^{-1}(j-l-k)}\frac{\gamma''(2^kt)2^kt}{\gamma'(2^kt)}\frac{\gamma'(2^kt)}{2^kt}2^{2k}\xi_2\right| \gtrsim 2^m\frac{\gamma'(2^{j-l})}{\gamma'(2^j)}.$$

By Theorem 1.2(i) and (2.1), we obtain

$$\begin{cases} (C_1)^{-l} \le \frac{\gamma'(2^{j-l})}{\gamma'(2^j)} \le e^{-\frac{C_2}{2}l} & \text{for } l > 0, \\ e^{-\frac{C_2}{2}l} \le \frac{\gamma'(2^{j-l})}{\gamma'(2^j)} \le (C_1)^{-l} & \text{for } l < 0, \end{cases}$$

which leads to

$$\left|\left(\varphi_{I,\gamma}^{j,k}\right)''(t)\right| \gtrsim \begin{cases} 2^m(C_1)^{-l} & \text{for } l > 0, \\ 2^me^{-\frac{C_2}{2}l} & \text{for } l < 0. \end{cases}$$

By van der Corput's lemma, we obtain

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim \begin{cases} 2^{-m/2}(C_1)^{l/2} & \text{for } l > 0, \\ 2^{-m/2}e^{\frac{C_2}{4}l} & \text{for } l < 0. \end{cases}$$

This boundedness can also be obtained for $\overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)$. Therefore,

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2) - \overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim \begin{cases} 2^{-m/2}(C_1)^{l/2} & \text{for } l > 0, \\ 2^{-m/2}e^{\frac{C_2}{4}l} & \text{for } l < 0. \end{cases}$$

To obtain (2.22) in this case, two different situations can arise.

◊ For $l > 0$, since $2^{-m/2}(C_1)^{l/2} = 2^{-m/8}2^{-3m/8}(C_1)^{l/2} \leq 2^{-m/8}(C_1)^{-1/4}$, we have (2.22) established with $\alpha := \frac{1}{4}\log_2 C_1$ and $\beta := 1/8$.

◊ For $l < 0$, we have (2.22) established with $\alpha := \frac{C_2}{4}\log_2 e$ and $\beta := 1/2$.

Case II: $m \geq 0$ *and* $\frac{m}{2\log_2 C_1} < l < \frac{4m}{C_2\log_2 e}$. From Theorem 1.2(i) and (2.10), we may conclude that

$$\left|\left(\varphi_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)'(t)\right| \geq 2^k|\xi_1| - \left|2^{(\psi_k)^{-1}(j-l-k)}\gamma'(2^k t)2^k\xi_2\right| \geq 2^m\left(\frac{1}{2} - 2C_1 e^{-\frac{C_2}{2}l}\right) \gtrsim 2^m$$

for l and m large enough. Notice the fact that $\left(\varphi_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)'(t)$ is monotonic on $[1,2)$ or $(-2,-1]$, by van der Corput's lemma, we may obtain

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 2^{-m}.$$

On the other hand, we also have that $\left|\overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right|$ can be bounded by 2^{-m}. Therefore,

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2) - \overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 2^{-m} \leq 2^{-m/2}2^{-\frac{C_2\log_2 e}{8}l},$$

which proves (2.22) with $\alpha := \frac{C_2\log_2 e}{8}$ and $\beta := 1/2$.

Case III: $m \geq 0$ *and* $l \geq \frac{4m}{C_2\log_2 e}$, *or* $m < 0$ *and* $\frac{m}{2\log_2 C_1} \leq l$. As in Case 4 in Subsection 2.1, by (2.2) and Theorem 1.2(i), we see that

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2) - \overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 2^m\frac{\gamma'(2^{j-l})}{\gamma'(2^j)} \lesssim \begin{cases} 2^m e^{-\frac{C_2}{2}l} & \text{for } l > 0, \\ 2^m(C_1)^{-l} & \text{for } l < 0. \end{cases}$$

For obtaining (2.22) in this case, we consider the following three cases.

◊ For the case of $m \geq 0$ and $l \geq \frac{4m}{C_2\log_2 e}$, note that $2^m e^{-\frac{C_2}{2}l} = 2^m e^{-\frac{3C_2}{8}l}e^{-\frac{C_2}{8}l} \leq 2^{-m/2}e^{-\frac{C_2}{8}l}$, we then establish (2.22) with $\alpha := \frac{C_2\log_2 e}{8}$ and $\beta := 1/2$.

◊ For the case of $m < 0$ and $l > 0$, it is easy to see that (2.22) established with $\alpha := \frac{C_2\log_2 e}{2}$ and $\beta := 1$.

◊ For the case of $m < 0$ and $\frac{m}{2\log_2 C_1} \leq l \leq 0$, we may take $|l|$ and $|m|$ large enough. Noting that $2^m(C_1)^{-l} = 2^{\frac{m}{4}}2^{\frac{3m}{4}}(C_1)^{-l} \leq 2^{\frac{m}{4}}(C_1)^{l/2}$, we then establish (2.22) with $\alpha := \frac{\log_2 C_1}{2}$ and $\beta := 1/4$.

Case IV: $m < 0$ *and* $\frac{4m}{C_2\log_2 e} \leq l \leq \frac{m}{2\log_2 C_1}$. As in Case 6 in Subsection 2.1, we have

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2) - \overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 2^m \leq 2^{m/2}e^{\frac{C_2}{8}l}.$$

We then establish (2.22) with $\alpha := \frac{C_2 \log_2 e}{8}$ and $\beta := 1/2$.

Case V: $m < 0$ *and* $l < \frac{4m}{C_2 \log_2 e}$. From Theorem 1.2(i) and (2.10), this implies that

$$\left|\left(\varphi_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}\right)'(t)\right| \geq \left|2^{(\psi_k)^{-1}(j-l-k)}\gamma'(2^k t)2^k \xi_2\right| - 2^k|\xi_1|$$

$$\geq 2^m\left(\frac{e^{-\frac{C_2}{2}l}}{2} - 2\right) \gtrsim 2^m e^{-\frac{C_2}{2}l}$$

for $|l|$ and $|m|$ large enough. We notice that $(\varphi_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k})'(t)$ is monotonic on $[1,2)$ or $(-2,-1]$, van der Corput's lemma yields

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 2^{-m}e^{\frac{C_2}{2}l}.$$

On the other hand, we also have $\left|\overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right|$ can be bounded by $2^{-m}e^{\frac{C_2}{2}l}$, which further leads to

$$\left|m_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2) - \overline{m}_{I,\gamma}^{(\psi_k)^{-1}(j-l-k),k}(\xi_1,\xi_2)\right| \lesssim 2^{-m}e^{\frac{C_2}{2}l} \leq 2^{m/2}e^{\frac{C_2}{8}l}.$$

We then establish (2.22) with $\alpha := \frac{C_2 \log_2 e}{8}$ and $\beta := 1/2$.

In total, we obtain (2.22) with $\alpha := \frac{C_2 \log_2 e}{8}$ and $\beta := 1/8$. This finishes the proof of Theorem 1.2.

3. Proof of Theorem 1.3

For $M_{II,\gamma}$, the method applied in $M_{I,\gamma}$ is no longer available, and therefore we must find a different method to prove Theorem 1.3. We first restrict $p \in (1,\infty)$ based on the trivial $L^\infty(\mathbb{R}^2)$-boundedness of $M_{II,\gamma}$, and also introduce some simple properties of the curve γ. Indeed, from Theorem 1.3(i) and (2.2), we have

$$(3.1) \qquad C_2 + 1 \leq \frac{t\gamma'(t)}{\gamma(t)} \leq C_1' \quad \text{for any } t \in (0,\infty).$$

As in (2.1), if consider $F(t) := \ln\gamma(t)$ for any $t \in (0,\infty)$, by (3.1), we have

$$(3.2) \qquad e^{(C_2+1)/2} \leq \frac{\gamma(2t)}{\gamma(t)} \leq e^{C_1'} \quad \text{for any } t \in (0,\infty).$$

As in $M_{I,\gamma}$, we may assume that f is non-negative. Let $\phi \colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $\{t \in \mathbb{R} : 1/2 \leq |t| \leq 2\}$ with the properties that $0 \leq \phi(t) \leq 1$ and $\sum_{l \in \mathbb{Z}} \phi_l(t) = 1$ for any $t \neq 0$, where $\phi_l(t) := \phi(2^{-l}t)$. Here and hereafter, for any $l \in \mathbb{Z}$, we denote $P_l^{(1)}$ and $P_l^{(2)}$ as the *Littlewood–Paley projection corresponding to* ϕ_l *in the first variable and the second variable*, respectively,

$$P_l^{(1)}f(x_1,x_2) := \int_{-\infty}^{\infty} f(x_1 - w, x_2)\check{\phi}_l(w)\,\mathrm{d}w,$$

$$P_l^{(2)}f(x_1,x_2) := \int_{-\infty}^{\infty} f(x_1, x_2 - w)\check{\phi}_l(w)\,\mathrm{d}w.$$

For any given $l \in \mathbb{Z}$, set

$$M_{II,l}^{i,j} f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1 - 2^i t, x_2 - 2^j \gamma(t)) \phi_l(t) \, \frac{\mathrm{d}t}{|t|}.$$

Then, as in $M_{I,\gamma}$, it suffices to prove the $L^p(\mathbb{R}^2)$-boundedness of the operator $\sup_{i,j,l \in \mathbb{Z}} M_{II,l}^{i,j}$. Furthermore, for simplicity, we let $z := (x_1, x_2)$, $i_z := i(x_1, x_2)$, $j_z := j(x_1, x_2)$ and $l_z := l(x_1, x_2)$ be measurable functions. By linearization, it suffices to prove the $L^p(\mathbb{R}^2)$-boundedness of the operator $M_{II,l_z}^{i_z,j_z}$.

Note that γ is strictly increasing on $\mathbb{R}^+$ with $\gamma(\mathbb{R}^+) = \mathbb{R}^+$. For these $j_z, l_z \in \mathbb{Z}$, we denote by $k_z \colon \mathbb{R}^2 \to \mathbb{R}$ the measurable function satisfying

$$(3.3) \qquad\qquad\qquad\qquad 2^{k_z} 2^{j_z} \gamma(2^{l_z}) = 1.$$

We then write $M_{II,l_z}^{i_z,j_z} f = \sum_{k \in \mathbb{Z}} M_{II,l_z}^{i_z,j_z} P_k^{(2)} f$, and further split it as the sum of the following *low-frequency part* $M_{II_a,l_z}^{i_z,j_z} f$ and the *high-frequency part* $M_{II_b,l_z}^{i_z,j_z} f$, where

$$M_{II_a,l_z}^{i_z,j_z} f := \sum_{k \le k_z} M_{II,l_z}^{i_z,j_z} P_k^{(2)} f \quad \text{and} \quad M_{II_b,l_z}^{i_z,j_z} f := \sum_{k > k_z} M_{II,l_z}^{i_z,j_z} P_k^{(2)} f.$$

Consider $M_{II_a,l_z}^{i_z,j_z} f$. We compare it with the following operator

$$\mathbb{M}_{II_a,l_z}^{i_z,j_z} f(x_1, x_2) := \sum_{k \le k_z} \int_{-\infty}^{\infty} P_k^{(2)} f(x_1 - 2^{i_z} t, x_2) \phi_{l_z}(t) \, \frac{\mathrm{d}t}{|t|}.$$

It is easy to see that $\left| \mathbb{M}_{II_a,l_z}^{i_z,j_z} f \right|$ can be bounded by $M^{(1)} \left(\sum_{k \le k_z} P_k^{(2)} f \right)$. Note that the operator $\sum_{k \le k_z} P_k^{(2)} f$ is bounded on $L^p(\mathbb{R}^2)$ for any given $p \in (1, \infty)$ by multiplier theory, then we can use the $L^p(\mathbb{R}^2)$-boundedness of $M^{(1)}$ to get

$$(3.4) \qquad\qquad\qquad \left\| \mathbb{M}_{II_a,l_z}^{i_z,j_z} f \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$. The difference between $M_{II_a,l_z}^{i_z,j_z} f(z)$ and $\mathbb{M}_{II_a,l_z}^{i_z,j_z} f(z)$ can be written as

$$(3.5) \qquad \sum_{k \le k_z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1 - 2^{i_z} t, x_2 - w) \left(\check{\phi}_k(w - 2^{j_z} \gamma(t)) - \check{\phi}_k(w) \right) \mathrm{d}w \, \phi_{l_z}(t) \, \frac{\mathrm{d}t}{|t|}.$$

By the Lagrange mean value theorem, we have

$$(3.6) \qquad\qquad\qquad \left| \check{\phi}_k(w - 2^{j_z} \gamma(t)) - \check{\phi}_k(w) \right| \lesssim 2^{j_z} |\gamma(t)| 2^{2k} 2^{-2n}$$

if $2^{j_z} |\gamma(t)| \le 2^{-k}$ and w is in the annulus $2^{-k+n-1} \le |w| \le 2^{-k+n}$ for $n \in \mathbb{N}$. For $n = 0$ the above estimate holds for all $|w| \le 2^{-k}$. For $t \in \operatorname{supp} \psi_{l_z}$, from $k \le k_z$, (3.2) and (3.3),

we have $2^{j_z}|\gamma(t)| \leq 2^{-k}$, which further implies (3.6). Noting $\frac{\gamma(t)}{t}$ is increasing on $\mathbb{R}^+$, (3.2) and (3.3), we then bound (3.5) by

$$\sum_{n\in\mathbb{N}}\sum_{k\leq k_z}\int_{-\infty}^{\infty}\int_{|w|\leq 2^{-k+n}}|f(x_1-2^{i_z}t,x_2-w)|\cdot 2^{j_z}|\gamma(t)|2^{2k}2^{-2n}\,\mathrm{d}w\,\phi_{l_z}(t)\frac{\mathrm{d}t}{|t|}$$

$$\lesssim \sum_{n\in\mathbb{N}}2^{-n}\sum_{k\leq k_z}2^k2^{j_z}\gamma(2^{l_z})\frac{1}{2^{l_z}}\int_{|t|\lesssim 2^{l_z}}\frac{1}{2^{-k+n}}\int_{|w|\leq 2^{-k+n}}|f(x_1-2^{i_z}t,x_2-w)|\,\mathrm{d}w\mathrm{d}t$$

$$\lesssim M^{(1)}M^{(2)}f(x_1,x_2).$$

From the $L^p(\mathbb{R})$-boundedness of $M^{(1)}$ and $M^{(2)}$, which trivially leads to the estimate

$$\left\|M_{II_a,l_z}^{i_z,j_z}f - \mathbb{M}_{II_a,l_z}^{i_z,j_z}f\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1,\infty)$. This, combined with (3.4), leads to

$$\left\|M_{II_a,l_z}^{i_z,j_z}f\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1,\infty)$.

Consider $M_{II_b,l_z}^{i_z,j_z}f$. We rewrite

$$M_{II_b,l_z}^{i_z,j_z}f = \sum_{k>0}M_{II,l_z}^{i_z,j_z}P_{k+k_z}^{(2)}f = \sum_{m\in\mathbb{Z}}\sum_{k>0}M_{II,l_z}^{i_z,j_z}P_m^{(1)}P_{k+k_z}^{(2)}f.$$

For these $i_z, l_z \in \mathbb{Z}$, we let $m_z\colon \mathbb{R}^2 \to \mathbb{R}$ be a measurable function satisfying

$$2^{m_z}2^{i_z}2^{l_z} = 1.$$

Then, we can split

$$M_{II_b,l_z}^{i_z,j_z}f = \sum_{m\leq m_z}\sum_{k>0}M_{II,l_z}^{i_z,j_z}P_m^{(1)}P_{k+k_z}^{(2)}f + \sum_{m>m_z}\sum_{k>0}M_{II,l_z}^{i_z,j_z}P_m^{(1)}P_{k+k_z}^{(2)}f$$

$$=: M_{II_b^1,l_z}^{i_z,j_z}f + M_{II_b^2,l_z}^{i_z,j_z}f.$$

Consider $M_{II_b^1,l_z}^{i_z,j_z}f$. As in $M_{II_a,l_z}^{i_z,j_z}f$, we will compare it with

$$\mathbb{M}_{II_b^1,l_z}^{i_z,j_z}f(x_1,x_2) := \sum_{m\leq m_z}\sum_{k>0}\int_{-\infty}^{\infty}P_m^{(1)}P_{k+k_z}^{(2)}f(x_1,x_2-2^{j_z}\gamma(t))\phi_{l_z}(t)\frac{\mathrm{d}t}{|t|}.$$

After changing of variable $2^{j_z}\gamma(t) =: w$, we use (3.1) and the fact that $(\gamma^{-1})'(t)\gamma'(\gamma^{-1}(t)) = 1$ to establish

$$\frac{1}{\gamma'\big(\gamma^{-1}(w/2^{j_z})\big)} = \frac{\gamma\big(\gamma^{-1}(w/2^{j_z})\big)}{\gamma'\big(\gamma^{-1}(w/2^{j_z})\big)\gamma^{-1}(w/2^{j_z})}\frac{\gamma^{-1}(w/2^{j_z})}{w/2^{j_z}} \lesssim \frac{\gamma^{-1}(w/2^{j_z})}{w/2^{j_z}}.$$

Then we have the following pointwise estimate

$$\left| \mathbb{M}_{II_b^1, l_z}^{i_z, j_z} f(x_1, x_2) \right| \lesssim M^{(2)} \left(\sum_{m \le m_z} \sum_{k>0} P_m^{(1)} P_{k+k_z}^{(2)} f \right)(x_1, x_2),$$

which trivially leads to

$$(3.7) \qquad\qquad \left\| \mathbb{M}_{II_b^1, l_z}^{i_z, j_z} f \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$. As the difference between $M_{II_a, l_z}^{i_z, j_z} f(z)$ and $\mathbb{M}_{II_a, l_z}^{i_z, j_z} f(z)$, we can bound $\left| M_{II_b^1, l_z}^{i_z, j_z} f(z) - \mathbb{M}_{II_b^1, l_z}^{i_z, j_z} f(z) \right|$ by $M^{(1)} M^{(2)} \left(\sum_{k>0} P_{k+k_z}^{(2)} f \right)(z)$, which further implies the $L^p(\mathbb{R})$-boundedness of the difference between $M_{II_b^1, l_z}^{i_z, j_z} f(z)$ and $\mathbb{M}_{II_b^1, l_z}^{i_z, j_z} f(z)$. Hence, from (3.7), we get

$$\left\| M_{II_b^1, l_z}^{i_z, j_z} f \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$.

Consider $M_{II_b^2, l_z}^{i_z, j_z} f$**.** We rewrite

$$M_{II_b^2, l_z}^{i_z, j_z} f = \sum_{m>0} \sum_{k>0} M_{II, l_z}^{i_z, j_z} P_{m+m_z}^{(1)} P_{k+k_z}^{(2)} f.$$

By interpolation, it suffices to prove that there exists a positive constant δ such that

$$(3.8) \qquad\qquad \left\| M_{II, l_z}^{i_z, j_z} P_{m+m_z}^{(1)} P_{k+k_z}^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\delta(m+k)} \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in [2, \infty)$, and

$$(3.9) \qquad\qquad \left\| M_{II, l_z}^{i_z, j_z} P_{m+m_z}^{(1)} P_{k+k_z}^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, 2)$.

Proof of (3.8). The expression inside the $L^p(\mathbb{R}^2)$-norm on the left-hand side of (3.8) can be dominated by

$$\left(\sum_{M, K \in \mathbb{Z}} \left| M_{II, l_z}^{i_z, j_z} P_{m+M}^{(1)} P_{k+K}^{(2)} f \right|^p \right)^{1/p},$$

where $2^M 2^{i_z} 2^{l_z} = 1$ and $2^K 2^{j_z} \gamma(2^{l_z}) = 1$. Noting that

$$\left\| \left(\sum_{M, K \in \mathbb{Z}} \left| P_{m+M}^{(1)} P_{k+K}^{(2)} f \right|^p \right)^{1/p} \right\|_{L^p(\mathbb{R}^2)} \lesssim \left\| \left(\sum_{M, K \in \mathbb{Z}} \left| P_{m+M}^{(1)} P_{k+K}^{(2)} f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)}$$

$$\lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in [2, \infty)$, and the $l^p(\mathbb{Z}^2)$ and $L^p(\mathbb{R}^2)$ norms commute, we prove only

$$(3.10) \qquad\qquad \left\| M_{II, l_z}^{i_z, j_z} P_{m+M}^{(1)} P_{k+K}^{(2)} f \right\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\delta(m+k)} \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in [2, \infty)$. Furthermore, if we bound the expression inside the $L^p(\mathbb{R}^2)$-norm on the left-hand side of (3.10) by

$$\left(\sum_{l \in \mathbb{Z}} \big| M_{II,l}^{i,j} P_{m+M}^{(1)} P_{k+K}^{(2)} f \big|^p \right)^{1/p},$$

where $2^M 2^i 2^l = 1$ and $2^K 2^j \gamma(2^l) = 1$, it is enough to show

$$(3.11) \qquad \big\| M_{II,l}^{i,j} P_{m+M}^{(1)} P_{k+K}^{(2)} f \big\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-\delta(m+k)} \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in [2, \infty)$. By Minkowski's inequality, then the following identity is valid,

$$(3.12) \qquad \big\| M_{II,l}^{i,j} P_{m+M}^{(1)} P_{k+K}^{(2)} f \big\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$. Therefore, by interpolating with (3.12), to prove (3.11) we need to show the following estimate

$$(3.13) \qquad \big\| M_{II,l}^{i,j} P_{m+M}^{(1)} P_{k+K}^{(2)} f \big\|_{L^2(\mathbb{R}^2)} \lesssim 2^{-\delta(m+k)} \|f\|_{L^2(\mathbb{R}^2)}.$$

To get (3.13), let us set $\Gamma_l(t) := \frac{\gamma(2^l t)}{\gamma(2^l)}$, and rewrite

$$M_{II,l}^{i,j} P_{m+M}^{(1)} P_{k+K}^{(2)} f(x_1, x_2) = \int_{-\infty}^{\infty} P_{m+M}^{(1)} P_{k+K}^{(2)} f(x_1 - 2^i 2^l t, x_2 - 2^j \gamma(2^l) \Gamma_l(t)) \phi(t) \frac{\mathrm{d}t}{|t|}.$$

Furthermore, let

$$T^l f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \Gamma_l(t)) \phi(t) \frac{\mathrm{d}t}{|t|},$$

and for any given real numbers a, b, define

$$\Delta_{a,b} f(x_1, x_2) := f(2^a x_1, 2^b x_2).$$

Therefore, it is easy to see that

$$M_{II,l}^{i,j} P_{m+M}^{(1)} P_{k+K}^{(2)} f = \Delta_{-i-l, -j-\log_2 \gamma(2^l)} T^l \Delta_{i+l, j+\log_2 \gamma(2^l)} P_{m+M}^{(1)} P_{k+K}^{(2)} f.$$

Note that $2^M 2^i 2^l = 1$ and $2^K 2^j \gamma(2^l) = 1$, then

$$\Delta_{i+l, j+\log_2 \gamma(2^l)} P_{m+M}^{(1)} P_{k+K}^{(2)} f = P_{m+M+i+l}^{(1)} P_{k+K+j+\log_2 \gamma(2^l)}^{(2)} \Delta_{i+l, j+\log_2 \gamma(2^l)} f$$
$$= P_m^{(1)} P_k^{(2)} \Delta_{i+l, j+\log_2 \gamma(2^l)} f.$$

So we can write

$$M_{II,l}^{i,j} P_{m+M}^{(1)} P_{k+K}^{(2)} f = \Delta_{-i-l, -j-\log_2 \gamma(2^l)} T^l P_m^{(1)} P_k^{(2)} \Delta_{i+l, j+\log_2 \gamma(2^l)} f.$$

Since conjugation by $\Delta_{a,b}$ is an $L^p(\mathbb{R}^2)$ isometry, we then have

$$\big\|M_{II,l}^{i,j}P_{m+M}^{(1)}P_{k+K}^{(2)}f\big\|_{L^2(\mathbb{R}^2)} = \big\|T^l P_m^{(1)}P_k^{(2)}f\big\|_{L^2(\mathbb{R}^2)}.$$

Let the multiplier of $T^l P_m^{(1)}P_k^{(2)}f$ be

$$m(\xi_1,\xi_2) := \psi_m(\xi_1)\psi_k(\xi_2)\int_{-\infty}^{\infty}e^{-i\xi_1 t - i\xi_2\Gamma_l(t)}\phi(t)\,\frac{\mathrm{d}t}{|t|},$$

and the corresponding phase function be $\Phi(t) := -\xi_1 t - \xi_2\Gamma_l(t)$, so

$$\Phi'(t) = -\xi_1 - \xi_2\Gamma_l'(t) \quad \text{and} \quad \Phi''(t) = -\xi_2\Gamma_l''(t).$$

We now consider the following two cases:

⋆ If $|\xi_1| \geq 4C_1'e^{C_1'}|\xi_2|$, from (3.1) and (3.2), this implies $|\Gamma_l'(t)| \leq 2C_1'e^{C_1'}$ with a bound independent of l, which further leads to $|\Phi'(t)| \geq |\xi_1| - |\xi_2\Gamma_l'(t)| \geq \frac{|\xi_1|}{2}$. Noting that $\xi_1 \in \mathrm{supp}\,\psi_m$, we have $|\Phi'(t)| \gtrsim 2^m$. This, by van der Corput's lemma and the fact that $\Phi'(t)$ is monotonic, implies $|m(\xi_1,\xi_2)| \lesssim 2^{-m}$. Furthermore, by $\xi_2 \in \mathrm{supp}\,\psi_k$ and $|\xi_1| \geq 4C_1'e^{C_1'}|\xi_2|$, we have $|m(\xi_1,\xi_2)| \lesssim 2^{-\frac{1}{2}(m+k)}$. Applying Plancherel's theorem, we obtain (3.13) with $\delta = 1/2$.

⋆ If $|\xi_1| \leq 4C_1'e^{C_1'}|\xi_2|$, from Theorem 1.3(ii), (3.1) and (3.2), we have $|\Gamma_l''(t)| \geq \frac{C_2(C_2+1)}{4e^{C_1'}}|\xi_2|$ with a bound independent of l. This, combined with the fact that $\xi_2 \in \mathrm{supp}\,\psi_k$, leads to $|\Phi''(t)| \gtrsim 2^k$. We then apply van der Corput's lemma to obtain $|m(\xi_1,\xi_2)| \lesssim 2^{-k/2}$. As in the first case, by Plancherel's theorem again, we obtain (3.13) with $\delta = 1/4$.

Putting things together, we obtain (3.13) with $\delta = 1/4$, this finishes the proof of (3.8). □

Proof of (3.9). It suffices to show
(3.14)
$$\left\|\left(\sum_{M,K\in\mathbb{Z}}\big|M_{II,l_z}^{i_z,j_z}P_{m+M}^{(1)}P_{k+K}^{(2)}f\big|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^2)} \lesssim \left\|\left(\sum_{M,K\in\mathbb{Z}}\big|P_{m+M}^{(1)}P_{k+K}^{(2)}f\big|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1,2)$, where $2^M 2^{i_z}2^{l_z} = 1$ and $2^K 2^{j_z}\gamma(2^{l_z}) = 1$. We consider the more general estimate
(3.15)
$$\left\|\left(\sum_{M,K\in\mathbb{Z}}\big|M_{II,l_z}^{i_z,j_z}P_{m+M}^{(1)}P_{k+K}^{(2)}f\big|^{p_1}\right)^{1/p_1}\right\|_{L^{p_2}(\mathbb{R}^2)} \lesssim \left\|\left(\sum_{M,K\in\mathbb{Z}}\big|P_{m+M}^{(1)}P_{k+K}^{(2)}f\big|^{p_1}\right)^{1/p_1}\right\|_{L^{p_2}(\mathbb{R}^2)}$$

for certain $p_1 \in (1,\infty]$ and $p_2 \in (1,\infty)$.

For the case of $p_1 = \infty$ and $p_2 = 2$, by linearization, we have that the expression inside the $L^{p_2}(\mathbb{R}^2)$-norm on the left-hand side of (3.15) can be bounded by $M_{II,\gamma}\big(\sup_{M,K\in\mathbb{Z}} |P^{(1)}_{m+M}P^{(2)}_{k+K}f|\big)$. It is clear that (3.8) implies the $L^2(\mathbb{R}^2)$-boundedness of $M_{II,\gamma}$. Therefore, (3.15) holds for $p_1 = \infty$ and $p_2 = 2$.

For the case of $p_1 = p_2$ and $p_1, p_2 \in (1, \infty)$, by replacing f by $P^{(1)}_{m+M}P^{(2)}_{k+K}f$, we have that (3.15) is equivalent to

$$(3.16) \qquad \big\|M^{i_z,j_z}_{II,l_z}P^{(1)}_{m+M}P^{(2)}_{k+K}f\big\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$. We admit it first, which proof will be submit later. Then the interpolation argument implies that (3.14) holds for all $p \in (4/3, 2)$. Repeating the interpolation argument and using $p_1 = \infty$ and $p_2 \in (4/3, 2)$, we can prove that (3.14) holds for all $p \in (8/7, 2)$. Reiterating this process sufficiently many times we can show that (3.14) holds for all $p \in (1, 2)$.

Therefore, it suffices to show (3.16). Furthermore, note that $M^{i_z,j_z}_{II,l_z}P^{(1)}_{m+M}P^{(2)}_{k+K}f$ can be bounded by $M_{II,\gamma}M^{(1)}M^{(2)}f$, and the $L^p(\mathbb{R}^2)$-boundedness of $M_{II,\gamma}$, $M^{(1)}$ and $M^{(2)}$ for all $p \in [2, \infty)$, thus we may restrict $p \in (1, 2)$ in the proof of (3.16). This further reduces to showing

$$\left\|\left(\sum_{l\in\mathbb{Z}}\big|M^{i,j}_{II,l}P^{(1)}_{m+M}P^{(2)}_{k+K}f\big|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^2)} \lesssim \left\|\left(\sum_{l\in\mathbb{Z}}\big|P^{(1)}_{m+M}f\big|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, 2)$, where $2^M 2^i 2^l = 1$ and $2^K 2^j \gamma(2^l) = 1$. Repeating the method of bootstrapping an iterated interpolation argument above, it suffices to prove

$$\big\|M^{i,j}_{II,l}P^{(1)}_{m+M}P^{(2)}_{k+K}f\big\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all $p \in (1, \infty)$, which can be followed from Minkowski's inequality. We then obtain (3.9) and complete the proof of Theorem 1.3. $\qquad\qquad\square$

Acknowledgments

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